

• **Equazione di Schrödinger**

$$i\hbar \frac{d}{dt} |\psi\rangle = H|\psi\rangle$$

• **Operatore  $P$**

$$P = -i\hbar \frac{d}{dx}$$

$$D_P = \{\psi \in L_2(a, b) \text{ ass. cont. } \wedge \psi' \in L_2(a, b)\}$$

$$[X, P] = i\hbar \mathbb{1}$$

$$[Q, P] = i\hbar \Rightarrow [Q, g(P)] = i\hbar g'(P)$$

• **Relazioni di indeterminazione**

$$(\Delta A)_{a,\psi} (\Delta B)_{b,\psi} \geq \frac{1}{2} \left| \langle [A, B] \rangle_\psi \right|$$

$$(\Delta X_i)_\psi (\Delta P_i)_\psi \geq \frac{\hbar}{2}$$

$$(\Delta t)_\psi (\Delta H)_\psi \geq \frac{\hbar}{2}$$

$$(\Delta t)_\psi := \inf_A \left| \frac{d}{dt} \langle A \rangle_\psi \right|$$

• **Base generalizzata del momento**

$$\langle \vec{x} | \vec{p} \rangle = \frac{e^{\frac{i}{\hbar} \vec{p} \cdot \vec{x}}}{(2\pi\hbar)^{\frac{3}{2}}}$$

• **Equazione di continuità**

$$\vec{j} := \frac{i\hbar}{2m} (\psi^* \vec{\nabla} \psi - \psi \vec{\nabla} \psi^*)$$

$$\frac{\partial}{\partial t} \rho(\vec{x}, t) + \vec{\nabla} \cdot \vec{j}(\vec{x}, t) = 0$$

• **Operatore di evoluzione temporale**

$$U(t) = e^{-\frac{i}{\hbar} H t}$$

• **Teo. di Wigner (trasformazione di simmetria)**

$$\begin{cases} \psi' = U\psi \\ A' = UAU^\dagger \end{cases}$$

• **Generatore infinitesimo della simmetria**

$$Q := i\hbar \frac{d}{ds} U(s) \Big|_{s=0}$$

$$U(s) = e^{-\frac{i}{\hbar} s Q}$$

• **Generatori delle rotazioni**

$$U(\varphi) e^{i\varphi \frac{L_3}{\hbar}}$$

$$R = \begin{pmatrix} \cos \varphi & -\sin \varphi & 0 \\ \sin \varphi & \cos \varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$[J_i, J_j] = i\hbar \varepsilon_{ijk} J_k$$

$$J_\pm := J_x \pm iJ_y$$

$$(J_\pm)^{-1} = J_\mp$$

$$[J_z, J_\pm] = \pm \hbar J_\pm \quad [J^2, J_\pm] = 0$$

$$[J_+, J_-] = 2\hbar J_z$$

$$\frac{J_\pm |j, m\rangle}{\hbar \sqrt{j(j+1) - m(m \pm 1)}} = |j, m \pm 1\rangle$$

$$\begin{cases} [J_i, X_j] = [L_i, X_j] = i\hbar \varepsilon_{ijk} X_k \\ [J_i, P_j] = [L_i, P_j] = i\hbar \varepsilon_{ijk} P_k \end{cases}$$

• **Particella senza spin 3D in polari**

$$L^2 \Psi_{l,m}(\vec{x}) = \hbar^2 l(l+1) \Psi_{l,m}(\vec{x})$$

$$L_z \Psi_{l,m}(\vec{x}) = \hbar m \Psi_{l,m}(\vec{x})$$

$$L_z = -i\hbar \frac{\partial}{\partial \phi}$$

$$Y_l^m(\theta, \phi) = \sqrt{\frac{2l+1}{4\pi}} \sqrt{\frac{(l+|m|)!}{(l-|m|)!}} \frac{1}{2^l l!} P_{l-m}(\theta) e^{im\phi}$$

