Name: ______________________________ Mark: ______________________________

Mini-math AP Calculus BC: Friday, October 22, 2021 (8 minutes)

1. (2 points) If the series $S = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ is approximated by the kth partial sum S_k, what is the least value of k for which the alternating series error bound guarantees that $|S - S_k| \leq \frac{1}{100}$?

Solution: The alternating series error bound guarantees that

$$|S - S_k| \leq b_{k+1} = \frac{1}{\sqrt{k+1}}$$

so we wish to find the least k for which

$$\begin{aligned} \frac{1}{\sqrt{k+1}} &\leq \frac{1}{1000} \\ \sqrt{k+1} &\geq 1000 \\ k+1 &\geq 1\,000\,000 \end{aligned}$$

Thus the least k is $999\,999$.

2. (2 points) For what values of p is the following series conditionally convergent?

$$\sum_{n=1}^{\infty} \frac{(-1)^n (n + \sqrt{n})}{n^{2p} - 4}$$

Solution: If $p \leq 1/2$, then $1 - 2p > 0$ and so

$$\lim_{n\to\infty} \frac{(-1)^n (n + \sqrt{n})}{n^{2p} - 4} = \lim_{n\to\infty} \frac{(-1)^n (n^{1-2p} + n^{1/2-2p})}{1 - 4n^{-2p}}$$

does not exist, so the nth term test shows the series diverges.

Let $b_n = \frac{n+\sqrt{n}}{n^{2p} - 4}$. Notice if $p > 1/2$, then b_n is decreasing and $\lim_{n\to 0} b_n = 0$. Then the series converges by the Alternating Series Test if $p > 1/2$.

Consider the series $\sum_{n=1}^{\infty} a_n$ where $a_n = |b_n|$. By the limit comparison test with $c_n = \frac{1}{n^{2p-1}}$,

$$\lim_{n\to\infty} \frac{a_n}{c_n} = \lim_{n\to\infty} \frac{n+\sqrt{n}}{n^{2p} - 4} \cdot n^{2p-1} = \lim_{n\to\infty} \frac{1 + n^{-1/2}}{1 - 4n^{-2p}} = 1$$

so the series converges if and only if $\sum_{n=1}^{\infty} \frac{1}{n^{2p-1}}$ converges. By p-series, this occurs if and only if $2p - 1 > 1$, so $p > 1$. Then the original series converges absolutely if $p > 1$.

Therefore, the original series is conditionally convergent if $1/2 < p \leq 1$.