COMPUTATIONAL ASTROPHYSICS

Observatorio
Astronómico
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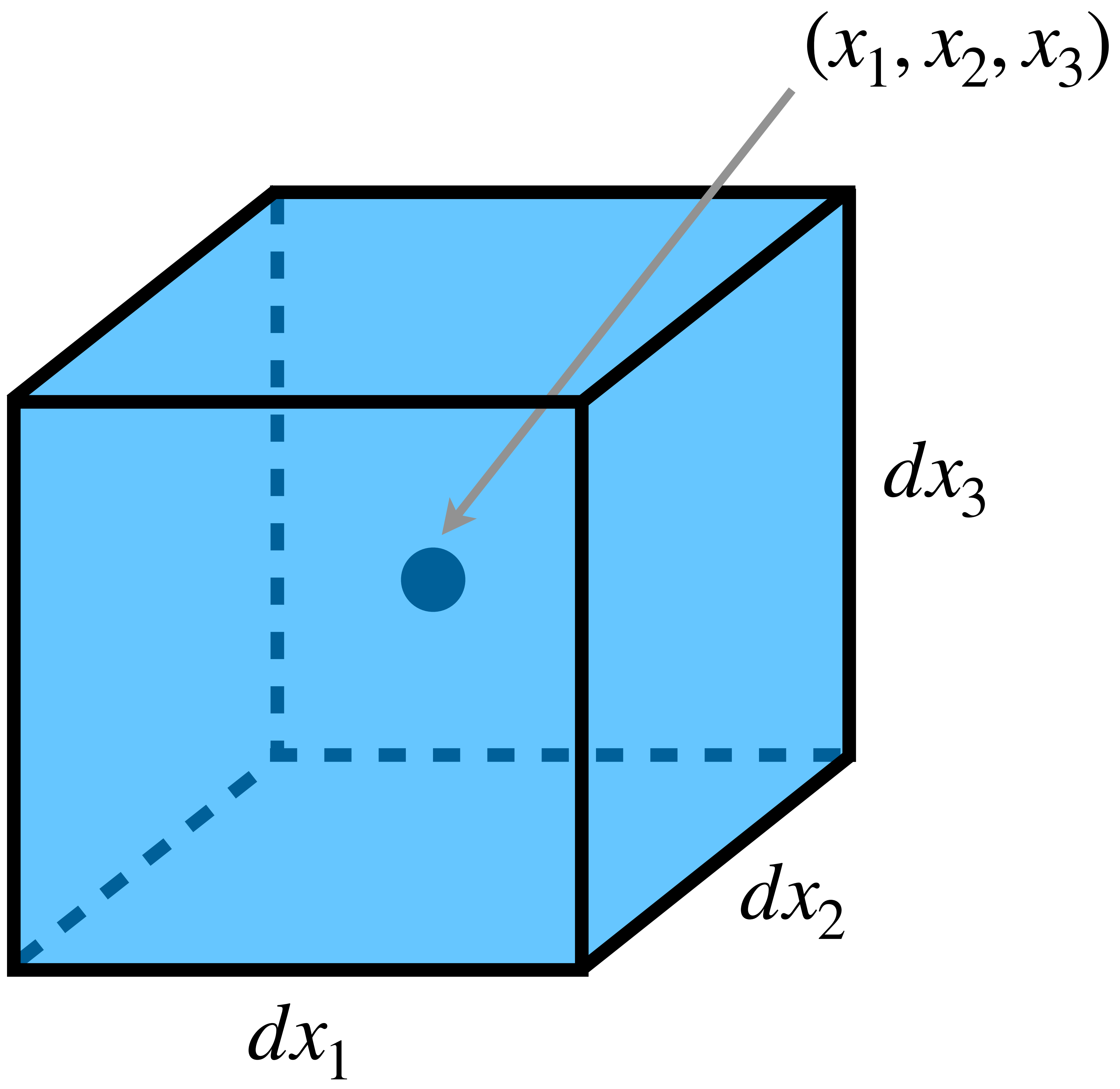
Computational Astrophysics

02. The Equation of Continuity and the Conservation of Mass

Eduard Larrañaga
Observatorio Astronómico Nacional
Universidad Nacional de Colombia

