Mini-Project DXV925 Daniel Valverde Menasalvas

1) $A \in \mathbb{R}^{m \times m}$ symmetric with distinct eigenvalues. $M \in \mathbb{R}^{m \times m}$ s.p.d.

GEVP: $A\underline{v} = \lambda M \underline{v}$

a) $\lambda_i \in \mathbb{R}$?

We will use the square-root factorisation for s.p.d. matrices covered in Examples sheet 4 (Q1), $M = M^{1/2} M^{1/2} = V D^{1/2} V^{-1} V D^{1/2} V^{-1}$ (where $V = [v_1 \dots v_m]$ is a matrix of eigenvectors of M and $D^{1/2}_{ii} = \sqrt{D_{ii}}$ $D^{1/2}_{ij} = 0, i \neq j$, being D the diagonal matrix with the eigenvalues of M).

By the examples sheet 4 Q2 we also have that V is ortogonal ($V^{-1} = V^T$).

Therefore, we have that $M^{1/2}$ is symmetric: $(M^{1/2})^T = (V D^{1/2} V^{-1})^T = (V^{-1})^T D^{1/2} V^T = (V^T)^T D^{1/2} V^{-1} =$
$= V D^{1/2} V^{-1} = M^{1/2}$. Also, $M^{1/2}$ is invertible with $(M^{1/2})^{-1} = M^{-1/2} = V D^{-1/2} V^{-1}$. ($D^{-1/2}_{ii} = \frac{1}{D^{1/2}_{ii}}$).

Now, consider this equivalent eigenvalue problem:

$$A\underline{v} = \lambda M \underline{v} \leftrightarrow A\underline{v} = \lambda M^{1/2} M^{1/2} \underline{v} \leftrightarrow M^{-1/2} A \underline{v} = \lambda M^{1/2} \underline{v} \leftrightarrow M^{-1/2} A M^{-1/2} M^{1/2} \underline{v} = \lambda M^{1/2} \underline{v}$$

Calling $\hat{A} = M^{-1/2} A M^{-1/2}$ and $\hat{\underline{v}} = M^{1/2} \underline{v}$ we have $A\underline{v} = \lambda M \underline{v} \leftrightarrow \hat{A} \hat{\underline{v}} = \lambda \hat{\underline{v}}$

And $\hat{A}$ is symmetric: $\hat{A}^T = ((M^{1/2})^{-1})^T A^T (((M^{1/2})^{-1})^T = ((M^{1/2})^T)^{-1} A ((M^{1/2})^T)^{-1} = (M^{1/2})^{-1} A (M^{1/2})^{-1} = \hat{A}$

Thus, by Q10 of the Examples sheet 4, the eigenvalues of $\hat{A}$ are real and the eigenvectors of $\hat{A}$ can be taken to be real. As $\hat{A}$ has the same eigenvalues of our GEVP and $\underline{v} = M^{-1/2} \hat{\underline{v}}$, that also applies to our GEVP.

b) $\{\underline{v}_i\}_{1\le i\le m}$ s.t. $\begin{cases} \underline{v}_i^T M \underline{v}_j = \delta_{ij} \\ \underline{v}_i^T A \underline{v}_j = \lambda_i \delta_{ij} \end{cases}$

Now we have $A\underline{v} = \lambda M \underline{v} \leftrightarrow \hat{A}\hat{\underline{v}} = \lambda \hat{\underline{v}}$ with $\hat{A}$ symmetric.

As $\hat{A}$ is symmetric, by the spectral theorem for symmetric real matrices$^{(*)}$, we know that $\hat{A}$ admits an ortogonal diagonalisation, i.e. there exists V s.t. $V^{-1} = V^T$ and $\hat{A} = VDV^T$ with D diagonal with the eigenvalues of $\hat{A}$ (which are the same as GEVP). So, $V^T\hat{A}V = D = V^T M^{-1/2} A M^{-1/2} V = V^T M^{-1/2} A (V^T M^{-1/2})^T$.

