COMPUTATIONAL ASTROPHYSICS

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03. Airy Diffraction

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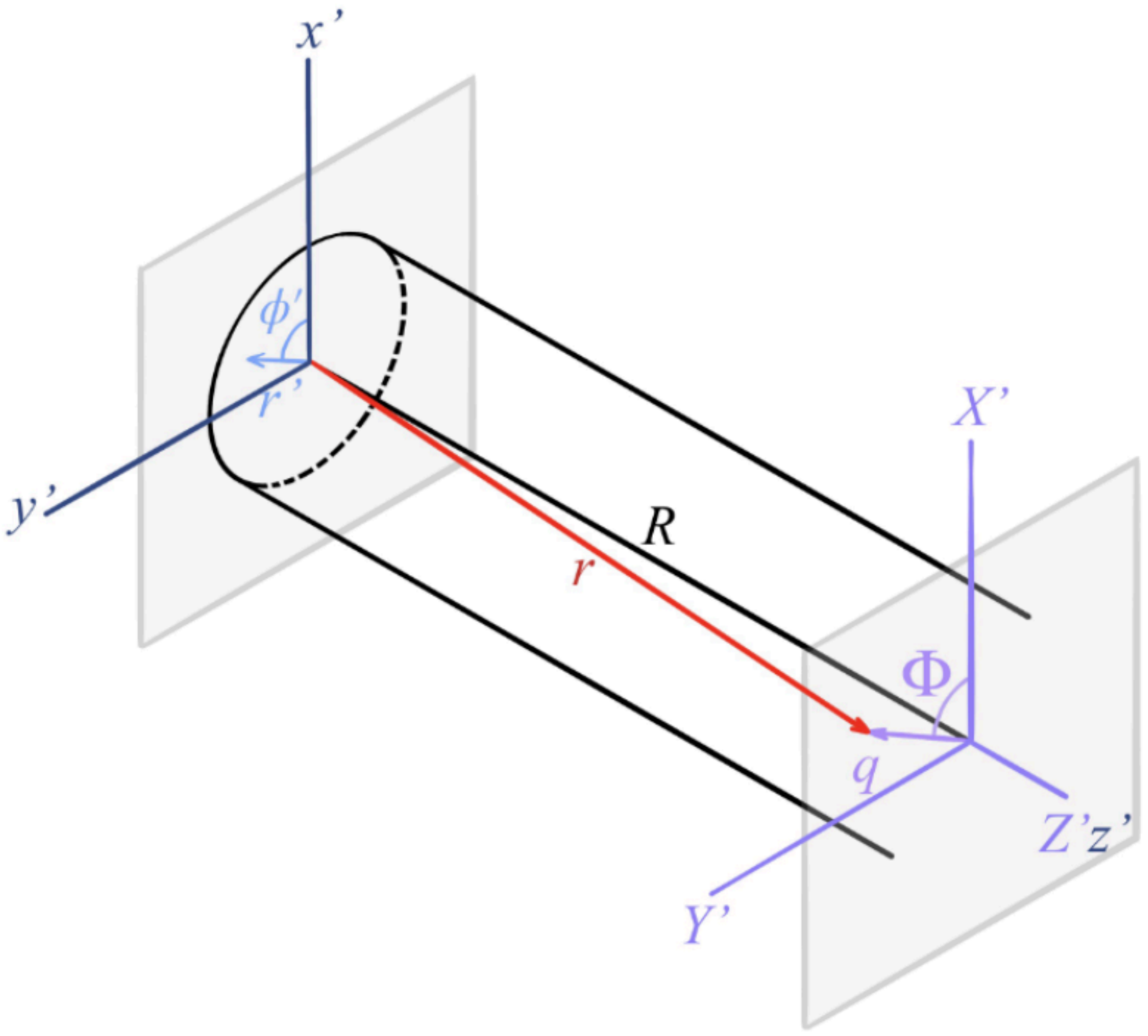
Diffraction



- Plane wave incident into a circular hole with radius a
- We want to calculate the intensity of light registered in a plane screen located at a distance $R \gg a$
- The electric field at the circular aperture is constant,

$$E(t, x', y') = E_0 e^{-i\omega t}$$

The Fourier transform of the electric field gives

$$E(t, k_x, k_y) = \frac{e^{ikr}}{\lambda r} \int_{-\infty}^{-\infty} \int_{-\infty}^{-\infty} E(t, x', y') e^{-i(k_x x' + k_y y')} dx' dy'$$

Using polar coordinates (r', ϕ') we have

$$E(t, k_x, k_y) = \frac{e^{ikr}}{\lambda r} \int_{0}^{a} \int_{0}^{2\pi} E(t, r', \phi') e^{-i(k_x r' \cos\phi' + k_y r' \sin\phi')} r' dr' d\phi'$$

Introducing the coordinates (X, Y) at the screen plane and approximating

$$k_x \approx k\frac{X}{r}$$

$$k_y \approx k\frac{Y}{r}$$

and the polar coordinates (q, Φ) at the screen plane, gives the relation

$$E(t,q) = \frac{e^{i(kr-\omega t)}}{\lambda r}\int_0^a \int_0^{2\pi} E_0 \exp\left[-\frac{ikr'q}{r}\cos(\phi' - \Phi)\right] r'dr'd\phi'$$

Due to the azimuthal symmetry, we can take $\Phi = 0$, without losing generality, giving

$$E(t,q) = \frac{e^{i(kr-\omega t)}}{\lambda r} \int_0^a \int_0^{2\pi} E_0 \exp\left[-\frac{ikr'q}{r}\cos(\phi')\right] r'dr'd\phi'$$

Introducing the Bessel functions of the first kind,

$$J_m(u) = \frac{1}{2\pi} \int_0^{2\pi} e^{i(mv+u\cos v})dv$$

We recognize, in the expression for the electric field, the $m = 0$ Bessel function,

$$\mathcal{J} = \int_0^{2\pi} \exp\left[-\frac{ikr'q}{r}\cos(\phi')\right] d\phi' = 2\pi J_0(u)\ ,$$

if $u = \dfrac{kr'q}{r}$.

Hence, we can write

$$E(t,q) = \frac{2\pi E_0 e^{i(kr-\omega t)}}{\lambda r} \int_0^a J_0\left(\frac{kr'q}{r}\right) r' dr'$$

where

$$J_0\left(\frac{kr'q}{r}\right) = \frac{1}{2\pi}\mathcal{J}.$$

Using the recurrence relation

$$\frac{d}{du}\left[u^m J_m(u)\right] = u^m J_{m-1}(u)$$

The integration gives

$$E(t,\theta) = \frac{AE_0 e^{i(kr-\omega t)}}{\lambda r}\frac{2J_1\left(ka\theta\right)}{ka\theta}$$

where $\theta = \dfrac{q}{r}$ and $A = \pi a^2$.

The intensity of the radiation is given by

$$I(ka\theta) = \frac{c}{8\pi}E(t,\theta)E^*(t,\theta)$$

which gives

$$I(k\rho) = \frac{cA^2E_0^2}{8\pi\lambda^2r^2}\left[\frac{2J_1\left(k\rho\right)}{k\rho}\right]^2$$

where $\rho = a\theta$.

Knowing that

$$\left.\frac{dJ_1}{du}\right|_{u=0} = \frac{1}{2}$$

We can write the final result as

$$\boxed{I(k\rho) = I_0 \left[\frac{2J_1\left(k\rho\right)}{k\rho}\right]^2}$$

where $I_0 = I(0) = \dfrac{cA^2E_0^2}{8\pi\lambda^2 r^2}$.

This equation gives the Airy's diffraction pattern for circular diffraction.