Eulerian and Lagrangian Frames

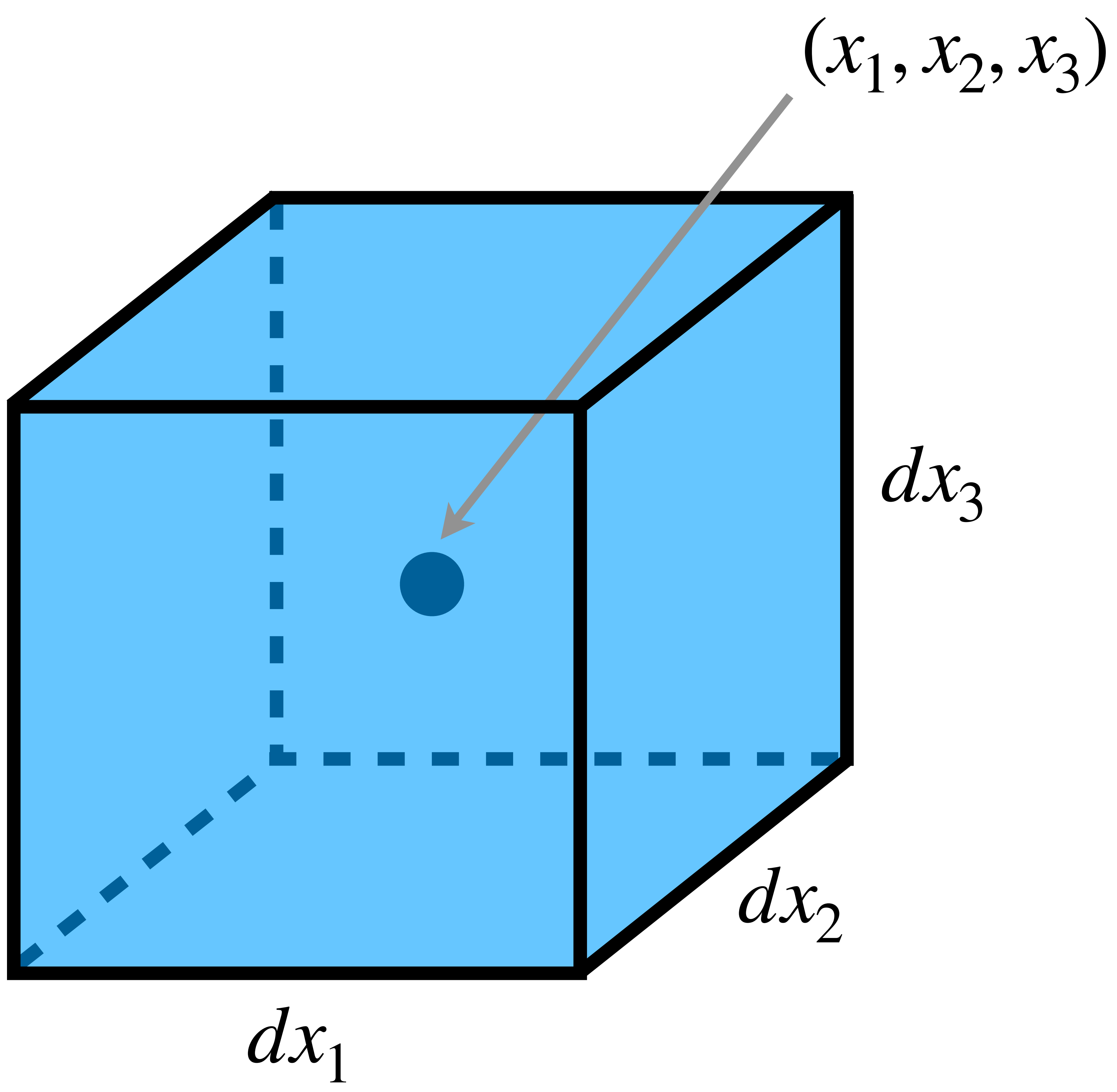
Eulerian and Lagrangian Frames



We consider a volume element of the fluid.