Let's call $\Phi^T := V^T M^{-1/2}$ and we have $\Phi^T M \Phi = V^T M^{-1/2} M M^{-1/2} V = V^T M^{-1/2} M^{1/2} M^{1/2} M^{-1/2} V =$

$= V^T V = I$. So: $\begin{cases} \Phi^T A \Phi = D \\ \Phi^T M \Phi = I \end{cases}$, calling $\underline{c}_j(\Phi) =: \underline{v}_j \quad 1 \le j \le m$

we have the vectors that we wanted fulfilling $\begin{cases} \underline{v}_i^T M \underline{v}_j = \delta_{ij} \\ \underline{v}_i^T A \underline{v}_j = \delta_{ij}\lambda_i \end{cases}$

(*) I know that we are not supposed to use this theorem. I guess there is a way to proof that $\hat{A}$ has an orthogonal diagonalisation using that A has distinct eigenvalues. As I couldn't get to it, I give this other reasoning which is equal to the other one excepts for this step.

c) $1 \le i \le n$ fixed, $\alpha \in \mathbb{R} \setminus \{0\}$ GEVP-mod: $\begin{bmatrix} A & \alpha M \underline{v_i} \\ \alpha \underline{v_i}^T M & 0 \end{bmatrix} \underline{u} = \tau \begin{bmatrix} M & \underline{0} \\ \underline{0}^T & 1 \end{bmatrix} \underline{u}$

$\underline{u} \overset{?}{=} \begin{bmatrix} v \\ a \end{bmatrix}$ $v \in \{v_1, \dots, v_n\}$ $a \in \mathbb{R}$. $\frac{\alpha^2}{\tau} \overset{?}{=} \tau - \lambda_i$, $\{(\tau_k, u_k)\}_{1 \le k \le n+1}$?

Let's take $\{v_1, \dots, v_n\}$ the vectors from (b). We'll start by studying the vectors of the form $\underline{u} = \begin{bmatrix} v_k \\ a \end{bmatrix}$ $k \neq i$:

$$\begin{bmatrix} A & \alpha M v_i \\ \alpha \underline{v_i}^T M & 0 \end{bmatrix} \begin{bmatrix} v_k \\ a \end{bmatrix} = \tau \begin{bmatrix} M & \underline{0} \\ \underline{0}^T & 1 \end{bmatrix} \begin{bmatrix} v_k \\ a \end{bmatrix}$$

$$\left\{ \begin{array}{l} A \underline{v_k} + \alpha a M \underline{v_i} = \tau M \underline{v_k} \\ \alpha \underline{v_i}^T M \underline{v_k} = \tau a \end{array} \right. \begin{array}{c} \xrightarrow{\text{GEVP}} \\ \xrightarrow{(b)} \end{array} \left\{ \begin{array}{l} \lambda_k M \underline{v_k} + \alpha a M \underline{v_i} = \tau M \underline{v_k} \\ \tau a = 0 \end{array} \right.$$

Multiplying by $\underline{v_k}^T$ on the left on the first equation:

$$\lambda_k \underline{v_k}^T M \underline{v_k} + \alpha a \underline{v_k}^T M \underline{v_i} = \tau \underline{v_k}^T M \underline{v_k} \xrightarrow{(b)} \boxed{\lambda_k = \tau}$$

Multiplying by $\underline{v_i}^T$ on the left on the first equation:

$$\lambda_k \underline{v_i}^T M \underline{v_k} + \alpha a \underline{v_i}^T M \underline{v_i} = \tau \underline{v_i}^T M \underline{v_k} \xrightarrow{(b)} \alpha a = 0 \xrightarrow[\alpha \neq 0]{} \boxed{a = 0}$$

Note that this values of τ and a also satisfies the second equation ($\tau a = 0$).

So we have that $\underline{u} = \begin{bmatrix} v_k \\ 0 \end{bmatrix}$ $k \neq i$ are $(n-1)$ eigenvectors with λ_k as its eigenvalue. As we want τ_k to be non-zero, if $\exists j \in \{1, 2, \dots, n\}$ s.t. $\lambda_j = 0$ we want i to be equal to j. Otherwise, zero would be an eigenvalue of the GEVP-mod. Notice that the algebraic multiplicity of 0 in the the GEVP will be the same as the algebraic multiplicity of 0 in A (the only eigenvectors of 0 of the GEVP are the eigenvectors of 0 of A).

As A has distinct eigenvalues, $\alpha(0) \leq 1$ in the GEVP. So we can make sure that our first $n-1$ eigenvalues are non-zero by choosing $i=j$.

