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A Review on Statistical Physics of Social Dynamics

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Abstract

This is a reading project on the application of statistical model in the field of social dynamics. There exists many models governed by a specific set of rules, which represents a certain social phenomena. The idea that many natural laws have a statistical basis has become so ingrained in modern physics that statistical physics has emerged as a distinct field of study. Recent years have seen an increase in the use of statistical physics in interdisciplinary domains including biology, medicine, information technology, computer science, etc. due to its popularity and its broad conceptual underpinnings. In this regard, physicists have demonstrated a quickly expanding interest in statistical physical modelling of phenomena that are blatantly outside of their typical fields of study.

Chapter 1

Introduction

In social phenomena, the basic constituents are not particles but humans, and every individual interacts with a limited number of peers, usually negligible compared to the total number of people in the system. In spite of that, human societies are characterized by stunning global regularities. There are transitions from disorder to order, like the spontaneous formation of a common language and culture or the emergence of consensus about a specific issue. There are examples of scaling and universality. These macroscopic phenomena naturally call for a statistical physics approach to social behavior, i.e., the attempt to understand regularities at large scale as collective effects of the interaction among single individuals, considered as relatively simple entities.

It may be surprising but the idea of a physical modeling of social phenomena is in some sense older than the idea of statistical modeling of physical phenomena. The discovery of quantitative laws in the collective properties of a large number of people, as revealed, for example, by birth and death rates or crime statistics, was one of the catalysts in the development of statistics, and it led many scientists and philosophers to call for some quantitative understanding of how such precise regularities arise out of the apparently erratic behavior of single individuals. Hobbes, Laplace, Comte, Stuart Mill, and many others shared, to a different extent, this line of thought . This point of view was well known to Maxwell and Boltzmann and probably played a role when they abandoned the idea of describing the trajectory of single particles and introduced a statistical description for gases, laying the foundations of modern statistical physics. The value of statistical laws for social sciences was foreseen also by Majorana.

But it is only in the past few years that the idea of approaching society within the framework of statistical physics has transformed from a philosophical declaration of principles to a concrete research effort involving a critical mass of physicists. The availability of new large databases as well as the appearance of brand new social phenomena (mostly related to the Internet), and the tendency of social scientists to move toward the formulation of simplified models and their quantitative analysis, have been instrumental in this change.

The review article basically discusses different aspects of a single basic question of social dynamics: how do the interactions between social agents create order out of an initial disordered situation? Order is a translation in the language of physics of what is denoted in social sciences as consensus, agreement, uniformity, while disorder stands for fragmentation or disagreement. It is reasonable to assume that without interactions, heterogeneity dominates: left alone, each agent would choose a personal response to a political question, a unique set of cultural features, his own special correspondence between objects and words. Still it is common experience that shared opinions, cultures, and languages do exist. The focus of the statistical physics approach to social dynamics is to understand how this comes about. The key factor is that agents interact and this generally tends to make people more similar. Repeated interactions in time lead to higher degrees of homogeneity, which can be partial or complete depending on the temporal or spatial scales. The investigation of this phenomenon is intrinsically dynamic in nature.

A conceptual difficulty immediately arises when trying to approach social dynamics from the point of view of statistical physics. In usual applications, the elementary components of the systems investigated, atoms and molecules, are relatively simple objects, whose behavior is well known: the macroscopic phenomena are not due to a complex behavior of single entities, but rather to nontrivial collective effects resulting from the interaction of a large number of "simple" elements. Humans are exactly the opposite of such simple entities: the detailed behavior of each of them is already the complex outcome of many physiological and psychological processes, still largely unknown. No one knows precisely the dynamics of a single individual, nor the way he interacts with others. Moreover, even if one knew the very nature of such dynamics and such interactions, they would be much more complicated than, say, the forces that atoms exert on each other. It would be impossible to describe them precisely with simple laws and few parameters. Therefore, any modeling of social agents inevitably involves a large and unwarranted simplification of the real problem. It is then clear that any investigation of models of social dynamics involves two levels of difficulty. The first is in the very definition of sensible and realistic microscopic models; the second is the usual problem of inferring the macroscopic phenomenology out of the microscopic dynamics of such models. Obtaining useful results out of these models may seem a hopeless task.

