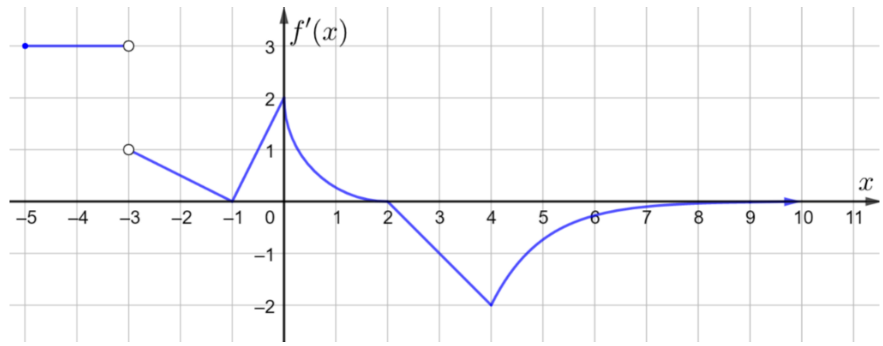
AP CALCULUS BC	YouTube Live Virtual Lessons	Mr. Bryan Passwater Mr. Anthony Record
Topic: All Units	**Free Response Question Stem Types** **Graphical**	**Date:** April 28, 2020

Free Response Questions Stem Types: Graphical

2020 FRQ Practice Problem BC1



BC1: The graph of f', the derivative of the continuous function f, is given above. For $-5 \le x < 4$, the graph of f' consists of four linear pieces and a quarter circle centered at (2,2). For $x \ge 4$, $f'(x) = -2e^{4-x}$. It is known that $f(0) = 6$

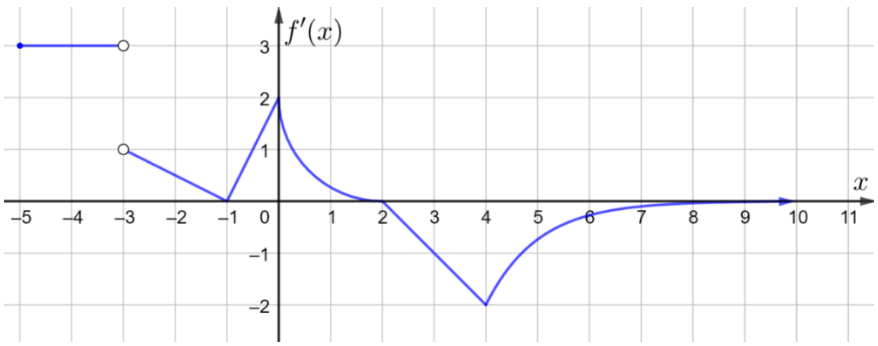
(**a**) Find $f(4)$.

$$f(4) = f(0) + \int_0^4 f'(x)\,dx \quad = 6 + \left[(2)(2) - \frac{1}{4}\pi(2)^2 - \frac{1}{2}(2)(2)\right] = 6 + [2 - \pi] = 8 - \pi$$

(**b**) Write, but do not evaluate, an integral expression that gives the arc length of the graph of f from $x = 4$ to $x = 10$.

$$L = \int_4^{10} \sqrt{1 + [f'(x)]^2}\,dx = \int_4^{10} \sqrt{1 + [-2e^{4-x}]^2}\,dx = \int_4^{10} \sqrt{1 + 4[e^{8-2x}]}\,dx$$

The problem has been restated.



BC1: The graph of f', the derivative of the continuous function f, is given above. For $-5 \le x < 4$, the graph of f' consists of four linear pieces and a quarter circle centered at $(2,2)$. For $x \ge 4$, $f'(x) = -2e^{4-x}$. It is known that $f(0) = 6$

(**c**) Find $\sum_{n=2}^{\infty} a_n$ where $a_n = f'(n)$.

