1. Random variables on the canonical space and Skorohod Representation

We recall that if X is a RV with values in $\mathbb{R}$ then we set $F_X(x) := \mathbb{P}(X \le x)$ for every $x \in \mathbb{R}$. Of course $F_X(x) = P_X((-\infty, x])$. We recall also that by canonical $(\Omega, \mathcal{F}, \mathbb{P})$ we mean $\Omega = [0,1]$, $\mathcal{F}$ are the Borel sets $\mathcal{B}([0,1])$ of $[0,1]$ and $\mathbb{P}$ is the (restriction of) the Lebesgue measure (to $[0,1]$). The choice of $[0,1]$ is arbitrary with respect to the choice of $(0,1)$, $[0,1)$ or $(0,1]$.

Ex. 1.1*: Suppose $F = F_X$. Then
(1) $F : \mathbb{R} \to [0,1]$, F is non decreasing (i.e., $x \le y \implies F(x) \le F(y)$);
(2) $F(-\infty) = 0$ and $F(+\infty) = 1$ (i.e., $\lim_{x\to-\infty} F(x) = 0$ and $\lim_{x\to+\infty} F(x) = 1$);
(3) F is CAD (Continue À Droite, i.e., right-continuous)
Moreover show that $\mathbb{P}(X = x) = F_X(x) - \lim_{y \nearrow x} F_X(y)$ for every $x \in \mathbb{R}$.

Ex. 1.2**(*Skorohod Representation*): F satisfies properties (1), (2) and (3) of **Ex. 1.1**. Set for $\omega \in \Omega$ with the canonical choice of $(\Omega, \mathcal{F}, \mathbb{P})$

$$X^+(\omega) := \inf\{z : F(z) > \omega\} = \sup\{z : F(z) \le \omega\},$$

and

$$X^-(\omega) := \inf\{z : F(z) \ge \omega\} = \sup\{z : F(z) < \omega\}.$$

(1) Show that $X^+ \ge X^-$ and that for every $\omega \in (0,1)$ we have $X^\pm(\omega) \in \mathbb{R}$.
(2) Show that $\{\omega \in \Omega : \omega \le F(y)\} = \{\omega \in \Omega : X^-(\omega) \le y\}$ for every $y \in \mathbb{R}$, so $F = F_{X^-}$.
(3) Show that $\mathbb{P}(X^+ = X^-) = 1$, so (in particular) $F = F_{X^+}$.

Hints. In (1) there are rather trivial issues, but note the use of the hypotheses for the second part. For (2) note that $(\omega \le F(y)) \implies X^-(\omega) \le y$ follows from the definition. The other implication is more difficult, but note that $(y > X^-(\omega)) \implies (F(y) \ge \omega)$ also follows from the definitions and implies (why?) $F(X^-(\omega)) \le \omega$: now you are really almost there. For (3) it may be useful to remark that $X^- \le X^+$ by definition and that $\{X^- \ne X^+\} = \cup_{q\in\mathbb{Q}}\{X^- < q < X^+\}$, hence it suffices to show that $\mathbb{P}(X^- < q < X^+) = 0$ for every q.

Ex. 1.3: Make explicit the Skorohod Representation in the following cases:
(1) Exp(λ). We say that $X \sim$Exp(λ), with $\lambda > 0$, if the law of X has a density $f_X(x) = \lambda \exp(-\lambda x)\mathbf{1}_{(0,\infty)}(x)$.
(2) Cauchy(γ). We say that $X \sim$Cauchy(γ), with $\gamma > 0$, if the law of X has a density $f_X(x) = 1/\left(\pi\gamma\left(1 + (x/\gamma)^2\right)\right)$.
(3) The law of X is $pP_Y + (1-p)P_0$, where $p \in [0,1]$, $Y \sim$Exp(1) and P_0 is the law of the (trivial) RV identically equal to 0.
(4) A random variable that takes only m different values, $a_1, \ldots, a_m$, respectively with probability $p_1, \ldots, p_m$ (so $\sum_{j=1}^m p_j$). In writing the solution it is practical to introduce $s_j := \sum_{k=1}^j p_k$ for $j = 1, \ldots, m$, and $s_0 := 0$. This is a boring, but important exercise, at least from the point of view of the numerical applications: in general, Skorohod representation is important (also) from the point of view of the numerical applications because computers generate (pseudo)random variables $Y \sim U([0,1])$, for $X^\pm(U) \sim X$.
(5) $F(x) = 0$ for $x < 0$ and $F(x) = 1 - p\exp(-x)$ for $x \ge 0$, with $p \in (0,1]$.
(6) A random variable with density $f(x) = (1/2)x^{-2}\mathbf{1}_{(-1,1)^\complement}(x)$, $x \in \mathbb{R}$.

