Transformations Part 1

CS GY-6533 / UY-4533

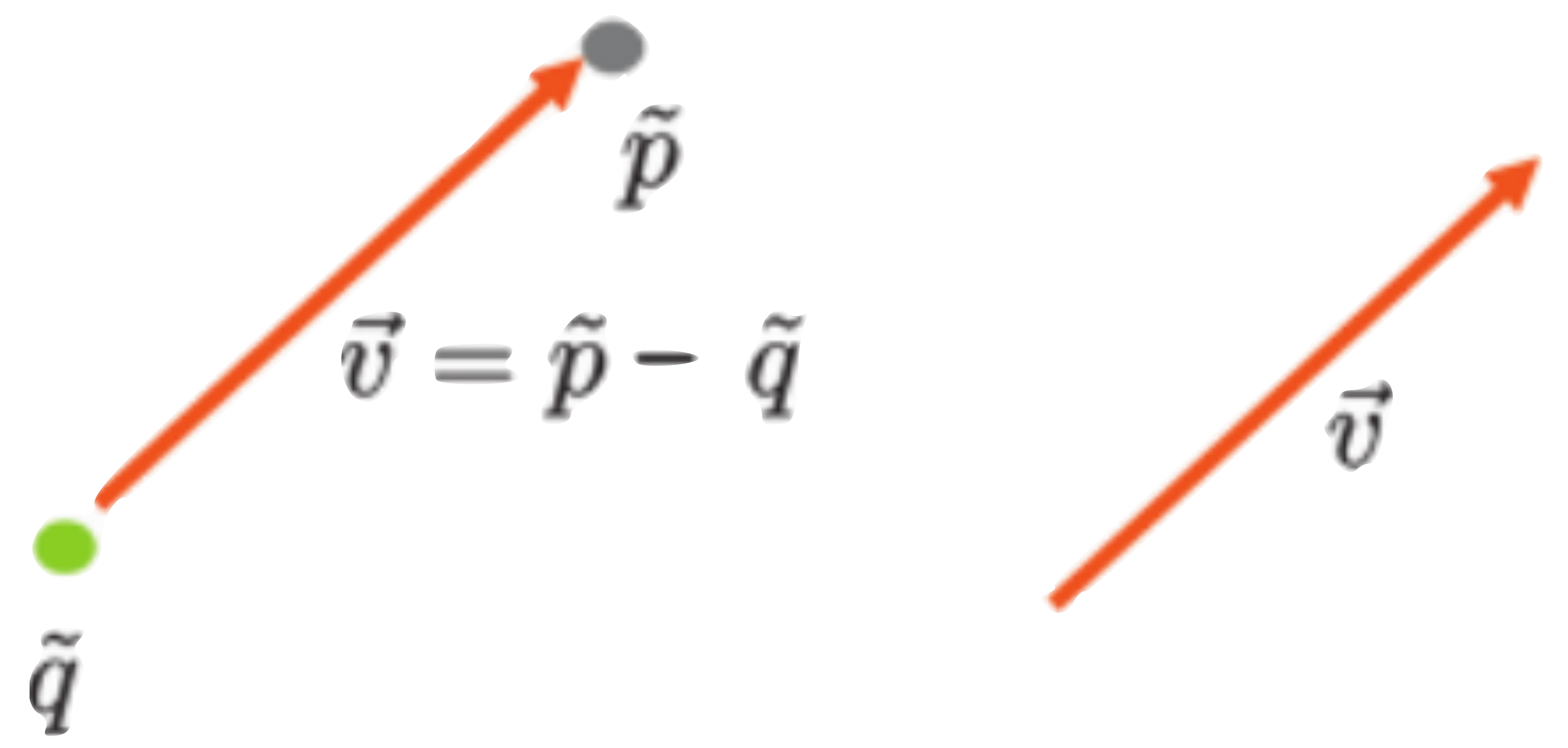
Vectors and Bases

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

A point in space can be represented with three real numbers, which we can call a **coordinate vector**.

$\mathbf{c}$ - a coordinate vector

$\tilde{p}$ - a point in space.



