COMPUTATIONAL ASTROPHYSICS

Observatorio
Astronómico
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Computational Astrophysics

07. Closure Equations

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MHD Equations (up to this point)

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$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v}) = 0$$

$$\rho \left(\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \vec{\nabla} \vec{v} \right) = - \vec{\nabla} p + \vec{\nabla} \cdot \boldsymbol{\sigma} + \vec{f}_g + \vec{f}_L$$

$$\frac{\partial (\rho e)}{\partial t} + \vec{\nabla} \cdot (e \rho \vec{v}) = - p \vec{\nabla} \cdot \vec{v} + \vec{v} \cdot \vec{f} - \vec{\nabla} \cdot \vec{F}_{rad} - \vec{\nabla} \cdot \vec{q} + \boldsymbol{\sigma} : \vec{\nabla} \vec{v} + \eta_e J^2$$

$$\frac{\partial \vec{B}}{\partial t} = \vec{\nabla} \times \left(\vec{v} \times \vec{B} - \frac{\eta_e}{\mu_0} \vec{\nabla} \times \vec{B} \right)$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\boxed{\vec{f}_L = \frac{1}{\mu_0} (\vec{\nabla} \times \vec{B}) \times \vec{B}}$$

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$$\vec{\nabla} \cdot \vec{B} = 0$$

We have 9 equations and 22 variables: $(\rho, e, p, \vec{v}, \eta_e, \vec{B}, \boldsymbol{\sigma}, \vec{q}, \vec{F}_{rad})$.

We need 13 equations to close the system!

Closure equations

Radiative Flux

The radiative flux vector is

$$\overrightarrow{F}_{rad} = \int d\nu \int d\Omega \hat{n} I_\nu(\hat{n}, \vec{r})$$

where $I_\nu(\hat{n}, \vec{r})$ is the specific intensity evaluated at $\vec{r}$ in the direction $\hat{n}$. This quantity is given by a conservation equation for the radiation field. In general it can be written as a function of the density, ρ, and the temperature of the gas, T. Hence, one of the closure equations, arising from the astrophysics description, is

$$\overrightarrow{F}_{rad} = \overrightarrow{F}_{rad}(\rho, T)$$

Conductive Flux of Heat

The conductive flux of heat is represented by the vector $\overrightarrow{q}$. From the kinetic theory of gases it is possible to obtain an expression for this vector as a function of the density and the temperature of the gas (and eventually the magnetic field). This gives another closure equation,

$$\overrightarrow{q} = \overrightarrow{q}(\rho, T, \overrightarrow{B})$$

Stress tensor

The stress tensor, $\boldsymbol{\sigma}$, has 6 independent components due to its symmetry properties. In general, we may expect a dependence of the form

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}(\rho, \overrightarrow{v}, \overrightarrow{B}),$$

which corresponds to the third closure equation.

Stress tensor

As an example, for a de-magnetized fluid, the viscous force produces the stress tensor

$$\sigma_{ij} = 2\eta\tau_{ij}$$

$$\tau_{ij} = \frac{1}{2}\left(\partial_i v_j + \partial_j v_i - \frac{2}{3}\partial_k v_k \delta_{ij}\right).$$

Equation of State

The final closure relation is the so-called *equation of state,* which relates pressure and density of the gas,

$$p = p(\rho)$$

Equation of State for an Ideal Fluid

Ideal Fluid

Consider the simplest relation between density and pressure,

$$\rho e = np$$

where we introduce the energy e and a constant n. Replacing this relation into the energy conservation equation in the Eulerian frame gives

$$\frac{dp}{dt} + \vec{v} \cdot \vec{\nabla} p = -\left(1 + \frac{1}{n}\right) p \vec{\nabla} \cdot \vec{v} + \frac{1}{n}\left[\vec{v} \cdot \vec{f} - \vec{\nabla} \cdot \vec{F}_{rad} - \vec{\nabla} \cdot \vec{q} + \boldsymbol{\sigma} : \vec{\nabla} \vec{v} + \eta_e J^2\right]$$

or in the Lagrangian frame as

$$\frac{dp}{dt} = -\left(1 + \frac{1}{n}\right) p \vec{\nabla} \cdot \vec{v} + \frac{1}{n}\left[\vec{v} \cdot \vec{f} - \vec{\nabla} \cdot \vec{F}_{rad} - \vec{\nabla} \cdot \vec{q} + \boldsymbol{\sigma} : \vec{\nabla} \vec{v} + \eta_e J^2\right]$$

