Exponent Laws
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In order to understand exponents a^n, we need to work in stages. We begin with $a \neq 0$ ($a = 0$ we will treat as a special case later) and n a positive integer, for which:

$$a^n \coloneqq \underbrace{a \cdot a \cdot \dots \cdot a}_{n \text{ copies of } a}$$

(Here, the symbol $\coloneqq$ is shorthand for "is defined as ...") We call a the *base* and n the *exponent* or *power*. In this way, we think of exponentiation as repeated multiplication, in a similar fashion to how multiplication is repeated addition. Note that if n is not one of $1, 2, 3, \dots$, then this is not well-defined! As a consequence of choosing to work with $a \neq 0$ for now, notice that $a^n \neq 0$ for any positive integer n.

With this definition in place, we can prove the following exponent properties:

(P1) $a^n \cdot a^m = \underbrace{a \cdot a \cdot \dots \cdot a}_{n \text{ copies of } a} \cdot \underbrace{a \cdot a \cdot \dots \cdot a}_{m \text{ copies of } a} = \underbrace{a \cdot a \cdot \dots \cdot a}_{n+m \text{ copies of } a} = a^{n+m}$ if n and m are positive integers

(P2) $\frac{a^n}{a^m} = \frac{\overbrace{a \cdot a \cdot \dots \cdot a}^{n \text{ copies of } a}}{\underbrace{a \cdot a \cdot \dots \cdot a}_{m \text{ copies of } a}} = \underbrace{a \cdot a \cdot \dots \cdot a}_{n-m \text{ copies of } a} = a^{n-m}$ if $n > m$ are positive integers

(P3) $(a^n)^m = \underbrace{\underbrace{a \cdot a \cdot \dots \cdot a}_{n \text{ copies of } a} \cdot \underbrace{a \cdot a \cdot \dots \cdot a}_{n \text{ copies of } a} \cdot \dots \cdot \underbrace{a \cdot a \cdot \dots \cdot a}_{n \text{ copies of } a}}_{m \text{ copies of } a \cdot a \cdot \dots \cdot a} = \underbrace{a \cdot a \cdot \dots \cdot a}_{nm \text{ copies of } a} = a^{nm}$ if n and m are positive integers

You may notice that at a minimum, we needed the exponent to be positive integers, which brings us to the question: what if the exponent is not a positive integer? Let's start with the simplest case, a^0. Notice that if we want property **(P1)** to hold, then we need

$$a^0 \cdot a^n = a^{0+n} = a^n$$

for all n. Since $a^n \neq 0$, we can divide and get $a^0 = 1$. It is important to note that this is *not* a proof that $a^0 = 1$, since we can only apply property 1 in the case when both exponents are positive integers, and 0 is not. Rather, this is *motivation* for why we should define

$$a^0 \coloneqq 1,$$

so that property **(P1)** can extend to the nonnegative integers.

Similarly, we can ask how negative exponents should be defined: for n a positive integer, if property **(P1)** holds, then

$$a^{-n} \cdot a^n = a^{-n+n} = a^0 = 1$$

where we have used our previous definition of $a^0 = 1$. Since $a^n \neq 0$, we want

$$a^{-n} := \frac{1}{a^n}$$

As an exercise, try seeing what the definition of a^{-n} must be in order for property **(P2)** to extend to integers.

To deal with rational exponents, we restrict our attention to $a > 0$ (we'll see later why negative bases are problematic). Since we have a method for extending from positive exponents to negative exponents, let's suppose n, m are positive integers and determine what happens with $a^{\frac{n}{m}}$. Extending property **(P3)** to rational numbers, we would want

$$(a^{\frac{n}{m}})^m = a^{\frac{n}{m} \cdot m} = a^n$$

that is, we want $a^{\frac{n}{m}}$ to be the unique number that, when raised to the power of m, gives a^n as the result. This is precisely what the mth root is! Thus, we define

$$(a^{\frac{n}{m}})^m := \sqrt[m]{a^n}$$

If $a < 0$, then $a^{\frac{1}{m}}$ has no real definition for even m for which property **(P3)** is satisfied. For example, if property **(P3)** does work, then

$$((-1)^2)^{\frac{1}{2}} = (1)^{\frac{1}{2}} = 1 \quad \text{but} \quad ((-1)^2)^{\frac{1}{2}} = (-1)^{2 \cdot \frac{1}{2}} = (-1)^1 = -1,$$

showing $1 = -1$, which is a contradiction. As an exercise, you can show that there is no issue when m is odd.

Earlier, we ignored what happens when $a = 0$. This is because for the most part, it's very uninteresting: $0^b := 0$ for any $b > 0$. As an exercise, check that the identities we have so far also work when extending to $a = 0$. Take a moment to consider why we need $b > 0$ (the case of 0^0 is especially tricky).

<u>Extension:</u> Finally, one might ask how we extend the definition of exponents to deal with real numbers. For instance, how might we define $2^{\sqrt{2}}$? This issue will be dealt with in a calculus course, but briefly, since we know how rational exponents work and $\sqrt{2} = 1.41421356237\ldots$ can be closely approximated with rationals (say, with the decimal approximations up to some number of digits), we look at what happens when we take the sequence of numbers:

$$\begin{aligned}
2^{1.4} &= 2^{\frac{14}{10}} = \sqrt[10]{2^{14}} \approx 2.63901582155 \\
2^{1.41} &= 2^{\frac{141}{100}} = \sqrt[100]{2^{141}} \approx 2.65737162819 \\
2^{1.414} &= 2^{\frac{1414}{1000}} = \sqrt[1000]{2^{1414}} \approx 2.66474965018 \\
2^{1.4142} &= 2^{\frac{14142}{10000}} = \sqrt[10000]{2^{14142}} \approx 2.66511908853 \\
&\vdots
\end{aligned}$$

By taking better and better approximations to $\sqrt{2}$ in the exponent, the answers we get seem to be approaching a specific number—this is how we would define $2^{\sqrt{2}}$. If you're interested, this turns out to be $2^{\sqrt{2}} = 2.66514414269\ldots$.