- **Equazione di Schrödinger**

$$i\hbar\frac{d}{dt}|\psi\rangle = H|\psi\rangle$$

- **Operatore P**

$$P = -i\hbar\frac{d}{dx} \qquad [X,P] = i\hbar\mathbb{1}$$

$$D_P = \{\psi \in L_2(a,b)\ ass.\ cont. \wedge \psi' \in L_2(a,b)\}$$

$$[Q,P] = i\hbar \ \Rightarrow\ [Q,g(P)] = i\hbar g'(P)$$

Fun fact:

$$\frac{P^2}{2m} = \frac{P_r^2}{2m} + \frac{L^2}{2mr^2} = -\frac{\hbar^2}{2m}\frac{1}{r}\frac{\partial^2}{\partial r^2}r + \frac{L^2}{2mr^2}$$

- **Relazioni di indeterminazione**

$$(\Delta A)_{a,\psi}(\Delta B)_{b,\psi} \geqslant \frac{1}{2}\left|\langle[A,B]\rangle_\psi\right|$$

$$(\Delta X_i)_\psi(\Delta P_i)_\psi \geqslant \frac{\hbar}{2}$$

$$(\Delta t)_\psi(\Delta H)_\psi \geqslant \frac{\hbar}{2}$$

$$(\Delta t)_\psi := \inf_A \frac{(\Delta A)_\psi}{\left|\frac{d}{dt}\langle A\rangle_\psi\right|}$$

- **Base generalizzata del momento**

$$\langle\vec{x}\,|\,\vec{p}\rangle = \frac{e^{\frac{i}{\hbar}\vec{p}\cdot\vec{x}}}{(2\pi\hbar)^{\frac{3}{2}}}$$

- **Equazione di continuità**

$$\vec{j} := \frac{i\hbar}{2m}\left(\psi^*\vec{\nabla}\psi - \psi\vec{\nabla}\psi^*\right)$$

$$\frac{\partial}{\partial t}\rho(\vec{x},t) + \vec{\nabla}\cdot\vec{j}(\vec{x},t) = 0$$

- **Operatore di evoluzione temporale**

$$U(t) = e^{-\frac{i}{\hbar}Ht}$$

- **Teo. di Wigner (trasformazione di simmetria)**

$$\begin{cases}\psi' = U\psi \\ A' = UAU^\dagger\end{cases}$$

- **Generatore infinitesimo della simmetria**

$$Q := i\hbar\frac{d}{ds}U(s)\Big|_{s=0}$$

$$U(s) = e^{-\frac{i}{\hbar}sQ}$$

- **Generatori delle rotazioni**

$$U(\varphi) = e^{i\varphi\frac{L_3}{\hbar}}$$

$$R = \begin{pmatrix}\cos\varphi & -\sin\varphi & 0\\ \sin\varphi & \cos\varphi & 0\\ 0 & 0 & 1\end{pmatrix}$$

$$[J_i,J_j] = i\hbar\varepsilon_{ijk}J_k$$

$$J_\pm := J_x \pm iJ_y \qquad (J_\pm)^{-1} = J_\mp$$

$$[J_z,J_\pm] = \pm\hbar J_\pm \qquad [J^2,J_\pm] = 0$$

$$[J_+,J_-] = 2\hbar J_z$$

$$\frac{J_\pm|j,m\rangle}{\hbar\sqrt{j(j+1)-m(m\pm1)}} = |j,m\pm1\rangle$$

$$\begin{cases}[J_i,X_j] = [L_i,X_j] = i\hbar\varepsilon_{ijk}X_k\\ [J_i,P_j] = [L_i,P_j] = i\hbar\varepsilon_{ijk}P_k\end{cases}$$

- **Particella senza spin 3D in polari**

$$L^2\Psi_{l,m}(\vec{x}) = \hbar^2 l(l+1)\Psi_{l,m}(\vec{x})$$

$$L_z\Psi_{l,m}(\vec{x}) = \hbar m\Psi_{l,m}(\vec{x})$$

$$L_z = -i\hbar\frac{\partial}{\partial\phi}$$

$$Y_l^m(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi}}\sqrt{\frac{(l+|m|)!}{(l-|m|)!}}\frac{1}{2^l l!}P_{l,m}(\theta)e^{im\phi}$$

