

1. RANDOM VARIABLES ON THE CANONICAL SPACE AND SKOROHOD REPRESENTATION

We recall that if X is a RV with values in \mathbb{R} then we set $F_X(x) := \mathbb{P}(X \leq x)$ for every $x \in \mathbb{R}$. Of course $F_X(x) = P_X((-\infty, x])$. We recall also that by canonical $(\Omega, \mathcal{F}, \mathbb{P})$ we mean $\Omega = [0, 1]$, \mathcal{F} are the Borel sets $\mathcal{B}([0, 1])$ of $[0, 1]$ and \mathbb{P} is the (restriction of) the Lebesgue measure (to $[0, 1]$). The choice of $[0, 1]$ is arbitrary with respect to the choice of $(0, 1)$, $[0, 1)$ or $(0, 1]$.

Ex. 1.1*: Suppose $F = F_X$. Then

- (1) $F : \mathbb{R} \rightarrow [0, 1]$, F is non decreasing (i.e., $x \leq y \implies F(x) \leq F(y)$);
- (2) $F(-\infty) = 0$ and $F(+\infty) = 1$ (i.e., $\lim_{x \rightarrow -\infty} F(x) = 0$ and $\lim_{x \rightarrow +\infty} F(x) = 1$);
- (3) F is CAD (Continue À Droite, i.e., right-continuous)

Moreover show that $\mathbb{P}(X = x) = F_X(x) - \lim_{y \nearrow x} F_X(y)$ for every $x \in \mathbb{R}$.

Ex. 1.2** (*Skorohod Representation*): F satisfies properties (1), (2) and (3) of **Ex. 1.1**. Set for $\omega \in \Omega$ with the canonical choice of $(\Omega, \mathcal{F}, \mathbb{P})$

$$X^+(\omega) := \inf \{z : F(z) > \omega\} = \sup \{z : F(z) \leq \omega\},$$

and

$$X^-(\omega) := \inf \{z : F(z) \geq \omega\} = \sup \{z : F(z) < \omega\}.$$

- (1) Show that $X^+ \geq X^-$ and that for every $\omega \in (0, 1)$ we have $X^\pm(\omega) \in \mathbb{R}$.
- (2) Show that $\{\omega \in \Omega : \omega \leq F(y)\} = \{\omega \in \Omega : X^-(\omega) \leq y\}$ for every $y \in \mathbb{R}$, so $F = F_{X^-}$.
- (3) Show that $\mathbb{P}(X^+ = X^-) = 1$, so (in particular) $F = F_{X^+}$.

Hints. In (1) there are rather trivial issues, but note the use of the hypotheses for the second part. For (2) note that $(\omega \leq F(y)) \implies X^-(\omega) \leq y$ follows from the definition. The other implication is more difficult, but note that $(y > X^-(\omega)) \implies (F(y) \geq \omega)$ also follows from the definitions and implies (why?) $F(X^-(\omega)) \leq \omega$: now you are really almost there. For (3) it may be useful to remark that $X^- \leq X^+$ by definition and that $\{X^- \neq X^+\} = \cup_{q \in \mathbb{Q}} \{X^- < q < X^+\}$, hence it suffices to show that $\mathbb{P}(X^- < q < X^+) = 0$ for every q .

Ex. 1.3: Make explicit the Skorohod Representation in the following cases:

- (1) $\text{Exp}(\lambda)$. We say that $X \sim \text{Exp}(\lambda)$, with $\lambda > 0$, if the law of X has a density $f_X(x) = \lambda \exp(-\lambda x) \mathbf{1}_{(0, \infty)}(x)$.
- (2) $\text{Cauchy}(\gamma)$. We say that $X \sim \text{Cauchy}(\gamma)$, with $\gamma > 0$, if the law of X has a density $f_X(x) = 1/(\pi\gamma(1 + (x/\gamma)^2))$.
- (3) The law of X is $pP_Y + (1-p)P_0$, where $p \in [0, 1]$, $Y \sim \text{Exp}(1)$ and P_0 is the law of the (trivial) RV identically equal to 0.
- (4) A random variable that takes only m different values, a_1, \dots, a_m , respectively with probability p_1, \dots, p_m (so $\sum_{j=1}^m p_j = 1$). In writing the solution it is practical to introduce $s_j := \sum_{k=1}^j p_k$ for $j = 1, \dots, m$, and $s_0 := 0$. This is a boring, but important exercise, at least from the point of view of the numerical applications: in general, Skorohod representation is important (also) from the point of view of the numerical applications because computers generate (pseudo)random variables $Y \sim U([0, 1])$, for $X^\pm(U) \sim X$.
- (5) $F(x) = 0$ for $x < 0$ and $F(x) = 1 - p \exp(-x)$ for $x \geq 0$, with $p \in (0, 1]$.
- (6) A random variable with density $f(x) = (1/2)x^{-2} \mathbf{1}_{(-1, 1)^c}(x)$, $x \in \mathbb{R}$.

2. INDEPENDENCE (AND MEASURABILITY)

One of the key tools is the following π -system Lemma:

Lemma (Lemma 1.6 in [Williams]). (E, \mathcal{E}) is a measurable space with $\mathcal{E} = \sigma(\Pi)$ and Π a set of subsets of E with the property of being \cap -stable (i.e., $B_1, B_2 \in \Pi$ implies $B_1 \cap B_2 \in \Pi$). Suppose now that μ_1 and μ_2 are measures on (E, \mathcal{E}) that coincide on Π and that $\mu_1(E) = \mu_2(E) < \infty$. Then $\mu_1 = \mu_2$.

Ex. 2.1*: We work on $(\Omega, \mathcal{A}, \mathbb{P})$. Suppose $\mathcal{G}_1 = \sigma(\Pi_1)$ and $\mathcal{G}_2 = \sigma(\Pi_2)$, with Π_1 and Π_2 π -systems of elements of \mathcal{A} . Show that $\mathbb{P}(A_1 \cap A_2) = \mathbb{P}(A_1)\mathbb{P}(A_2)$ for every choice of $A_1 \in \Pi_1$ and $A_2 \in \Pi_2$ implies $\mathcal{G}_1 \perp\!\!\!\perp \mathcal{G}_2$.

Hint. Fix $A_2 \in \Pi_2$ and define the two finite measures $\nu_1(A) := \mu_1(A \cap A_2)$ and $\nu_2(A) := \mu_2(A \cap A_2)$ for every $A \in \mathcal{A}$ and use the π -system Lemma to conclude that $\nu_1 = \nu_2$. Then repeat the other way around.

Ex. 2.2*: Show that the real random variables X_1, \dots, X_n are independent iff for every $x_1, \dots, x_n \in \mathbb{R}$

$$\mathbb{P}(X_j \leq x_j \text{ for } j = 1, \dots, n) = \prod_{j=1}^n \mathbb{P}(X_j \leq x_j).$$

Ex. 2.3*: $(X_j)_{j \in \mathbb{N}}$ are independent. Show that for $k \in \mathbb{N}$

$$\sigma(X_1, \dots, X_k) \perp\!\!\!\perp \sigma(X_{k+1}, X_{k+2}, \dots).$$

Ex. 2.4: We consider $\Omega = [0, 1]$, $\mathcal{A} = \mathcal{B}(\Omega)$ and \mathbb{P} is the Lebesgue measure. For $n = 1, 2, \dots$ we consider the σ -algebra

$$\mathcal{D}_n := \sigma([(j-1)2^{-n}, j2^{-n}] : j = 1, \dots, 2^n),$$

of subsets of Ω . We set also $A_n := \sqcup_{j=1}^{2^{n-1}} [(2j-1)2^{-n}, 2j2^{-n}]$, for $n = 1, 2, \dots$

- (1) Show that, for every $B \in \mathcal{D}_n$, $A_{n+1} \perp\!\!\!\perp B$.
- (2) Show that $(\mathbf{1}_{A_n})_{n=1,2,\dots}$ is an IID $B(1/2)$ sequence.

Ex. 2.5: (X_n) is an IID $B(1/2)$ sequence and we set $X(\omega) := \sum_{n=1}^{\infty} 2^{-n} X_n(\omega)$.

