Butterworth Filters

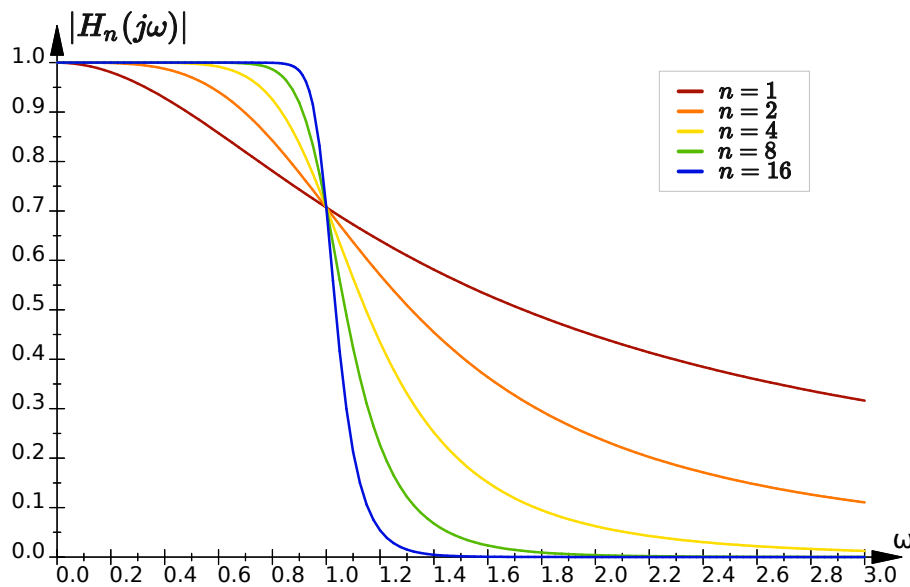*Pieter P*

This page will cover the derivation of the transfer functions of low-pass and high-pass <u>Butterworth filters</u>. Butterworth filters are designed to have a very flat frequency response in the passband.

Definition

Normalized Butterworth filters are defined in the frequency domain as follows:

$$|H_n(j\omega)| \triangleq \frac{1}{\sqrt{1 + \omega^{2n}}} \tag{1}$$

In order to determine the transfer function, we'll start from the frequency response squared. We'll assume that the transfer function $H_n(s)$ is a rational function with real coefficients. Therefore, $\overline{H_n(s)} = H_n\left(\overline{s}\right)$.

$$
\begin{aligned}
|H_n(j\omega)|^2 &= H_n(j\omega)\overline{H_n(j\omega)} \\
&= H_n(j\omega)H_n(\overline{j\omega}) \\
&= H_n(j\omega)H_n(-j\omega) \\
&= \frac{1}{1 + \omega^{2n}}
\end{aligned}
$$

We're looking for the transfer function $H_n(s)$, so we'll use the identity $s = j\omega \Leftrightarrow \omega = \frac{s}{j}$.

$$H_n(s)H_n(-s) = \frac{1}{1 + \left(\frac{s}{j}\right)^{2n}}$$

Poles of $H_n(s)H_n(-s)$

The poles of this transfer function are given by:

$$
\begin{aligned}
&\quad\ \left(\frac{s}{j}\right)^{2n} = -1 \\
\Leftrightarrow &\quad s^{2n} = -1(j)^{2n} \\
\Leftrightarrow &\quad s^{2n} = -1(-1)^n \\
\Leftrightarrow &\quad s^{2n} = (-1)^{n+1} \\
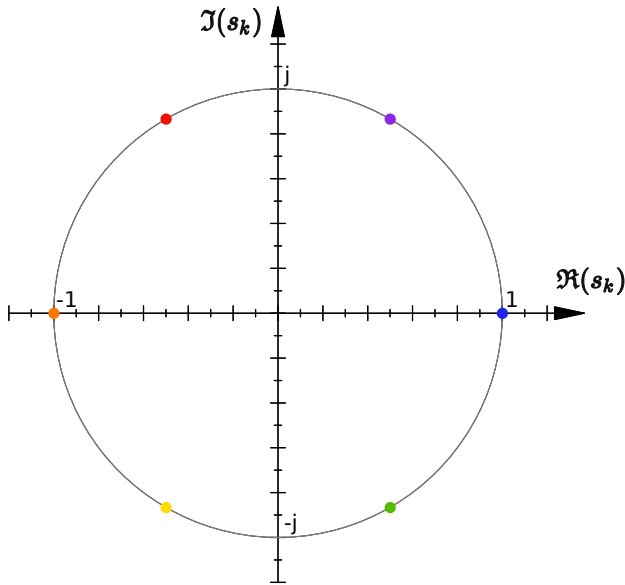\Leftrightarrow &\quad s^{2n} = e^{j\pi(n+1)}
\end{aligned}
$$

