COMPUTATIONAL ASTROPHYSICS

Observatorio
Astronómico
Nacional

Computational Astrophysics

08. Final Equations of the MHD

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MHD Equations

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v}) = 0$$

$$\rho \left(\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \vec{\nabla} \vec{v} \right) = -\vec{\nabla} p + \vec{\nabla} \cdot \boldsymbol{\sigma} + \vec{f}_g + \frac{1}{\mu_0} (\vec{\nabla} \times \vec{B}) \times \vec{B}$$

$$\frac{\partial (\rho e)}{\partial t} + \vec{\nabla} \cdot (e \rho \vec{v}) = -p \vec{\nabla} \cdot \vec{v} + \vec{v} \cdot \vec{f} - \vec{\nabla} \cdot \vec{F}_{rad} - \vec{\nabla} \cdot \vec{q} + \boldsymbol{\sigma} : \vec{\nabla} \vec{v} + \eta_e J^2$$

$$\frac{\partial \vec{B}}{\partial t} = \vec{\nabla} \times \left(\vec{v} \times \vec{B} - \frac{\eta_e}{\mu_0} \vec{\nabla} \times \vec{B} \right)$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$P = P(\rho)$$

In general, 22 variables: $(\rho, e, p, \vec{v}, \eta_e, \vec{B}, \boldsymbol{\sigma}, \vec{q}, \vec{F}_{rad})$.

Equation of Motion

Equation of Motion

$$\rho\left(\frac{\partial\vec{v}}{\partial t}+\vec{v}\cdot\vec{\nabla}\vec{v}\right)=-\vec{\nabla}p+\vec{\nabla}\cdot\boldsymbol{\sigma}+\vec{f}_g+\frac{1}{\mu_0}(\vec{\nabla}\times\vec{B})\times\vec{B}$$

$$\rho\left(\frac{\partial\vec{v}}{\partial t}+\vec{v}\cdot\vec{\nabla}\vec{v}\right)=-\vec{\nabla}p+\vec{\nabla}\cdot\boldsymbol{\sigma}+\vec{f}_g-\frac{1}{\mu_0}\vec{\nabla}\left(B^2\right)+\frac{1}{\mu_0}\left(\vec{B}\cdot\vec{\nabla}\right)\vec{B}$$

Identity : $\vec{a}\times(\vec{b}\times\vec{c})=-(\vec{b}\times\vec{c})\times\vec{a}=\vec{b}(\vec{a}\cdot\vec{c})-\vec{c}(\vec{a}\cdot\vec{b})$

Equation of Motion

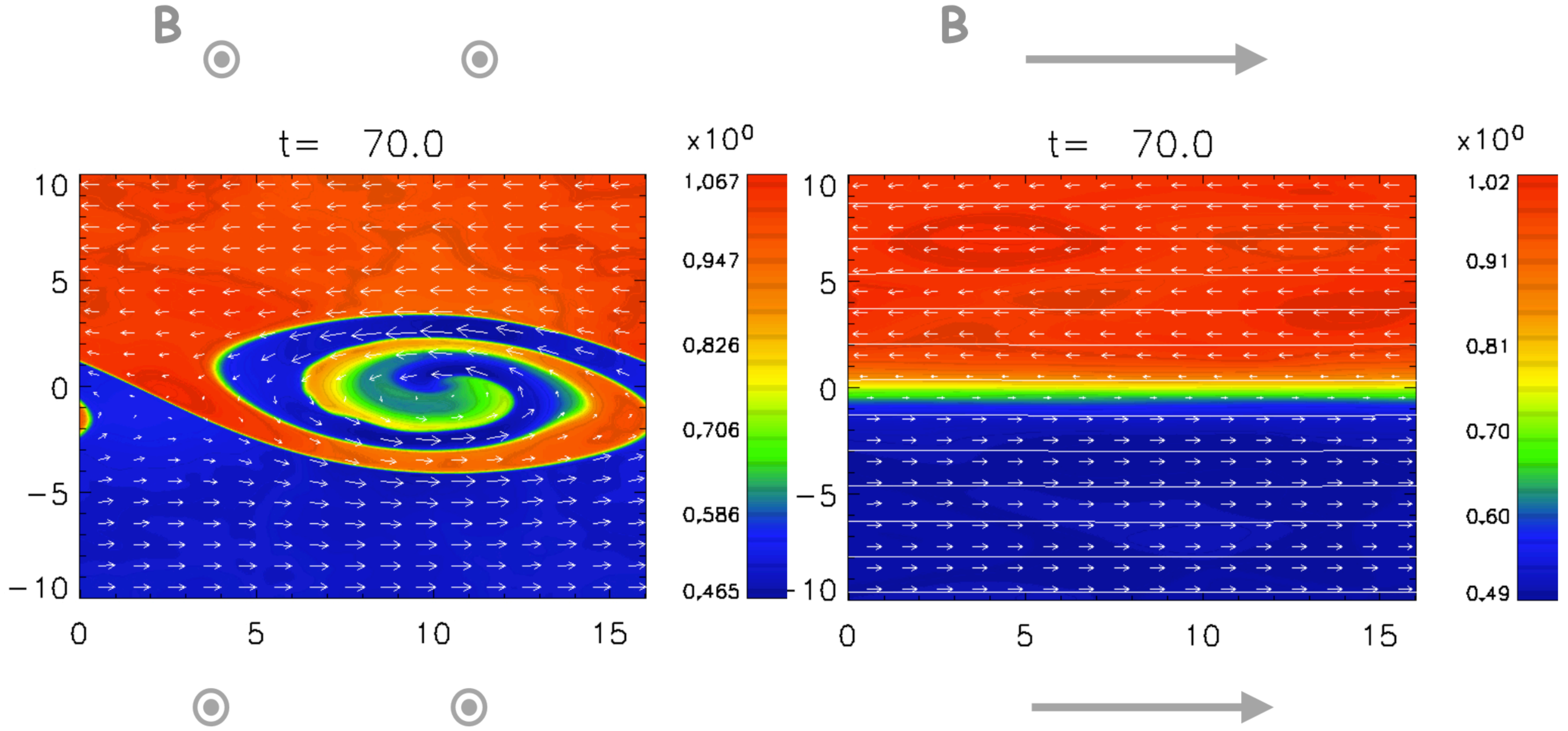
$$\rho\left(\frac{\partial \vec{v}}{\partial t} + \vec{v}\cdot\vec{\nabla}\vec{v}\right) = -\vec{\nabla}p + \vec{\nabla}\cdot\boldsymbol{\sigma} + \vec{f}_g + \frac{1}{\mu_0}(\vec{\nabla}\times\vec{B})\times\vec{B}$$