1.1 Ising model

Consider a collection of N spins s_i that can assume two values ± 1. Each spin is energetically pushed to be aligned with its nearest neighbors. The total energy is

$$H = -\frac{1}{2}\sum_{\langle i,j \rangle} s_i s_j$$

where the sum runs on the pairs of nearest-neighbors spins. Among the possible types of dynamics, the most common (Metropolis)(Landau and Binder, 2005) takes as an elementary move a single spin flip that is accepted with probability exp(E/kBT), where E is the change in energy and T is the temperature. Ferromagnetic interactions drive the system toward one the two possible ordered states, with all positive or all negative spins. At the same time, thermal noise injects fluctuations that tend to destroy order. For low temperature T, the ordering tendency wins and long-range order is established in the system, while above a critical temperature Tc the system remains macroscopically disordered. The transition point is characterized by the average magnetization m=(1/N)$\sum$<si> passing from 0 for T>Tc to a value m(T)>0 for T>Tc. The brackets denote the average over different realizations of the dynamics. This kind of transition is exhibited by a variety of systems. We mention, for its similarity with many of the social dynamic models discussed, the Potts model (Wu, 1982), where each spin can assume one out of q values and equal nearest-neighbor values are energetically favored. The Ising model corresponds to the special case q=2. It is important to stress that above Tc no infinite-range order is established, but on short spatial scales spins are correlated: there are domains of +1 spins (and others of 1 spins) extended over regions of finite size. Below Tc instead these ordered regions extend to infinity (they span the whole system), although at finite temperature some disordered fluctuations are present on short scales (Shown in figure below).

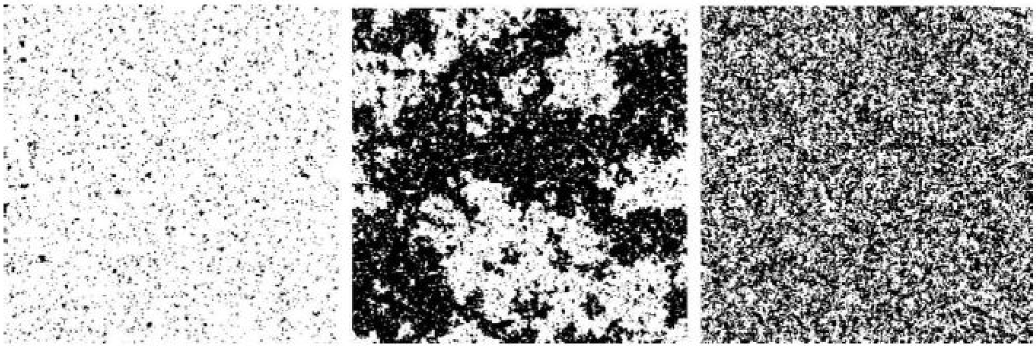

Figure 1.1: Equilibrium configurations of the Ising model (from left to right) below, at, and above Tc [1]

The equilibrium properties just described, which are attained in the long run, are not the only interesting ones. A much investigated and nontrivial issue (Bray, 1994) is the way the final ordered state at T<Tc is reached, when the system is initially prepared in a fully disordered state. This ordering dynamics is a prototype for the analogous processes occurring in many models of social dynamics. On short time scales, coexisting ordered domains of small size (both positive and negative) are formed. The subsequent evolution occurs through a coarsening process of such domains, which grow larger and larger while their global statistical features remain unchanged over time. This is the dynamic scaling phenomenon: the morphology remains statistically the same if rescaled by the typical domain size, which is the only relevant length in the system and grows over time as a power law.

1.2 Topology

An important aspect always present in social dynamics is topology, i.e., the structure of the interaction network describing who is interacting with whom, how frequently, and with which intensity. Agents are thus supposed to sit on vertices (nodes) of a network, and the edges (links) define the possible interaction patterns. The prototype of homogeneous networks is the uncorrelated random graph model proposed by Erdös and Rényi (ER model) (Erdös and Rényi, 1959, 1960)[2], whose construction consists in drawing an (undirected) edge with a fixed probability p between each possible pair out of N given vertices. The resulting graph shows a binomial degree distribution, the degree of a node being the number of its connections, with average $\langle k \rangle \simeq Np$. The degree distribution converges to a Poissonian for large N. If p is sufficiently small (order 1/N), the graph is sparse and presents locally treelike structures. In order to account for degree heterogeneity, other constructions have been proposed for random graphs with arbitrary degree distributions.
A well known paradigm, especially for social sciences, is that of "small-world" networks, in which, on the one hand, the average distance between two agents is small (Milgram, 1967), growing only logarithmically with the network size, and, on the other hand, many triangles are present, unlike ER graphs. In order to reconcile both properties, Watts and Strogatz introduced the small world network model (Watts and Strogatz, 1998)[3], which allows one to interpolate between regular low dimensional lattices and random networks, by introducing a certain fraction p of random long-range connections into an initially regular lattice (Newman and Watts, 1999). Watts and Strogatz (1998) considered the following two quantities: the characteristic path length L(p), defined as the number of edges in the shortest path between two vertices, averaged over all pairs of vertices, and the clustering coefficient C(p), defined as follows. If a node i has k connections, then at most k(k-1)/2 edges can exist between its neighbors (this occurs when every neighbor of i is connected to every other neighbor). The clustering coefficient C(p) denotes the fraction of these allowable edges that actually exist, averaged over all nodes. Small-world networks feature high values of C(p) and

