**Continuous Additive Functions**

**Definition 1.** A function $f$ defined on the real numbers is said *additive* if $f(x+y) = f(x) + f(y)$ for all real numbers $x$ and $y$.

(1) Give an example of an additive function and show that it is additive.

(2) Give an example of a function which is not additive and show that it is not additive.

(3) Suppose $f$ is an additive function and $m$ is a real number. Define $g(x) = f(x) - mx$. Prove that $g$ is an additive function.

(4) Suppose $f$ is an additive function. Define $g(x) = f(x) - mx$ where $m = f(1)$. Prove $g(x+1) = g(x)$ for all real numbers $x$.

**Definition 2.** A function $f$ is *bounded on* $[a, b]$ if there is a real number $M$ such that $|f(x)| \leq M$ for all $x$ in $[a, b]$. We could say $f$ is *bounded on* $[a, b]$ *by* $M$.

A function $f$ is *bounded* (or for emphasis, *bounded everywhere*) if there is a real number $M$ such that $|f(x)| \leq M$ for all real numbers $x$. We could say $f$ is *bounded by* $M$ (or for emphasis, *bounded everywhere by* $M$).

(5) Suppose $f$ is an additive function that is bounded on $[0, 1]$. Define $g(x) = f(x) - mx$ where $m = f(1)$. Prove that $g$ is bounded everywhere.

(6) Let $g$ be an additive function which is bounded by $M$. Prove that if there is a real number $a$ such that $g(a) \neq 0$, then there must be a real number $b$ such that $|g(b)| > M$ (a contradiction). What can you conclude about $g$? (Hint: consider $g(2a)$ and $g(3a)$...)

(7) Prove that if $f$ is additive and continuous, there is a real number $m$ such that $f(x) = mx$ for all $x$. (Hint: use the previous parts!)

**Bonus 1.** Prove that if $f$ is additive and continuous at $x = a$, there is a real number $m$ such that $f(x) = mx$ for all $x$.

**Bonus 2.** Prove that if $f$ is additive and monotonic, there is a real number $m$ such that $f(x) = mx$ for all $x$.

**Bonus 3.** Prove that if $f$ is additive and $f(x) \geq 0$ for $x \geq 0$, there is a real number $m$ such that $f(x) = mx$ for all $x$.