

Mini-math Div 3/4: Friday, January 12, 2024 (9.1-9.5) - 20 minutes

SOLUTIONS

1. (3 points) Write an equation for the line tangent to the curve defined by $r(t) = \langle 2^t, 1/t \rangle$ at the point where $x = 8$.

Solution: $2^t = 8$ gives $t = 3$. At this value, $y(3) = 1/3$. Now,

$$\left. \frac{dy}{dx} \right|_{t=3} = \left. \frac{dy/dt}{dx/dt} \right|_{t=3} = \left. \frac{-t^{-2}}{2^t \ln 2} \right|_{t=3} = \frac{-1/9}{8 \ln 2} = -\frac{1}{72 \ln 2}$$

by point-slope, an equation of the tangent line is

$$y - \frac{1}{3} = -\frac{1}{72 \ln 2}(x - 8)$$

2. (4 points) If $x(\theta) = \tan 2\theta$ and $y(\theta) = \sec 2\theta$, find the concavity at $\theta = \pi/6$.

Solution:

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{2 \sec 2\theta \tan 2\theta}{2 \sec^2 2\theta} = \sin 2\theta$$

Then

$$\left. \frac{d^2y}{dx^2} \right|_{\theta=\pi/6} = \left. \frac{\frac{d}{d\theta} \left(\frac{dy}{dx} \right)}{dx/d\theta} \right|_{\theta=\pi/6} = \left. \frac{\frac{d}{d\theta} (\sin 2\theta)}{2 \sec^2 2\theta} \right|_{\theta=\pi/6} = \left. \frac{2 \cos 2\theta}{2 \sec^2 2\theta} \right|_{\theta=\pi/6} = (\cos \pi/3)^3 = \frac{1}{8}$$

Concave up.

3. (2 points) Write down (but do not evaluate) an integral which represents the length of the curve described by the parametric equations $x = t^3/3$ and $y = t^2/2$ from $t = 0$ to $t = 1$. (Extra challenge: find the exact value.)

Solution: First, note that $x'(t) = t^2$ and $y'(t) = t$

$$L = \int_0^1 \sqrt{t^4 + t^2} dt \quad \left(= \left. \frac{(t^2 + 1)^{3/2}}{3} \right|_0^1 = \frac{2\sqrt{2} - 1}{3} \right)$$

4. (3 points) If f is a vector-valued function defined by $f(t) = \langle 2 \sin t, \cos 2t \rangle$, then what is $f''(\pi/3)$?

Solution:

$$\begin{aligned}f'(t) &= \langle 2 \cos t, -2 \sin 2t \rangle, \\f''(t) &= \langle -2 \sin t, -4 \cos 2t \rangle, \\f''(\pi/3) &= \langle -\sqrt{3}, 2 \rangle\end{aligned}$$

5. (3 points) Find the vector-valued function $f(t)$ that satisfies the initial conditions $f(1) = \langle 4, 5 \rangle$, and $f'(t) = \langle 6t, 7 \rangle$.

Solution:

$$\begin{aligned}f(t) &= \langle 4 + \int_1^t 6u \, du, 5 + \int_1^t 7 \, du \rangle \\&= \langle 4 + 3(t^2 - 1), 5 + 7(t - 1) \rangle\end{aligned}$$

6. (4 points) (Calculator-active) At time $t \geq 0$, a particle moving in the xy -plane has velocity vector given by $v(t) = \langle \sin(t^2), 2\sqrt{t} \rangle$. If the particle is at point $(-3, 1)$ at time $t = 0$, how far is the particle from the origin at time $t = 3$?

Solution: We calculate

$$\begin{aligned}x(3) &= -3 + \int_0^3 \sin(t^2) \, dt \approx -2.22644 \\y(3) &= 1 + \int_0^3 2\sqrt{t} \, dt \approx 7.93628\end{aligned}$$

Then

$$d = \sqrt{[x(3)]^2 + [y(3)]^2} \approx 8.243$$