Ideal Fluid

In both cases, the term $p\overrightarrow{\nabla} \cdot \overrightarrow{v}$ represents the reversible work due to the factor pV. All the other terms in the right hand side represent *heating irreversible processes.*

If there are no irreversible processes, the gas is said to be an *Ideal Fluid* and its energy equation will be

$$\frac{dp}{dt} = -\left(1 + \frac{1}{n}\right) p\overrightarrow{\nabla} \cdot \overrightarrow{v}$$

Ideal Fluid

Using the equation of continuity (in the Lagrangian frame), it is possible to integrate the energy equation to obtain the *Polytropic State Equation,*

$$p = \kappa \rho^{\gamma}$$

where κ is an integration constant and $\gamma = 1 + \dfrac{1}{n}$ is called the *adiabatic index*.

Examples of Polytropic Equations of State

Ideal Gas

The simplest equation of state is the *ideal gas law,*

$$p = \frac{kT}{\mu m_H}\rho,$$

where

k : Boltzmann constant

$m_H \sim m_p$: mass of the Hydrogen atom (approx. the mass of the proton)

μ : mean molecular wight, i.e. the mean mass per particle in the gas (measured in units of m_H)

- $\mu = 1$: for neutral H
- $\mu = \frac{1}{2}$: for completely ionized H
- $\frac{1}{2} \leq \mu \leq 1$: for Hydrogen depending on the ionization state

Isothermal Flow

An isothermal flow has $T =$ constant. Hence, the ideal gas law becomes

$$p \propto \rho.$$

This means that the isothermal flow satisfies a polytropic equation of state with an adiabatic index $\gamma = 1$.

General Ideal Gas

From the kinetic theory and the equipartition of energy, it is possible to find a more general equation for the internal energy of a gas, ε, in the form

$$\varepsilon = \frac{\beta}{2}\frac{kT}{\mu m_H},$$

where the coefficient β determines the nature of the gas:

$\beta = 3$: Monoatomic

$\beta = 5$: Diatomic

$\beta = 6$: Multiatomic

Adiabatic Flow

For an adiabatic flow, $dQ = 0$, the first law gives

$$d\varepsilon = d\left(\frac{\beta}{2}\frac{kT}{\mu m_H}\right) = -pd\left(\frac{1}{\rho}\right)$$

$$\frac{\beta}{2}\frac{kdT}{\mu m_H} = \frac{p}{\rho^2}d\rho.$$

Introducing the ideal gas law to eliminate the pressure gives

$$\frac{\beta}{2}\frac{dT}{T} = \frac{d\rho}{\rho}.$$

Adiabatic Flow

Integrating this equation, we obtain the relation

$$T^{\beta/2}\rho^{-1} = \text{constant}$$

and using again the ideal gas law to replace the temperature in terms of p,

$$p \propto \rho^{(2+\beta)/\beta}.$$

For a monoatomic gas, $\beta = 3$, this gives

$$p \propto \rho^{5/3}.$$

Gas dominated by radiation pressure

Radiation pressure is described by the equation

$$p = \frac{1}{3} a T^4$$

and the internal energy of the gas is

$$\epsilon \rho = 3p = aT^4.$$

Gas dominated by radiation pressure

Considering again an adiabatic flow, $dQ = 0$, the first law gives now

$$d\varepsilon = d\left(\frac{aT^4}{\rho}\right) = -\,pd\left(\frac{1}{\rho}\right).$$

Integration of this equation gives the relation

$$p \propto \rho^{4/3}.$$

Degenerate Gas (non-relativistic)

In this case, the gas pressure is not produced by the microscopic motion of particles nor by the radiation but is due to the degeneracy pressure of quantum particles (e.g. electrons or neutrons). The model states that

$$p \propto \rho E_F$$

where the *Fermi energy* in the *non-relativistic case* is given by

$$E_F = \frac{p^2}{2m} = \frac{1}{2m}\left(\frac{h}{\lambda}\right)^2$$

with h the Planck's constant and λ the wavelength associated with the degenerate particles. For a completely degenerated gas, this wavelength is equal to the mean space between particles,

$$\lambda \sim n^{-1/3} \propto \rho^{-1/3}.$$

Degenerate Gas (non-relativistic)

Replacing the Fermi energy for a completely degenerate non-relativistic gas, gives the relation

$$p \propto \rho^{5/3}$$

Degenerate Gas (relativistic)

The Fermi energy for a completely degenerate relativistic gas is given by

$$E_F \sim Pc = c\left(\frac{h}{\lambda}\right)$$

which produces an equation of state with the form

$$p \propto \rho^{4/3}$$