low values of L(p).
Since many real networks are not static but evolving, with new nodes entering and establishing connections to already existing nodes, many models of growing networks have also been introduced. The Barabási-Albert model (Barabási and Albert, 1999) has become one of the most famous models for complex heterogeneous networks, and is constructed as follows: starting from a small set of m fully interconnected nodes, new nodes are introduced one by one. Each new node selects m older nodes according to the preferential attachment rule, i.e., with probability proportional to their degree, and creates links with them. The procedure stops when the required network size N is reached. The obtained network has average degree $<k>=2m$, small clustering coefficient (of order 1/N), and a power-law degree distribution $P(k) \simeq k^{-\gamma}$, with $\gamma=3$. Graphs with power-law degree distributions with $\gamma<3$ are referred to as scalefree networks.

1.3 Agent-based modelling

The origin of agent-based modeling can be traced back to the 1940s, to the introduction by Von Neumann and Ulam of the notion of a cellular automaton (Ulam, 1960; Neumann, 1966), e.g., a machine composed of a collection of cells on a grid. Each cell can be found in a discrete set of states and its update occurs on discrete time steps according to the state of the neighboring cells. A well known example is Conway's game of life, defined in terms of simple rules in a virtual world shaped as a two-dimensional checkerboard. This kind of algorithm became popular in population biology (Matsuda et al, 1992).
The notion of agent has been important in the development of the concept of artificial intelligence (McCarthy, 1959; Minsky, 1961), which traditionally focuses on the individual and on rule-based paradigms inspired by psychology. In this framework, the term actors was used to indicate interactive objects characterized by a certain number of internal states, acting in parallel and exchanging messages (Hewitt, 1970). In computer science, the notion of an actor turned in that of an agent and more emphasis has been put on the interaction level instead of autonomous actions.
Agent-based models were primarily used for social systems by Reynolds, who tried to model the reality of living biological agents, known as artificial life, a term coined by Langton (1996). Reynolds introduced the notion of individual-based models, in which one investigates the global consequences of local interactions of members of a population. In these models, individual agents (possibly heterogeneous) interact in a given environment according to procedural rules tuned by characteristic parameters. One thus focuses on the features of each individual instead of looking at some global quantity averaged over the whole population.
The artificial life community has been the first in developing agent-based models (Meyer and Wilson, 1990; Maes, 1991; Varela and Bourgine, 1992; Steels, 1995; Weiss, 1999), but since then agent-based simulations have become an important tool in other scientific. Epstein and Axtell (1996) introduced, by focusing on a bottom-up approach, the first large-scale agent model (the Sugarscape) to simulate and explore the role of social phenomena such as seasonal migrations, pollution, sexual reproduction, combat, trade and transmission of disease, and culture.

Chapter 2

Opinion Dynamics

Consensus(order) and disorder are the most important aspects of social group dynamics. In our day to day life, we often come across situations where we need a group to reach the same decisions. Agreement makes that opinion more influential and thus have an impact on the society(if we talk in large scale).

Society is made up of people and an individual is complex(may have varied opinions and can have their own bias), which makes the dynamics complex and hard to follow. In statistical physics, the aim is to define the opinion states of a population and the elementary processes that determine transitions between such states. The main question is whether this is possible and whether this approach can shed new light on the process of popular opinion formation.

Consider it as a mathematical model, where the opinion is taken as a variable or a set of variables(a tuple). This may appear as much simpler form of opinion dynamics in reality but in daily life, we see that people are sometimes confronted with a limited number of choices on a specific issue, which often are as few as two: right or left, Windows or Linux, buying or selling, etc. If opinions can be represented by numbers, the challenge is to find an adequate set of mathematical rules to describe the mechanisms responsible for their evolution and changes.

