LU Factorization

Pg. 43 familiar version of Gaussian elimination
Square matrix, $\mathbb{A}$,

$$\mathbb{A} = \begin{bmatrix} 6 & -2 & 2 & 4 \\ 12 & -8 & 6 & 10 \\ 3 & -13 & 9 & 3 \\ -6 & 4 & 1 & -18 \end{bmatrix}$$

Row 2 $\leftarrow$ Row 2 $-$ (12/6) $*$ Row 1

$\mathbb{M}_{21}\,\mathbb{A} = \mathbb{M}_{21} * \mathbb{A}$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} 6 & -2 & 2 & 4 \\ 12 & -8 & 6 & 10 \\ 3 & -13 & 9 & 3 \\ -6 & 4 & 1 & -18 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & -2 & 2 & 4 \\ 0 & -4 & 2 & 2 \\ 3 & -13 & 9 & 3 \\ -6 & 4 & 1 & -18 \end{bmatrix}$$

Each $\mathbb{M}$ an *elementary matrix*.

Row 3 $\leftarrow$ Row 3 $-$ (3/6) $*$ Row 1

$\mathbb{M}_{31}\ \mathbb{M}_{21}\ \mathbb{A} = \mathbb{M}_{31} * \mathbb{M}_{21}\ \mathbb{A}$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1/2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} 6 & -2 & 2 & 4 \\ 0 & -4 & 2 & 2 \\ 3 & -13 & 9 & 3 \\ -6 & 4 & 1 & -18 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & -2 & 2 & 4 \\ 0 & -4 & 2 & 2 \\ 0 & -12 & 8 & 1 \\ -6 & 4 & 1 & -18 \end{bmatrix}$$

Row 4 $\leftarrow$ Row 4 $-$ ($-$6/6) $*$ Row 1

$\mathbb{M}_{41}\ \mathbb{M}_{31}\ \mathbb{M}_{21}\ \mathbb{A} = \mathbb{M}_{41} * \mathbb{M}_{31}\ \mathbb{M}_{21}\ \mathbb{A}$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} 6 & -2 & 2 & 4 \\ 0 & -4 & 2 & 2 \\ 0 & -12 & 8 & 1 \\ -6 & 4 & 1 & -18 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & -2 & 2 & 4 \\ 0 & -4 & 2 & 2 \\ 0 & -12 & 8 & 1 \\ 0 & 2 & 3 & -14 \end{bmatrix}$$

$$\text{Row } 3 \leftarrow \text{Row } 3 - (-12/-4) * \text{ Row } 2$$

$$\mathbb{M}_{32}\,\mathbb{M}_{41}\,\mathbb{M}_{31}\,\mathbb{M}_{21}\,\mathbb{A} = \mathbb{M}_{32} * \mathbb{M}_{41}\,\mathbb{M}_{31}\,\mathbb{M}_{21}\,\mathbb{A}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -3 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} 6 & -2 & 2 & 4 \\ 0 & -4 & 2 & 2 \\ 0 & -12 & 8 & 1 \\ 0 & 2 & 3 & -14 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & -2 & 2 & 4 \\ 0 & -4 & 2 & 2 \\ 0 & 0 & 2 & -5 \\ 0 & 2 & 3 & -14 \end{bmatrix}$$

$$\text{Row } 4 \leftarrow \text{Row } 4 - (2/-4) * \text{ Row } 2$$

$$\mathbb{M}_{42}\,\mathbb{M}_{32}\,\mathbb{M}_{41}\,\mathbb{M}_{31}\,\mathbb{M}_{21}\,\mathbb{A} = \mathbb{M}_{42} * \mathbb{M}_{32}\,\mathbb{M}_{41}\,\mathbb{M}_{31}\,\mathbb{M}_{21}\,\mathbb{A}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1/2 & 0 & 1 \end{bmatrix} * \begin{bmatrix} 6 & -2 & 2 & 4 \\ 0 & -4 & 2 & 2 \\ 0 & 0 & 2 & -5 \\ 0 & 2 & 3 & -14 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & -2 & 2 & 4 \\ 0 & -4 & 2 & 2 \\ 0 & 0 & 2 & -5 \\ 0 & 0 & 4 & -13 \end{bmatrix}$$

Row 4 $\leftarrow$ Row 4 $-$ (4/2) $*$ Row 3

$$\mathbb{M}_{43}\,\mathbb{M}_{42}\,\mathbb{M}_{32}\,\mathbb{M}_{41}\,\mathbb{M}_{31}\,\mathbb{M}_{21}\,\mathbb{A}$$

$$= \mathbb{M}_{43} * \mathbb{M}_{42}\,\mathbb{M}_{32}\,\mathbb{M}_{41}\,\mathbb{M}_{31}\,\mathbb{M}_{21}\,\mathbb{A}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -2 & 1 \end{bmatrix} * \begin{bmatrix} 6 & -2 & 2 & 4 \\ 0 & -4 & 2 & 2 \\ 0 & 0 & 2 & -5 \\ 0 & 0 & 4 & -13 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & -2 & 2 & 4 \\ 0 & -4 & 2 & 2 \\ 0 & 0 & 2 & -5 \\ 0 & 0 & 0 & -3 \end{bmatrix}$$

$$= \mathbb{U}$$

So

$$\mathbb{A} = \underbrace{\mathbb{M}_{21}^{-1}\,\mathbb{M}_{31}^{-1}\,\mathbb{M}_{32}^{-1}\,\mathbb{M}_{41}^{-1}\,\mathbb{M}_{42}^{-1}\,\mathbb{M}_{43}^{-1}}_{\mathbb{L}}\,\mathbb{U}$$