- (1) Show that X is a RV and that $P(X \in (0, 1)) = 1$.
- (2) show that

$$2F_X(x) = \begin{cases} F_X(2x) & \text{if } x \in [0, 1/2], \\ 1 + F_X(2x - 1) & \text{if } x \in [1/2, 1]. \end{cases} \quad (\star)$$

- (3) Show that (\star) determines the value of $F_X(x)$ for x dyadic rational in $[0, 1]$ (i.e., x is rational with denominator that is a power of 2). Conclude that (\star) determines F_X , so $X \sim U([0, 1])$. *Obs.*: Note that this and **Ex. 2.3** show that the law of fair coin tossing and Lebesgue on $[0, 1]$ are *essentially* the same measure, in the sense that they are the same measure up to a measurable bijection of the spaces.
- (4) Give, in the canonical space, X_1 and X_2 independent and $U([0, 1])$.
- (5) Give, in the canonical space, a sequence (X_n) of IID $U([0, 1])$ random variables.

3. CONVERGENCE ISSUES

Ex. 3.1*: X is a RV and $(X_n)_{n \in \mathbb{N}}$ is a sequence of RVs on the same probability space. Show that if $\sum_n \mathbb{P}(|X_n - X| > \varepsilon) < \infty$ for every $\varepsilon > 0$, then $X_n \rightarrow X$ a.s..

Obs.: this is a direct application of BC1, but make the effort of writing *as explicitly as possible* an event Ω_0 such that $\mathbb{P}(\Omega_0) = 1$ and such that $\lim_n X_n(\omega) = X(\omega)$ for every $\omega \in \Omega_0$.

Ex. 3.2*: In this exercise we give a sequence of RV that converges (to zero) in probability, but not almost surely. On the canonical space we consider for $k = 2, 3, \dots$ the partition

$$I_{k,1} := (0, 1/k), \quad I_{k,2} := [1/k, 2/k), \dots, \quad I_{k,k} := ((k-1)/k, 1),$$

in intervals of length $1/k$. For $n = 2, 3, \dots$ we set $X_n := \mathbf{1}_{I_{k(n),j(n)}}$ with

$$k(n) := 1 + \max \left\{ m = 1, 2, \dots : \frac{m(m+1)}{2} < n \right\} \quad \text{and} \quad j(n) := n - \frac{k(n)(k(n)-1)}{2}.$$

- (1) Show that $\lim_{n \rightarrow \infty} X_n = 0$ in probability.
- (2) Show that for every $\omega \in \Omega$ we have $\sum_n \mathbf{1}_{\{1\}}(X(\omega)) = \infty$ and conclude that (X_n) does not converge a.s..

Important observation: The only complicated point is understanding that $(I_{k(n),j(n)})$ is a “rotating” sequence of intervals that repeatedly cover all of $(0, 1)$.

Ex. 3.3: $(X_n)_{n \in \mathbb{N}}$ is an IID $\text{Exp}(1)$ sequence ($X \sim \text{Exp}(1)$ if $\mathbb{P}(X > t) = \exp(-t)$ for $t > 0$). Let $Y_n := X_n / \log(n)$. Show that $Y_n \rightarrow 0$ in \mathbb{L}^p for every $p \geq 1$ (hence in probability too), but the sequence (Y_n) does not converge a.s..

Ex. 3.4*: $(X_n)_{n \in \mathbb{N}}$ is an IID sequence. Show that $X_1 \in \mathbb{L}^1$ iff $\lim_n X_n/n = 0$ a.s..

Hint. Start by observing that it suffices to show that $X_1 \in \mathbb{L}^1$ iff $\mathbb{P}(|X_n| > 1, \text{ i.o.}) = 0$. It may be useful to keep in mind that, if $X \geq 0$, then $\mathbb{E}[X] = \int_0^\infty \mathbb{P}(X > x) dx$.