Con  $P_{l,m}$  polinomi di Legendre di grado  $l$ ; parità dell'armonica sferica uguale a quella di  $l$ :

$$Y_0^0 = \frac{1}{2} \sqrt{\frac{1}{\pi}}$$

$$Y_1^- = \frac{1}{2} \sqrt{\frac{3}{2\pi}} e^{-i\phi} \sin \theta$$

$$Y_1^0 = \frac{1}{2} \sqrt{\frac{3}{\pi}} \cos \theta$$

$$Y_1^1 = -\frac{1}{2} \sqrt{\frac{3}{2\pi}} e^{i\phi} \sin \theta$$

• **Spin**

$$\vec{S} := \vec{J} - \vec{L}$$

Ha stessa algebra di  $\vec{J}$  e  $\vec{L}$ . Non commuta con  $\vec{X}$  e  $\vec{P}$ .

$$S_x = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad S_y = \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$S_z = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\sigma_i \sigma_j = i \varepsilon_{ijk} \sigma_k$$

$$[\sigma_i, \sigma_j] = 2i \varepsilon_{ijk} \sigma_k$$

• **Spin di 2 particelle**

Singoletto:

$$|0\ 0\rangle = \frac{1}{\sqrt{2}} (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$$

Tripletto:

$$\begin{cases} |1\ 1\rangle = |\uparrow\uparrow\rangle \\ |1\ 0\rangle = \frac{1}{\sqrt{2}} (|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) \\ |1\ -1\rangle = |\downarrow\downarrow\rangle \end{cases}$$

• **Simmetrie discrete**

– Parità  $\mathbb{P}$ :

$$\begin{cases} \mathbb{P} \vec{X} \mathbb{P}^\dagger = -\vec{X} \\ \mathbb{P} \vec{P} \mathbb{P}^\dagger = -\vec{P} \\ \mathbb{P} \vec{L} \mathbb{P}^\dagger = \vec{L} \\ \psi'(\vec{x}) = \psi(-\vec{x}) \end{cases}$$

$$\mathbb{P} = \mathbb{P}^{-1} = \mathbb{P}^\dagger$$

Unici autovalori possibili:  $\pm 1$

– Inversione temporale  $\mathbb{T}$ :

$$\begin{cases} \mathbb{T} \vec{X} \mathbb{T}^\dagger = \vec{X} \\ \mathbb{T} \vec{P} \mathbb{T}^\dagger = -\vec{P} \\ \mathbb{T} \vec{L} \mathbb{T}^\dagger = -\vec{L} \\ \psi'(\vec{x}, t) = \psi^*(\vec{x}, -t) \end{cases}$$

• **Visuale di Heisenberg**

$$\begin{cases} |\psi_h(t)\rangle = U^\dagger |\psi_s(t)\rangle \\ A_h(t) = U^\dagger A_s(t) U \end{cases}$$

Nota:  $U$  è l'operatore di evoluzione temporale.

$$\begin{cases} \frac{d}{dt} |\psi_h(t)\rangle = 0 \\ \frac{dA_h(t)}{dt} = -\frac{i}{\hbar} [A_h, H_h] + \left( \frac{\partial A}{\partial t} \right)_h \end{cases}$$

• **Teo. di Ehrenfest**

$$\begin{cases} \frac{d}{dt} \langle Q \rangle_\psi = \frac{1}{m} \langle P \rangle_\psi \\ \frac{d}{dt} \langle P \rangle_\psi = -\langle V'(Q) \rangle_\psi \end{cases}$$

• **Matrici densità**

$$\rho_\psi := |\psi\rangle\langle\psi|$$

$$\langle A \rangle_\psi = \text{tr}(\rho A)$$

Definizione di operatore densità:  $\rho$  tale che:

$$\begin{cases} \rho \text{ a.a.} \\ \rho \geq 0 \\ \text{tr}(\rho) = 1 \end{cases}$$

$$\text{tr}(\rho^2) \leq 1$$

• **Oscillatore armonico 1D**

$$H = \frac{P^2}{2m} + \frac{1}{2} m \omega^2 X^2$$

$$\hat{X} := \sqrt{\frac{m\omega}{\hbar}} X \quad \hat{P} := \frac{P}{\sqrt{m\omega\hbar}}$$

$$[\hat{X}, \hat{P}] = i$$

$$a := \frac{\hat{X} + i\hat{P}}{\sqrt{2}} \quad a^\dagger = \frac{\hat{X} - i\hat{P}}{\sqrt{2}}$$

