Quantifying Chaos

Title

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Contents

Introduction:

Aim:

The aim of this experiment is to determine a quantitative measure of the chaos in the motion of a triple pendulum, and use this to determine a relationship between the extent of chaos exhibited by the pendulum, and the the total energy of the pendulum arm.

Hypothesis:

> *The extent of chaos exhibited by the triple pendulum will be directly proportional to the total energy of the pendulum at that point in time.*

As the impact of frictional effects on this pendulum is very minimal, it is valid to assume that the total energy of the pendulum will remain approximately constant throughout a run with duration of ~15 seconds. Hence, as the initial energy of the pendulum arm is entirely in the form of gravitational potential energy, we can adapt the hypothesis to suit this easier to measure parameter as follows:

> *The extent of chaos exhibited by the triple pendulum will be directly proportional to the initial energy of the pendulum.*

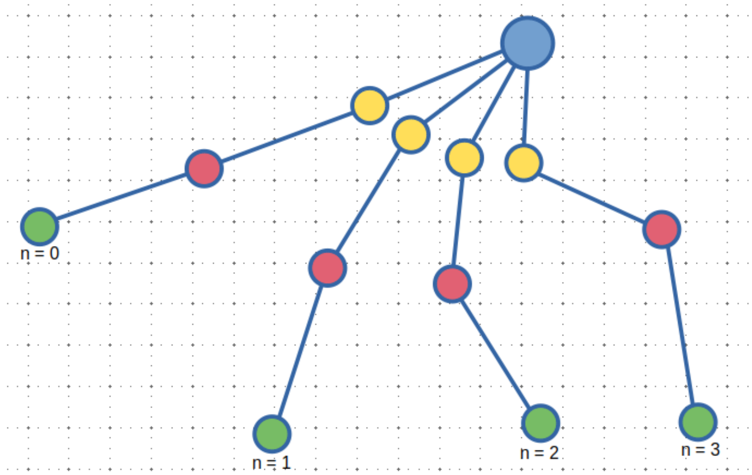
Scientific basis:

Edward Lorenz defined chaos as:

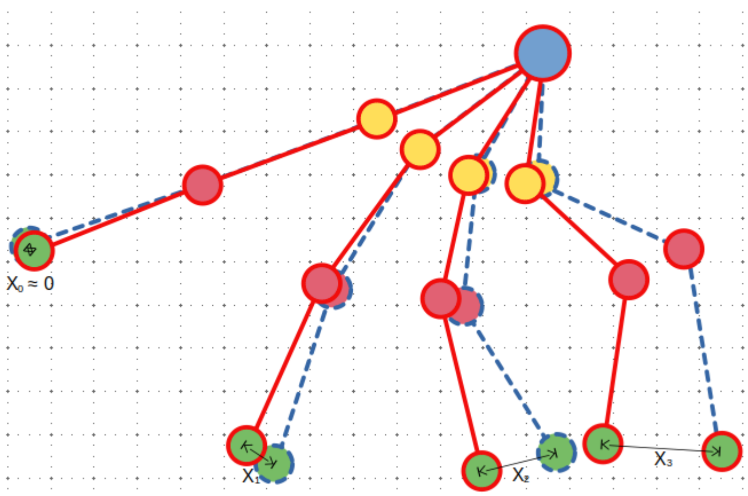
> ***When the present determines the future, but the approximate present does not approximately determine the future.***

Based on this definition, we can conclude that the extent of chaos expressed by a system over time can be measured as the difference in paths taken by a system over multiple trials whose initial conditions are approximately the same.