Keep in mind that this is a polynomial of order $2n$, so it has $2n$ complex roots.

$$s_k = e^{j2\pi\frac{2k+n+1}{4n}} \quad k \in \{0, 1, \ldots 2n-1\}$$

For example, for $n = 3$, the poles are:

$$
\begin{aligned}
s_0 &= e^{j2\pi\frac{0+3+1}{12}} &= e^{j2\pi\frac{2}{6}} \\
s_1 &= e^{j2\pi\frac{2+3+1}{12}} &= e^{j2\pi\frac{3}{6}} \\
s_2 &= e^{j2\pi\frac{4+3+1}{12}} &= e^{j2\pi\frac{4}{6}} \\
s_3 &= e^{j2\pi\frac{6+3+1}{12}} &= e^{j2\pi\frac{5}{6}} \\
s_4 &= e^{j2\pi\frac{8+3+1}{12}} &= e^{j2\pi\frac{6}{6}} \\
s_5 &= e^{j2\pi\frac{10+3+1}{12}} &= e^{j2\pi\frac{1}{6}}
\end{aligned}
$$

These are all points on the unit circle, $\pi/3 = 60°$ apart.

The poles are stable if they are in the left half plane, if their complex argument is between 90° and 270°:

$$
\begin{aligned}
& 2\pi\frac{2k+n+1}{4n} \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right) \\
\Leftrightarrow \quad & 2k+n+1 \in (n, 3n) \\
\Leftrightarrow \quad & k \in \left(-\frac{1}{2}, n-\frac{1}{2}\right) \\
\Rightarrow \quad & k \in \left(-\frac{1}{2}, n-\frac{1}{2}\right) \cup \{0, 1, \ldots 2n-1\} \\
\Leftrightarrow \quad & k \in \{0, 1, \ldots n-1\} \\
& s_{k,stable} = e^{j2\pi\frac{2k+n+1}{4n}} \quad k \in \{0, 1, \ldots n-1\}
\end{aligned}
\tag{2}
$$

Poles of $H_n(s)$

We want our filter $H_n(s)$ to be stable, so we pick the poles in the left half plane to be the poles of $H_n(s)$. The unstable poles, for $k \in \{n, n+1, \ldots 2n-1\}$ are the poles of $H_n(-s)$. They are the opposites of the poles of $H_n(s)$:

$$
\begin{aligned}
s_{k,unstable} &= e^{j2\pi\frac{2k+n+1}{4n}} & k \in \{n, n+1, \ldots 2n-1\} \\
l &\triangleq k - n \\
&= e^{j2\pi\frac{2(l+n)+n+1}{4n}} & l \in \{0, 1, \ldots n-1\} \\
&= e^{j\left(2\pi\frac{2l+n+1}{4n}+\pi\right)} \\
&= e^{j\pi} \cdot e^{j2\pi\frac{2l+n+1}{4n}} \\
&= -1 \cdot e^{j2\pi\frac{2l+n+1}{4n}} \\
&= -s_{l,stable}
\end{aligned}
$$

Butterworth Polynomials

We'll define the normalized Butterworth polynomial as follows:

$$B_n(s) \triangleq \prod_{k=0}^{n-1} \left(s - e^{j2\pi \frac{2k+n+1}{4n}} \right) \tag{3}$$

We'll rearrange the product to group each pole with its complex conjugate. Then, using the identity $e^{j\theta} + e^{-j\theta} = 2\cos\theta$, we can further simplify this expression:

Even order n:

$$
\begin{aligned}
B_n(s) &= \prod_{k=0}^{n-1} \left(s - e^{j2\pi \frac{2k+n+1}{4n}} \right) \\
&= \prod_{k=0}^{\frac{n}{2}-1} \left(s - e^{j2\pi \frac{2k+n+1}{4n}} \right) \prod_{l=\frac{n}{2}}^{n-1} \left(s - e^{j2\pi \frac{2l+n+1}{4n}} \right) \\
&\quad l = n - k - 1 \\
&= \prod_{k=0}^{\frac{n}{2}-1} \left(s - e^{j2\pi \frac{2k+n+1}{4n}} \right) \left(s - e^{j2\pi \frac{2(n-k-1)+n+1}{4n}} \right) \\
&= \prod_{k=0}^{\frac{n}{2}-1} \left(s - e^{j2\pi \frac{2k+n+1}{4n}} \right) \left(s - \frac{e^{j2\pi \frac{-2k+3n-1}{4n}}}{1} \right) \\
&= \prod_{k=0}^{\frac{n}{2}-1} \left(s - e^{j2\pi \frac{2k+n+1}{4n}} \right) \left(s - \frac{e^{j2\pi \frac{-2k+3n-1}{4n}}}{e^{j2\pi}} \right) \\
&= \prod_{k=0}^{\frac{n}{2}-1} \left(s - e^{j2\pi \frac{2k+n+1}{4n}} \right) \left(s - e^{j2\pi \left(\frac{-2k+3n-1}{4n} - 1 \right)} \right) \\
&= \prod_{k=0}^{\frac{n}{2}-1} \left(s - e^{j2\pi \frac{2k+n+1}{4n}} \right) \left(s - e^{j2\pi \frac{-2k+3n-4n-1}{4n}} \right) \\
&= \prod_{k=0}^{\frac{n}{2}-1} \left(s - e^{j2\pi \frac{2k+n+1}{4n}} \right) \left(s - e^{j2\pi \frac{-2k-n-1}{4n}} \right) \\
&= \prod_{k=0}^{\frac{n}{2}-1} \left(s - e^{j2\pi \frac{2k+n+1}{4n}} \right) \left(s - e^{-j2\pi \frac{2k+n+1}{4n}} \right) \\
&= \prod_{k=0}^{\frac{n}{2}-1} \left(s^2 - s e^{j2\pi \frac{2k+n+1}{4n}} - s e^{-j2\pi \frac{2k+n+1}{4n}} + 1 \right) \\
&= \prod_{k=0}^{\frac{n}{2}-1} \left(s^2 - 2\cos\left(2\pi \frac{2k+n+1}{4n} \right) s + 1 \right)
\end{aligned}
$$

Odd order n:

In this case, $n-1$ is even, and you get a special pole for $k = \frac{n-1}{2}$:

$$
\begin{aligned}
s_{\frac{n-1}{2}} &= e^{j2\pi \frac{2\frac{n-1}{2}+n+1}{4n}} \\
&= e^{j2\pi \frac{2n}{4n}} \\
&= e^{j\pi} \\
&= -1
\end{aligned}
$$

After isolating this pole, we're left with an even number of complex conjugate poles, just like in the case where n was even.

In conclusion, the normalized Butterworth polynomial of degree n is given by:

$$
B_n(s) =
\begin{cases}
\prod_{k=0}^{\frac{n}{2}-1} \left(s^2 - 2\cos\left(2\pi \frac{2k+n+1}{4n} \right) s + 1 \right) & \text{even } n \\
(s+1) \prod_{k=0}^{\frac{n-1}{2}-1} \left(s^2 - 2\cos\left(2\pi \frac{2k+n+1}{4n} \right) s + 1 \right) & \text{odd } n
\end{cases}
\tag{4}
$$

Butterworth Transfer Function $H_n(s)$

The transfer function $H_n(s)$ has no zeros, so the numerator is a constant. The poles of $H_n(s)$ are given by Equation (2), so the denominator is given by Equation (3).

$$H_n(s) = \frac{c}{B_n(s)}$$

We wanted a DC gain of $1 \ (= 0dB)$ for $\omega = 0$:

$$
\begin{aligned}
|H_n(0j)| &= 1 \\
\Leftrightarrow \left| \frac{c}{B_n(0)} \right| &= 1 \\
\Leftrightarrow \left| \frac{c}{\prod_{k=0}^{n-1} \left(0 - e^{j2\pi \frac{2k+n+1}{4n}} \right)} \right| &= 1 \\
\Leftrightarrow \frac{|c|}{\prod_{k=0}^{n-1} \left| -e^{j2\pi \frac{2k+n+1}{4n}} \right|} &= 1 \\
\Leftrightarrow \frac{|c|}{1} &= 1
\end{aligned}
$$

If we want no phase offset for low frequencies, we can postulate that $\angle H_n(0j) = 0$:

$$
\begin{aligned}
\angle H_n(0j) &= 0 \\
\Leftrightarrow \angle \left(\frac{c}{B_n(0)} \right) &= 0 \\
\Leftrightarrow \angle c - \angle \left(\prod_{k=0}^{\frac{n}{2}-1} \left(0^2 - 2\cos\left(2\pi \frac{2k+n+1}{4n} \right) \cdot 0 + 1 \right) \right) &= 0 \\
\Leftrightarrow \angle c - \angle 1 &= 0
\end{aligned}
$$
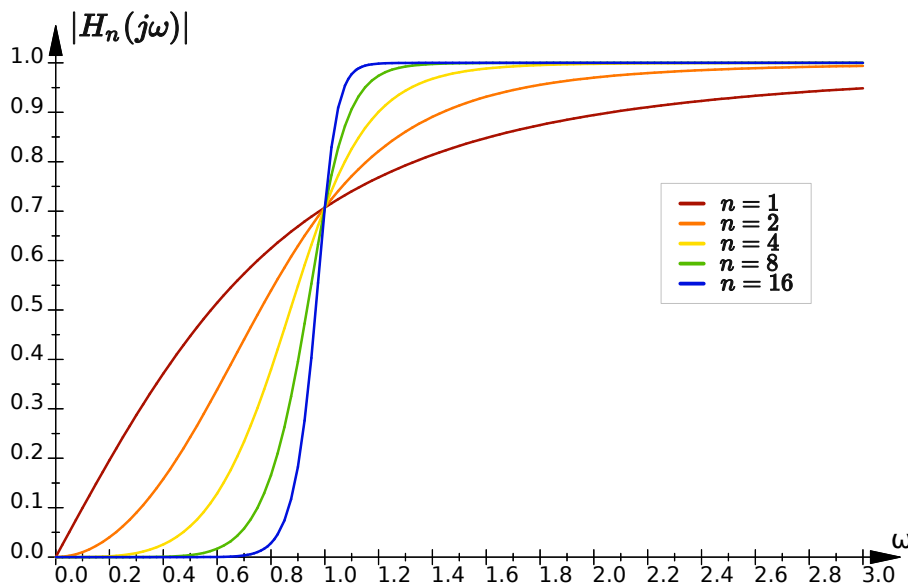
The derivation is analogous for odd n.
Therefore, $c = 1$, and we've eliminated all unknown parameters from the transfer function:

$$
H_n(s) = \frac{1}{B_n(s)} \tag{5}
$$

High-Pass Butterworth filters

Up until now, we only looked at the low-pass Butterworth filter. There's also a high-pass version:

$$
|H_{n,hp}(j\omega)| \triangleq \frac{1}{\sqrt{1 + \omega^{-2n}}} \tag{6}
$$

We can just multiply the numerator and the denominator by ω^n to get a more familiar form:

$$
|H_{n,hp}(j\omega)| = \frac{\omega^n}{\sqrt{1 + \omega^{2n}}}
$$

As you can see, the poles will be the same as for the low-pass version. On top of that, there now are n zeros for $s = 0$.
So the transfer function becomes:

$$H_{n,hp}(s) = \frac{s^n}{B_n(s)} \tag{7}$$

Non-normalized Butterworth Filters

Up until now, we only looked at normalized Butterworth filters, that have a corner frequency of $1 \ rad/s$. To get a specific corner frequency ω_c, we can just scale ω, so the definitions become:

$$|H_{n,lp}(j\omega)| \triangleq \frac{1}{\sqrt{1 + \left(\frac{\omega}{\omega_c}\right)^{2n}}} \tag{8}$$

$$|H_{n,hp}(j\omega)| \triangleq \frac{1}{\sqrt{1 + \left(\frac{\omega_c}{\omega}\right)^{2n}}} \tag{9}$$

If you start recalculating the transfer functions, you'll quickly realize that this just scales everything by a factor of ω_c. The poles no longer lie on the unit circle, but on a circle with radius $|s_k| = \omega_c$.

This results in the following transfer functions:

$$H_{n,lp}(s) = \frac{1}{B_n\left(\frac{s}{\omega_c}\right)} \tag{10}$$

$$H_{n,hp}(s) = \frac{s^n}{\omega_c^n \ B_n\left(\frac{s}{\omega_c}\right)} \tag{11}$$

The gain at the corner frequency can easily be determined from the definitions:

$$
\begin{aligned}
|H_{n,lp}(j\omega_c)| \ = \ |H_{n,hp}(j\omega_c)| \ &= \ \frac{1}{\sqrt{1 + \left(\frac{\omega_c}{\omega_c}\right)^{2n}}} \\
&= \frac{1}{\sqrt{2}} \\
&= \frac{\sqrt{2}}{2} \\
&\approx 0.707 \\
20 \log_{10}|H_n(j\omega_c)| \ &= 20 \log_{10}\left(\frac{\sqrt{2}}{2}\right) \\
&= 10 \log_{10}\left(\frac{1}{2}\right) \\
&\approx -3.01 \ dB
\end{aligned}
$$

This is often called the $-3 \ dB$-point or the half-power point, because a sinusoidal input signal at that frequency will result in an output signal that has only half of the power of the input signal: $|H_n(j\omega_c)|^2 = \frac{1}{2}$.