Now, let's study what happens with the vectors of the form $\underline{u} = \begin{bmatrix} \underline{v}_i \\ a \end{bmatrix}$:

$$\begin{pmatrix} A & \underline{c} \\ c^T & 0 \end{pmatrix} \begin{bmatrix} \underline{v}_i \\ a \end{bmatrix} = \begin{pmatrix} M & 0 \\ 0 & 1 \end{pmatrix} \begin{bmatrix} \underline{v}_i \\ a \end{bmatrix} \rightarrow \begin{cases} A\underline{v}_i + \alpha a M \underline{v}_i = \tau M v_i \\ \alpha v_i^T M v_i = \tau a \xrightarrow{(l)} \tau a = \alpha \end{cases}$$

$$\Rightarrow \begin{cases} \lambda_i M \underline{v}_i + \alpha a M \underline{v}_i = \tau M \underline{v}_i \rightarrow (\lambda_i + \alpha a - \tau) M \underline{v}_i = 0 \rightarrow \lambda_i + \alpha a = \tau \\ \qquad (M \underline{v}_i \neq 0 \text{ as } \underline{v}_i^T M \underline{v}_i = 1) \\ \tau a = \alpha \end{cases}$$

$$\Rightarrow \begin{cases} \tau = \lambda_i + \alpha a \rightarrow \alpha = (\lambda_i + \alpha a) a \rightarrow \alpha a^2 + \lambda_i a - \alpha = 0 \rightarrow a = \dfrac{-\lambda_i \pm \sqrt{\lambda_i^2 + 4\alpha^2}}{2\alpha} \\ \alpha = \tau a \end{cases}$$

$$\rightarrow \tau = \lambda_i + \frac{-\lambda_i \pm \sqrt{\lambda_i^2 + 4\alpha^2}}{2} = \frac{\lambda_i \pm \sqrt{\lambda_i^2 + 4\alpha^2}}{2}$$

So $\begin{bmatrix} v_i \\ \dfrac{-\lambda_i \pm \sqrt{\lambda_i^2 + 4\alpha^2}}{2} \end{bmatrix}$ are eigenvectors with $\tau_i = \dfrac{\lambda_i \pm \sqrt{\lambda_i^2 + 4\alpha^2}}{2}$ as its eigenvalues.

Notice that, if $\lambda_i = 0$, then $\underline{u}_i = \begin{bmatrix} \underline{v}_i \\ \pm \alpha \end{bmatrix}$ and $\tau_i = \pm \alpha$.

Also, $\tau_i \neq 0$ in any case as $\alpha \neq 0$, so all the eigenvalue of GEVP-mod are non-zero.

Now that we know that, we can do the following:

$$\begin{cases} \alpha = \tau a \xrightarrow{\tau \neq 0} a = \dfrac{\alpha}{\tau} \\ \tau = \lambda_i + \alpha a \end{cases} \longrightarrow \boxed{\tau - \lambda_i = \frac{\alpha^2}{\tau}}$$

As wanted. This is just true for the last two eigenvalues.

Finally, the solutions of the GEVP-mod are:

$$\{(\lambda_k, \begin{bmatrix} V_k \\ 0 \end{bmatrix})\}_{\substack{1 \leq k \leq n \\ k \neq i}} \cup \{(\begin{bmatrix} \underline{v}_i \\ \dfrac{-\lambda_i + \sqrt{\lambda_i^2 + 4\alpha^2}}{2} \end{bmatrix}, \frac{\lambda_i + \sqrt{\lambda_i^2 + 4\alpha^2}}{2})\} \cup$$

$$\cup \{(\begin{bmatrix} \underline{v}_i \\ \dfrac{-\lambda_i - \sqrt{\lambda_i^2 + 4\alpha^2}}{2} \end{bmatrix}, \frac{\lambda_i - \sqrt{\lambda_i^2 + 4\alpha^2}}{2})\}$$

(2) FEM-EVP: $A\underline{v} = \lambda M \underline{v}$ M symmetric and positive definite

A symmetric positive semi-definite. The eigenvalues are real and non-negative.

$A\underline{1} = \underline{0}$

a) Setup from Q1(c) with non-zero eigenvalues. It's smallest eigenvalue should be equal to the second smallest eigenvalue of FEM-EVP(λ_{m-1}).

As we saw in Q1(c), we want $\underline{v}_i$ to be equal to the eigenvector of 0.

So we will make $\underline{v}_i := \underline{1}$. The eigenvalues of our new problem will be the same as FEM-EVP without 0 and adding $\pm\alpha$. So we just need $|\alpha| \geq \lambda_{m-1} > 0$ to satisfy the statement.

Let's consider the eigenvectors $\{\underline{v}_1, \dots, \underline{v}_m\}$ of FEM-EVP chosen as in Q1(b).

Let's take an arbitrary $\underline{x} \in \mathbb{R}^m$. As $\{\underline{v}_k\}_{1\le k\le m}$ are linearly independent, we can write $\underline{x}$ as: $\underline{x} = \alpha_1 \underline{v}_1 + \alpha_2 \underline{v}_2 + \dots + \alpha_m \underline{v}_m$ where $\alpha_i \in \mathbb{R}$ $\forall i \in \{1,2,\dots,m\}$.