Consider the first four frames of a pendulum run, taken at equal intervals:



We can the conduct a second run with approximate the same initial conditions, and measure the difference between the end node positions x at each frame x_n:



Hence, based of our above definition, it is valid to describe a measure of chaos for this system as:

$$Chaos = \Delta t \cdot \sum_{k=0}^{n} x_k$$

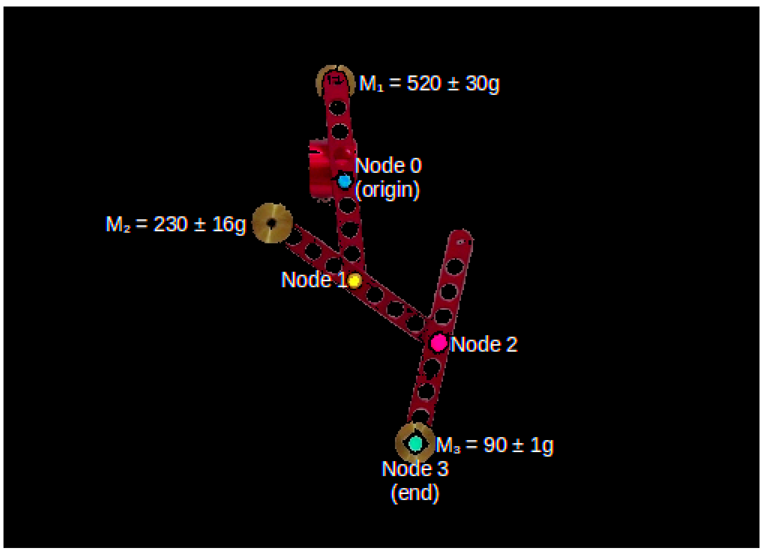
Where $\Delta t = \frac{1}{120}$ is the time between each frame.

Experimental Setup:

Equipment list:

The following equipment was used in this experiment:

- Weighted triple pendulum *(shown below)*, with an arm length of 120mm, as measured between the centers of any two consecutive nodes.
- Four different brightly colored stickers.
- Sony A7C full-camera using a 24mm, f2.8 lens, at aperture f5.6. Capable of recording footage at 4K resolution, with 120 frames per second.
- Scales with 0.1g precision.
- Protractor with 0.5° precision, and a hanging bob to reference the vertical direction.
- Tape measure & meter ruler with 1mm precision.
- White paper and masking tape to cover reflective parts of pendulum *(see experimental setup diagram)*.



Method:

The method used to collect data in this experiment was as follows:

1. Camera and pendulum were set up in front of blank white wall, with good lighting, such that the camera's field of view was centered on the origin node (to reduce parallax error), and the lens of the camera was placed 1.2m horizontally from the origin node, such that the whole range of motion of the pendulum was within the camera's field of view.

2. Blue, yellow, pink, and green markers were placed on nodes 0 to 3 respectively. These colors were chosen as they stood out easily from the other colors present on the pendulum apparatus, such as the red of the arms.

3. Pendulum arm was brought to the desired initial angle (measured relative to the downwards vertical), as measure using protractor, and held until all nodes were motionless.

4. Recording was started and pendulum released after a brief countdown.

5. During recording, the initial conditions were verbally announced, so that they could be manually determined from the data.

6. Pendulum was allowed to run for 15 seconds (or 45 seconds for propagation trials), then recording was stopped.

7. The above process was repeated from step 3 with initial angle adjusted in 15° increments from 0°, a

motionless control run, to 180°, where the arm was oriented vertically upward and the initial energy was a maximum. For each distinct initial angle, three runs were conducted.

8. Recordings of each run were saved, and the data analysed using the method described under *Data & Analysis.*

Data and Results:

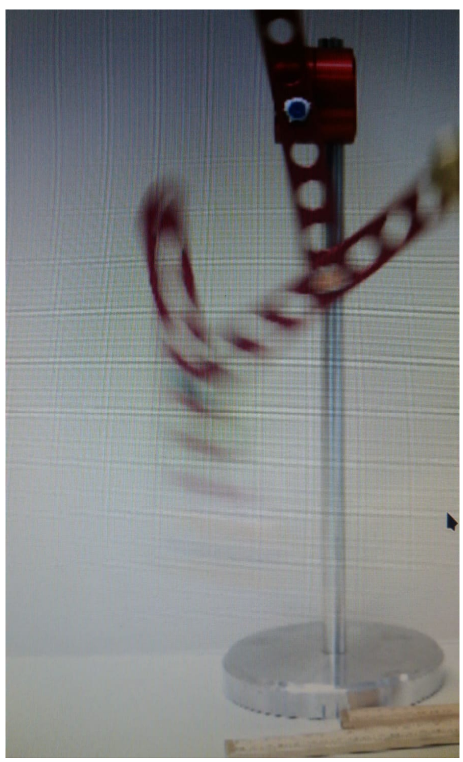
Tracking the nodes:

Our data collection process produced approximately 10Gb of recordings.