Ex. 3.5*: The aim of this exercise is to show the (a priori) very surprising result that if (X_n) converges to X in law then $\lim_n X_n^+ = X^+$, where $X_n^+(\omega) := \inf \{z : F_{X_n}(z) > \omega\}$ for $\omega \in \Omega = (0, 1)$ (we are of course making the canonical choice for $(\Omega, \mathcal{A}, \mathbb{P})$: recall the Skorohod Representation, Ex. 1.2) and analogous definition for X^+ . Said otherwise, if $X_n \xrightarrow{\mathcal{L}} X$ then we can find a probability space and RV's (Y_n) and Y such that $Y_n \sim X_n$ for every n , $Y \sim X$ and $Y_n \rightarrow Y$ a.s..

For this we propose to establish in order: (we write $F = F_X$ and $F_n = F_{X_n}$)

- (1) Show that the complement of C_F^0 , i.e. the set of jumps of F , is at most countable and conclude that C_F^0 is dense in \mathbb{R} .
- (2) For every $\omega \in (0, 1)$ choose $z \in C_F^0$ such that $z > X^+(\omega)$. Show that $F_n(z) > \omega$ for n large enough and conclude that $X_n^+(\omega) \leq z$ for the same values of n .
- (3) Conclude that $\limsup_n X_n^+(\omega) \leq X^+(\omega)$ for every ω .
- (4) Repeat/modify the argument to show that $\liminf_n X_n^-(\omega) \geq X^-(\omega)$ for every ω .
- (5) Conclude that $X_n^\pm \rightarrow X^\pm$ a.s. (we could write a bit informally $X_n^\pm \rightarrow X^\pm$ a.s.).

Ex. 3.6*: $(X_n)_{n \in \mathbb{N}}$ is an IID sequence with X_1 not trivial, i.e. there exists x such that $F_X(x) \in (0, 1)$ (one of the various ways of saying that X is not a.s. a constant).

- (1) Explain why $(X_n)_{n \in \mathbb{N}}$ converges in law to X_1 (or to X_5 if you prefer!).
- (2) Show that there exists $a < b$ such that $\mathbb{P}(X_1 \leq a) > 0$ and $\mathbb{P}(X_1 \geq b) > 0$ and conclude that $\mathbb{P}(X_n \leq a, \text{i.o.}) = 1$ and $\mathbb{P}(X_n \geq b, \text{i.o.}) = 1$.
- (3) Conclude that $(X_n)_{n \in \mathbb{N}}$ does not converge a.s..
- (4) Upgrade the argument to conclude that $(X_n)_{n \in \mathbb{N}}$ does not converge in probability either.

Remark: in this case it is immediate to construct a sequence (Y_n) and a random variable Y such that $Y_n \sim X_n$ for every n , $Y \sim X$ and $Y_n \rightarrow Y$ a.s.. Just choose $Y_n = Y = X_1$ for every n !

Ex. 3.7*: we did not give a proof of Kolmogorov LLN, that says that if $(X_n)_{n \in \mathbb{N}}$ with $X_1 \in \mathbb{L}^1$, then $\lim_n S_n/n = \mathbb{E}[X_1]$ a.s. and in \mathbb{L}^1 , con $S_n := X_1 + \dots + X_n$. But can you show that this statement cannot be improved? Namely show that

- (1) if $\mathbb{E}[|X_1|] = \infty$ then $(S_n/n)_{n \in \mathbb{N}}$ does not converge a.s. nor in \mathbb{L}^1 ;
- (2) in fact, if $\mathbb{E}[|X_1|] = \infty$ we have $\limsup_n |S_n|/n = \infty$ a.s..

Hint. Do not forget of **Ex. 3.4**.

4. MORE ABOUT AND AROUND CONVERGENCE IN LAW

Ex. 4.1: Show that if $X_n \xrightarrow{\mathcal{L}} X$ in law, then $(X_n)_{n \in \mathbb{N}}$ is tight.

Ex. 4.2: Show that if there exists an increasing function $h : [0, \infty) \rightarrow [0, \infty)$ such that $\lim_{x \rightarrow \infty} h(x) = \infty$ and $\sup_n \mathbb{E}[h(|X_n|)] < \infty$, then (X_n) is tight.

Ex. 4.3*: this exercise is an interesting application of theorems from measure theory (Fubini-Tonelli!). It yields an inversion formula for the characteristic function of a RV. We use $\varphi = \varphi_X$, where of course $\varphi_X(t) = \mathbb{E}[\exp(itX)]$.

