Mini-math Div 3/4: Friday, March 19, 2025 (10.10-10.15) - (25 minutes)

SOLUTIONS

1. (3 points) The series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges to S. If S_n is used to approximate S, what is the least value of n for which the alternating series error bound guarantees an error to strictly within 0.01?

Solution:

$$\frac{1}{n+1} < \frac{1}{100}$$
$$n+1 > 100$$

so $n = 100$ is the least value of n needed.

2. (3 points) Let $P(x)$ be the fifth-degree Taylor Polynomial for a function f about $x = 1$. Information about the maximum of the absolute value of selected derivatives of f over various intervals is given below.

$$\max_{0\le x\le 1.5} |f^{(4)}(x)| = 4.6, \quad \max_{0\le x\le 1.5} |f^{(5)}(x)| = 7.2, \quad \max_{0\le x\le 1.5} |f^{(6)}(x)| = 6.8,$$
$$\max_{1\le x\le 1.5} |f^{(4)}(x)| = 3.2, \quad \max_{1\le x\le 1.5} |f^{(5)}(x)| = 4.7, \quad \max_{1\le x\le 1.5} |f^{(6)}(x)| = 5.1$$

Find the smallest value of k for which the Lagrange error bound guarantees that

$$|f(1.5) - P(1.5)| \le k$$

Solution: We use the bound on the 6th derivative on the interval $[1, 1.5]$, since this is the smallest interval which contains the centre and the point we are approximating. The Lagrange error bound gives a maximum absolute error of

$$\frac{\max_{1\le x\le 1.5} |f^{(6)}(x)|}{6!} \cdot |1.5-1|^6 = \frac{5.1 \cdot 0.5^6}{6!}$$

3. (4 points) Find the interval of convergence for the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n(x-3)^n}{n2^n}$

Solution:

$$\lim_{n\to\infty} \left| \frac{(-1)^{n+1}(x-3)^{n+1}}{(n+1)2^{n+1}} \cdot \frac{n2^n}{(-1)^n(x-3)^n} \right| = \lim_{n\to\infty} \frac{n}{n+1} \cdot \frac{|x-3|}{2} = \frac{|x-3|}{2}$$

Then the radius of convergence is 2. Testing $|x-3| = 2$, we have at $x-3=2$

$$\sum_{n=1}^{\infty} \frac{(-1)^n 2^n}{n2^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

converges by AST and

$$\sum_{n=1}^{\infty} \frac{(-1)^n(-2)^n}{n2^n} = \sum_{n=1}^{\infty} \frac{1}{n}$$

diverges by p-series. Therefore, the interval of convergence is $(3-2, 3+2] = (1,5]$.

4. (3 points) What is the Maclaurin series for $\dfrac{\cos x - 1}{x}$? Assume differentiability at 0 (e.g. the function has a value at 0 which makes it differentiable). You may, but are not required to, express your answer in summation notation.

Solution:

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \cdots$$

so

$$\begin{aligned} \frac{\cos x - 1}{x} &= \frac{1}{x} \cdot \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = \frac{-\frac{x^2}{2} + \frac{x^4}{4!} - \cdots}{x} \\ &= \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n-1}}{(2n)!} = -\frac{x}{2} + \frac{x^3}{4!} - \cdots \end{aligned}$$

5. (4 points) Let f be a function with $f(0) = 2$ and $f'(x) = \arctan x$. Write the first three non-zero terms of the Maclaurin series for f.

Solution:

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$$

$$\begin{aligned} f(x) = \int \arctan x \, dx &= \int (x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots) \, dx \\ &= C + \frac{x^2}{2} - \frac{x^4}{4 \cdot 3} + \frac{x^6}{6 \cdot 5} - \cdots \end{aligned}$$

Since $f(0) = 2$, we get $C = 2$ so the first three non-zero terms are:

$$2 + \frac{x^2}{2} - \frac{x^4}{12}$$