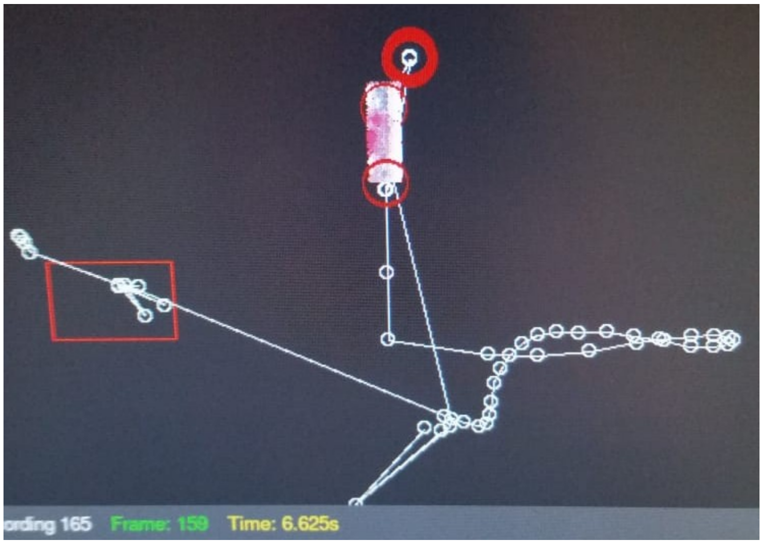
To extract the node position data from these recordings, we used a Python script of my own design. This script operated as follows:

1. The OpenCV python module was used to create a bit-wise HSV colour mask for each of the colored node stickers (and later, also the arms and masses).
2. The nodes selected b these masks were displayed at every frame to verify correct calibration.
3. The SciPy module was used to determine the median centers of the masks.
4. The x & y coordinates of each center, as will as the number of selected pixels, were saved to a raw data file. The value of -1 was used to indicate that a node could not be found on a particular frame. This occurred frequently when nodes were covered by other parts of the pendulum arms.

Our initial dataset, recorded at 24 frames per second (FPS), had a severe amount of motion blur, as shown below.



Whilst this was not a major issue for low angle tests, the blur made it almost impossible to track the nodes through the pendulum's faster motions. One attempt at this tracking is shown below.



After re-recording our entire data-set at 120 FPS, the nodes were significantly easier to track, as the blur was now minimal. A successful tracking of the first 3 seconds of a 120° run is shown below:

Cleaning the data:

After running the node tracking script for a day and a half, we had successfully extracted the raw data from the videos. A second python script, and some R code, was then used to clean this data by:

- Averaging the origin node position over the entire run, and centering the coordinate system about this origin node.

- Normalizing the units of the coordinate system to be in SI units, instead of the pixel & frame coordinates output by the tracking script.
- Interpolating node data for times when a particular node was hidden behind other parts of the pendulum arm.
- Removing discrepancies caused by partially covered nodes, by only considering nodes where the number of pixels detected to be part of the node was above a certain threshold.
- Applying time shifts to some of the runs to account for the pendulum's release not precisely aligning with the start of the recording.