The development of opinion dynamics so far has been uncoordinated and based on individual attempts, where social mechanisms considered reasonable turned into mathematical rules, without a general shared framework and often with no reference to real sociological studies. The first opinion dynamics designed by a physicist was a model proposed by Weidlich (1971). The model is based on the probabilistic framework of sociodynamics. Later on, the Ising model made its first appearance in opinion dynamics (Galam et al, 1982; Galam and Moscovici, 1991). The spin-spin coupling represents the pairwise interaction between agents, the magnetic field represents the cultural majority, or propaganda. Moreover, individual fields are introduced that determine personal preferences toward either orientation. Depending on the strength of the individual fields, the system may reach total consensus toward one of the two possible opinions, or a state in which both opinions coexist.

2.1 Voter model

The voter model has been named in this way for the very natural interpretation of its rules in terms of opinion dynamics; for its extremely simple definition, however, the model has also been thoroughly investigated in fields quite far from social dynamics, such as probability theory and population genetics. Voter dynamics was first considered by Clifford and Sudbury (1973) as a model for the competition of species and named "voter model" by Holley and Liggett (1975). It soon became popular because, despite being a rather crude description of any real process, it is one of the very few nonequilibrium stochastic processes that can be solved exactly in any dimension (Redner, 2001). It can also be seen as a model for dimer-dimer heterogeneous catalysis in the reaction controlled limit

(Evans and Ray, 1993).
The definition is extremely simple: each agent is endowed with a binary variable s= ±1. At each time step, an agent i is selected along with one of its neighbors j and si=sj, i.e., the agent takes the opinion of the neighbor. This update rule implies that agents imitate their neighbors. They feel the pressure of the majority of their peers only in an average sense: the state of the majority does not play a direct role and more fluctuations may be expected with respect to the zero-temperature Glauber dynamics. Bulk noise is absent in the model, so the states with all sites equal (consensus) are absorbing. Starting from a disordered initial condition, voter dynamics tends to increase the order of the system, as in usual coarsening processes (Scheucher and Spohn, 1988). The question is whether full consensus is reached in a system of infinite size. In one-dimensional lattices, the dynamics is exactly the same as the zero temperature Glauber dynamics.

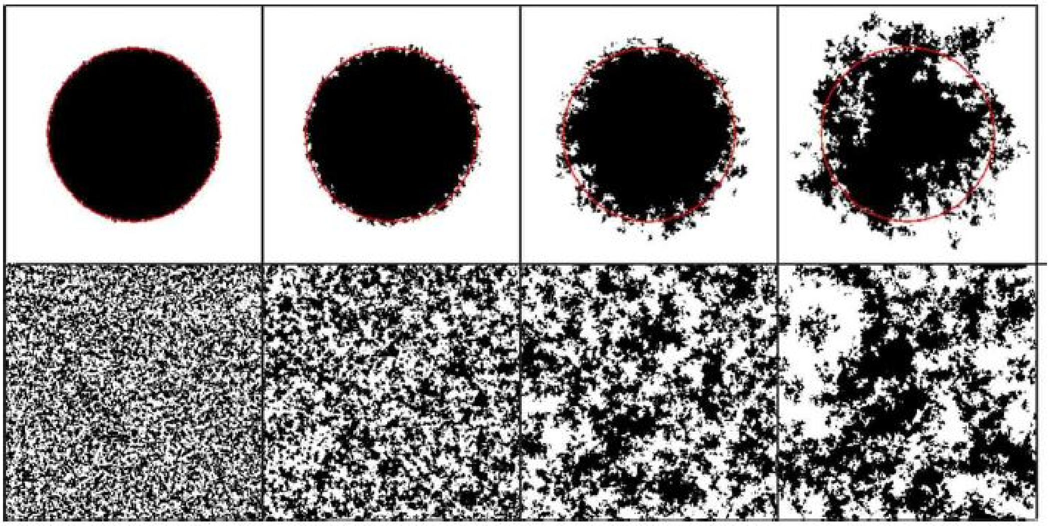

Figure 2.1: Evolution of a two-dimensional voter model starting from a droplet (top) or a fully disordered configuration (bottom) [4]

Considering a d-dimensional hypercubic lattice and denoting with S= s_i the state of the system, the transition rate for a spin k to flip is

$$W_k(S) \equiv W\left(s_k \rightarrow -s_k\right) = \frac{d}{4}\left(1 - \frac{1}{2d} s_k \sum_j s_j\right) \tag{2.1}$$

where j runs over all 2d nearest neighbors and the prefactor, setting the overall temporal scale, is chosen for convenience. The probability distribution function P(S,t) obeys the master equation

$$dP(S,t)/dt = \sum_k \left[W_k\left(S^k\right) P\left(S^k, t\right) - W_k(S) P(S,t)\right]$$

where S^k is equal to S except for the flipped spin s_k. The linear structure of the rates has the consequence that the equations for correlation functions of any order $\langle s_k \cdots s_l \rangle \equiv \Sigma_S P(S,t) s_k \cdots s_l$ can be closed, i.e., they do not depend on higher-order functions and hence can be solved.