$$\sum_{n=2}^{\infty} a_n = \sum_{n=2}^{\infty} f'(n) = f'(2) + f'(3) + \sum_{n=4}^{\infty}\left(-2e^{4-n}\right) = -1 + \sum_{n=4}^{\infty}\left(-2e^{4-n}\right) = -1 + \sum_{n=4}^{\infty}\left(-2e^{4}\left(\frac{1}{e}\right)^{n}\right)$$

$$\sum_{n=4}^{\infty}\left(-2e^{4}\left(\frac{1}{e}\right)^{n}\right) \Rightarrow \text{geometric with } r = \frac{1}{e}, a = \frac{-2e^4}{e^4} = -2 \Rightarrow \sum_{n=4}^{\infty}\left(-2e^{4}\left(\frac{1}{e}\right)^{n}\right) = \frac{-2}{1-\frac{1}{e}} = \frac{-2e}{e-1}$$

$$\sum_{n=2}^{\infty} a_n = -1 + \frac{-2e}{e-1} = \frac{-(e-1)-2e}{e-1} = \frac{1-3e}{e-1}$$

For parts (d) and (e), let $g(x) = 3 - \int_{-1}^{x} [2f'(2t) + 1]dt$.

(**d**) Write an expression for $g'(x)$ and $g''(x)$.

$$g'(x) = -\left[2f'(2x) + 1\right] \qquad g''(x) = -\left[2f''(2x)(2)\right] = -4f''(2x)$$

(**e**) Does the graph of g have a point of inflection at $x = 2$? Give a reason for your answer.

$$g''(2) = -4f''(4) \qquad \lim_{x\to 4^-} f''(x) = -1 \qquad \lim_{x\to 4^+} f''(x) = \lim_{x\to 4^+}\left(2e^{4-x}\right) = 2$$

$$\lim_{x\to 2^-} g''(x) = (-4)\left[\lim_{x\to 4^-} f'(x)\right] = 4 \text{ and } \lim_{x\to 2^+} g''(x) = (-4)\left[\lim_{x\to 4^+} f'(x)\right] = -8$$

g does have a point of inflection at $x = 2$, $g''(2)$ does not exist and there is sign change.

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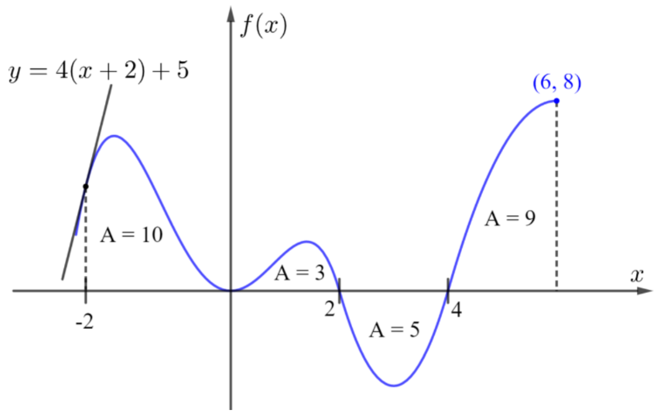
2020 FRQ Practice Problem BC2



BC2: A portion of the graph of the twice differentiable function f is shown in the figure above. The areas of the regions bounded by the graph of f and the x axis for $[-2, 6]$ are shown above. At $x = -2$, the line tangent to the graph of f is shown along with its equation.

The function H is defined by $H(x) = \int_0^x f(t)dt$.

(**a**) Evaluate $\int_{-1}^{2} f(4-2x)dx$.

$$-\frac{1}{2}\int_{-1}^{2} f\big(\underbrace{4-2x}_{u}\big)\big(\underbrace{-2dx}_{du}\big) = -\frac{1}{2}\int_{6}^{0} f(u)du = \frac{1}{2}\int_{0}^{6} f(u)du = \frac{1}{2}[3-5+9] = \frac{7}{2}$$

(**b**) Find $\lim\limits_{x\to 2} \dfrac{H(3x)-7}{\sin(\pi x)}$.