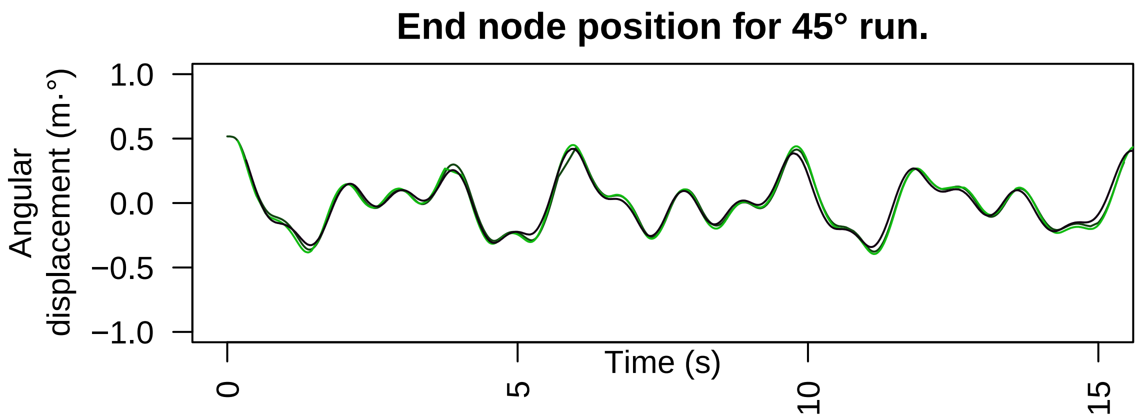
Once the data for each run was cleaned, it was compiled into a single 50Mb .csv data file, containing node coordinates, in both Cartesian and polar form, for every node at every frame of every run. The full structure of this data-set, including all measured and calculated variables, is shown below:

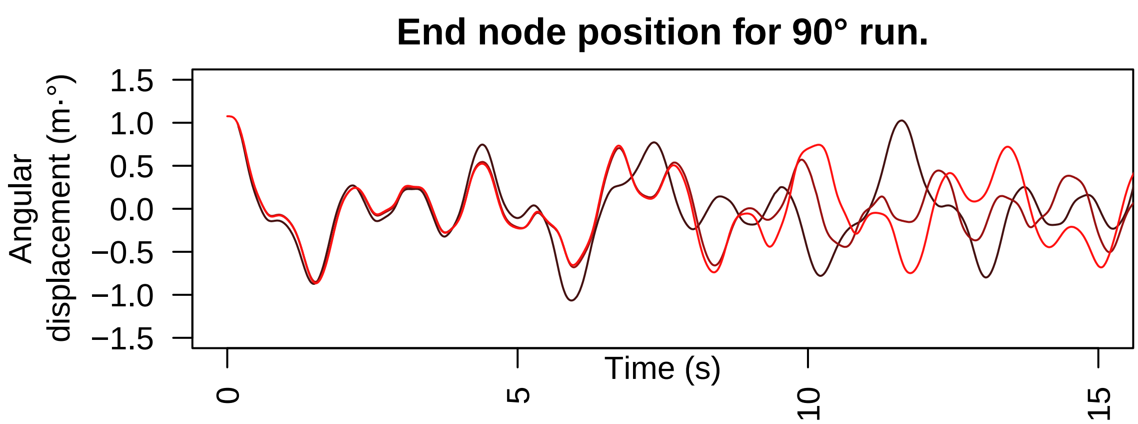
```
## 'data.frame':    148434 obs. of  24 variables:
##  $ Recording  : int  223 223 223 223 223 223 223 223 223 223 ...
##  $ Time       : num  0 0.00833 0.01667 0.025 0.03333 ...
##  $ Type       : chr  "prop" "prop" "prop" "prop" ...
##  $ InitAngle  : num  45 45 45 45 45 45 45 45 45 45 ...
##  $ AddedMass  : int  0 0 0 0 0 0 0 0 0 0 ...
##  $ LateStart  : int  0 0 0 0 0 0 0 0 0 0 ...
##  $ BlueX      : num  -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 ...
##  $ BlueY      : num  -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 ...
##  $ BlueAngle  : num  -45 -45 -45 -45 -45 ...
##  $ BlueDist   : num  1.41 1.41 1.41 1.41 1.41 ...
##  $ YellowX    : num  0.0176 0.0174 0.0176 0.0176 0.0174 ...
##  $ YellowY    : num  -0.359 -0.358 -0.359 -0.358 -0.358 ...
##  $ YellowAngle: num  2.82 2.78 2.82 2.81 2.78 ...
##  $ YellowDist : num  0.359 0.359 0.359 0.359 0.359 ...
##  $ PinkX      : num  0.0417 0.0423 0.0412 0.0401 0.0395 ...
##  $ PinkY      : num  -0.731 -0.731 -0.731 -0.731 -0.731 ...
##  $ PinkAngle  : num  3.26 3.31 3.22 3.14 3.09 ...
##  $ PinkDist   : num  0.732 0.733 0.732 0.732 0.732 ...
##  $ GreenX     : num  0.338 0.338 0.338 0.337 0.337 ...
##  $ GreenY     : num  -0.957 -0.956 -0.955 -0.954 -0.955 ...
##  $ GreenAngle : num  19.4 19.5 19.5 19.5 19.5 ...
##  $ GreenDist  : num  1.01 1.01 1.01 1.01 1.01 ...
##  $ Timeshifts : num  0 0 0 0 0 0 0 0 0 0 ...
##  $ AdjTime    : num  0 0.00833 0.01667 0.025 0.03333 ...
```