Con $P_{l,m}$ polinomi di Legendre di grado l; parità dell'armonica sferica uguale a quella di l:

$$Y_0^0 = \frac{1}{2}\sqrt{\frac{1}{\pi}}$$

$$Y_1^{-1} = \frac{1}{2}\sqrt{\frac{3}{2\pi}}e^{-i\phi}\sin\theta$$

$$Y_1^0 = \frac{1}{2}\sqrt{\frac{3}{\pi}}\cos\theta$$

$$Y_1^{+1} = -\frac{1}{2}\sqrt{\frac{3}{2\pi}}e^{i\phi}\sin\theta$$

- **Spin**

$$\vec{S} := \vec{J} - \vec{L}$$

Ha stessa algebra di $\vec{J}$ e $\vec{L}$. Commuta con $\vec{X}$ e $\vec{P}$.

$$S_x = \frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \qquad S_y = \frac{\hbar}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}$$

$$S_z = \frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

$$\vec{S}\cdot\hat{n} = \frac{\hbar}{2}\begin{pmatrix}\cos\theta & e^{-i\phi}\sin\theta\\ e^{i\phi}\sin\theta & -\cos\theta\end{pmatrix}$$

$$|+\rangle_{\hat{n}} = \begin{pmatrix}\cos\frac{\theta}{2}\\ e^{i\phi}\sin\frac{\theta}{2}\end{pmatrix} \qquad |-\rangle_{\hat{n}} = \begin{pmatrix}-\sin\frac{\theta}{2}\\ e^{i\phi}\cos\frac{\theta}{2}\end{pmatrix}$$

$$\sigma_i\sigma_j = i\varepsilon_{ijk}\sigma_k$$

$$[\sigma_i,\sigma_j] = 2i\varepsilon_{ijk}\sigma_k$$

Con $s=1$:

$$L_x = \frac{\hbar}{\sqrt{2}}\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 1\\ 0 & 1 & 0\end{pmatrix} \qquad L_y = \frac{\hbar}{\sqrt{2}}\begin{pmatrix}0 & -i & 0\\ i & 0 & -i\\ 0 & i & 0\end{pmatrix}$$

$$L_z = \hbar\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -1\end{pmatrix}$$

- **Simmetrie discrete**
  - Parità $\mathbb{P}$:

$$\begin{cases}\mathbb{P}\vec{X}\mathbb{P}^\dagger = -\vec{X}\\ \mathbb{P}\vec{P}\mathbb{P}^\dagger = -\vec{P}\\ \mathbb{P}\vec{L}\mathbb{P}^\dagger = \vec{L}\\ \psi'(\vec{x}) = \psi(-\vec{x})\end{cases}$$

$$\mathbb{P} = \mathbb{P}^{-1} = \mathbb{P}^\dagger$$

  Unici autovalori possibili: ±1

  - Inversione temporale $\mathbb{T}$:

$$\begin{cases}\mathbb{T}\vec{X}\mathbb{T}^\dagger = \vec{X}\\ \mathbb{T}\vec{P}\mathbb{T}^\dagger = -\vec{P}\\ \mathbb{T}\vec{L}\mathbb{T}^\dagger = -\vec{L}\\ \psi'(\vec{x},t) = \psi^*(\vec{x},-t)\end{cases}$$

- **Visuale di Heisenberg**

$$\begin{cases}|\psi_h(t)\rangle = U^\dagger|\psi_s(t)\rangle\\ A_h(t) = U^\dagger A_s(t)U\end{cases}$$

Nota: U è l'operatore di evoluzione temporale.