Now, consider the following operation:

$$\frac{\underline{x}^T A \underline{x}}{\underline{x}^T M \underline{x}} = \frac{(\alpha_1 \underline{v}_1^T + \alpha_2 \underline{v}_2^T + \dots + \alpha_m \underline{v}_m^T) A (\alpha_1 \underline{v}_1 + \alpha_2 \underline{v}_2 + \dots + \alpha_m \underline{v}_m)}{\underline{x}^T M \underline{x}}$$

Remember from Q1(b) that all the products of the form $\underline{v}_i^T A \underline{v}_j$ $i \neq j$ or $\underline{v}_i^T M \underline{v}_j$ $i \neq j$ will be equal to zero, so we get:

$$\frac{\underline{x}^T A \underline{x}}{\underline{x}^T M \underline{x}} = \frac{\alpha_1^2 \underline{v}_1^T A \underline{v}_1 + \dots + \alpha_m^2 \underline{v}_m^T A \underline{v}_m}{\alpha_1^2 \underline{v}_1^T M \underline{v}_1 + \dots + \alpha_m^2 \underline{v}_m^T M \underline{v}_m} \overset{Q1(b)}{=} \frac{\alpha_1^2 \lambda_1 + \dots + \alpha_m^2 \lambda_m}{\alpha_1^2 + \alpha_2^2 + \dots + \alpha_m^2}$$

Using that $\{\lambda_i\}_{1\le i\le m}$ are all non-negative and assuming w.l.o.g. that $\lambda_m \le \lambda_{m-1} \le \dots \le \lambda_1$

$$\frac{\underline{x}^T A \underline{x}}{\underline{x}^T M \underline{x}} = \frac{\alpha_1^2 \lambda_1 + \dots + \alpha_m^2 \lambda_m}{\alpha_1^2 + \dots + \alpha_m^2} \overset{\text{if } \alpha_m = 0}{=} \frac{\alpha_1^2 \lambda_1 + \dots + \alpha_{m-1}^2 \lambda_{m-1}}{\alpha_1^2 + \dots + \alpha_{m-1}^2} \geq \frac{(\alpha_1^2 + \dots + \alpha_{m-1}^2)\lambda_{m-1}}{\alpha_1^2 + \dots + \alpha_{m-1}^2} = \lambda_{m-1}$$

So we need that $\alpha_m = 0$. We can achieve it if we get a vector s.t. $\underline{x} \perp \underline{1} = \underline{v}_m$.

A possible choice would be $\underline{x} = \begin{bmatrix} m-1 \\ \underline{1} \end{bmatrix}$. So, with $\alpha := \frac{\underline{x}^T A \underline{x}}{\underline{x}^T M \underline{x}}$, $\underline{x} \perp \underline{1}$ we will get an adequate α.

b) Find a preconditioner for $\begin{bmatrix} A & \underline{c} \\ \underline{c}^T & 0 \end{bmatrix} \underline{v} = \underline{w}$. Prove it is positive-definite.

My choice for the preconditioner is $P = \begin{bmatrix} A + M & \underline{0} \\ \underline{0}^T & 1 \end{bmatrix}$

As M is a sparse matrix, $A + M$ will be fairly similar to A, so P will be similar to $\begin{bmatrix} A & \underline{c} \\ \underline{c}^T & 0 \end{bmatrix}$. This preconditioner works really well in practice.

Let's proof it is positive definite:

Take ~~an~~ arbitrary $\underline{x}$, $\underline{x} = \begin{bmatrix} \underline{x_1} \\ x_2 \end{bmatrix}$,

$$\begin{bmatrix} \underline{x_1}^T & x_2 \end{bmatrix} \begin{bmatrix} A + M & \underline{0} \\ \underline{0}^T & 1 \end{bmatrix} \begin{bmatrix} \underline{x_1} \\ x_2 \end{bmatrix} = \underline{x_1}^T A \underline{x_1} + \underline{x_1}^T M \underline{x_1} + x_2^2$$

As $A \geq 0$ and $M > 0 \rightarrow \underline{x_1}^T A \underline{x_1} \geq 0$ and $\underline{x_1}^T M \underline{x_1} > 0$, and $x_2^2 > 0$

so $x^T P x > 0$. As x is arbitrary, P is positive definite.

Note that, as A and M are symmetric, P is symmetric too.