$\vec{v}$ - a vector

$\vec{f}^{\,t}$ - a coordinate system (horizontal collection of vectors)

$$\vec{v} = \sum_i c_i \vec{b}_i .$$

$$\vec{v} = \sum_i c_i \vec{b}_i = \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} .$$

Basis

$$\vec{v} = \vec{\mathbf{b}}^t \mathbf{c},$$

Linear Transformations by 3x3 matrices

A transformation is said to be linear if it preserves the operations of addition and scalar multiplication.

$$\mathcal{L}(\vec{v} + \vec{u}) = \mathcal{L}(\vec{v}) + \mathcal{L}(\vec{u})$$

$$\mathcal{L}(\alpha\vec{v}) = \alpha\mathcal{L}(\vec{v}).$$

$$\vec{v} \Rightarrow \mathcal{L}(\vec{v}) = \mathcal{L}\left(\sum_i c_i \vec{b}_i\right) = \sum_i c_i \mathcal{L}(\vec{b}_i),$$

$$\begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \Rightarrow \begin{bmatrix} \mathcal{L}(\vec{b}_1) & \mathcal{L}(\vec{b}_2) & \mathcal{L}(\vec{b}_3) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}.$$

$$\begin{bmatrix} \mathcal{L}(\vec{b}_1) & \mathcal{L}(\vec{b}_2) & \mathcal{L}(\vec{b}_3) \end{bmatrix} = \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 \end{bmatrix} \begin{bmatrix} M_{1,1} & M_{1,2} & M_{1,3} \\ M_{2,1} & M_{2,2} & M_{2,3} \\ M_{3,1} & M_{3,2} & M_{3,3} \end{bmatrix}$$

$$\vec{\mathbf{b}}^t\mathbf{c} \Rightarrow \vec{\mathbf{b}}^t M\mathbf{c}$$

$$\begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 \end{bmatrix} \Rightarrow \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 \end{bmatrix} \begin{bmatrix} M_{1,1} & M_{1,2} & M_{1,3} \\ M_{2,1} & M_{2,2} & M_{2,3} \\ M_{3,1} & M_{3,2} & M_{3,3} \end{bmatrix}$$

$$\vec{\mathbf{b}}^t \Rightarrow \vec{\mathbf{b}}^t M$$

Identity

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Scaling

Scale basis vectors by factor of α

$$\begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 \end{bmatrix} \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

Scale along each axis.

$$\begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 \end{bmatrix} \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

Rotation

2D rotation.

$$\vec{v} = \begin{bmatrix} \vec{b}_1 & \vec{b}_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

$$x' = x\cos\theta - y\sin\theta$$

$$y' = x\sin\theta + y\cos\theta.$$

$$\begin{bmatrix} \vec{b}_1 & \vec{b}_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \vec{b}_1 & \vec{b}_2 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Expressing the same rotation as a 3D rotation around the Z axis.

$$\begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 \end{bmatrix} \begin{bmatrix} c & -s & 0 \\ s & c & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Rotation around the X axis.

$$\begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

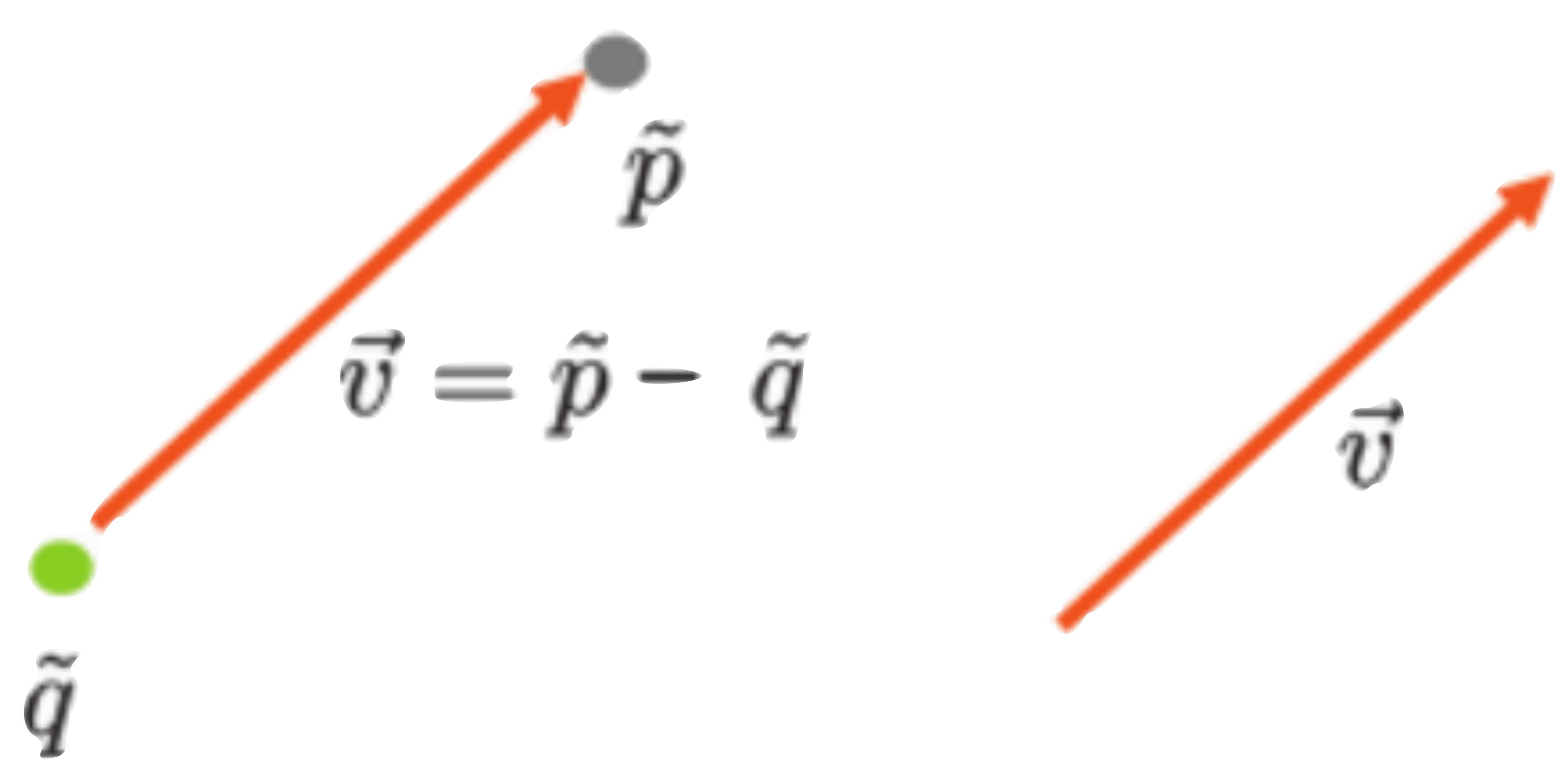
$$\Rightarrow \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c & -s \\ 0 & s & c \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Rotation around the Y axis.