2.2 Majority model

In a population of N agents, endowed with binary opinions, a fraction p+ of agents has opinion +1 while a fraction p-=1-p+ has opinion -1. For simplicity, suppose that all agents can communicate

with each other, so that the social network of contacts is a complete graph. At each iteration, a group of r agents is selected at random (discussion group) as a consequence of the interaction, all agents take the majority opinion inside the group. This is the basic principle of the majority rule (MR) model, which was proposed to describe public debates (Galam, 2002)[5].
The group size r is not fixed, but is selected at each step from a given distribution. If r is odd, there is always a majority in favor of either opinion. If r is even, instead, there is the possibility of a tie, i.e., that either opinion is supported by exactly r /2 agents. In this case, one introduces a bias in favor of one of the opinions, say +1, and that opinion prevails in the group. This prescription is inspired by the principle of social inertia, for which people are reluctant to accept a reform if there is no clear majority in its favor. Majority rule with opinion bias was originally applied within a simple model describing hierarchical voting in a society (Galam, 1986, 1990, 1999, 2000). Defined as p_+^0 the initial fraction of agents with the opinion +1, the dynamics is characterized by a threshold pc such that, for p_+^0 < pc, all agents will have opinion +1 (-1) in the long run. The time to reach consensus scales like log N . If the group sizes are odd, pc(r) =1/2, due to the symmetry of the two opinions. If there are groups with r even, pc<1/2, i.e., the favored opinion will eventually be the dominant one, even if it is initially shared by a minority of agents.
The MR model with a fixed group size r was analytically solved in the mean-field limit (Krapivsky and Redner, 2003). The group size r is odd, to keep the symmetry of the two opinions. The solution can be derived both for a finite population of N agents and in the continuum limit of N→∞. The latter derivation is simpler (Chen and Redner, 2005a), and is sketched here.
Let s_k= ±1 be the opinion of agent k; the average opinion (magnetization) of the system is $m = (1/N)\sum_k s_k = p_+ - p_-$. The size of each discussion group is 3. At each update step, the number N+ of agents in state + increases by one unit if the group state is ++-, while it decreases by one unit if the group state is +–. One thus has

$$dN_+ = 3\left(p_+^2 p_- - p_+ p_-^2\right) = -6p_+\left(p_+ - \frac{1}{2}\right)(p_+ - 1)$$

where the factor of 3 is due to the different permutations of the configurations ++- and +–. The equation can be rewritten as

$$\frac{dN_+}{N}\frac{N}{3} = \dot{p}_+ = -2p_+\left(p_+ - \frac{1}{2}\right)(p_+ - 1)$$

with the time increment dt=3/N, so that each agent is updated once per unit of time. The fixed points are determined by the condition $\dot{p}$+=0, and from the above equation we see that this happens when p+=0, 1/2, and 1, respectively. The point p+=1/2 is unstable, whereas the others are stable: starting from any p+≠1/2, all agents will converge to the state of initial majority, recovering Galam's result. The integration of this equation yields that the consensus time grows as log N.

2.3 Sznajd model

We saw that the impact exerted by a social group on an individual increases with the size of the group. We would not pay attention to a single guy staring at a blank wall; however, if a group of people stares at that wall, we may be tempted to do the same. Convincing somebody is easier for two or more people than for a single individual. This is the basic principle behind the Sznajd model, which we call Sznajd B, agents occupy the sites of a linear chain, and have binary opinions, denoted by Ising spin variables. A pair of neighboring agents i and i+1 determines the opinions of their two nearest neighbors i-1 and i+2, according to the following rules:
$s_i = s_{i+1}$, then $s_{i-1} = s_i = s_{i+1} = s_{i+2}$
$s_i \neq s_{i+1}$, then $s_{i-1} = s_{i+1}$ and $s_{i+2} = s_i$

So, if the agents of the pair share the same opinion, they successfully impose their opinion on their neighbors. If, instead, the two agents disagree, each agent imposes its opinion on the other agent's neighbor. Opinions are updated in a random sequential order. Starting from a totally random initial configuration, where both opinions are equally distributed, two types of stationary states are found, corresponding to consensus, with all spins up (m=1) or all spins down (m=-1), and to a stalemate, with the same number of up and down spins in antiferromagnetic order (m=0). The latter state is a consequence of the second rule, which favors antiferromagnetic configurations, and has a probability 1/2 to be reached. Each of the two (ferromagnetic) consensus states occurs with a probability 1/4. The values of the probability can be easily deduced from the up-down symmetry of the model. The relaxation time of the system into one of the possible attractors has a log-normal distribution (Behera and Schweitzer, 2003). The number of agents that never changed opinion first decays as a power law of time, and then reaches a constant but finite value, at odds with the Ising model (Stauffer and de Oliveira, 2002). The exit probability has been calculated analytically for both random and correlated initial conditions (Lambiotte and Redner, 2008; Slanina et al, 2008).