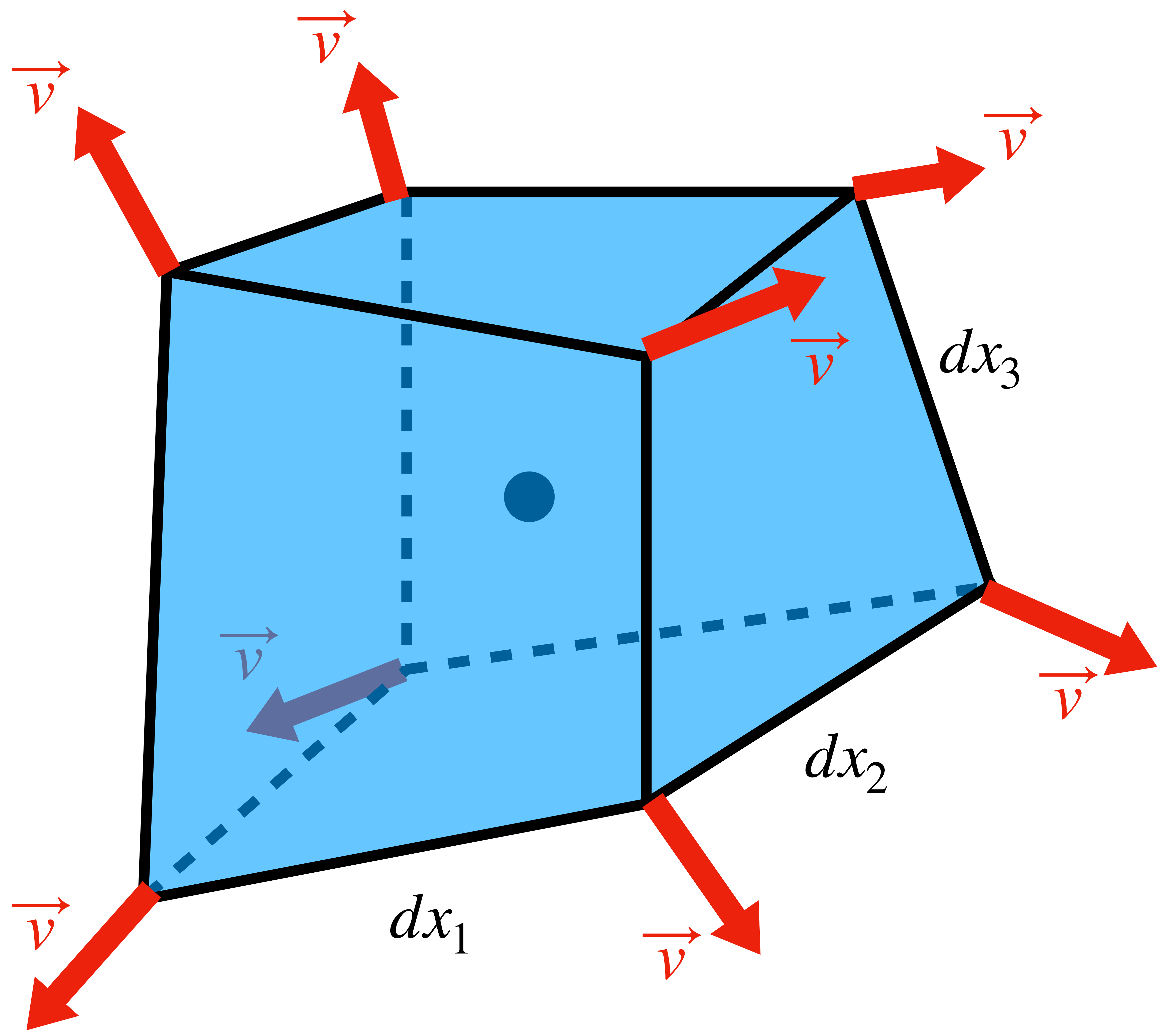
There are two *pictures* to describe its characteristics and evolution.

Eulerian and Lagrangian Frames



In the *Eulerian frame*, the surfaces defining the volume element are fixed in the laboratory system of reference.