Analyzing the data:

In order to attempt to quantify the chaos expressed by this triple pendulum, it was important to first verify that the system was, in fact, chaotic.

Below are the angular displacements (calculated as angle $\cdot$ distance from origin)of the end node over time, for two sets of initial conditions. At 45°, we can see that the system exhibits very little chaos over the 15 second span. However, when the initial angle was set to 90°, the system became evidently chaotic after ~5 seconds.





Measuring Chaos:

We can now proceed to quantify the extent of the chaos between two runs with the same initial conditions as:

$$Chaos(A,B) = \frac{t}{120} \cdot \sum_{k=0}^{120t} \sqrt{(x_{A,k} - x_{B,k})^2 + (y_{A,k} - y_{B,k})^2}$$

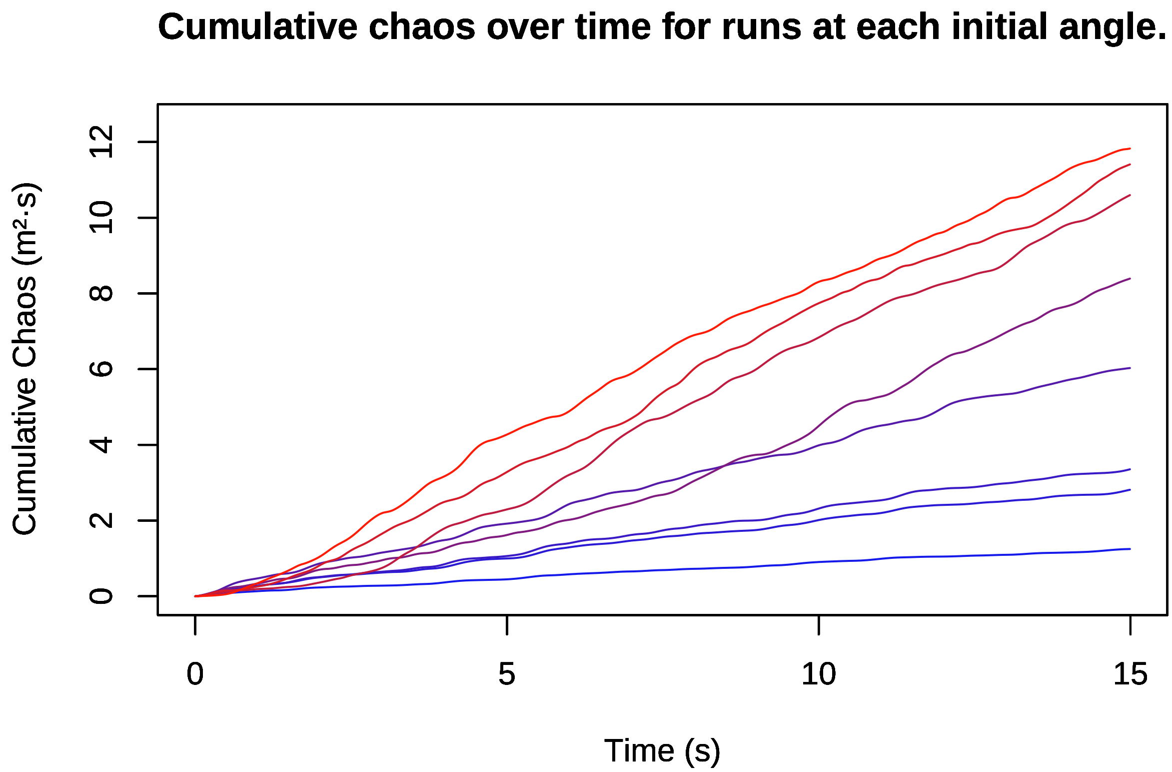
Where:

- 120 was the frame-rate used.
- k is the current frame.
- t is the length of the run in seconds,
- $x_{A,k}$ is the x coordinate of the end node, relative to the origin node, at frame k of the first of the two runs being compared.

