COMPUTATIONAL ASTROPHYSICS

Observatorio
Astronómico
Nacional

Computational Astrophysics

03. Equation of Motion

Eduard Larrañaga
Observatorio Astronómico Nacional
Universidad Nacional de Colombia

Equation of Motion for the Fluid

Equation of Motion

The equation of motion for the fluid may be written as

$$\rho \frac{d\overrightarrow{v}}{dt} = \vec{f} + \vec{f}_s$$

where $\vec{f}$ and $\vec{f}_s$ represent the forces acting on the fluid per unit volume. We divide these into two types: volumetric forces and surface forces.

Volumetric Forces

The term $\vec{f}$ in the equation of motion represents the *volumetric forces* that act on the centroid of the volume element of the fluid. Some of them are gravitational and electromagnetic interactions.

- **Gravitational Force.** This term can be written as

$$\vec{f}_g = \rho \overrightarrow{g}$$

where we write the gravitational field $\overrightarrow{g}$ in terms of the gravitational potential ϕ as

$$\overrightarrow{g} = -\vec{\nabla}\phi$$

Volumetric Forces

- **Electromagnetic Force.** The electromagnetic force (a.k.a. Lorentz force) is

$$\vec{f}_L = \rho_q \overrightarrow{E} + \overrightarrow{J} \times \overrightarrow{B}$$

where we use

ρ_q : electric charge density

$\overrightarrow{J}$: electric current density

$\overrightarrow{E}$: electric field

$\overrightarrow{B}$: magnetic field

Surface Forces

The term $\vec{f}_s$ in the equation of motion represents the *surface forces* which are applied to the boundary of the volume element in the fluid. These are usually defined using the surface vector, $\overrightarrow{S}$, and a 3×3 stress tensor, $\boldsymbol{P}$.

Hence, the total surface force acting on a volume element is

$$\overrightarrow{F}_s = -\oint_{\partial V} \boldsymbol{P} \cdot d\overrightarrow{S}.$$

Surface Forces

Using Guass' theorem we transform the surface integral into a volume integral,

$$\vec{F}_s = -\int_V \vec{\nabla} \cdot \boldsymbol{P} dV$$

from which we identify the surface force per unit volume as

$$\vec{f}_s = -\vec{\nabla} \cdot \boldsymbol{P}.$$

Surface Forces

The divergence of the stress tensor is understood, when using components, as

$$f_s^i = -\frac{\partial P^{ij}}{\partial x^j} = -\left(\frac{\partial P^{i1}}{\partial x^1} + \frac{\partial P^{i2}}{\partial x^2} + \frac{\partial P^{i3}}{\partial x^3}\right).$$

The Equation of Motion revisited

Using the above results, we write the equation of motion for the fluid as

$$\rho \frac{d\overrightarrow{v}}{dt} = \vec{f}_g + \vec{f}_L - \overrightarrow{\nabla} \cdot \boldsymbol{P}$$

Theorem. The stress tensor is symmetric

We consider the angular momentum (per unit volume) of an element of the fluid,

$$\overrightarrow{\ell} = \vec{r} \times \rho \overrightarrow{v}.$$

Considering a constant density, i.e. $\dfrac{d\rho}{dt} = 0$, differentiation w.r.t time gives

$$\dot{\overrightarrow{\ell}} = \rho \vec{r} \times \frac{d\overrightarrow{v}}{dt}.$$

Theorem. The stress tensor is symmetric

Using the equation of motion with only the surface force term, we have

$$\dot{\overrightarrow{\ell}} = -\,\vec{r} \times \overrightarrow{\nabla} \cdot \boldsymbol{P}.$$

Integrating over the whole volume of the fluid gives the result

$$\dot{\overrightarrow{L}} = -\int_{V_T} \vec{r} \times \overrightarrow{\nabla} \cdot \boldsymbol{P} dV$$

Theorem. The stress tensor is symmetric

Writing this expression in cartesian components we have

$$\dot{L}^i = -\int_{V_T} \varepsilon^{ijk} r^j \partial_l P^{kl} dV.$$

Using Leibniz rule, the integrand is written as

$$\dot{L}^i = -\int_{V_T} \varepsilon^{ijk} \left[\partial_l (r^j P^{kl}) - P^{kj}\right] dV$$

$$\dot{L}^i = -\int_{V_T} \varepsilon^{ijk} \partial_l (r^j P^{kl}) dV + \int_{V_T} \varepsilon^{ijk} P^{kj} dV$$

Theorem. The stress tensor is symmetric

Using Gauss' theorem, the first term becomes a surface integral,

$$\dot{L}^i = -\oint_{\partial V_T} \varepsilon^{ijk} r^j P^{kl} dS^l + \int_{V_T} \varepsilon^{ijk} P^{kj} dV.$$

Here, the first term represents the total external torque applied to the surface while the second term represents the rate of change the internal angular momentum.

If the total external torque is zero, the total angular momentum must be a conserved quantity, $\dot{\vec{L}} = 0$, and therefore,

$$\int_{V_T} \varepsilon^{ijk} P^{kj} dV = 0$$

Theorem. The stress tensor is symmetric

Using the properties of the Levi-Civita symbol, we can write this relation as

$$\frac{1}{2}\int_{V_T} \varepsilon^{ijk}(P^{kj} - P^{jk})dV = 0,$$

givinig the desired result: $P^{jk} = P^{kj}$.

This symmetry property is general and prevents a divergence of the internal angular momentum of the fluid.