$$\begin{cases}\dfrac{d}{dt}|\psi_h(t)\rangle = 0\\ \dfrac{dA_h(t)}{dt} = -\dfrac{i}{\hbar}[A_h,H_h] + \dfrac{\partial A_h}{\partial t}\end{cases}$$

- **Teo. di Ehrenfest**

$$\begin{cases}\dfrac{d}{dt}\langle Q\rangle_\psi = \dfrac{1}{m}\langle P\rangle_\psi\\ \dfrac{d}{dt}\langle P\rangle_\psi = -\langle V'(Q)\rangle_\psi\end{cases}$$

- **Matrici densità**

$$\rho_\psi := |\psi\rangle\langle\psi| \qquad \langle A\rangle_\psi = tr(\rho A)$$

Definizione di operatore densità: ρ tale che:

$$\begin{cases}\rho \quad a.a.\\ \rho \geqslant 0 \qquad\qquad tr(\rho^2) \leqslant 1\\ tr(\rho) = 1\end{cases}$$

- **Oscillatore armonico 1D**

$$H = \frac{P^2}{2m} + \frac{1}{2}m\omega^2X^2$$

$$\hat{X} := \sqrt{\frac{m\omega}{\hbar}}X \qquad \hat{P} := \frac{P}{\sqrt{m\omega\hbar}}$$

$$\left[\hat{X},\hat{P}\right] = i$$

$$a := \frac{\hat{X}+i\hat{P}}{\sqrt{2}} \qquad a^\dagger = \frac{\hat{X}-i\hat{P}}{\sqrt{2}}$$

$$H = \hbar\omega\left(N+\frac{1}{2}\right)$$

$$N := a^\dagger a \qquad aa^\dagger - a^\dagger a = 1$$

$$aa^\dagger = N+1$$

$$u_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}}e^{-\frac{m\omega}{\hbar}\frac{x^2}{2}}$$

$$|\nu\rangle = \frac{\left(a^\dagger\right)^\nu|0\rangle}{\sqrt{\nu!}}$$

$$\begin{cases}a|\nu\rangle = \sqrt{\nu}\,|\nu-1\rangle\\ a^\dagger|\nu\rangle = \sqrt{\nu+1}\,|\nu+1\rangle\end{cases}$$

- **Atomo H**

$$H_r = \frac{\vec{P_\mu}^{\,2}}{2\mu} - \frac{e^2}{r}$$

$$E = -\frac{E_I}{n^2}$$

$$\Psi_{n,l,m} = \frac{N_{n,l}}{a_0^{\frac{3}{2}}}e^{-\frac{r}{a_0 n}}\left(\frac{2r}{na_0}\right)^l L\left(\frac{2r}{na_0}\right)Y_l^m(\theta,\phi)$$

$$e^2 := 2,3\cdot10^{-28}\,\mathrm{J\cdot m}$$

$$a_0 := \frac{\hbar^2}{e^2\mu} \simeq 0,5\cdot10^{-10}\,\mathrm{m}$$

$$E_I := \frac{e^2}{2a_0} = \frac{\hbar^2}{2\mu a_0} = 13,6\,\mathrm{eV} = 1\,\mathrm{Ry}$$

- **Perturbazioni stazionarie**

Problema:

$$H = H_0 + W = H_0 + \lambda\hat{W}$$

$$\begin{cases}E_a^\lambda = \mathcal{E}_0 + \lambda\mathcal{E}_1 + \lambda^2\mathcal{E}_2 + O(\lambda^3)\\ |\lambda\rangle = |0\rangle + \lambda|1\rangle + \lambda^2|2\rangle + O(\lambda^3)\end{cases}$$

Convenzione:

$$\begin{cases}|0\rangle = \left|E_a^0\right\rangle\\ \langle\lambda\,|\,\lambda\rangle = 1 \quad \forall\lambda\in\mathbb{R}\\ \langle0\,|\,\lambda\rangle\in\mathbb{R} \quad \forall\lambda\in\mathbb{R}\end{cases}$$

