Vieta jumping
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Example 1 (2007 IMO Q5)**.** Let a and b be positive integers such that $ab+1$ divides a^2+b^2. Show that

$$\frac{a^2+b^2}{ab+1}$$

is the square of an integer.

Solution:

Assume there exists at least one pair of positive integers a and b such that $k = \dfrac{a^2+b^2}{ab+1} > 0$ is not a perfect square. For this fixed value of k, define

$$S(k) = \left\{ (a,b) \in \mathbb{Z}^+ \times \mathbb{Z}^+ : \ k = \frac{a^2+b^2}{ab+1} \right\}$$

Since $S(k)$ is a subset of $\mathbb{Z}^+ \times \mathbb{Z}^+$, let $(A,B) \in S(k)$ have minimal sum $A+B$. Without loss of generality, we may assume $A \le B$, as the equation is symmetric in A and B.

$$k = \frac{A^2+B^2}{AB+1} \tag{1}$$

Replacing B with the variable x and rearranging, we have

$$\begin{aligned} k &= \frac{A^2+x^2}{Ax+1} \\ (kA)x + k &= A^2 + x^2 \\ 0 &= x^2 - (kA)x + (A^2 - k) \end{aligned}$$

By construction, $x_1 = B$ is one of the roots of this equation. Let x_2 be the other root. By Vieta's formulas,

$$\begin{aligned} B + x_2 &= kA \\ Bx_2 &= A^2 - k \end{aligned}$$

yielding

$$x_2 = kA - B \tag{2}$$

$$x_2 = \frac{A^2-k}{B} \tag{3}$$

Since $k, A, B \in \mathbb{Z}$, (2) implies $x_2 \in \mathbb{Z}$. Since k is not a perfect square, (3) implies $x_2 \neq 0$. Furthermore, $k > 0$ implies

$$0 < \frac{A^2+x_2^2}{Ax_2+1} \quad \Longrightarrow \quad Ax_2 + 1 > 0$$

since $A^2 + x_2^2 > 0$. However, A is a positive integer and $x_2 \neq 0$, and hence $x_2 > 0$. Therefore, we have $x_2 \in \mathbb{Z}^+$, and so $(A, x_2) \in S(k)$.

However, $A \leq B$ gives

$$x_2 = \frac{A^2 - k}{B} < \frac{A^2}{B} \leq B$$

so $A + x_2 < A + B$, contradicting minimality of $A + B$. Therefore, our assumption that there exists some pair (a, b) with $k = \dfrac{a^2 + b^2}{ab + 1}$ is not a perfect square is false.

$\square$

Example 2 (2019 Euclid Q9(c)(d)). Let a and b be positive integers such that ab divides $a^2 + b^2 + 1$. Show that

$$\frac{a^2 + b^2 + 1}{ab}$$

is a multiple of 3.

Solution:

We prove the more challenging result that

$$\frac{a^2 + b^2 + 1}{ab} = 3$$

Assume there exists at least one pair of positive integers a and b such that $k = \dfrac{a^2 + b^2 + 1}{ab} > 0$. For this fixed value of k, define

$$S(k) = \left\{ (a, b) \in \mathbb{Z}^+ \times \mathbb{Z}^+ : \ k = \frac{a^2 + b^2 + 1}{ab} \right\}$$

Since $S(k)$ is a subset of $\mathbb{Z}^+ \times \mathbb{Z}^+$, let $(A, B) \in S(k)$ have minimal sum $A + B$. Assume $A \neq B$. Then without loss of generality, we may assume $A < B$, as the equation is symmetric in A and B.

$$k = \frac{A^2 + B^2 + 1}{AB} \tag{4}$$

Replacing B with the variable x and rearranging, we have

$$\begin{aligned} k &= \frac{A^2 + x^2 + 1}{Ax} \\ (kA)x &= A^2 + x^2 + 1 \\ 0 &= x^2 - (kA)x + (A^2 + 1) \end{aligned}$$

By construction, $x_1 = B$ is one of the roots of this equation. Let x_2 be the other root. By Vieta's formulas,

$$\begin{aligned} B + x_2 &= kA \\ Bx_2 &= A^2 + 1 \end{aligned}$$

yielding

$$x_2 = kA - B \tag{5}$$

$$x_2 = \frac{A^2+1}{B} \tag{6}$$

Since $k, A, B \in \mathbb{Z}$, (5) implies $x_2 \in \mathbb{Z}$. Furthermore, $k > 0$ implies

$$0 < \frac{x_2^2 + B^2 + 1}{x_2 B} \quad \implies \quad x_2 B > 0$$

However, B is a positive integer and hence $x_2 > 0$. Therefore, we have $x_2 \in \mathbb{Z}^+$, and so $(A, x_2) \in S(k)$.

However, $A < B$ gives

$$x_2 = \frac{A^2+1}{B} < \frac{B^2}{B} = B$$

so $A + x_2 < A + B$, contradicting minimality of $A + B$. Therefore, our assumption that $A \neq B$ is false.

Therefore $A = B$, so that

$$k = \frac{A^2 + A^2 + 1}{A^2} \in \mathbb{Z}$$

so $A^2 \mid 1$, giving $A = 1$. Then

$$k = \frac{1+1+1}{1} = 3.$$

Since k was arbitrary, this shows the quotient is always 3.

□