- (1) Show (do justify the steps!) that for every $T > 0$ and $a, b \in \mathbb{R}$

$$\int_{-T}^T \frac{e^{-ita} - e^{-itb}}{it} \varphi(t) dt = \int_{\mathbb{R}} \left(\int_{-T}^T \frac{e^{it(x-a)} - e^{it(x-b)}}{it} dt \right) P_X(dx).$$

- (2) Just a computation: show that

$$\frac{1}{2\pi} \int_{-T}^T \frac{e^{it(x-a)} - e^{it(x-b)}}{it} dt = \frac{1}{\pi} [\text{sign}(x-a)S(|x-a|T) - \text{sign}(x-b)S(|x-b|T)],$$

where $\text{sign}(x) := 0$ if $x = 0$ and $S : [0, \infty) \rightarrow \mathbb{R}$ is the integral sine function, i.e. $S(y) := \int_0^y (\sin(x)/x) dx$ for $y \geq 0$. Note that $\int_0^\infty (|\sin(x)|/x) dx = \infty$, but $\lim_{y \rightarrow \infty} S(y)$ exists and it is equal to $\pi/2$. The convergence can be established for example via Dirichlet's test for convergence of series, and the fact that the limit is $\pi/2$ is best seen via integration in the complex plane (but for this exercise we can just accept it as a fact).

- (3) Show Lévy Inversion Formula: for $a < b$

$$\lim_{T \nearrow \infty} \frac{1}{2\pi} \int_{-T}^T \frac{e^{-ita} - e^{-itb}}{it} \varphi(t) dt = P_X((a, b)) + \frac{1}{2}P_X(\{a\}) + \frac{1}{2}P_X(\{b\}).$$

Hint. Study first the limit of the expression in point (2) separating the cases $x < a$, $x = a$, $x \in (a, b)$, $x = b$ and $x > b$.

- (4) Explain why Lévy Inversion Formula allows to recover F_X if we know φ_X .
 (5) Assume that $\varphi \in \mathbb{L}^1$, i.e. that $\int_{\mathbb{R}} |\varphi(t)| dt < \infty$. Show that

$$F_X(b) - F_X(a) = \frac{1}{2\pi} \int_{\mathbb{R}} \frac{e^{-ita} - e^{-itb}}{it} \varphi(t) dt.$$

Obs.: the proof of this result normally goes along with showing that $P_X(\{a\}) = P_X(\{b\}) = 0$. Do justify this point, i.e. that $F_X \in C^0$.

- (6) In fact, if $\varphi \in \mathbb{L}^1$ then $F_X \in C^1$. Show it and show also that for every $x \in \mathbb{R}$

$$F'_X(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-itx} \varphi(t) dt.$$

Food for thought. This shows that X is a continuous RV with density $f_X = F'_X$ and recall that $\varphi_X(t) = \int_{\mathbb{R}} e^{itx} f_X(x) dx$, recovering thus the (possibly already) known inversion formula for the Fourier transform in the \mathbb{L}^1 context. If you are interested, meditate about the fact that the Fourier transform is naturally given in \mathbb{L}^2 , where a priori is even ill-defined, but (once properly defined) it does map \mathbb{L}^2 into and onto \mathbb{L}^2 . Instead the characteristic function of a continuous RV, i.e. the Fourier transform of an \mathbb{L}^1 function, is not necessarily in \mathbb{L}^1 (just try with a uniform RV): this is consistent with the fact that requiring $\varphi_X \in \mathbb{L}^1$ implies that X has a continuous density.

Ex. 4.4: Not any functions $\varphi : \mathbb{R} \rightarrow \mathbb{C}$ is a characteristic function. What about

- (1) $\sin(t)$;
- (2) $\cos(t)$;
- (3) $\cos(t^2)$;
- (4) $e^{-t^2} \cos(t)^3 / (1 - it)$.

Hint. Once a function satisfies the basic properties of a characteristic function, it may be of help to look for a RV whose characteristic function does the job.

Ex. 4.5: Show that if φ is a characteristic function, then $\exp(\varphi - 1)$ is too (of which random variable?).