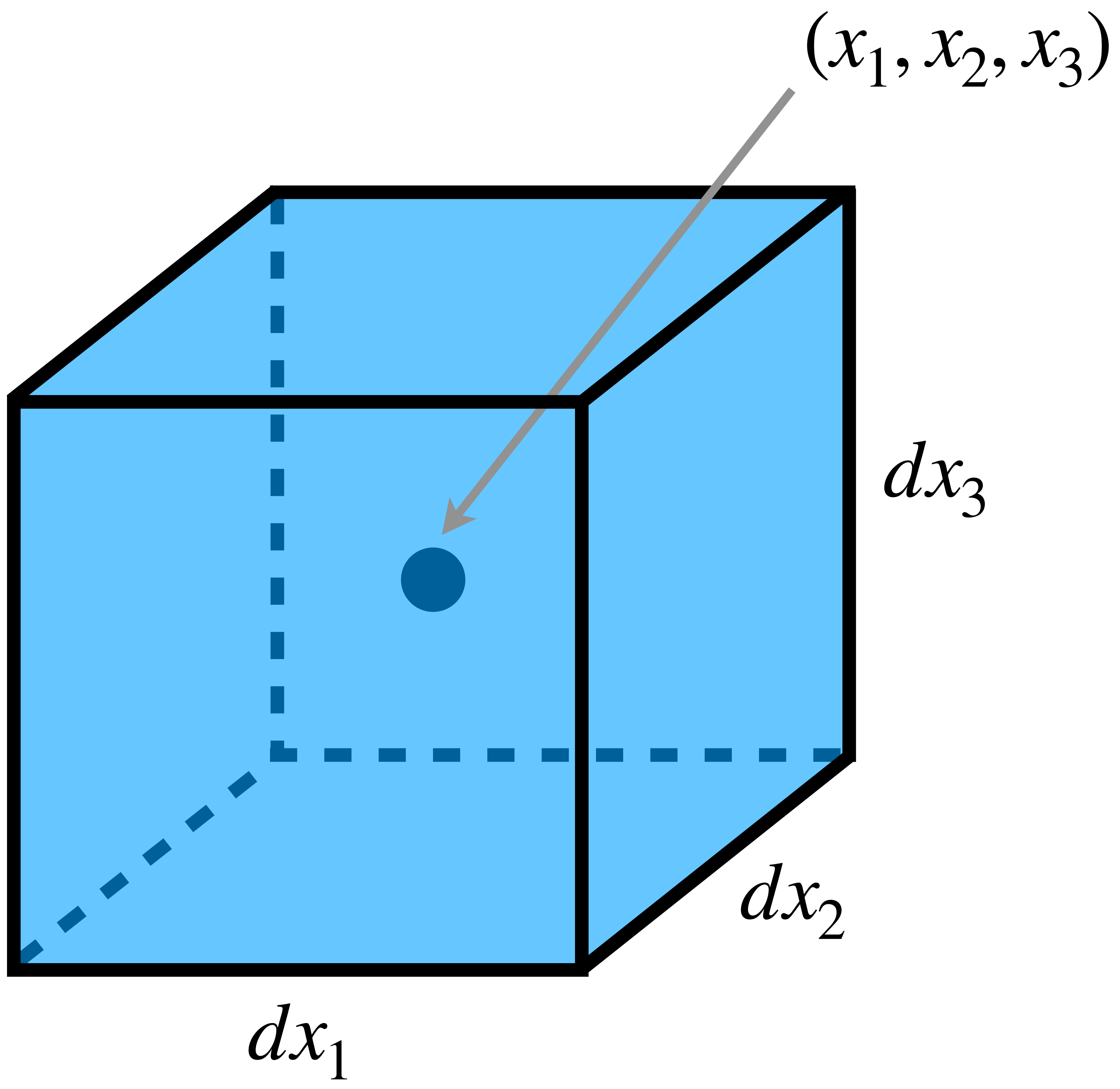
Eulerian and Lagrangian Frames



In the *Lagrangian frame*, the surfaces defining the volume element move with the fluid (they are co-moving in the fluid system of reference).

Mass Conservation in the Eulerian Frame

Continuity Equation in the Eulerian Frame



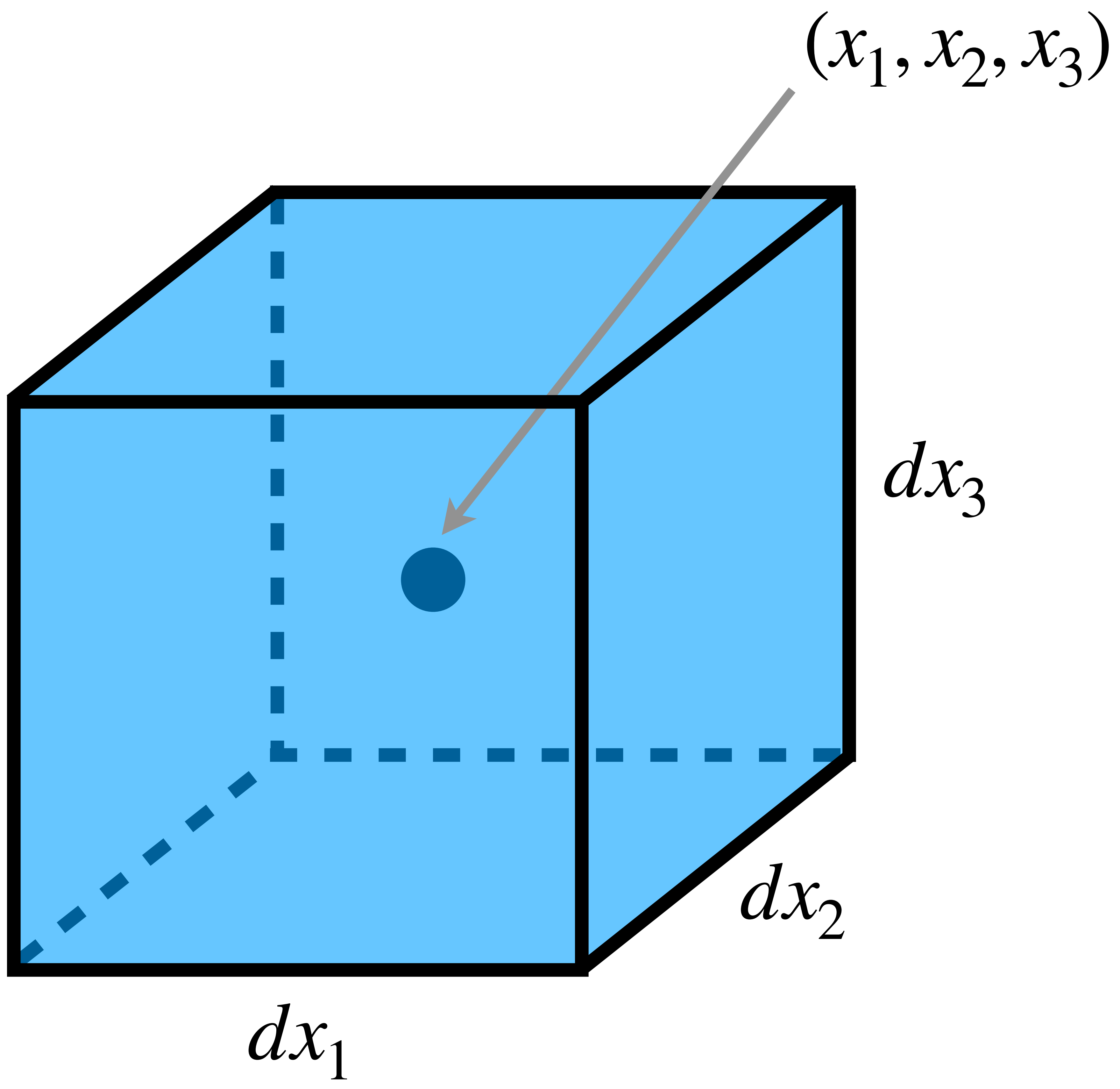
The centroid of the volume element is the point (x_1, x_2, x_3)