2. Independence (and measurability)

One of the key tools is the following π-*system Lemma*:

Lemma (Lemma 1.6 in [Williams]). *$(E, \mathcal{E})$ is a measurable space with $\mathcal{E} = \sigma(\Pi)$ and Π a set of subsets of E with the property of being $\cap$-stable (i.e., $B_1, B_2 \in \Pi$ implies $B_1 \cap B_2 \in \Pi$). Suppose now that μ_1 and μ_2 are measures on $(E, \mathcal{E})$ that coincide on Π and that $\mu_1(E) = \mu_2(E) < \infty$. Then $\mu_1 = \mu_2$.*

Ex. 2.1*: We work on $(\Omega, \mathcal{A}, \mathbb{P})$. Suppose $\mathcal{G}_1 = \sigma(\Pi_1)$ and $\mathcal{G}_2 = \sigma(\Pi_2)$, with Π_1 and Π_2 π-systems of elements of $\mathcal{A}$. Show that $\mathbb{P}(A_1 \cap A_2) = \mathbb{P}(A_1)\mathbb{P}(A_2)$ for every choice of $A_1 \in \Pi_1$ and $A_2 \in \Pi_2$ implies $\mathcal{G}_1 \perp\!\!\!\perp \mathcal{G}_2$.

Hint. Fix $A_2 \in \Pi_2$ and define the two finite measures $\nu_1(A) := \mu_1(A \cap A_2)$ and $\nu_2(A) := \mu_2(A \cap A_2)$ for every $A \in \mathcal{A}$ and use the π-system Lemma to conclude that $\nu_1 = \nu_2$. Then repeat the other way around.

Ex. 2.2*: Show that the real random variables $X_1, \ldots, X_n$ are independent iff for every $x_1, \ldots, x_n \in \mathbb{R}$

$$\mathbb{P}\left(X_j \le x_j \text{ for } j = 1, \ldots, n\right) = \prod_{j=1}^{n} \mathbb{P}(X_j \le x_j)\,.$$

Ex. 2.3*: $(X_j)_{j \in \mathbb{N}}$ are independent. Show that for $k \in \mathbb{N}$

$$\sigma(X_1, \ldots, X_k) \perp\!\!\!\perp \sigma(X_{k+1}, X_{k+2}, \ldots).$$

Ex. 2.4: We consider $\Omega = [0,1)$, $\mathcal{A} = \mathcal{B}(\Omega)$ and $\mathbb{P}$ is the Lebesgue measure. For $n = 1, 2, \ldots$ we consider the σ-algebra

$$\mathcal{D}_n := \sigma\Big(\big[(j-1)2^{-n}, j2^{-n}\big) : j = 1, \ldots, 2^n\Big)\,,$$

of subsets of Ω. We set also $A_n := \sqcup_{j=1}^{2^{n-1}} [(2j-1)2^{-n}, 2j\, 2^{-n})$, for $n = 1, 2, \ldots$

(1) Show that, for every $B \in \mathcal{D}_n$, $A_{n+1} \perp\!\!\!\perp B$.
(2) Show that $(\mathbf{1}_{A_n})_{n=1,2,\ldots}$ is an IID B$(1/2)$ sequence.

Ex. 2.5: (X_n) is an IID B$(1/2)$ sequence and we set $X(\omega) := \sum_{n=1}^{\infty} 2^{-n} X_n(\omega)$.

(1) Show that X is a RV and that $P(X \in (0,1)) = 1$.
(2) show that

$$2\, F_X(x) = \begin{cases} F_X(2x) & \text{if } x \in [0, 1/2], \\ 1 + F_X(2x-1) & \text{if } x \in [1/2, 1]. \end{cases} \qquad (\star)$$

(3) Show that $(\star)$ determines the value of $F_X(x)$ for x dyadic rational in $[0,1]$ (i.e., x is rational with denominator that is a power of 2). Conclude that $(\star)$ determines F_X, so $X \sim U([0,1])$. *Obs.:* Note that this and **Ex. 2.3** show that the law of fair coin tossing and Lebesgue on $[0,1]$ are *essentially* the same measure, in the sense that they are the same measure up to a measurable bijection of the spaces.
(4) Give, in the canonical space, X_1 and X_2 independent and $U([0,1])$.
(5) Give, in the canonical space, a sequence (X_n) of IID $U([0,1])$ random variables.