We can then find the chaos for all 3 runs conducted with a given set of initial conditions as:

$$Chaos(A,B,C) = \frac{1}{3}(Chaos(A,B) + Chaos(B,C) + Chaos(A,C))$$

Below is the cumulative chaos, calculated using the above formula, for the first $t = 15$ seconds of the run set for each of the initial angles tested. Large angles are denoted in red, smaller angles in blue.



The uncertainty ΔC for this measurement of chaos can be calculated as follows:

$$\Delta C = \Delta t \cdot \sqrt{(2\Delta x)^2 + (2\Delta y)^2}$$

Where:

- $\Delta t = \frac{0.5}{120} \approx 0.004s$ is the uncertainty in the current time, due to the 120 frames / second frame rate of the camera used.
- Δx and Δy are the uncertainties in the coordinates of the nodes. The python script used was able to pinpoint the node locations to an precicion of much less than one pixel by taxing the average of all ~16000 pixels masked as part of the node. Hence, this uncertainty, and by extension ΔC is negligible.

Measuring Initial Energy:

Initial energy was measured on a scale from 0, where the pendulum was at rest (0°), to 1, where the pendulum was raised vertically (180°), and hence had a maximum initial gravitational potential energy. Using this unit scale, initial energy E_0 could be calculated as:

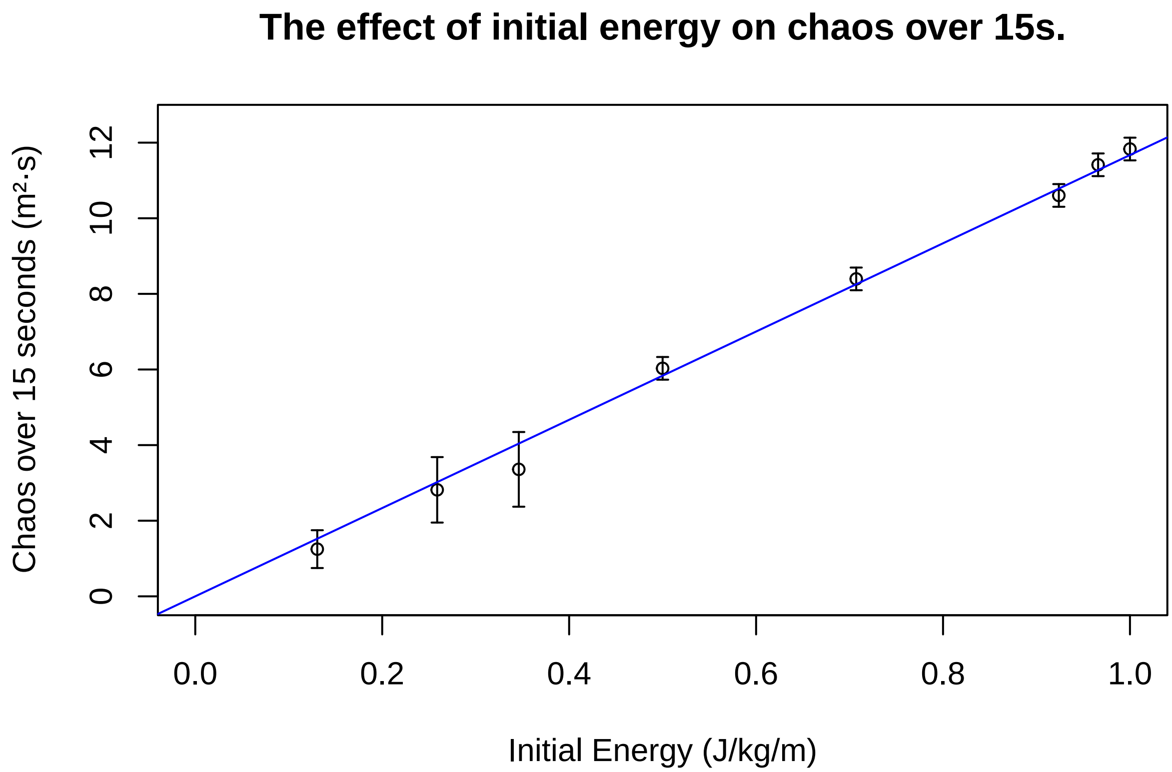
$$E_0 = \sin(\frac{\theta}{2})$$

The uncertainty ΔE_0 in this initial energy is then simply $\Delta E_0 = \sin(\frac{\Delta\theta}{2})$, where $\Delta\theta$ is the uncertainty in the initial angle θ.

For our lower angle runs ($\theta \leq 45°$) this uncertainty was quite significant, as the protractor we used had a secondary scale that was not in degrees, but rather mapped 180° to a value of 200. We did not realize this until the first of the 60° trials, and hence we were unsure which scale we had used for the earlier runs. After realising this, we made certain to use the degrees scale, which had half degree increments. So:

$$\Delta\theta = \begin{cases} 1° + \frac{200}{180°}\theta & \text{if } \theta \leq 45° \\ 1° & \text{if } \theta > 45° \end{cases}$$

After mapping these uncertainties to the y axis, we can produce the following graph of the cumulative chaos over 15 seconds vs. initial energy of the pendulum:



From the above graph, it is clear that there is a linear relationship between initial energy and chaos, for this triple pendulum with constant moments of inertia, as we had originally hypothesized.

Discussion and Interpretation:

Analyzing the results:

The aim of this experiment was to determine a relationship between chaos exhibited by a triple pendulum, and total energy of the pendulum. Ignoring the minimal effect of friction, we have taken the energy of the pendulum throughout the run as a constant, equal to the initial gravitational potential energy of the arm, and proven that there is a clear, linear relationship between chaos and energy.