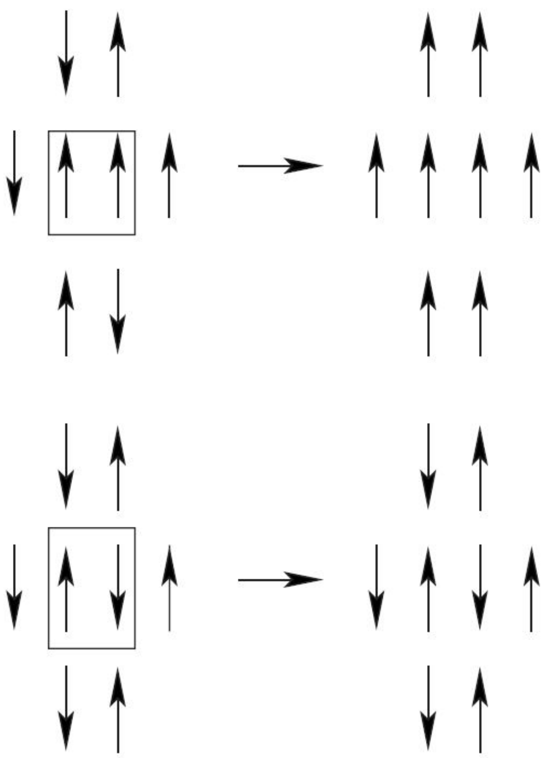

Figure 2.2: Sznajd model. In the most common version of the model (Sznajd A), a pair of neighboring agents with the same opinion convince all their neighbors (top), while they have no influence if they disagree (bottom) [1]

Since the very introduction of the Sznajd model, it has been argued that a distinctive feature of its dynamics is the fact that information flows from the initial pair of agents to their neighbors, at variance with the other opinion dynamics models, in which instead agents are influenced by their neighbors. Because of that, the Sznajd model was supposed to describe how opinions spread in a society. On the other hand, Behera and Schweitzer (2003) showed that, in one dimension, the direction of the information flow is actually irrelevant, and that the Sznajd B dynamics is equivalent to a voter dynamics. The only difference with the classic voter model is that an agent is not influenced by its nearest neighbors but by its next-to-nearest neighbors. Indeed, the dynamics of Sznajd B on a linear chain can be summarized by the simple sentence "just follow your nextto- nearest neighbor." The fact that in the Sznajd model one pair of agents is updated at a time, whereas in the voter model the dynamics affects a single spin, introduces a factor of 2 in the average relaxation time of the equivalent voter dynamics; all other features are exactly the same, from the probability to hit the attractors to the distributions of decision and relaxation times. Therefore, the Sznajd B model does not respect the principle of social validation which motivated its introduction, as each spin is influenced only by a single spin, not by a pair.

Chapter 3

Cultural Dynamics

The border between the field of opinion dynamics and cultural dynamics is not sharp and the distinction is not clear-cut. The general attitude is to consider opinion as a scalar variable, while the more faceted culture of an individual is modeled as a vector of variables, whose dynamics is inextricably coupled. This definition is largely arbitrary, but we adopt it here.
The typical questions asked with respect to cultural influence are similar to those related to the dynamics of opinions: What are the microscopic mechanisms that drive the formation of cultural domains? What is the ultimate fate of diversity? Is it bound to persist or do all differences eventually disappear in the long run? What is the role of the social network structure?

3.1 Axelrod model

From the point of view of statistical physicists, the Axelrod model is a simple and natural "vectorial" generalization of models of opinion dynamics that gives rise to a very rich and nontrivial phenomenology, with some genuinely novel behavior. The model is defined as follows. Individuals are located on the nodes of a network (or on the sites of a regular lattice) and are endowed with F integer variables (σ_1 , . . . ,σ_F) that can assume q values, σ_f=0,1, . . . ,q-1. The variables are called cultural features and q is the number of the possible traits allowed per feature. They are supposed to model the different "beliefs, attitudes, and behavior" of individuals. In an elementary dynamic step, an individual i and one of his neighbors j are selected and the overlap between them,