Soluzioni:

$$\mathcal{E}_1 = \left\langle E_a^0\middle|\hat{W}\middle|E_a^0\right\rangle$$

$$|1\rangle = \sum_{b\neq a}\left|E_b^0\right\rangle\frac{\left\langle E_b^0\middle|\hat{W}\middle|E_a^0\right\rangle}{E_a^0-E_b^0}$$

$$\mathcal{E}_2 = \sum_{b\neq a}\frac{\left|\left\langle E_a^0\middle|\hat{W}\middle|E_b^0\right\rangle\right|^2}{E_a^0-E_b^0}$$

- **Perturbazioni: caso degenere**

Sia $\{|\phi_t\rangle\}_{t=1,\dots,d}$ BON di $\mathcal{H}_0$

Ordine λ^q:

$$(H_0-\mathcal{E}_0)|q,r\rangle + \left(\hat{W}-\mathcal{E}_1^r\right)|q-1,r\rangle$$
$$-\mathcal{E}_2^r|q-2,r\rangle - \cdots - \mathcal{E}_q^r|0,r\rangle = 0$$

Ordine λ:

$$\sum_{t=1}^d\left\langle\phi_s\middle|\hat{W}\middle|\phi_t\right\rangle\langle\phi_t\,|\,0,r\rangle = \mathcal{E}_1^{(r)}\langle\phi_s\,|\,0,r\rangle$$

che è equazione ad autovalori/autovettori da risolvere $\forall\phi_s$

- **EM**

$$H = \frac{1}{2m}\left(\vec{P} - \frac{q}{c}\vec{A}(\vec{X},t)\right)^2 + q\varphi(\vec{X},t)$$

Trasformazione di Gauge:

$$\begin{cases} \vec{A}' = \vec{A} + \vec{\nabla}\Lambda(\vec{x},t) \\ \varphi' = \varphi - \frac{1}{c}\frac{\partial}{\partial t}\Lambda(\vec{x},t) \\ \psi'(\vec{x},t) = e^{i\frac{q}{\hbar c}\Lambda(\vec{x},t)}\psi(\vec{x},t) =: U_\Lambda \psi(\vec{x},t) \end{cases}$$

Trasformazione di Gauge è trasformazione di simmetria: solo osservabili covarianti sono fisici: bisogna imporre:

$$f_{\vec{A}',\varphi'} = U_\Lambda f_{\vec{A},\varphi} U_\Lambda^\dagger$$

Gauge di Landau (con $\vec{E} = 0, \quad \vec{B} = -B_0\hat{u}_z$):

$$\begin{cases} \varphi \equiv 0 \\ \vec{A} = (B_0 y,\, 0,\, 0) \\ H = \frac{1}{2m}\left(P_x - \frac{q}{c}B_0 Y\right)^2 + \frac{P_y^2}{2m} + \frac{P_z^2}{2m} \end{cases}$$

Asse z è indipendente ed è onda 1D libera.

Autostati di onda xy (u_n sono le autofunzioni di oscillatore armonico):

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right) = \frac{\hbar q B_0}{mc}\left(n + \frac{1}{2}\right)$$

$$\Psi_{n,p_x}(x,y) = e^{i\frac{p_x}{\hbar}x}u_n(y - y_0)$$

$$\omega := \frac{qB_0}{mc} \qquad y_0 := \frac{cp_x}{qB_0}$$

Gauge simmetrica:

$$\vec{A} = \left(\frac{B_0 y}{2},\, -\frac{B_0 x}{2},\, 0\right)$$

- **Effetto Aharonov-Bohm**

$$\frac{1}{2m}\left(P_s - \frac{q}{c}\frac{\Phi}{2\pi R}\right)^2$$

Doppia fenditura con solenoide acceso:

$$|\psi(x)|^2 \propto \left|\psi_1(x) + e^{\frac{iq}{\hbar c}\int_{\gamma_2 - \gamma_1}\vec{A}\cdot d\vec{l}}\psi_2(x)\right|^2$$
$$= \left|\psi_1(x) + e^{\frac{iq}{\hbar c}\Phi}\psi_2(x)\right|^2$$