The fluid may flow-in or flow-out through the surface.

$$dV = dx_1 dx_2 dx_3$$

$$\rho = \rho(x_1, x_2, x_3)$$

Continuity Equation in the Eulerian Frame



The mass inside this volume element is

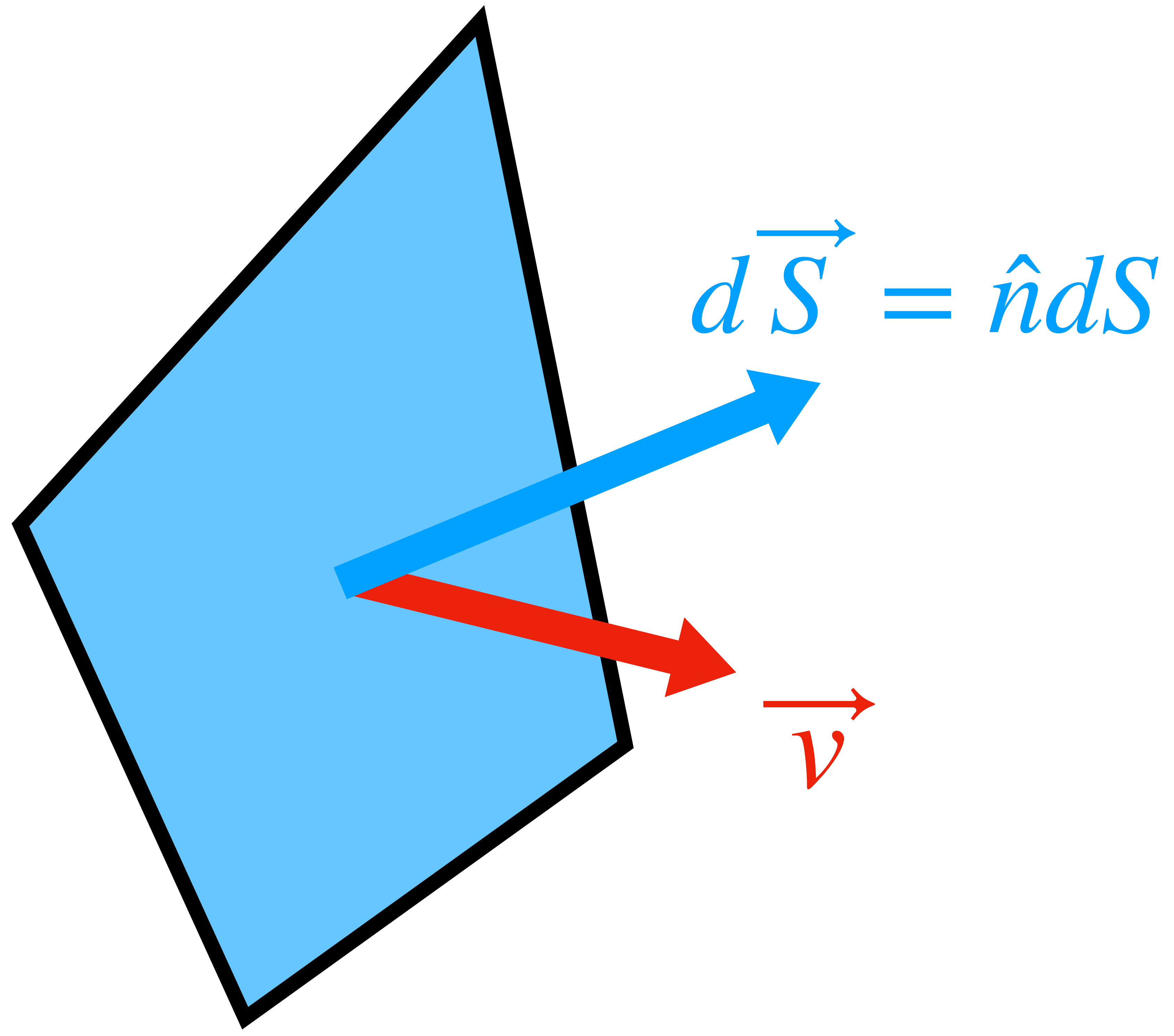
$$M = \int_V \rho dV = \int_V \rho(x_1 \,.\, x_2, x_3) dx_1 dx_2 dx_3$$

Continuity Equation in the Eulerian Frame

The rate of change in the mass inside the volume element is

$$\frac{dM}{dt} = \frac{d}{dt}\int_V \rho dV = -\oint_{\partial V} \rho \vec{v} \cdot d\vec{S},$$

where we introduced the flux of mass across the boundary surface and the surface element $d\vec{S}$ as usual.