$$H(x) \text{ is continuous} \Rightarrow \lim_{x\to 2}\big(H(3x)-7\big) = H(6)-7 = -7+\int_0^6 f(t)dt = -7+[7] = 0$$

$$\lim_{x\to 2}\sin(\pi x) = \sin(2\pi) = 0$$

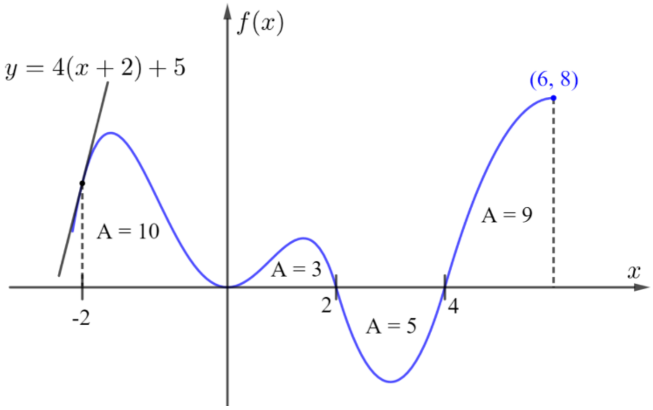
$$\lim_{x\to 2}\frac{H(3x)-7}{\sin(\pi x)} = \underbrace{\lim_{x\to 2}\frac{H'(3x)(3)}{\cos(\pi x)(\pi)}}_{\text{l'Hospital's Rule}} = \frac{3H'(6)}{\pi} = \frac{3f(6)}{\pi} = \frac{3(8)}{\pi} = \frac{24}{\pi}$$

(**c**) Find any x value(s) where $H(x)$ has a relative maximum. Give a reason for your answer.

$H'(x) = f(x) = 0 \Rightarrow x = 0, x = 2, x = 4$

$H(x)$ has a relative maximum at $x = 2$ because $H'(x) = f(x)$ changes from positive to negative at $x = 2$.

The problem has been restated.



BC2: A portion of the graph of the twice differentiable function f is shown in the figure above. The areas of the regions bounded by the graph of f and the x axis for $[-2, 6]$ are shown above. At $x = -2$, the line tangent to the graph of f is shown along with its equation.

The function H is defined by $H(x) = \int_0^x f(t)dt$.

(**d**) Find the second degree Taylor polynomial to $H(x)$ centered at $x = -2$.

$$H(-2) = \int_0^{-2} f(t)dt = -10 \qquad H'(-2) = \underbrace{f(-2) = 5}_{\text{point of tangency}} \qquad H''(-2) = \underbrace{f'(-2) = 4}_{\text{slope of tangent line when } x=-2}$$

$$P_2(x) = H(-2) + H'(-2)(x+2) + \frac{H''(-2)}{2!}(x+2)^2$$

$$P_2(x) = -10 + 5(x+2) + \frac{4}{2!}(x+2)^2 = 10 + 5(x+2) + 2(x+2)^2$$

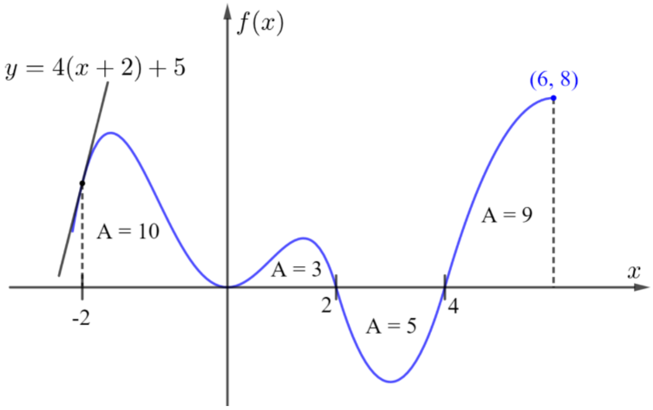
(**e**) Consider the curve $y^2 + 2xy - x = H(x)$. Find the slope of the line tangent to the curve at the point (6,1).

$$2y\frac{dy}{dx} + 2\left(y + x\frac{dy}{dx}\right) - 1 = H'(x) \qquad (6,1) \Rightarrow 2(1)\frac{dy}{dx} + 2\left(1 + (6)\frac{dy}{dx}\right) - 1 = H'(6) = f(6) = 8$$

$$2\frac{dy}{dx} + 2\left(1 + (6)\frac{dy}{dx}\right) - 1 = 8 \Rightarrow 2\frac{dy}{dx} + 2 + (12)\frac{dy}{dx} = 9 \Rightarrow 14\frac{dy}{dx} = 7 \Rightarrow \frac{dy}{dx} = \frac{1}{2}$$

The slope of the tangent line to the curve at the point $(6,1)$ is $\left.\frac{dy}{dx}\right|_{(6,1)} = \frac{1}{2}$

The problem has been restated.