- **Scattering**

Sezione d'urto differenziale generico:

$$f_K(\theta,\phi) = -\frac{1}{2\pi}\frac{\mu}{\hbar^2}\int d^3r'\, e^{-i\vec{x}'\cdot\vec{K}}V(\vec{x})$$

$$\frac{d\sigma}{d\Omega}(\theta,\phi) = \frac{\mu^2}{4\pi^2\hbar^4}\left|\int d^3x'\, e^{-i\vec{K}\cdot\vec{x}'}V(\vec{x}')\right|^2$$

Con $\vec{K} := k\left(\hat{u}_{\theta,\phi} - \hat{u}_z\right)$

Potenziale centrale: $K := \left|\vec{K}\right| = 2\sin\frac{\theta}{2}k$

$$f_K^B(\theta,\phi) = -\frac{2\mu}{\hbar^2 K}\int_0^\infty dr'\, r'\sin(Kr')V(r')$$

Con potenziale centrale di Yukawa:

$$V(r) = \beta\frac{e^{-\alpha r}}{r}$$

$$f_K^B(\theta,\phi) = -\frac{2\mu\beta}{\hbar^2\left(K^2 + \alpha^2\right)}$$

$$\frac{d\sigma_B}{d\Omega}(\theta) = \frac{4\mu^2\beta^2}{\hbar^4}\frac{1}{\left(\alpha^2 + 4k^2\sin^2\frac{\theta}{2}\right)^2}$$

$$\frac{d\sigma_B}{d\Omega} \xrightarrow{\alpha\to 0} \frac{\mu^2\beta^2}{4\hbar^4 k^4\sin^4\frac{\theta}{2}}$$

- **Buca di potenziale**

Con potenziale $V(x) = +\infty\,\mathbb{1}_{[0,a]^c}$:

$$E_n = \frac{\pi^2\hbar^2}{2ma^2}n^2$$

$$\psi_n(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi}{a}x\right)$$

Con potenziale $V(x) = +\infty\,\mathbb{1}_{\{x>|a|/2\}}$

$$E_n = \frac{\hbar^2 n^2\pi^2}{2ma^2}$$

$$\psi_n(x) = \begin{cases} \sqrt{\frac{2}{a}}\cos\left(\frac{n\pi x}{a}\right) & n \text{ dispari} \\ \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right) & n \text{ pari} \end{cases}$$

- **Polari**

$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{cases}$$

$$\vec{\nabla}f = \left(\frac{\partial f}{\partial r},\ \frac{1}{r}\frac{\partial f}{\partial\theta},\ \frac{1}{r\sin\theta}\frac{\partial f}{\partial\phi}\right)$$

$$\vec{\nabla}\cdot\vec{F} = \frac{1}{r^2}\frac{\partial r^2 F_r}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial F_\theta\sin\theta}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial F_\phi}{\partial\phi}$$

$$\nabla^2 f = \frac{1}{r}\frac{\partial^2}{\partial r^2}(rf) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}$$

$$\int_{\mathbb{R}^3} d^3x = \int_0^\infty dr\int_0^\pi d\theta\int_0^{2\pi}d\phi\ r^2\sin\theta$$

- **Formule goniometriche**

$$\sin^2(\theta/2) = (1 - \cos\theta)/2$$
$$\cos^2(\theta/2) = (1 + \cos\theta)/2$$
$$\tan^2(\theta/2) = (1 - \cos\theta)/(1 + \cos\theta)$$
$$\tan(2\theta) = 2\tan\theta/(1 - \tan^2\theta)$$
$$\sin\alpha\cos\beta = [\sin(\alpha + \beta) + \sin(\alpha - \beta)]/2$$
$$\cos\alpha\cos\beta = [\cos(\alpha + \beta) + \cos(\alpha - \beta)]/2$$
$$\sin\alpha\sin\beta = [\cos(\alpha - \beta) + \cos(\alpha + \beta)]/2$$