$$\begin{bmatrix} c & 0 & s \\ 0 & 1 & 0 \\ -s & 0 & c \end{bmatrix}$$

Affine transformations

Points and frames



$$\tilde{p} - \tilde{q} = \vec{v}.$$

$$\tilde{q} + \vec{v} = \tilde{p}.$$

$$\tilde{p} = \tilde{o} + \sum_i c_i \vec{b}_i = \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 & \tilde{o} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ 1 \end{bmatrix} = \vec{\mathbf{f}}^t \mathbf{c},$$

Affine **frame**

$$\begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 & \tilde{o} \end{bmatrix} = \vec{\mathbf{f}}^t$$

To specify a point using a frame, we use a coordinate 4-vector with four entries, with the last entry always being a one. To express a vector using an affine frame, we use a coordinate vector with a zero as the fourth coordinate (i.e., it is simply a sum of the basis vectors).

Affine transformation by 4x4 matrices.

$$\begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

$$
\begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 & \tilde{o} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ 1 \end{bmatrix}
$$

$$
\Rightarrow \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 & \tilde{o} \end{bmatrix} \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ 1 \end{bmatrix},
$$

$$
\vec{\mathbf{f}}^t \mathbf{c} \Rightarrow \vec{\mathbf{f}}^t A \mathbf{c}.
$$

$$\begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 & \tilde{o} \end{bmatrix} \Rightarrow \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 & \tilde{o} \end{bmatrix} \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\vec{\mathbf{f}}^t \Rightarrow \vec{\mathbf{f}}^t A.$$

Linear transformations in an affine matrix.

$$
\begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 & \tilde{o} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ 1 \end{bmatrix}
$$

$$
\Rightarrow \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 & \tilde{o} \end{bmatrix} \begin{bmatrix} a & b & c & 0 \\ e & f & g & 0 \\ i & j & k & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ 1 \end{bmatrix}.
$$

$$
L = \begin{bmatrix} l & 0 \\ 0 & 1 \end{bmatrix}
$$

Translation.

$$\begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 & \tilde{o} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 & \tilde{o} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ 1 \end{bmatrix},$$

$$T = \begin{bmatrix} i & t \\ 0 & 1 \end{bmatrix},$$

Putting them together

$$\begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & d \\ 0 & 1 & 0 & h \\ 0 & 0 & 1 & l \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c & 0 \\ e & f & g & 0 \\ h & i & j & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\begin{bmatrix} l & t \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} i & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} l & 0 \\ 0 & 1 \end{bmatrix}$$

$$A = TL.$$

Read chapters 2 and 3.