$$\rho\left(\frac{\partial \vec{v}}{\partial t} + \vec{v}\cdot\vec{\nabla}\vec{v}\right) = -\vec{\nabla}p + \vec{\nabla}\cdot\boldsymbol{\sigma} + \vec{f}_g - \frac{1}{\mu_0}\vec{\nabla}\left(B^2\right) + \frac{1}{\mu_0}\left(\vec{B}\cdot\vec{\nabla}\right)\vec{B}$$

Magnetic pressure

Magnetic tension

Effect of Magnetic Tension



Energy Equation

MHD Equations

$$\frac{\partial(\rho e)}{\partial t} + \overrightarrow{\nabla} \cdot (e\rho \overrightarrow{v}) = -p\overrightarrow{\nabla} \cdot \overrightarrow{v} + \overrightarrow{v} \cdot \vec{f} - \overrightarrow{\nabla} \cdot \overrightarrow{F}_{rad} - \overrightarrow{\nabla} \cdot \overrightarrow{q} + \boldsymbol{\sigma} : \overrightarrow{\nabla}\overrightarrow{v} + \eta_e J^2$$

Using the Ampere's law, $\overrightarrow{\nabla} \times \overrightarrow{B} = \mu_0 \overrightarrow{J}$, we have

$$\frac{\partial(\rho e)}{\partial t} + \overrightarrow{\nabla} \cdot (e\rho \overrightarrow{v}) = -p\overrightarrow{\nabla} \cdot \overrightarrow{v} + \overrightarrow{v} \cdot \vec{f} - \overrightarrow{\nabla} \cdot \overrightarrow{F}_{rad} - \overrightarrow{\nabla} \cdot \overrightarrow{q} + \boldsymbol{\sigma} : \overrightarrow{\nabla}\overrightarrow{v} + \frac{\eta_e}{\mu_0^2}\left(\overrightarrow{\nabla} \times \overrightarrow{B}\right)^2$$

Finally, the complete set of equations …

MHD Equations

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$$\rho \left(\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \vec{\nabla} \vec{v} \right) = -\vec{\nabla} p + \vec{\nabla} \cdot \boldsymbol{\sigma} + \vec{f}_g - \frac{1}{\mu_0} \vec{\nabla} \left(B^2 \right) + \frac{1}{\mu_0} \left(\vec{B} \cdot \vec{\nabla} \right) \vec{B}$$

$$\frac{\partial (\rho e)}{\partial t} + \vec{\nabla} \cdot (e \rho \vec{v}) = -p \vec{\nabla} \cdot \vec{v} + \vec{v} \cdot \vec{f} - \vec{\nabla} \cdot \vec{F}_{rad} - \vec{\nabla} \cdot \vec{q} + \boldsymbol{\sigma} : \vec{\nabla} \vec{v} + \frac{\eta_e}{\mu_0^2} \left(\vec{\nabla} \times \vec{B} \right)^2$$

$$\frac{\partial \vec{B}}{\partial t} = \vec{\nabla} \times \left(\vec{v} \times \vec{B} - \frac{\eta_e}{\mu_0} \vec{\nabla} \times \vec{B} \right)$$

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In general, 22 variables: $(\rho, e, p, \vec{v}, \eta_e, \vec{B}, \boldsymbol{\sigma}, \vec{q}, \vec{F}_{rad})$.