3. Convergence issues

Ex. 3.1*: X is a RV and $(X_n)_{n\in\mathbb{N}}$ is a sequence of RVs on the same probability space. Show that if $\sum_n \mathbb{P}(|X_n - X| > \varepsilon) < \infty$ for every $\varepsilon > 0$, then $X_n \to X$ a.s..

Obs.: this is a direct application of BC1, but make the effort of writing *as explicitly as possible* an event Ω_0 such that $\mathbb{P}(\Omega_0) = 1$ and such that $\lim_n X_n(\omega) = X(\omega)$ for every $\omega \in \Omega_0$.

Ex. 3.2*: In this exercise we give a sequence of RV that converges (to zero) in probability, but not almost surely. On the canonical space we consider for $k = 2, 3, \ldots$ the partition

$$I_{k,1} := (0, 1/k), \quad I_{k,2} := [1/k, 2/k), \ldots, \quad I_{k,k} := ((k-1)/k, 1),$$

in intervals of length $1/k$. For $n = 2, 3 \ldots$ we set $X_n := \mathbf{1}_{I_{k(n),j(n)}}$ with

$$k(n) := 1 + \max\left\{ m = 1, 2, \ldots : \frac{m(m+1)}{2} < n \right\} \quad \text{and} \quad j(n) := n - \frac{k(n)(k(n)-1)}{2}.$$

(1) Show that $\lim_{n\to\infty} X_n = 0$ in probability.
(2) Show that for every $\omega \in \Omega$ we have $\sum_n \mathbf{1}_{\{1\}}(X(\omega)) = \infty$ and conclude that (X_n) does not converge a.s..

Important observation: The only complicated point is understanding that $(I_{k(n),j(n)})$ is a "rotating" sequence of intervals that repeatedly cover all of $(0, 1)$.

Ex. 3.3: $(X_n)_{n\in\mathbb{N}}$ is an IID Exp(1) sequence ($X \sim \mathrm{Exp}(1)$ if $\mathbb{P}(X > t) = \exp(-t)$ for $t > 0$). Let $Y_n := X_n/\log(n)$. Show that $Y_n \to 0$ in $\mathbb{L}^p$ for every $p \geq 1$ (hence in probability too), but the sequence (Y_n) does not converge a.s..

Ex. 3.4*: $(X_n)_{n\in\mathbb{N}}$ is an IID sequence. Show that $X_1 \in \mathbb{L}^1$ iff $\lim_n X_n/n = 0$ a.s..

Hint. Start by observing that it suffices to show that $X_1 \in \mathbb{L}^1$ iff $\mathbb{P}(|X_n| > 1, \text{ i.o.}) = 0$. It may be useful to keep in mind that, if $X \geq 0$, then $\mathbb{E}[X] = \int_0^\infty \mathbb{P}(X > x)\,\mathrm{d}x$.

Ex. 3.5*: The aim of this exercise is to show the (a priori) very surprising result that if (X_n) converges to X in law then $\lim_n X_n^+ = X^+$, where $X_n^+(\omega) := \inf\{z : F_{X_n}(z) > \omega\}$ for $\omega \in \Omega = (0, 1)$ (we are of course making the canonical choice for $(\Omega, \mathcal{A}, \mathbb{P})$: recall the Skorohod Representation, Ex. 1.2) and analogous definition for X^+. Said otherwise, if $X_n \xrightarrow{\mathcal{L}} X$ then we can find a probability space and RV's (Y_n) and Y such that $Y_n \sim X_n$ for every n, $Y \sim X$ and $Y_n \to Y$ a.s..