Ex. 4.6: $(X_n)_{n \in \mathbb{N}}$ is a sequence of independent random variables and $S_n = X_1 + \dots + X_n$. Assume that there exists two sequences of real numbers $(a_n)_{n \in \mathbb{N}}$ and $(b_n)_{n \in \mathbb{N}}$, with $b_n > 0$ for every n and $\lim_{n \rightarrow \infty} b_n = \infty$, such that $((S_n - a_n)/b_n)_{n \in \mathbb{N}}$ converges in law to Y , with Y not trivial. Then the same sequence does not converge in probability.

Hint. Use Kolmogorov 0-1 law.

Ex. 4.7: Let us reconsider the (probably already known) problem of showing that if X_n is a binomial random variable of parameters $n \in \mathbb{N}$ and $p_n := \min(\lambda/n, 1)$ (with $\lambda > 0$), i.e. X_n is the sum of n independent Bernoulli RV of parameter p_n .

- (1) Show that for $k = 0, 1, \dots$ we have $\lim_{n \rightarrow \infty} \mathbb{P}(X_n = k) = \lambda^k e^{-\lambda} / k!$ and conclude that (X_n) converges in law to a Poisson random variable of parameter λ .
- (2) Show the same convergence in law by using Lévy Continuity Theorem.

Ex. 4.8: (X_n) is an IID sequence and $\mathbb{P}(X_1 > x) = 1/x^\alpha$ for every $x \geq 1$, with $\alpha > 0$. Show that the sequence

$$\left(\frac{1}{n^{1/\alpha}} \max_{j=1, \dots, n} X_j \right)_{n=1, 2, \dots}$$

converges in law. Show that the limit law is absolutely continuous with respect to the Lebesgue measure and give its density.

Obs.: also in this case the convergence cannot be upgraded to convergence in probability. Do you see why?

5. UNIFORM INTEGRABILITY (UI)

Ex. 5.1: show that $\mathcal{C}_{p,c} := \{X \text{ r.v.} : \|X\|_p \leq c\}$, $p > 1$ and c a positive constant, is UI.

Ex. 5.2: there is an interesting alternative characterization of uniform integrability, namely \mathcal{C} is UI if and only if $\sup_{X \in \mathcal{C}} \|X\|_1 < \infty$ and for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\sup_{X \in \mathcal{C}} \mathbb{E}[|X|; G] < \varepsilon$ whenever $\mathbb{P}(G) < \delta$. Prove it.

This characterization provides a rather straightforward proof of the fact that if \mathcal{C}_1 and \mathcal{C}_2 are UI, then $\{X + Y : X \in \mathcal{C}_1 \text{ and } Y \in \mathcal{C}_2\}$ is also UI.

Ex. 5.3*: show that if $X_n \rightarrow X$ in law and $(X_n)_{n \in \mathbb{N}}$ is UI, then $X \in \mathbb{L}^1$ and $\lim_n \mathbb{E}[X_n] = \mathbb{E}[X]$ (or, rather, $\lim \int_{\mathbb{R}} x P_{X_n}(dx) = \int_{\mathbb{R}} x P_X(dx)$).

Hint. The Skorohod representation is a way to approach the problem.

Ex. 5.4: $(X_n)_{n \in \mathbb{N}}$ is IID with $X_1 \in \mathbb{L}^4$, $\mathbb{E}[X_1] = 0$ and $\mathbb{E}[X_1^2] = 1$. We use the standard notation $S_n := X_1 + \dots + X_n$. Show that if $f : \mathbb{R} \rightarrow \mathbb{R}$ is C^0 and $f(x) = O(|x|^p)$ for a value $p \in (0, 4)$ and $|x| \rightarrow \infty$, then $\lim_n \mathbb{E}[f(S_n/\sqrt{n})] = \mathbb{E}[f(Z)]$, where $Z \sim \mathcal{N}(0, 1)$.

Ex. 5.5: (X_n) is IID with $X_1 \in \mathbb{L}^1$. As usual $S_n := \sum_{j=1}^n X_j$. Show that $(S_n/n)_{n \in \mathbb{N}}$ is UI.