$$\omega_{i,j} = \frac{1}{F}\sum_{f=1}^{F}\delta_{\sigma_f(i),\sigma_f(j)}$$

is computed, where $\delta_{i,j}$ is Kronecker's delta. With probability $\omega_{i,j}$ the interaction takes place: one of the features for which traits are different $\sigma_f(i) \neq \sigma_f(j)$ is selected and the trait of the neighbor is set equal to $\sigma_f(i)$. Otherwise nothing happens. It is immediately clear that the dynamics tends to make interacting individuals more similar, but the interaction is more likely for neighbors already sharing many traits (homophily) and it becomes impossible when no trait is the same. There are two stable configurations for a pair of neighbors: when they are exactly equal, so that they belong to the same cultural region, or when they are completely different, i.e., they sit at the border between cultural regions.
Starting from a disordered initial condition (for example, with uniform random distribution of the traits), the evolution on any finite system leads unavoidably to one of the many absorbing states, which belong to two classes: the qF ordered states, in which all individuals have the same set of variables, or the other, more numerous, frozen states with coexistence of different cultural regions.

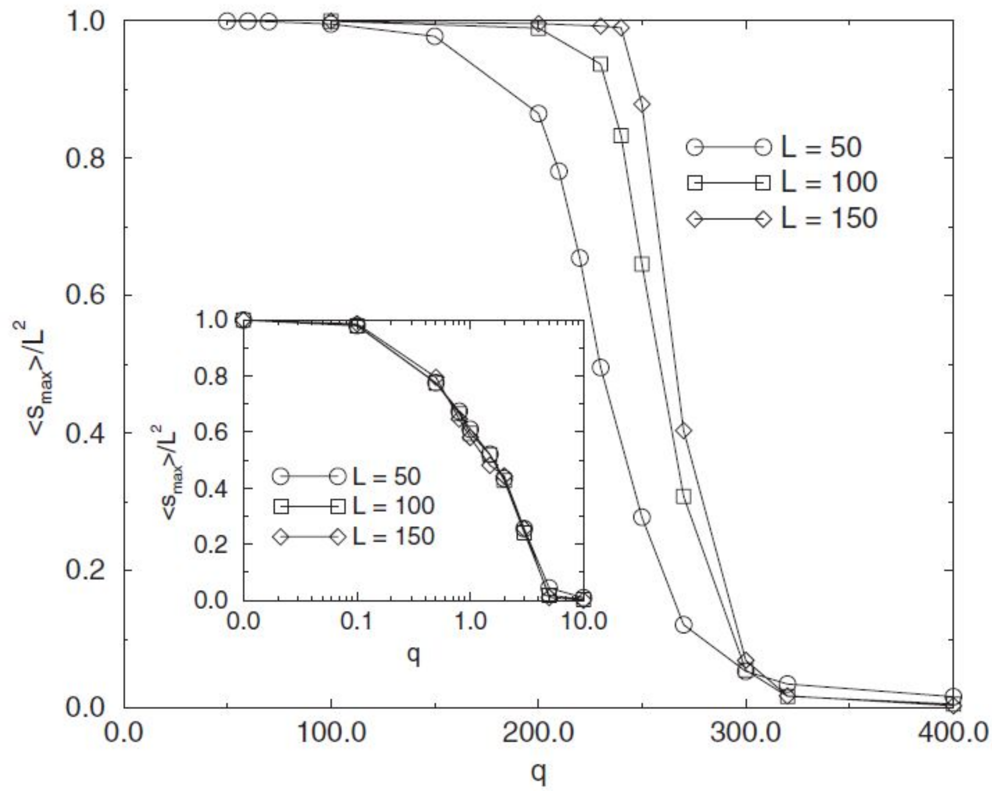

Figure 3.1: Axelrod model. Behavior of the order parameter <Smax> $/L^2$ vs q for three different system sizes and F=10. In the inset the same quantity is reported for F=2. From Castellano et al., 2000. [6]

It turns out that which of the two classes is reached depends on the number of possible traits q in the initial condition (Castellano et al., 2000). For small q, individuals share many traits with their neighbors, interactions are possible, and quickly full consensus is achieved. For large q instead, very few individuals share traits. Few interactions occur, leading to the formation of small cultural domains that are not able to grow: a disordered frozen state. On regular lattices, the two regimes are separated by a phase transition at a critical value qc, depending on F.