For this we propose to establish in order: (we write $F = F_X$ and $F_n = F_{X_n}$)

(1) Show that the complement of C_F^0, i.e. the set of jumps of F, is at most countable and conclude that C_F^0 is dense in $\mathbb{R}$.
(2) For every $\omega \in (0, 1)$ choose $z \in C_F^0$ such that $z > X^+(\omega)$. Show that $F_n(z) > \omega$ for n large enough and conclude that $X_n^+(\omega) \leq z$ for the same values of n.
(3) Conclude that $\limsup_n X_n^+(\omega) \leq X^+(\omega)$ for every ω.
(4) Repeat/modify the argument to show that $\liminf_n X_n^-(\omega) \geq X^-(\omega)$ for every ω.
(5) Conclude that $X_n^+ \to X^+$ a.s. (we could write a bit informally $X_n^\pm \to X^\pm$ a.s).

Ex. 3.6*: $(X_n)_{n\in\mathbb{N}}$ is an IID sequence with X_1 not trivial, i.e. there exists x such that $F_X(x) \in (0,1)$ (one of the various ways of saying that X is not a.s. a constant).

(1) Explain why $(X_n)_{n\in\mathbb{N}}$ converges in law to X_1 (or to X_5 if you prefer!).
(2) Show that there exists $a < b$ such that $\mathbb{P}(X_1 \le a) > 0$ and $\mathbb{P}(X_1 \ge b) > 0$ and conclude that $\mathbb{P}(X_n \le a, \text{ i.o.}) = 1$ and $\mathbb{P}(X_n \ge b, \text{ i.o.}) = 1$.
(3) Conclude that $(X_n)_{n\in\mathbb{N}}$ does not converge a.s..
(4) Upgrade the argument to conclude that $(X_n)_{n\in\mathbb{N}}$ does not converge in probability either.

Remark: in this case it is immediate to construct a sequence (Y_n) and a random variable Y such that $Y_n \sim X_n$ for every n, $Y \sim X$ and $Y_n \to Y$ a.s.. Just choose $Y_n = Y = X_1$ for every n!

4. More about and around convergence in law

Ex. 4.1: Show that if $X_n \xrightarrow{\mathcal{L}} X$ in law, then $(X_n)_{n\in\mathbb{N}}$ is tight.

Ex. 4.2: Show that if there exists an increasing function $h : [0,\infty) \to [0,\infty)$ such that $\lim_{x\to\infty} h(x) = \infty$ and $\sup_n \mathbb{E}[h(|X_n|)] < \infty$, then (X_n) is tight.

Ex. 4.3*: this exercise is an interesting application of theorems from measure theory (Fubini-Tonelli!). It yields an inversion formula for the characteristic function of a RV. We use $\varphi = \varphi_X$, where of course $\varphi_X(t) = \mathbb{E}[\exp(itX)]$.

(1) Show (do justify the steps!) that for every $T > 0$ and $a, b \in \mathbb{R}$

$$\int_{-T}^{T} \frac{e^{-ita} - e^{-itb}}{it} \varphi(t)\,\mathrm{d}t \;=\; \int_{\mathbb{R}} \left(\int_{-T}^{T} \frac{e^{it(x-a)} - e^{it(x-b)}}{it}\,\mathrm{d}t \right) P_X(\,\mathrm{d}x)\,.$$

(2) Just a computation: show that

$$\frac{1}{2\pi} \int_{-T}^{T} \frac{e^{it(x-a)} - e^{it(x-b)}}{it}\,\mathrm{d}t \;=\; \frac{1}{\pi} \left[\mathrm{sign}(x-a) S\left(|x-a|T\right) - \mathrm{sign}(x-b) S\left(|x-b|T\right)\right]\,,$$

where $\mathrm{sign}(x) := 0$ if $x = 0$ and $S : [0,\infty) \to \mathbb{R}$ is the integral sine function, i.e. $S(y) := \int_0^y (\sin(x)/x)\,\mathrm{d}x$ for $y \geq 0$. Note that $\int_0^\infty (|\sin(x)|/x)\,\mathrm{d}x = \infty$, but $\lim_{y\to\infty} S(y)$ exists and it is equal to $\pi/2$. The convergence can be established for example via Dirichlet's test for convergence of series, and the fact that the limit is $\pi/2$ is best seen via integration in the complex plane (but for this exercise we can just accept it as a fact).

(3) Show Lévy Inversion Formula: for $a < b$

$$\lim_{T \nearrow \infty} \frac{1}{2\pi} \int_{-T}^{T} \frac{e^{-ita} - e^{-itb}}{it} \varphi(t)\,\mathrm{d}t \;=\; P_X((a,b)) + \frac{1}{2} P_X(\{a\}) + \frac{1}{2} P_X(\{b\})\,.$$

Hint. Study first the limit of the expression in point (2) separating the cases $x < a$, $x = a$, $x \in (a,b)$, $x = b$ and $x > b$.