Continuity Equation in the Eulerian Frame

Using Gauss' theorem we obtain

$$\frac{dM}{dt} = \int_V \frac{\partial \rho}{\partial t} dV = - \int_V \vec{\nabla} \cdot (\rho \vec{v}) dV.$$

Hence,

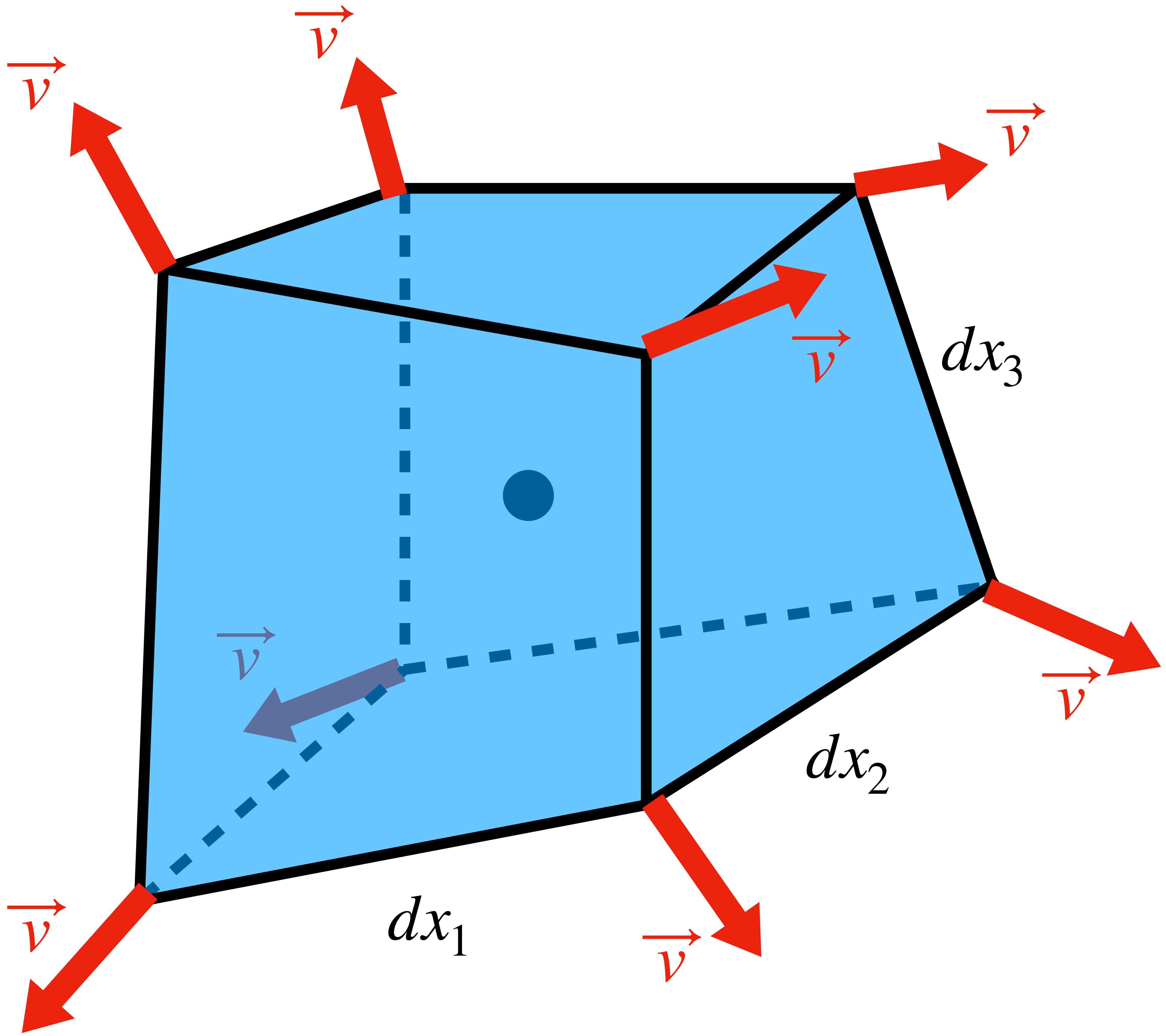
$$\int_V \left[\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v}) \right] dV = 0$$

