

Mini-math Div 3/4: Wednesday, January 22, 2025 (9.6-9.9) - (24 minutes)

Calculator active

SOLUTIONS

1. (4 points) The velocity vector of a particle moving in the plane is given by

$$\langle 5 - 2 \cos(t^2), 8 \sin(t^2) \cos(e^t) \rangle, \text{ for } 0 \leq t \leq 2$$

At time $t = 0$, the particle is at position $(3, -1)$. Write an equation for the line tangent to the path of the particle at $t = 1$.

Solution:

$$\left. \frac{dy}{dx} \right|_{t=1} = \frac{y'(1)}{x'(1)} = -1.566$$

$$x(1) = 3 + \int_0^1 x'(t) dt = 6.191$$

$$y(1) = -1 + \int_0^1 y'(t) dt = -2.241$$

so an equation is given by

$$y + 2.241 = -1.566(x - 6.191)$$

2. (4 points) Where does the graph $r = 1 - \sin \theta$, $0 \leq \theta \leq 2\pi$, have a vertical tangent?

Solution: To have a vertical tangent, we want $x'(\theta) = 0$ (and $y'(\theta) \neq 0$). We get

$$0 = x'(\theta) = \frac{d}{d\theta}(r \cos \theta) = \frac{d}{d\theta}(\cos \theta - \sin \theta \cos \theta) = -\sin \theta - \cos^2 \theta + \sin^2 \theta$$

With a calculator, $\theta = \pi/2, 7\pi/6, 11\pi/6$ (or 1.571, 3.665, 5.760). However, we check

$$0 = y'(\theta) = \frac{d}{d\theta}(r \sin \theta) = \frac{d}{d\theta}(\sin \theta - \sin^2 \theta) = \cos \theta - 2 \sin \theta \cos \theta$$

Checking our three values, $y'(\pi/2) = 0$, but the other two yield non-zero y' . Therefore, the answer is $\theta = 7\pi/6, 11\pi/6$ (or 3.665, 5.760).

3. (4 points) Find the area of the inner loop of $r = 4\sqrt{3} - 8\cos\theta$

Solution: The point of intersection with the origin is found by solving:

$$\begin{aligned} 0 &= 4\sqrt{3} - 8\cos\theta \\ \cos\theta &= \frac{\sqrt{3}}{2} \\ \theta &= \frac{\pi}{6} + 2\pi k, \frac{11\pi}{6} + 2\pi k \end{aligned}$$

Since we want the integration bound to be increasing, we use $\frac{11\pi}{6} - 2\pi = -\frac{\pi}{6}$ and calculate

$$\int_{-\pi/6}^{\pi/6} \frac{1}{2} [\sqrt{3} - 2\cos\theta]^2 d\theta \approx 0.319$$

4. (4 points) Find the area of the region common to $r = 1 - \sin\theta$ and $r = 2\sin\theta$.

Solution: The graphs intersect at

$$\begin{aligned} 1 - \sin\theta &= 2\sin\theta \\ \theta &\approx 0.33984, 2.80176 \end{aligned}$$

With a graph, we see that $1 - \sin\theta$ is the desired bounding function only on $[0.33984, 2.80176]$, while $2\sin\theta$ is the desired bounding function on $[0, 0.33984] \cup [2.80176, \pi]$. Then the area is

$$\frac{1}{2} \int_0^{0.33984} (2\sin\theta)^2 d\theta + \frac{1}{2} \int_{0.33984}^{2.80176} (1 - \sin\theta)^2 d\theta + \frac{1}{2} \int_{2.80176}^{\pi} (2\sin\theta)^2 d\theta \approx 0.169$$

Alternatively, we can use symmetry and get the area as

$$2 \left(\int_0^{0.33984} (2\sin\theta)^2 d\theta + \frac{1}{2} \int_{0.33984}^{\pi/2} (1 - \sin\theta)^2 d\theta \right) \approx 0.169$$