(4) Explain why Lévy Inversion Formula allows to recover F_X if we know φ_X.

(5) Assume that $\varphi \in \mathbb{L}^1$, i.e. that $\int_{\mathbb{R}} |\varphi(t)|\,\mathrm{d}t < \infty$. Show that

$$F_X(b) - F_X(a) \;=\; \frac{1}{2\pi} \int_{\mathbb{R}} \frac{e^{-ita} - e^{-itb}}{it} \varphi(t)\,\mathrm{d}t\,.$$

*O*bs.: the proof of this result normally goes along with showing that $P_X(\{a\}) = P_X(\{b\}) = 0$. Do justify this point, i.e. that $F_X \in C^0$.

(6) In fact, if $\varphi \in \mathbb{L}^1$ then $F_X \in C^1$. Show it and show also that for every $x \in \mathbb{R}$

$$F_X'(x) \;=\; \frac{1}{2\pi} \int_{\mathbb{R}} e^{-itx} \varphi(t)\,\mathrm{d}t\,.$$

Food for thought. This shows that X is a continuous RV with density $f_X = F_X'$ and recall that $\varphi_X(t) = \int_{\mathbb{R}} e^{itx} f_X(x)\,\mathrm{d}x$, recovering thus the (possibly already) known inversion formula for the Fourier transform in the $\mathbb{L}^1$ context. If you are interested, meditate about the fact that the Fourier transform is naturally given in $\mathbb{L}^2$, where a priori is even ill-defined, but it does map $\mathbb{L}^2$ into and onto $\mathbb{L}^2$. Instead the characteristic function of a continuous RV, i.e. the Fourier transform of an $\mathbb{L}^1$ function, is not necessarily in $\mathbb{L}^1$ (just try with a uniform RV): this is consistent with the fact that requiring $\varphi_X \in \mathbb{L}^1$ implies that X has a continuous density.

Ex. 4.4: Not any functions $\varphi : \mathbb{R} \to \mathbb{C}$ is a characteristic function. What about

(1) $\sin(t)$;
(2) $\cos(t)$;
(3) $\cos(t^2)$;
(4) $e^{-t^2} \cos(t)^3/(1 - it)$.

Hint. Once a function satisfies the basic properties of a characteristic function, it may be of help to look for a RV whose characteristic function does the job.

Ex. 4.5: Show that if φ is a characteristic function, then $\exp(\varphi - 1)$ is too (of which random variable?).

Ex. 4.6: $(X_n)_{n\in\mathbb{N}}$ is a sequence of independent random variables and $S_n = X_1 + \ldots + X_n$. Assume that there exists two sequences of real numbers $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$, with $b_n > 0$ for every n and $\lim_{n\to\infty} b_n = \infty$, such that $((S_n - a_n)/b_n)_{n\in\mathbb{N}}$ converges in law to Y, with Y not trivial. Then the same sequence does not converge in probability.

Hint. Use Kolmogorov 0-1 law.

Ex. 4.7: Let us reconsider the (probably already known) problem of showing that if X_n is a binomial random variable of parameters $n \in \mathbb{N}$ and $p_n := \min(\lambda/n, 1)$ (with $\lambda > 0$), i.e. X_n is the sum of n independent Bernoulli RV of parameter p_n.

(1) Show that for $k = 0, 1, \ldots$ we have $\lim_{n\to\infty} \mathbb{P}(X_n = k) = \lambda^k e^{-\lambda}/k!$ and conclude that (X_n) converges in law to a Poisson random variable of parameter λ.
(2) Show the same convergence in law by using Lévy Continuity Theorem.

Ex. 4.8: (X_n) is an IID sequence and $\mathbb{P}(X_1 > x) = 1/x^\alpha$ for every $x \geq 1$, with $\alpha > 0$. Show that the sequence

$$\left(\frac{1}{n^{1/\alpha}} \max_{j=1,\ldots,n} X_j \right)_{n=1,2,\ldots}$$

converges in law. Show that the limit law is absolutely continuous with respect to the Lebesgue measure and give its density.

Obs.: also in this case the convergence cannot be upgraded to convergence in probability. Do you see why?