From which we obtain the continuity equation in the Eulerian frame,

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v}) = 0.$$

Mass Conservation in the Lagrangian Frame

Continuity Equation in the Lagrangian Frame



Each point in the surface moves with the local velocity of the flow. This implies that *there is no flow across the surface and therefore the mass in the volume element is constant !*

$$\frac{dM}{dt} = 0$$

Continuity Equation in the Lagrangian Frame

However, the volume is changing because the surface is moving,

$$\frac{dM}{dt} = 0 = \frac{d}{dt}\int_V \rho(t, x_1, x_2, x_3)\delta x_1 \delta x_2 \delta x_3$$

Where introduced the symbol δ to denote the infinitesimal changes in the spatial coordinates. Hence

$$\int_V \frac{d\rho}{dt}\delta x_1 \delta x_2 \delta x_3 + \int_V \rho \frac{d}{dt}(\delta x_1 \delta x_2 \delta x_3) = 0$$

Continuity Equation in the Lagrangian Frame

The time derivative in the second integral may be written as

$$\int_V \rho \frac{d}{dt}(\delta x_1 \delta x_2 \delta x_3) = \int_V \rho(\delta v_1 \delta x_2 \delta x_3 + \delta x_1 \delta v_2 \delta x_3 + \delta x_1 \delta x_2 \delta v_3)$$

Where we introduced the velocities using

$$\delta v_i = \delta \left(\frac{dx_i}{dt} \right) = \frac{d}{dt}(\delta x_i)$$

Continuity Equation in the Lagrangian Frame

This integrand can be written as

$$\int_V \rho \frac{d}{dt}(\delta x_1 \delta x_2 \delta x_3) = \int_V \rho \left(\frac{\delta v_1}{\delta x_1} + \frac{\delta v_2}{\delta x_2} + \frac{\delta v_3}{\delta x_3} \right) \delta x_1 \delta x_2 \delta x_3$$

$$\int_V \rho \frac{d}{dt}(\delta x_1 \delta x_2 \delta x_3) = \int_V \rho \left(\vec{\nabla} \cdot \vec{v} \right) \delta x_1 \delta x_2 \delta x_3$$

Continuity Equation in the Lagrangian Frame

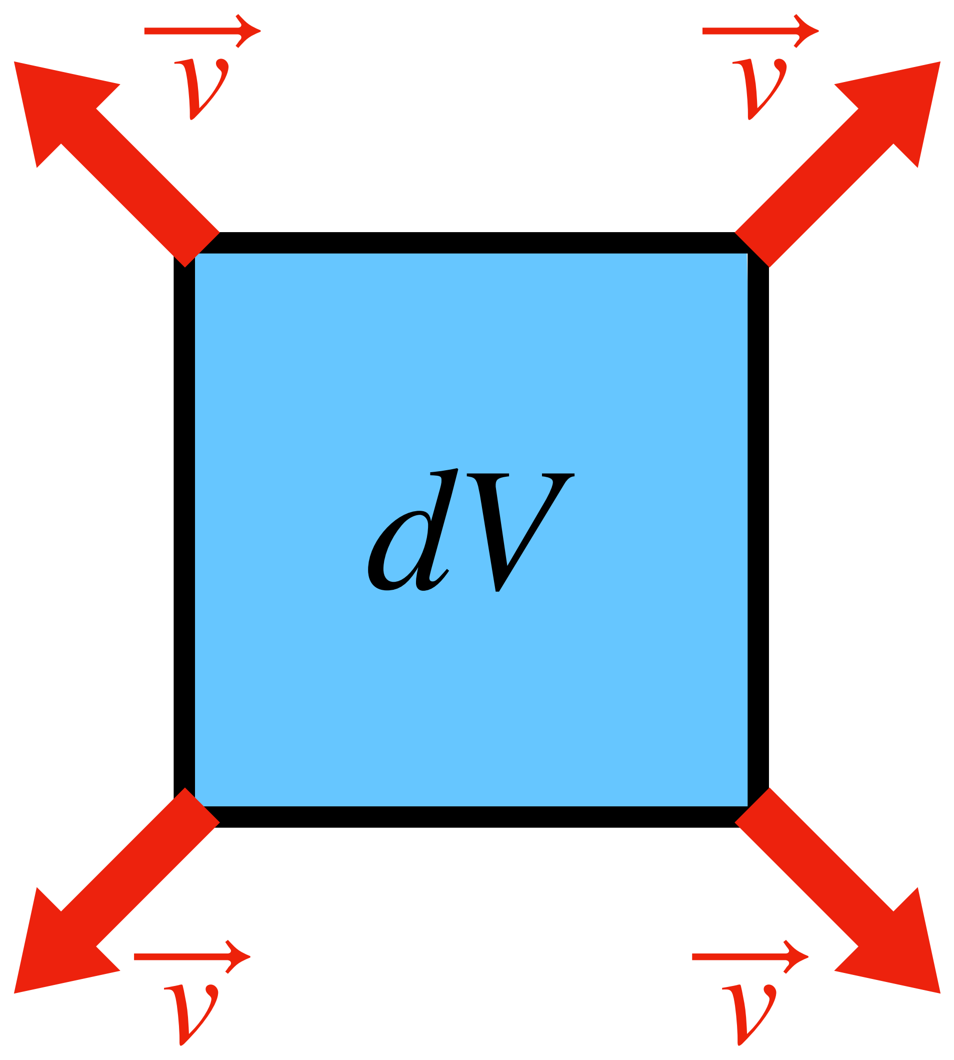
Hence, the conservation of mass takes the form

$$\int_V \left[\frac{d\rho}{dt} + \rho \overrightarrow{\nabla} \cdot \overrightarrow{v} \right] \delta x_1 \delta x_2 \delta x_3 = 0$$