We can quantify the statistical significance of this relationship by performing a t test for correlation:

```
##
## Call:
## lm(formula = difSet ~ 0 + energySet)
##
## Residuals:
##      Min       1Q   Median       3Q      Max
## -0.68179 -0.22182 -0.02039  0.14636  0.19417
##
## Coefficients:
##           Estimate Std. Error t value Pr(>|t|)
## energySet  11.6724     0.1655   70.52 3.03e-11 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.3201 on 7 degrees of freedom
## Multiple R-squared:  0.9986, Adjusted R-squared:  0.9984
## F-statistic:  4973 on 1 and 7 DF,  p-value: 3.031e-11
```

This test shows us a P value $\approx 3 \cdot 10^{-11}$, indicating an extremely strong linear correlation between chaos and energy. This result aligns with our hypothesis.

Validity, reliability and accuracy of the data:

Whilst we do not have a predicted theoretical value of formula with which to compare our results, as any such formula would have to be highly specific to the system in question, it is reasonable to conclude that out data is accurate, as our uncertainties *(discussed under measurement)* were minimal, and our experimental method was carefully considered to ensure that our data was collected in a valid manner.

Additionally, by conducting three tests for each set of initial conditions, we were able to observe that our results were relatively consistent between 3 comparisons of chaos, and ensure our results were reliable.

Limitations of experimental approach:

Our final experimental approach was highly successful at collecting reliable and valid data for use in our analysis. We did, however, modify aspects of this approach over the course of this experiment in order to mitigate the following limitations of our preliminary data set:

- The frame rate of the recording was raised from 24 FPS to 120 FPS to reduce the effects of motion blur, discussed further under *Tracking the Nodes*. All of our data was retaken after this modification.
- Our protractor was changed for one with only a single scale in degrees, to avoid confusion in the initial angle measurement. Discussed further under *Measuring Initial Energy*.

Conclusion:

In conclusion, our experiment was highly successful at quantifying the relationship between chaos and system energy for a triple pendulum. We were able to define a robust mathematical measure of chaos in the system, and relate this linearly to initial energy with a correlation coefficient of 0.9986.

Further aspects for investigation could include the point in time at which the system becomes chaotic, on analysis of the effect of changing the moment of inertia of the arm.

References