BC2: A portion of the graph of the twice differentiable function f is shown in the figure above. The areas of the regions bounded by the graph of f and the x axis for $[-2, 6]$ are shown above. At $x = -2$, the line tangent to the graph of f is shown along with its equation.

The function H is defined by $H(x) = \int_0^x f(t)dt$.

For values of x greater than or equal to 6 , the function H can also be modeled by the function:

$$H(x) = 64 - \frac{128}{\sqrt{x-2}}.$$

(**f**) Find $\int_6^\infty f(t)dt$.

$$H(x) = \int_0^x f(t)dt = \int_0^6 f(t)dt + \int_6^x f(t)dt = (3-5+9) + \int_6^x f(t)dt = 7 + \int_6^x f(t)dt$$

$$\lim_{x\to\infty} H(x) = 7 + \int_6^\infty f(t)dt \qquad \lim_{x\to\infty} H(x) = \lim_{x\to\infty}\left[64 - \frac{128}{\sqrt{x-2}}\right] \to 64 - \frac{128}{\infty} = 64$$

$$64 = 7 + \int_6^\infty f(t)dt \Rightarrow \int_6^\infty f(t)dt = 57$$

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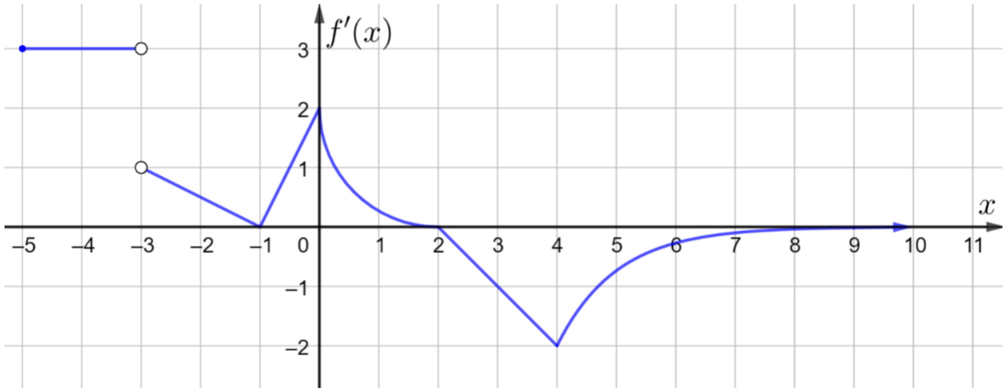
2020 FRQ Practice Problem BC3



BC3: The graph of f', the derivative of the continuous function f, is given above.
For $-5 \le x < 4$, the graph of f' consists of four linear pieces and a quarter circle centered at $(2,2)$. For $x \ge 4$, $f'(x) = -2e^{4-x}$. It is known that $f(0) = 6$

(**a**) Find any x value(s) where the graph of f has a point of inflection. Explain your reasoning.

point of inflection occurs where f' changes from increasing to decreasing or vice versa
points of inflection at $x = -1, 0,$ and 4.

(**b**) Find the maximum value of f on the closed interval $[-1, 4]$. Justify your answer.

local maximum when $x = 2$ endpoints of interval: $x = -1, 4$

x	$f(x)$
-1	$6 - \int_{-1}^{0} f'(x)\,dx = 6 - \left(\frac{1}{2}(1)(2)\right) = 5$
2	$6 + \int_{0}^{2} f'(x)\,dx = 6 + \left[(2)(2) - \frac{1}{4}\pi(2)^2\right] = 10 - \pi$
4	$6 + \int_{0}^{4} f'(x)\,dx = 6 + \left[(2)(2) - \frac{1}{4}\pi(2)^2 - \frac{1}{2}(2)(2)\right] = 6 + [2 - \pi] = 8 - \pi$