Con $t = \tan(\theta/2)$, con $\theta \neq (2k+1)\pi$

$$\sin\theta = \frac{2t}{1 + t^2} \qquad \cos\theta = \frac{1 - t^2}{1 + t^2}$$

$$d\theta = \frac{2}{1 + t^2}dt$$

$$\sin\alpha + \sin\beta = 2\sin\left(\frac{\alpha + \beta}{2}\right)\cos\left(\frac{\alpha - \beta}{2}\right)$$

$$\cos\alpha + \cos\beta = 2\cos\left(\frac{\alpha - \beta}{2}\right)\cos\left(\frac{\alpha + \beta}{2}\right)$$

- **Integrali**

$$\int\sin^2 x\,dx = \frac{1}{2}(x - \sin x\cos x)$$

$$\int\cos^2 x\,dx = \frac{1}{2}(x + \sin x\cos x)$$

$$\int\cos^3 x\,dx = \sin x - \frac{\sin^3 x}{3}$$

$$\int\sin^3 x\,dx = -\cos x + \frac{\cos^3 x}{3}$$

$$\int_{-\infty}^{+\infty}e^{-ax^2+bx}\,dx = \sqrt{\frac{\pi}{a}}\exp\left(\frac{b^2}{4a}\right)$$

$$\int_0^{+\infty}x^2 e^{-x^2}\,dx = \frac{\sqrt{\pi}}{4}$$

$$\int_0^\infty x^n e^{-x}\,dx = n!$$

- **Taylor**

$$\sin x = \sum_{n=0}^{+\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}$$

$$\cos x = \sum_{n=0}^{+\infty}(-1)^n\frac{x^{2n}}{(2n)!}$$

$$\tan x = x + \frac{x^3}{3} + 2\frac{x^5}{15} + O(x^7)$$

- **Eq. differenziale**

$$x''(t) - \omega^2 x(t) = 0:$$
$$x(t) = Ae^{\omega t} + Be^{-\omega t}$$

$$\dot{y}(t) + a(t)y(t) = b(t):$$
$$y(t) = e^{-A(t)}\left[c + \int b(t)e^{A(t)}\,dt\right]$$

con $A(t)$ una primitiva di $a(t)$.

$$\ddot{y} + a\dot{y} + by = 0 \quad a, b \in \mathbb{R}$$

In base a Δ di eq. di secondo grado associata:

$$y(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} \qquad \Delta > 0$$
$$y(t) = c_1 e^{\lambda t} + tc_2 e^{\lambda t} \qquad \Delta = 0$$
$$y(t) = c_1 e^{\alpha t}\cos(\beta t) + c_2 e^{\alpha t}\sin(\beta t) \qquad \Delta < 0$$

Con $\alpha := \mathfrak{Re}(\lambda)$, $\beta := \mathfrak{Im}(\lambda)$

- **Coefficienti di Clebsch-Gordan**

1/2×1/2					
		1			
		+1	1	0	
+1/2	+1/2	1	0	0	
+1/2	−1/2		1/2	1/2	1
−1/2	+1/2		1/2	−1/2	−1
−1/2	−1/2				1

1×1/2							
		3/2					
		+3/2	3/2	1/2			
+1	+1/2	1	+1/2	+1/2			
+1	−1/2		1/3	2/3	3/2	1/2	
0	+1/2		2/3	−1/3	−1/2	−1/2	
0	−1/2				2/3	1/3	3/2
−1	+1/2				1/3	−2/3	−3/2
−1	−1/2						1

1×1										
		2								
		+2	2	1						
+1	+1	1	+1	+1						
+1	0		1/2	1/2	2	1	0			
0	+1		1/2	−1/2	0	0	0			
+1	−1				1/6	1/2	1/3			
0	0				2/3	0	−1/3	2	1	
−1	+1				1/6	−1/2	1/3	−1	−1	
0	−1							1/2	1/2	2
−1	0							1/2	−1/2	−2
−1	−1									1

REMINDER: Coefficienti vanno messi sotto radice!!!