COMPUTATIONAL ASTROPHYSICS

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04. Viscous Stress Tensor

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Viscous Stress Tensor

The Viscous Stress Tensor

The stress tensor for a fluid is usually decomposed into two parts,

$$\boldsymbol{P} = p\boldsymbol{I} - \boldsymbol{\sigma}$$

where p is the scalar pressure, $\boldsymbol{I}$ is the 3×3 identity matrix and $\boldsymbol{\sigma}$ is called the *viscous stress tensor.*

This tensor gives the stress due to internal friction between the various parts of the fluid produced by differential motion.

The Viscous Stress Tensor

For a non-magnetized isotropic fluid, the viscous stress term is defined by imposing the following conditions,

- $\boldsymbol{\sigma}$ is a symmetric tensor.
- If there is no relative motion between the different parts of the fluid, there is no stress, i.e. $\boldsymbol{\sigma} = 0$. This condition implies that the viscous stress tensor must be proportional to the gradient of velocities in the fluid. If this gradient is small, we expect that the relation is lineal, i.e.

$$\sigma_{ij} \propto \partial_i v_j$$

The Viscous Stress Tensor

- We do not expect terms independent of the velocity gradient in $\boldsymbol{\sigma}$, i.e. if $\partial_i v_j = 0$, then $\sigma_{ij} = 0$.
- $\boldsymbol{\sigma} = 0$ for rigid body rotation, i.e. when $\overrightarrow{v} = \overrightarrow{\Omega} \times \vec{r}$ with $\overrightarrow{\Omega}$ a constant.

The Viscous Stress Tensor

All these conditions let us define the viscous stress tensor as*

$$\sigma_{ij} = 2\eta\tau_{ij}$$
$$\tau_{ij} = \frac{1}{2}\left(\partial_i v_j + \partial_j v_i - \frac{2}{3}\partial_k v_k \delta_{ij}\right)$$

donde $\eta = \eta(p, \rho)$ is the *dynamic coefficient of viscosity* of the medium.

It is also usual to define a *kinematic coefficient of viscosity, ν, by the relation $\eta = \rho\nu$.*

* See L. D. Landau and E. M. Lifschitz, Fluid Mechanics, Pergamon Press, London, UK (1959) for details of this definition.

Viscous Force

The total force due to the stress tensor is

$$\vec{f}_s = -\overrightarrow{\nabla} \cdot \boldsymbol{P} = -\overrightarrow{\nabla} \cdot (p\boldsymbol{I}) + \overrightarrow{\nabla} \cdot \boldsymbol{\sigma}$$

where the first term corresponds to the pressure gradient,

$$-\partial_j(pI^{ij})) = -\partial_j(p\delta^{ij})) = -\partial_i p$$

and the second term is the viscous force

$$\vec{f}^{(vis)} = \overrightarrow{\nabla} \cdot \boldsymbol{\sigma}.$$

Viscous Force

Using the definition of the viscous stress tensor in components, and considering a constant dynamic coefficient of viscosity, we obtain the force

$$f_i^{(vis)} = \eta \left(\partial_j \partial_j v_i - \frac{1}{3} \partial_i \partial_k v_k \right).$$

Considering now that the fluid is *incompressible,* $\vec{\nabla} \cdot \vec{v} = 0$ or equivalently $\partial_k v_k = 0$, we have the force

$$f_i^{(vis)} = \eta \partial_j \partial_j v_i.$$

Equation of Motion

The Equation of Motion (once more)

Using the viscous stress tensor, the equation of motion for the fluid (in the Lagrangian frame) will be

$$\rho \frac{d\vec{v}}{dt} = -\vec{\nabla} p + \vec{\nabla} \cdot \boldsymbol{\sigma} + \vec{f}_g + \rho_q \vec{E} + \vec{J} \times \vec{B}$$

or in the Eulerian frame,

$$\rho \left(\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \vec{\nabla} \vec{v} \right) = -\vec{\nabla} p + \vec{\nabla} \cdot \boldsymbol{\sigma} + \vec{f}_g + \rho_q \vec{E} + \vec{J} \times \vec{B}$$

Work done by the Viscosity Force

Work done by the viscosity force

The work done by the viscosity force is

$$W^{(vis)} = \vec{v} \cdot \vec{f}^{(vis)} = \vec{v} \cdot \left(\vec{\nabla} \cdot \boldsymbol{\sigma} \right)$$

or in components,

$$W^{(vis)} = \partial_j \left(v_i \sigma_{ij} \right) - \sigma_{ij} \partial_j v_i.$$

This relation may be written, symbolically, as

$$W^{(vis)} = \vec{\nabla} \cdot \left(\vec{v} \cdot \boldsymbol{\sigma} \right) - \boldsymbol{\sigma} : \vec{\nabla} \vec{v}.$$

Work done by the viscosity force

Integrating the work done by the viscosity force in the total volume of the fluid, we have

$$\int_{V_T} W^{(vis)} dV = \oint_{\partial V_T} \left(\overrightarrow{v} \cdot \boldsymbol{\sigma}\right) \cdot d\overrightarrow{S} - \int_{V_T} \boldsymbol{\sigma} : \overrightarrow{\nabla} \overrightarrow{v}\, dV$$

The first term corresponds to the work done on the surface while the second corresponds to the work done on the volume. Hence, if the first term is zero, the second term will represent the lost of kinetic energy that from the fluid (if $\boldsymbol{\sigma} : \overrightarrow{\nabla} \overrightarrow{v} > 0$) due to viscosity.

Work done by the viscosity force

It is defined the *rate of volumetric heating due to viscosity* as

$$Q^{(vis)} = \boldsymbol{\sigma} : \vec{\nabla}\,\vec{v}.$$