COMPUTATIONAL ASTROPHYSICS

Observatorio
Astronómico
Nacional

Computational Astrophysics

06. Electrodynamics

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Maxwell Equations

Maxwell Equations

$$\overrightarrow{\nabla} \cdot \overrightarrow{E} = \frac{\rho_q}{\epsilon_0}$$

$$\overrightarrow{\nabla} \cdot \overrightarrow{B} = 0$$

$$\overrightarrow{\nabla} \times \overrightarrow{E} = -\frac{\partial \overrightarrow{B}}{\partial t}$$

$$\overrightarrow{\nabla} \times \overrightarrow{B} = \mu_0 \overrightarrow{J} + \frac{1}{c^2}\frac{\partial \overrightarrow{E}}{\partial t}$$

where

ϵ_0 : electric permitivity of vacuum

μ_0 : magnetic permeability of vacuum

Ohm’s Law

The dynamics of the fields and the fluid are coupled by *Ohm’s law*,

$$\overrightarrow{E}' = \eta_e \overrightarrow{J}$$

where

η_e : electric resistivity of the fluid

$\overrightarrow{E}'$: electric field measured in the co-moving frame (moving with velocity $\overrightarrow{v}$).

Using a Lorentz transformation we obtain the relation with the fields in the stationary frame,

$$\overrightarrow{E}' = \frac{\overrightarrow{E} + \overrightarrow{v} \times \overrightarrow{B}}{\sqrt{1 - \frac{v^2}{c^2}}}.$$

Non-relativistic Limit

Ohm's Law (Non-relativistic form)

Since we will consider non-relativistic MHD, we use the approximation

$$\left(1-\frac{v^2}{c^2}\right)^{-1/2} = 1 - \frac{1}{2}\frac{v^2}{c^2} + \ldots$$

and therefore we may write Ohm's law in the approximated form

$$\overrightarrow{E} + \overrightarrow{v} \times \overrightarrow{B} \approx \eta_e \overrightarrow{J}.$$

Ampere's Law (Non-relativistic form)

Ampere's law is also approximated by considering a non-relativistic limit (small velocities) and consequently low frequency motion. First, note that

$$\frac{\left|\frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}\right|}{\left|\vec{\nabla}\times\vec{B}\right|} \sim \frac{\frac{E_0\omega}{c^2}}{\frac{B_0}{L}} \sim \frac{v_0\omega L}{c^2} \sim \frac{v_0^2}{c^2} \ll 1$$

where we have considered an ideal MHD mode, with $\eta_e = 0$, in which Ohm's law implies $E_0 \sim v_0 B_0$. Under this approximation, Ampere's law becomes

$$\vec{\nabla}\times\vec{B} = \mu_0 \vec{J}$$

Lorentz Force (Non-relativistic form)

Lorentz force (per unit volume) establishes

$$\vec{f}_L = \rho_q \overrightarrow{E} + \overrightarrow{J} \times \overrightarrow{B}.$$

However, in the non-relativistic limit we have

$$\frac{\left| \rho_q \overrightarrow{E} \right|}{\left| \overrightarrow{J} \times \overrightarrow{B} \right|} \sim \frac{\rho_q E_0}{J B_0} \sim \frac{\rho_q v_0}{J}$$

Lorentz Force (Non-relativistic form)

From Gauss' law we have $\rho_q = \epsilon_0 \vec{\nabla} \cdot \vec{E}$ and from Ampere's law we get $\vec{J} = \dfrac{\vec{\nabla} \times \vec{B}}{\mu_0}$. Hence

$$\frac{\left|\rho_q \vec{E}\right|}{\left|\vec{J} \times \vec{B}\right|} \sim \frac{\epsilon_0 \mu_0 E_0 v_0}{B_0} \sim \frac{v_0^2}{c^2} \ll 1$$

Therefore, the Lorentz force is approximated in this formulation of the MHD as

$$\vec{f}_L \approx \vec{J} \times \vec{B}.$$

Electrodynamics equations

Electrodynamics equations

A summary of the results shows a set of 11 equations with 14 variables: $(\overrightarrow{v}, \overrightarrow{E}, \overrightarrow{B}, \overrightarrow{J}, \rho_q, \eta_e)$

$$\overrightarrow{\nabla} \cdot \overrightarrow{E} = \frac{\rho_q}{\epsilon_0}$$

$$\overrightarrow{\nabla} \cdot \overrightarrow{B} = 0$$

$$\overrightarrow{\nabla} \times \overrightarrow{E} = -\frac{\partial \overrightarrow{B}}{\partial t}$$

$$\overrightarrow{\nabla} \times \overrightarrow{B} \approx \mu_0 \overrightarrow{J}$$

$$\overrightarrow{E} + \overrightarrow{v} \times \overrightarrow{B} \approx \eta_e \overrightarrow{J}$$

Electrodynamics equations

We can re-write the system as follows: First, we eliminate the electric field from Gauss' law using Ohm's law,

$$\rho_q = \epsilon_0 \vec{\nabla} \cdot \vec{E} = \epsilon_0 \vec{\nabla} \cdot \left(\eta_e \vec{J} - \vec{v} \times \vec{B} \right).$$

It is also possible to re-write Faraday's law using Ohm's and Ampere's laws to eliminate $\vec{E}$ and $\vec{J}$. This gives

$$\frac{\partial \vec{B}}{\partial t} = \vec{\nabla} \times \left(\vec{v} \times \vec{B} - \frac{\eta_e}{\mu_0} \vec{\nabla} \times \vec{B} \right).$$

Electrodynamics equations

Hence, the electrodynamics system will be stated as a set of only 4 equations and 7 variables: $(\overrightarrow{v}, \overrightarrow{B}, \eta_e)$,

$$\frac{\partial \overrightarrow{B}}{\partial t} = \overrightarrow{\nabla} \times \left(\overrightarrow{v} \times \overrightarrow{B} - \frac{\eta_e}{\mu_0} \overrightarrow{\nabla} \times \overrightarrow{B} \right)$$

$$\overrightarrow{\nabla} \cdot \overrightarrow{B} = 0$$

Finally, we will also include in the MHD formulation the approximated form of the Lorentz law,

$$\vec{f}_L = \overrightarrow{J} \times \overrightarrow{B} = \frac{1}{\mu_0} (\overrightarrow{\nabla} \times \overrightarrow{B}) \times \overrightarrow{B}.$$