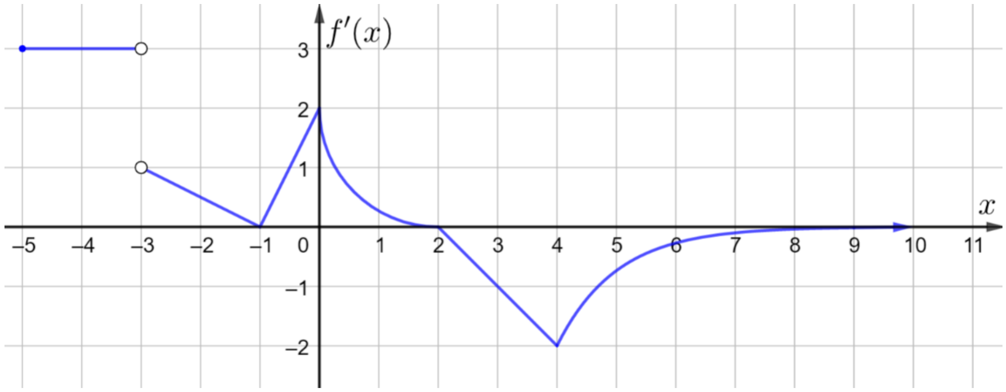
The maximum value is $10 - \pi$

(**c**) Find $\lim\limits_{x\to\infty} f(x)$.

$$x \ge 4 \Rightarrow f'(x) = -2e^{4-x} \Rightarrow f(x) = f(4) + \int_{4}^{x} -2e^{4-t}\,dt = (8-\pi) + \left[2e^{4-t}\right]_4^x$$

$$= 2e^{4-x} - 2e^{0} + (8-\pi) = 2e^{4-x} + 6 - \pi \qquad \lim_{x\to\infty} f(x) = \lim_{x\to\infty}\left(2e^{4-x} + 6 - \pi\right) = 6 - \pi$$

The problem has been restated.



BC3: The graph of f', the derivative of the continuous function f, is given above.
For $-5 \leq x < 4$, the graph of f' consists of four linear pieces and a quarter circle centered at $(2,2)$. For $x \geq 4$, $f'(x) = -2e^{4-x}$. It is known that $f(0) = 6$

For parts (d) and (e), let $g(x) = 3 - \int_{-1}^{x} [2f'(2t) + 1]dt$.

(**d**) Does the graph of g have a local minimum, local maximum, or neither at $x = 2$?

Give a reason for your answer.

$$g'(x) = -[2f'(2x)+1]$$

$$g'(2) = -[2f'(4)+1] = -[2(-2)+1] = 3 \Rightarrow \text{neither since } g'(2) \neq 0$$

(**e**) Find $P_2(x)$, the second degree Taylor polynomial to $g(x)$ centered at $x = -1$.

$$g(-1) = 3 \qquad g'(-1) = -[2f'(-2)+1] = -\left[2\left(\frac{1}{2}\right)+1\right] = -2$$

$$g''(-1) = -4f''(-2) = (-4)\left(-\frac{1}{2}\right) = 2$$

$$P_2(x) = g(-1) + g'(-1)(x+1) + \frac{g''(-1)}{2!}(x+1)^2$$

$$P_2(x) = 3 - 2(x+1) + \frac{2}{2!}(x+1)^2 = 3 - 2(x+1) + (x+1)^2$$

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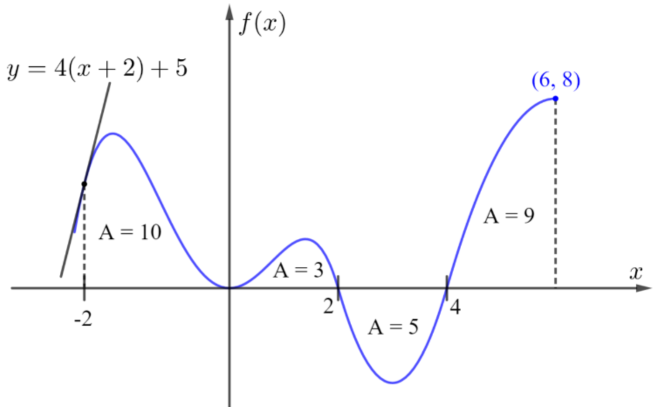
2020 FRQ Practice Problem BC4



BC4: A portion of the graph of the twice differentiable function f is shown in the figure above. The areas of the regions bounded by the graph of f and the x axis for $[-2, 6]$ are shown above. At $x = -2$, the line tangent to the graph of f is shown along with its equation.