$$H = \hbar\omega \left( N + \frac{1}{2} \right)$$

$$N := a^\dagger a \quad aa^\dagger - a^\dagger a = 1$$

$$aa^\dagger = N + 1$$

$$u_0(x) = \left( \frac{m\omega}{\pi\hbar} \right)^{\frac{1}{4}} e^{-\frac{m\omega}{\hbar} \frac{x^2}{2}}$$

$$|\nu\rangle = \frac{(a^\dagger)^\nu |0\rangle}{\sqrt{\nu!}}$$

$$\begin{cases} a|\nu\rangle = \sqrt{\nu} |\nu-1\rangle \\ a^\dagger|\nu\rangle = \sqrt{\nu+1} |\nu+1\rangle \end{cases}$$

• **Atomo H**

$$H_r = \frac{\vec{P}_\mu^2}{2\mu} - \frac{e^2}{r}$$

$$E = -\frac{E_I}{n^2}$$

$$\Psi_{n,l,m} = \frac{N_{n,l}}{a_0^{\frac{3}{2}}} e^{-\frac{r}{a_0 n}} \left( \frac{2r}{na_0} \right)^l$$

$$L \left( \frac{2r}{na_0} \right) Y_l^m(\theta, \phi)$$

$$e^2 := 2,3 \cdot 10^{-28} \text{ J} \cdot \text{m}$$

$$a_0 := \frac{\hbar^2}{e^2 \mu} \simeq 0,5 \cdot 10^{-10} \text{ m}$$

$$E_I := \frac{e^2}{2a_0} = \frac{\hbar^2}{2\mu a_0} = 13,6 \text{ eV} = 1 \text{ Ry}$$

## • Perturbazioni stazionarie

Problema:

$$H = H_0 + W = H_0 + \lambda \hat{W}$$

$$\begin{cases} E_a^\lambda = \mathcal{E}_0 + \lambda \mathcal{E}_1 + \lambda^2 \mathcal{E}_2 + O(\lambda^3) \\ |\lambda\rangle = |0\rangle + \lambda |1\rangle + \lambda^2 |2\rangle + O(\lambda^3) \end{cases}$$

Convenzione:

$$\begin{cases} |0\rangle = |E_a^0\rangle \\ \langle \lambda | \lambda \rangle = 1 \quad \forall \lambda \in \mathbb{R} \\ \langle 0 | \lambda \rangle \in \mathbb{R} \quad \forall \lambda \in \mathbb{R} \end{cases}$$

Soluzioni:

$$\mathcal{E}_1 = \langle E_a^0 | \hat{W} | E_a^0 \rangle$$

$$\langle E_b^0 | 1 \rangle = \frac{\langle E_b^0 | \hat{W} | E_a^0 \rangle}{-E_b^0 + E_a^0}$$

$$\mathcal{E}_n^k = \langle \mathcal{E}_n^0 | \hat{W} | \mathcal{E}_n^{k-1} \rangle$$

$$\mathcal{E}_2 = \sum_{b \neq a} \frac{\left| \langle E_a^0 | \hat{W} | E_b^0 \rangle \right|^2}{E_a^0 - E_b^0}$$

## • Perturbazioni: caso degenero

Sia  $\{|\phi_r\rangle\}_{r=1,\dots,d}$  BON di  $\mathcal{H}_0$

Ordine  $\lambda^q$ :

$$\begin{aligned} (H_0 - \mathcal{E}_0) |q, k\rangle + (\hat{W} - \mathcal{E}_1^k) |q-1, k\rangle \\ - \mathcal{E}_2^k |q-2, k\rangle - \dots - \mathcal{E}_q^k |0, k\rangle = 0 \end{aligned}$$

Ordine  $\lambda$ :

$$\sum_{s=1}^d \langle \phi_r | \hat{W} | \phi_s \rangle \langle \phi_s | 0, k \rangle = \mathcal{E}_1^k \langle \phi_r | 0, k \rangle$$

che è equazione ad autovalori/autovettori da risolvere  $\forall \phi_r$

## • EM

$$H = \frac{1}{2m} \left( \vec{P} - \frac{q}{c} \vec{A}(\vec{X}, t) \right)^2 + q\varphi(\vec{X}, t)$$