3.2 Variants of Axelrod model

In his seminal paper, Axelrod mentioned many possible variants of his model, to be studied in order to investigate the effect of additional ingredients as the topology of the interactions, random noise, the effect of mass media, and many others. Over the years this program has been followed by many researchers. The possibility of one individual to change spontaneously one of his traits, independently of his neighborhood, is denoted as "cultural drift" in social science and corresponds to the addition of flipping events driven by random noise. Klemm et al. (2003a) demonstrated that the inclusion of noise at rate r has a profound influence on the model, resulting in a noise-induced order disorder transition, practically independent of the value of the parameter q.
For small noise the state of the system is monocultural for any q, because disordered configurations are unstable with respect to the perturbation introduced by the noise: the random variation of a trait unfreezes in some cases the boundary between two domains leading to the disappearance of one in favor of the other. However, when the noise rate is large, the disappearance of domains is compensated by the rapid formation of new ones, so that the steady state is disordered. The threshold between the two behaviors is set by the inverse of the average relaxation time for a perturbation

T(N), so that the transition occurs for $r_c T(N) = O(1)$. An approximate evaluation of the relaxation in d=2 gives T=N ln(N), in good agreement with simulations, while T≃N^2 in one dimension (Klemm et al., 2005) [7]. Since T(N) diverges with N, the conclusion is that, no matter how small the rate of cultural drift is, in the thermodynamic limit the system remains always disordered for any q.

The discovery of the fragility of the Axelrod model with respect to the presence of noise immediately raises the question: What is the simplest modification of the original model that preserves the existence of a transition in the presence of noise? Kuperman (2006), introduced two modified Axelrod-like dynamics, where the interaction between individuals is also influenced by which trait is adopted by the majority of agents in the local neighborhood. Similar ingredients are present in two other variants of the Axelrod model recently proposed (Flache and Macy, 2007). A convincing illustration that these modifications lead to a robust phenomenology with respect to the addition of (at least weak) noise is still lacking.

Another variant to the original definition of the model is the introduction of a threshold such that, if the overlap is smaller than a certain value, no interaction takes place (Flache and Macy, 2007). Unsurprisingly, no qualitative change occurs, except for a reduction of the ordered region of the phase diagram (De Sanctis and Galla, 2007). Another possibility, called "interaction noise," is that for smaller than the threshold the interaction takes place with probability δ. This kind of noise favors ordering but again does not lead to drastic changes of the model behavior (De Sanctis and Galla, 2007).

In order to understand the effect of complex interaction topologies on its behavior, the Axelrod model has been studied on small-world and scale-free graphs (Klemm et al., 2003b) [8]. In the first case, the transition between consensus and a disordered multicultural phase is still observed, for values of the control parameter qc that grow as a function of the rewiring parameter p. Since the WS network for p=1 is a random network (and then practically an infinite-dimensional system), this is consistent with the observation of the transition also in the mean-field approaches (Castellano et al., 2000; Vázquez and Redner, 2007) [9]. The scale-free nature of the BA network dramatically changes the picture. For a given network of size N, a somewhat smeared-out transition is found for a value qc, with bistability of the order parameter, the signature of a first-order transition. However, numerical simulations show that the transition threshold grows with N as qc N0.39, so that in the thermodynamic limit the transition disappears and only ordered states are possible. This is similar to what occurs for the Ising model on scale-free networks, where the transition temperature diverges with system size (Leone et al., 2002).

3.3 Other models

In the original paper on the Deffuant model (Deffuant et al., 2000)[10], a generalization to vectorial opinions was introduced, considering in this case binary variables instead of continuous ones. This gives a model similar to the Axelrod model with q=2 traits per feature, with the difference that the probability of interaction between two agents as a function of their overlap is a step function at a bounded confidence threshold d. In mean field, a transition between full consensus for large threshold and fragmentation for small d is found.

A similar model has been studied by Laguna et al. (2003). In this case, when two agents are sufficiently close to interact, each pair of different variables may become equal with a probability μ. Again a transition between consensus and fragmentation is found as a function of the bounded confidence threshold, but its properties change depending on whether μ =1 or μ<1.

A generalization of continuous opinions (the HK model) to the vectorial (two-dimensional) case has been reported by Fortunato et al. (2005) [11] for a square opinion space, with both opinions ranging between 0 and 1, and square or circular confidence ranges. Assuming homogeneous mixing and solving the rate equations, it turns out that no drastic change occurs with respect to the ordinary HK model. The consensus threshold is practically the same. When there is no consensus, the position of coexisting clusters is determined by the shape of the opinion space. An extension of Deffuant and HK models to vectorial opinions was proposed by Lorenz. Here opinions sit on a hypercubic space or on a simplex, i.e., the components of the opinion vectors sum up to 1. It turns out that consensus is easier to attain if the opinion space is a simplex rather than hypercubic.

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