Trigonometry — Optimization

1. A rain gutter is to be constructed from a metal sheet of width 36 cm by bending up one-third of the sheet on each side through an angle θ. How should angle θ be chosen so that the gutter will carry the maximum amount of water?

   Solution: Since one-quarter of the sheet is bent up through an angle θ, the gutter will be a trapezoid with short base 12 cm and arms 12 cm through an angle θ. Clearly, $0 \le \theta \le \pi/2$, since a greater angle will reduce the area. By simple trigonometry, the height of the trapezoid is $12\sin\theta$, and the long base is $12 + 2 \cdot 12\cos\theta$. Then we wish to maximize

   $$A(\theta) = \frac{1}{2}\left(12 + 12 + 12\cos\theta\right)\left(10\sin\theta\right) = 12^2(\sin\theta + \sin\theta\cos\theta)$$

   Differentiating,

   $$\begin{aligned} A'(\theta) &= 12^2(\cos\theta + \cos\theta\cos\theta - \sin\theta\sin\theta) \\ &= 12^2(\cos\theta + 2\cos^2\theta - 1) \end{aligned}$$

   A' exists everywhere, so we check where $A' = 0$:

   $$\begin{aligned} 0 &= 12^2(\cos\theta + 2\cos^2\theta - 1) \\ 0 &= 2\cos^2\theta + \cos\theta - 1 = (2\cos\theta - 1)(\cos\theta + 1) \end{aligned}$$

   so either $\cos\theta = 1/2$ or $\cos\theta = -1$. In our domain, we get $\theta = \pi/3$. Finally,

   $$\begin{aligned} A(0) &= 0, \\ A(\pi/3) &= 12^2\left(\sin\frac{\pi}{3} + \sin\frac{\pi}{3}\cos\frac{\pi}{3}\right) = 12^2\left(\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}\cdot\frac{1}{2}\right) = \frac{3 \cdot 12^2\sqrt{3}}{4} = 108\sqrt{3}, \\ A(\pi/2) &= 12^2\left(\sin\frac{\pi}{2} + \sin\frac{\pi}{2}\cos\frac{\pi}{2}\right) = 12^2 = 144 \end{aligned}$$

   Since $144 < 108\sqrt{3}$ (indeed, squaring yields $2^4 \cdot 6^4$ vs $3^3 \cdot 6^4$, and $16 < 27$), $\theta = \pi/3$ is the maximum.