

Mini-math Div 3/4: Friday, April 7, 2023 (25 minutes)

SOLUTIONS

1. (3 points) The series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges to S . If S_n is used to approximate S , what is the least value of n for which the alternating series error bound guarantees an error to strictly within 0.01?

Solution:

$$\begin{aligned}\frac{1}{n} &< \frac{1}{100} \\ n &> 100\end{aligned}$$

so $n = 101$ is the least value of n needed.

2. (3 points) Let $P(x)$ be the fifth-degree Taylor Polynomial for a function f about $x = 1$. Information about the maximum of the absolute value of selected derivatives of f over various intervals is given below.

$$\begin{aligned}\max_{0 \leq x \leq 1.5} |f^{(4)}(x)| &= 4.6, & \max_{0 \leq x \leq 1.5} |f^{(5)}(x)| &= 7.2, & \max_{0 \leq x \leq 1.5} |f^{(6)}(x)| &= 6.8, \\ \max_{1 \leq x \leq 1.5} |f^{(4)}(x)| &= 3.2, & \max_{1 \leq x \leq 1.5} |f^{(5)}(x)| &= 4.7, & \max_{1 \leq x \leq 1.5} |f^{(6)}(x)| &= 5.1\end{aligned}$$

Find the smallest value of k for which the Lagrange error bound guarantees that

$$|f(1.5) - P(1.5)| \leq k$$

Solution: We use the bound on the 6th derivative on the interval $[1, 1.5]$, since this is the smallest interval which contains the centre and the point we are approximating. The Lagrange error bound gives a maximum absolute error of

$$\frac{\max_{1 \leq x \leq 1.5} |f^{(6)}(x)|}{6!} \cdot |1.5 - 1|^6 = \frac{5.1 \cdot 0.5^6}{6!}$$

3. (4 points) Find the interval of convergence for the series $\sum_{n=1}^{\infty} \frac{(-1)^n (x-3)^n}{n2^n}$

Solution:

$$\lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+1} (x-3)^{n+1}}{(n+1)2^{n+1}} \cdot \frac{n2^n}{(-1)^n (x-3)^n} \right| = \lim_{n \rightarrow \infty} \frac{n}{n+1} \cdot \frac{|x-3|}{2} = \frac{|x-3|}{2}$$

Then the radius of convergence is 2. Testing $|x-3| = 2$, we have at $x-3 = 2$

$$\sum_{n=1}^{\infty} \frac{(-1)^n 2^n}{n2^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

converges by AST and

$$\sum_{n=1}^{\infty} \frac{(-1)^n (-2)^n}{n2^n} = \sum_{n=1}^{\infty} \frac{1}{n}$$

diverges by p -series. Therefore, the interval of convergence is $(3-2, 3+2] = (1, 5]$.

4. (3 points) What is the Maclaurin series for $\frac{\cos x - 1}{x}$? (You may, but are not required to, express your answer in summation notation.)

Solution:

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \dots$$

so

$$\begin{aligned} \frac{\cos x - 1}{x} &= \frac{1}{x} \cdot \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = \frac{-\frac{x^2}{2} + \frac{x^4}{4!} - \dots}{x} \\ &= \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n-1}}{(2n)!} = -\frac{x}{2} + \frac{x^3}{4!} - \dots \end{aligned}$$

5. (4 points) Let f be a function with $f(0) = 2$ and $f'(x) = \arctan x$. Write the first three non-zero terms of the Maclaurin series for f .

Solution:

$$\begin{aligned}\arctan x &= x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots \\ f(x) &= \int \arctan x \, dx = \int \left(x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots \right) dx \\ &= C + \frac{x^2}{2} - \frac{x^4}{4 \cdot 3} + \frac{x^6}{6 \cdot 5} - \cdots\end{aligned}$$

Since $f(0) = 2$, we get $C = 2$ so the first three non-zero terms are:

$$2 + \frac{x^2}{2} - \frac{x^4}{12}$$