Trasformazione di Gauge:

$$\begin{cases} \vec{A}' = \vec{A} + \vec{\nabla} \Lambda(\vec{x}, t) \\ \varphi' = \varphi - \frac{1}{c} \frac{\partial}{\partial t} \Lambda(\vec{x}, t) \\ \psi'(\vec{x}, t) = e^{i \frac{q}{\hbar c} \Lambda(\vec{x}, t)} \psi(\vec{x}, t) =: U_\Lambda \psi(\vec{x}, t) \end{cases}$$

Trasformazione di Gauge è trasformazione di simmetria: solo osservabili covarianti sono fisici: bisogna imporre:

$$f_{\vec{A}', \varphi'} = U_\Lambda f_{\vec{A}, \varphi} U_\Lambda^\dagger$$

Gauge di Landau (con  $\vec{E} = 0$ ,  $\vec{B} = -B_0 \hat{u}_z$ ):

$$\begin{cases} \varphi = 0 \\ \vec{A} = (B_0 y, 0, 0) \\ H = \frac{1}{2m} \left( P_x - \frac{q}{c} B_0 Y \right)^2 + \frac{P_y^2}{2m} + \frac{P_z^2}{2m} \end{cases}$$

Asse z è indipendente ed è onda 1D libera.

Autostati di onda xy ( $u_n$  sono le autofunzioni di oscillatore armonico):

$$E_n = \hbar \omega \left( n + \frac{1}{2} \right) = \frac{\hbar q B_0}{mc} \left( n + \frac{1}{2} \right)$$

$$\Psi_{n, p_x}(x, y) = e^{i \frac{p_x}{\hbar} x} u_n(y - y_0)$$

$$\omega := \frac{q B_0}{mc} \quad y_0 := \frac{c p_x}{q B_0}$$

Gauge simmetrica:

$$\vec{A} = \left( \frac{B_0 y}{2}, -\frac{B_0 x}{2}, 0 \right)$$

## • Effetto Aharonov-Bohm

$$\frac{1}{2m} \left( P_s - \frac{q}{c} \frac{\Phi}{2\pi R} \right)^2$$

Doppia fenditura con solenoide acceso:

$$\begin{aligned} |\psi(x)|^2 \propto \left| \psi_1(x) + e^{i \frac{q}{\hbar c} \int_{\gamma_2 - \gamma_1} \vec{A} \cdot d\vec{l}} \psi_2(x) \right|^2 \\ = \left| \psi_1(x) + e^{i \frac{q}{\hbar c} \Phi} \psi_2(x) \right|^2 \end{aligned}$$

## • Scattering

$$\frac{d\sigma_B}{d\Omega}(\theta) = \frac{4\mu^2 \beta^2}{\hbar^4} \frac{1}{(\alpha^2 + 4k^2 \sin^2 \frac{\theta}{2})^2}$$

$$\frac{d\sigma_B}{d\Omega} \xrightarrow{\alpha \rightarrow 0} \frac{\mu^2 \beta^2}{4\hbar^4 k^4 \sin^4 \frac{\theta}{2}}$$

## • Buca di potenziale

Con potenziale  $V(x) = +\infty \mathbb{1}_{[0, a]c}$ :

$$E_n = \frac{\pi^2 \hbar^2}{2ma^2} n^2$$

$$\psi_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right)$$

Con potenziale  $V(x) = +\infty \mathbb{1}_{\{x > |a|/2\}}$

$$E_n = \frac{\hbar^2 n^2 \pi^2}{2ma^2}$$

$$\psi_n(x) = \begin{cases} \sqrt{\frac{2}{a}} \cos\left(\frac{n\pi x}{a}\right) & n \text{ dispari} \\ \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right) & n \text{ pari} \end{cases}$$