Example: What inverses of these elementary matrices look like:

$$\mathbb{M}_{21}^{-1} * \mathbb{M}_{21} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Example: Multiply elementary matrices:

$$\mathbb{M}_{42}^{-1} * \mathbb{M}_{43}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & \frac{-1}{2} & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & \frac{-1}{2} & 2 & 1 \end{bmatrix}$$

$$\mathbb{M}_{21}^{-1}\,\mathbb{M}_{31}^{-1}\,\mathbb{M}_{32}^{-1}\,\mathbb{M}_{41}^{-1}\,\mathbb{M}_{42}^{-1}\,\mathbb{M}_{43}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ \frac{1}{2} & 3 & 1 & 0 \\ -1 & \frac{-1}{2} & 2 & 1 \end{bmatrix}$$

$$= \mathbb{L}$$

$\mathbb{L}$ is the collection of multipliers used in Gaussian Elimination; in the spots where they were employed to eliminate entries of $\mathbb{A}$.

$$\mathbb{A} = \begin{bmatrix} 6 & -2 & 2 & 4 \\ 12 & -8 & 6 & 10 \\ 3 & -13 & 9 & 3 \\ -6 & 4 & 1 & -18 \end{bmatrix}$$

$$= \mathbb{L}\,\mathbb{U}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ \frac{1}{2} & 3 & 1 & 0 \\ -1 & \frac{-1}{2} & 2 & 1 \end{bmatrix} \begin{bmatrix} 6 & -2 & 2 & 4 \\ 0 & -4 & 2 & 2 \\ 0 & 0 & 2 & -5 \\ 0 & 0 & 0 & -3 \end{bmatrix}$$

With pivoting, Matlab returns a permutation matrix, $\mathbb{P}$, as well as $\mathbb{L}$ and $\mathbb{U}$.

```
>> A = [ 6 -2 2 4; 12 -8 6 10; 3 -13 9 3; -6 4 1 -18]
A =
     6    -2     2     4
    12    -8     6    10
     3   -13     9     3
    -6     4     1   -18
>> [L,U,P]=lu(A);
>> L
L =
    1.0000         0         0         0
    0.2500    1.0000         0         0
   -0.5000         0    1.0000         0
    0.5000   -0.1818    0.0909    1.0000
>> U
U =
   12.0000   -8.0000    6.0000   10.0000
         0  -11.0000    7.5000    0.5000
         0         0    4.0000  -13.0000
         0         0         0    0.2727

>> P
```

```
P =
     0     1     0     0
     0     0     1     0
     0     0     0     1
     1     0     0     0
>> L*U
ans =
    12    -8     6    10
     3   -13     9     3
    -6     4     1   -18
     6    -2     2     4
>> P*A
ans =
    12    -8     6    10
     3   -13     9     3
    -6     4     1   -18
     6    -2     2     4
```

```
GEPP ‘‘by hand’’ ...
A =
     6    -2     2     4
    12    -8     6    10
     3   -13     9     3
    -6     4     1   -18
permute row 1 and row 2: [rows now 2 1 3 4]
A =
    12    -8     6    10
     6    -2     2     4
     3   -13     9     3
    -6     4     1   -18
eliminate on column 1
A =
   12.0000   -8.0000    6.0000   10.0000
         0    2.0000   -1.0000   -1.0000
         0  -11.0000    7.5000    0.5000
         0         0    4.0000  -13.0000
permute row 2 and row 3 [rows now 2 3 1 4]
A =
   12.0000   -8.0000    6.0000   10.0000
         0  -11.0000    7.5000    0.5000
         0    2.0000   -1.0000   -1.0000
         0         0    4.0000  -13.0000
```

```
eliminate on column 2
A =
   12.0000   -8.0000    6.0000   10.0000
         0  -11.0000    7.5000    0.5000
         0         0    0.3636   -0.9091
         0         0    4.0000  -13.0000
permute row 3 and row 4 [rows now 2 3 4 1]
A =
   12.0000   -8.0000    6.0000   10.0000
         0  -11.0000    7.5000    0.5000
         0         0    4.0000  -13.0000
         0         0    0.3636   -0.9091
eliminate on column 3
A =
   12.0000   -8.0000    6.0000   10.0000
         0  -11.0000    7.5000    0.5000
         0         0    4.0000  -13.0000
         0         0         0    0.2727
P =
     0     1     0     0
     0     0     1     0
     0     0     0     1
     1     0     0     0
```