Divisibility
Dr. Vince

Definition 1. If $a, b \in \mathbb{Z}$, then we say *a divides* b and write $a \mid b$ if there exists $n \in \mathbb{Z}$ such that $an = b$. If there is no integer n such that $an = b$, then we say *a does not divide* b and write $a \nmid b$. Notice the similarity between "a divides b" and "b is divisible by a."

Example 2. $2 \mid 18$ because $2 \times 9 = 18$.

Example 3. $2 \nmid 19$ because $2 \times 9 = 18$ and $2 \times 10 = 20$, so there is no integer n with $2n = 19$.

Example 4. $4 \mid 0$ because $4 \times 0 = 0$.

In fact, notice that $a \mid 0$ for every integer a, because $a \cdot 0 = 0$. Of special note is the case of 0; $0 \mid 0$ because $0 \cdot 0 = 0$, so 0 does divide 0. This is the only difference between "a divides b" and "b is divisible by a;" of course, 0 is not divisible by 0.

Theorem 5 (Properties)**.** *Let* $a, b \in \mathbb{Z}$.

(1) If $a \mid b$ *and* $b \mid c$*, then* $a \mid c$.

(2) If $a \mid b$*, then* $a \mid kb$ *for any* $k \in \mathbb{Z}$.

(3) If $a \mid b$ *and* $a \mid c$*, then* $a \mid (b + c)$.

(4) If $a \mid b$ *and* $a \mid c$*, then* $a \mid (kb + \ell c)$ *for any* $k, \ell \in \mathbb{Z}$.

Proof. (1) Since $a \mid b$ and $b \mid c$, there exist $m, n \in \mathbb{Z}$ such that $am = b$ and $bn = c$. Then $a(mn) = (am)n = c$, so $a \mid c$.

(2) Since $a \mid b$, there exists $n \in \mathbb{Z}$ such that $an = b$. Then $a(nk) = kb$, so $a \mid kb$.

(3) Since $a \mid b$ and $a \mid c$, there exist $m, n \in \mathbb{Z}$ such that $am = b$ and $an = c$. Then $a(m + n) = am + an = b + c$, so $a \mid (b + c)$.

(4) (Proof 1) Since $a \mid b$ and $a \mid c$, there exist $m, n \in \mathbb{Z}$ such that $am = b$ and $an = c$. Then $a(km + \ell n) = kam + \ell an = kb + \ell c$, so $a \mid (kb + \ell c)$ for any $k, \ell \in \mathbb{Z}$.

(Proof 2) Using property (2), $a \mid kb$ and $a \mid \ell c$. Using property (3), $a \mid (kb + \ell c)$.

$\square$

Example 6 (IMO 1959)**.** Prove that the fraction $\dfrac{21n + 4}{14n + 3}$ is irreducible for every natural number n.

Solution: If $d \mid (21n + 4)$ and $d \mid (14n + 3)$, then property (4) tells us that $d \mid [3(14n + 3) - 2(21n + 4)]$. But $[3(14n + 3) - 2(21n + 4)] = 1$, so $d \mid 1$. The only positive integer d which satisfies this is $d = 1$, so we have proved that the only positive common factor in the numerator and denominator is 1, that is, the fraction is irreducible.

$\square$