## • Polari

$$\begin{cases} x = r \sin \theta \cos \phi \\ y = r \sin \theta \sin \phi \\ z = r \cos \theta \end{cases}$$

$$\vec{\nabla} f = \left( \frac{\partial f}{\partial r}, \frac{1}{r} \frac{\partial f}{\partial \theta}, \frac{1}{r \sin \theta} \frac{\partial f}{\partial \phi} \right)$$

$$\begin{aligned} \vec{\nabla} \cdot \vec{F} &= \frac{1}{r^2} \frac{\partial r^2 F_r}{\partial r} + \frac{1}{r \sin \theta} \frac{\partial F_\theta \sin \theta}{\partial \theta} \\ &\quad + \frac{1}{r \sin \theta} \frac{\partial F_\phi}{\partial \phi} \end{aligned}$$

$$\begin{aligned} \int_{\mathbb{R}^3} d^3 x &= \int_0^\infty dr \int_0^\pi d\theta \\ &\quad \int_0^{2\pi} d\phi \, r^2 \sin \theta \end{aligned}$$

## • Formule goniometriche

$$\sin^2(\theta/2) = (1 - \cos \theta)/2$$

$$\cos^2(\theta/2) = (1 + \cos \theta)/2$$

$$\tan^2(\theta/2) = (1 - \cos \theta)/(1 + \cos \theta)$$

$$\tan(2\theta) = 2 \tan \theta / (1 - \tan^2 \theta)$$

$$\sin \alpha \cos \beta = [\sin(\alpha + \beta) + \sin(\alpha - \beta)]/2$$

$$\cos \alpha \cos \beta = [\cos(\alpha + \beta) + \cos(\alpha - \beta)]/2$$

$$\sin \alpha \sin \beta = [\cos(\alpha - \beta) - \cos(\alpha + \beta)]/2$$

Con  $t = \tan(\theta/2)$ , con  $\theta \neq (2k+1)\pi$

$$\sin \theta = \frac{2t}{1+t^2} \quad \cos \theta = \frac{1-t^2}{1+t^2}$$

$$\sin \alpha + \sin \beta = 2 \sin\left(\frac{\alpha + \beta}{2}\right) \cos\left(\frac{\alpha - \beta}{2}\right)$$

$$\cos \alpha + \cos \beta = 2 \cos\left(\frac{\alpha + \beta}{2}\right) \cos\left(\frac{\alpha - \beta}{2}\right)$$

## • Integrali

$$\int \sin^2 x \, dx = \frac{1}{2} (x - \sin x \cos x) + C$$

$$\int \cos^2 x \, dx = \frac{1}{2} (x + \sin x \cos x) + C$$

$$\int_{-\infty}^{+\infty} e^{-ax^2+bx} \, dx = \sqrt{\frac{\pi}{a}} \exp\left(\frac{b^2}{4a}\right)$$

$$\int_0^{+\infty} x^2 e^{-x^2} \, dx = \frac{\sqrt{\pi}}{4}$$

$$\int \cos^3 x \, dx = \sin x - \frac{\sin^3 x}{3}$$

$$\int \sin^3 x \, dx = -\cos x + \frac{\cos^3 x}{3}$$

## • Taylor

$$\sin x = \sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$\cos x = \sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

$$\tan x = x + \frac{x^3}{3} + 2\frac{x^5}{15} + O(x^7)$$

## • Eq. differenziale

$$x''(t) - \omega^2 x(t) = 0 :$$

$$x(t) = A e^{\omega t} + B e^{-\omega t}$$

$$\dot{y}^t + a(t)y(t) = b(t)$$

$$y(t) = e^{-A(t)} \left[ c + \int b(t) e^{A(t)} dt \right]$$

con  $A(t)$  una primitiva di  $a(t)$ .

$$\ddot{y} + a\dot{y} + by = 0 \quad a, b \in \mathbb{R}$$

In base a  $\Delta$  di eq. di secondo grado associata:

$$y(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} \quad \Delta > 0$$

$$y(t) = c_1 e^{\lambda t} + t c_2 e^{\lambda t} \quad \Delta = 0$$

$$y(t) = c_1 e^{\alpha t} \cos(\beta t) + c_2 e^{\alpha t} \sin(\beta t) \quad \Delta < 0$$

Con  $\alpha := \Re(\lambda)$ ,  $\beta := \Im(\lambda)$