The function H is defined by $H(x) = \int_0^x f(t)dt$.

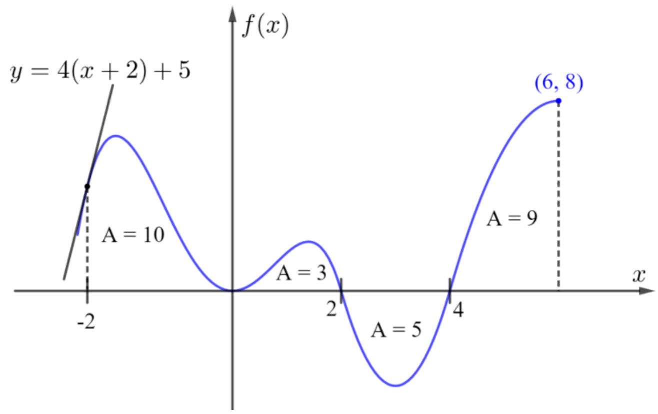
(**a**) Evaluate $\int_0^4 H(x)f(x)dx$

$$\int_0^4 \underbrace{H(x)}_{u}\underbrace{f(x)dx}_{du} = \int_0^{-2} u\,du = \left[\frac{1}{2}u^2\right]_0^{-2} = \left[\frac{1}{2}(-2)^2\right] - \left[\frac{1}{2}(0)^2\right] = 2$$

(**b**) Let $k(x) = H(x)e^{2x}$. Find $k'(6)$.

$$k'(x) = H'(x)e^{2x} + H(x)\left(2e^{2x}\right)$$
$$k'(6) = H'(6)e^{12} + H(6)\left(2e^{12}\right) = f(6)e^{12} + 7\left(2e^{12}\right) = 8e^{12} + 14e^{12} = 22e^{12}$$

The problem has been restated.



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The function H is defined by $H(x) = \int_0^x f(t)dt$.

For values of x greater than or equal to 6 , the function H can also be modeled by the function:

$$H(x) = 64 - \frac{128}{\sqrt{x-2}}.$$

(**c**) Consider the series $\sum_{n=6}^{\infty} a_n$ where $a_n = H'(x)$. Determine if $\sum_{n=6}^{\infty} a_n$ converges or diverges.

$$H(x) = 64 - 128(x-2)^{-1/2} \qquad H'(x) = -128\left(-\frac{1}{2}\right)(x-2)^{-3/2} = \frac{64}{(x-2)^{3/2}}$$

$$\sum_{n=6}^{\infty} a_n = \sum_{n=6}^{\infty} \frac{64}{(n-2)^{3/2}} = 64\sum_{n=6}^{\infty} \frac{1}{(n-2)^{3/2}}$$

$$\sum_{n=6}^{\infty} \frac{1}{(n-2)^{3/2}} = \frac{1}{(6-2)^{3/2}} + \frac{1}{(7-2)^{3/2}} + \frac{1}{(8-2)^{3/2}} + \cdots = \frac{1}{(4)^{3/2}} + \frac{1}{(5)^{3/2}} + \frac{1}{(6)^{3/2}} + \cdots p-\text{series}$$

$$\sum_{n=6}^{\infty} \frac{1}{(n-2)^{3/2}} \text{ is a } p-\text{series with } p = \frac{3}{2} > 1 \Rightarrow \sum_{n=6}^{\infty} \frac{1}{(n-2)^{3/2}} \text{ converges so } 64\sum_{n=6}^{\infty} \frac{1}{(n-2)^{3/2}} \text{ converges}$$

(**d**) Write, but do not evaluate, an expression with one or more integrals in terms of x and $f(x)$ that gives the length of the curve $H(x)$ from $x = 0$ to $x = 10$.

$$0 \le x \le 6 \Rightarrow H'(x) = f(x) \qquad 6 < x \le 10 \Rightarrow H'(x) = \frac{64}{(x-2)^{3/2}}$$

$$L = \int_0^{10} \sqrt{1 + \left[H'(x)\right]^2}\, dt = \int_0^{6} \sqrt{1 + \left[f(x)\right]^2}\, dt + \int_6^{10} \sqrt{1 + \left[\frac{64}{(x-2)^{3/2}}\right]^2}\, dt$$