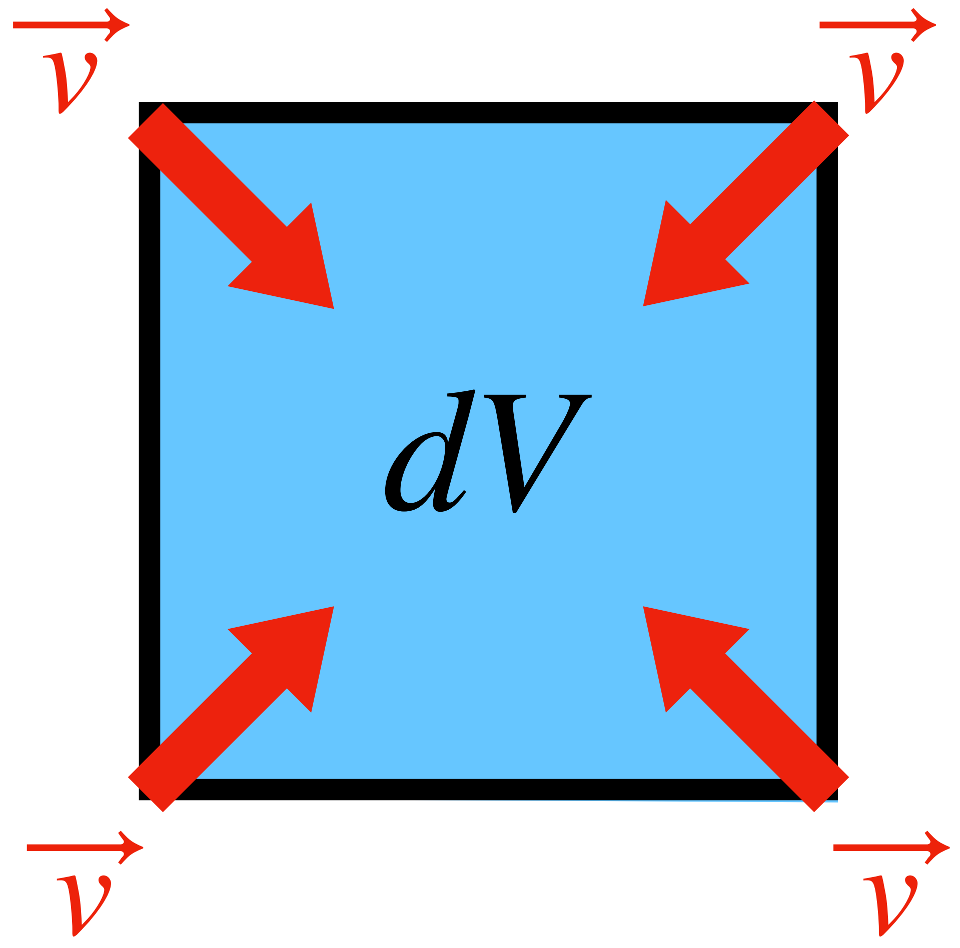
From which we obtain the continuity equation in the Lagrangian frame,

$$\frac{d\rho}{dt} + \rho \overrightarrow{\nabla} \cdot \overrightarrow{v} = 0$$

The term $\rho \vec{\nabla} \cdot \vec{v}$ measures the change in density due to the expansion or compression of the volume element. Graphically,



$$\vec{\nabla} \cdot \vec{v} > 0$$



$$\vec{\nabla} \cdot \vec{v} < 0$$

The Lagrangian Derivative

The Lagrangian derivative

Consider the continuity equation in the Eulerian and Lagrangian frames

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v}) = 0$$

Eulerian frame

$$\frac{d\rho}{dt} + \rho \vec{\nabla} \cdot \vec{v} = 0$$

Lagrangian frame

Comparison of these equations gives the condition

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \vec{v} \cdot \vec{\nabla} \rho$$

The Lagrangian derivative

The operator

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \overrightarrow{v} \cdot \overrightarrow{\nabla}$$

is known as the *Lagrangian derivative* or as the *total time derivative*.

The Lagrangian derivative

The term $\overrightarrow{v} \cdot \overrightarrow{\nabla}$ applied to a physical quantity ψ, i.e.

$$\overrightarrow{v} \cdot \overrightarrow{\nabla} \psi$$

is known as the *advective derivative* and measures the change in ψ along $\overrightarrow{v}$ (measures the transport of ψ due to the velocity field).