Renert School: Integration Bee 2025–2026

Problem 1.

$$\int_1^{mower} \frac{1}{u}\, du$$

Solution:

$$\int_1^{mower} \frac{1}{u}\, du = \ln|u|\Big|_1^{mower} = \ln|mower|$$

Problems 2-17 are great review.

Problem 2.

$$\int_{-8}^{-1} \left(\frac{2}{x^{1/3}} + 4x\right) dx$$

Solution:

$$\begin{aligned}
&\int_{-8}^{-1} \left(\frac{2}{x^{1/3}} + 4x\right) dx \\
=& \left(3x^{2/3} + 2x^2\right)\Big|_{-8}^{-1} \\
=& 3((-1)^{2/3} - (-8)^{2/3}) + 2((-1)^2 - (-8)^2) \\
=& 3(1-4) + 2(1-64) \\
=& -9 - 126 = -135
\end{aligned}$$

Problem 3.

$$\int \frac{\sqrt{r}-5}{\sqrt{r}}\, dr$$

Solution: Divide to get

$$\int \frac{\sqrt{r}-5}{\sqrt{r}}\, dr = \int (1 - 5r^{-1/2})\, dr = r - 10\sqrt{r} + C$$

Problem 4.

$$\int s(\sqrt{s}+1)\, ds$$

Solution: Expand to get

$$\int s(\sqrt{s}+1)\, ds = \int (s^{3/2} + s)\, ds = \frac{2}{5}s^{5/2} + \frac{1}{2}s^2 + C$$

Problem 5.

$$\int_{\pi/6}^{\pi/3} \sin(2x)\, dx$$

Solution:

Method 1: Using $u = 2x$, $du = 2\, dx$, $\pi/6 \mapsto \pi/3$, $\pi/3 \mapsto 2\pi/3$,

$$\begin{aligned}\int_{\pi/6}^{\pi/3} \sin(2x)\, dx &= \int_{\pi/3}^{2\pi/3} \frac{1}{2} \sin u\, du \\ &= -\frac{1}{2} \cos u \Big|_{\pi/3}^{2\pi/3} = -\frac{1}{2}\left(-\frac{1}{2} - \frac{1}{2}\right) = \frac{1}{2}\end{aligned}$$

Method 2: Integrating directly,

$$\int_{\pi/6}^{\pi/3} \sin(2x)\, dx = -\frac{1}{2}\cos(2x)\Big|_{\pi/6}^{\pi/3} = -\frac{1}{2}\left(-\frac{1}{2} - \frac{1}{2}\right) = \frac{1}{2}$$

Problem 6.

$$\int x^3 e^{x^4}\, dx$$

Solution: Using $u = x^4$,

$$\int x^3 e^{x^4}\, dx = \int \frac{1}{4} e^u\, du = \frac{1}{4} e^{x^4} + C$$

Problem 7.

$$\int (2x-1)\sin(4x)\, dx$$

Solution: Use $f = 2x - 1$, $g' = \sin(4x)$, so $f' = 2$, $g = -\frac{1}{4}\cos(4x)$. By IBP,

$$\begin{aligned}\int (2x-1)\sin(4x)\, dx &= -\frac{2x-1}{4}\cos(4x) + \frac{1}{2}\int \cos(4x)\, dx \\ &= -\frac{2x-1}{4}\cos(4x) + \frac{1}{8}\sin(4x) + C\end{aligned}$$

Problem 8.

$$\int \frac{\sec\theta \tan\theta}{\sqrt{\sec\theta + 1}}\, d\theta$$

Solution: Using $u = \sec\theta + 1$, $du = \sec\theta\tan\theta\, d\theta$, so

$$\int \frac{\sec\theta\tan\theta}{\sqrt{\sec\theta+1}}\, d\theta = \int \frac{du}{\sqrt{u}} = 2u^{1/2} + C = 2\sqrt{\sec x + 1} + C$$

Problem 9.

$$\int \ln x^3\, dx$$

Solution:

Method 1:

Use $f = \ln x^3, g' = 1$, so $f' = \frac{3x^2}{x^3} = \frac{3}{x}, g = x$. By IBP,

$$\int \ln x^3\, dx = x\ln x^3 - \int 3\, dx = x\ln x^3 - 3x + C$$

Method 2:

Simplify first, then

$$\int \ln x^3\, dx = \int 3\ln x\, dx = 3(x\ln x - x) + C$$

Problem 10.

$$\int \frac{1}{t^2 - 6t + 10}\, dt$$

Solution:

$$\int \frac{1}{t^2-6t+10}\, dt = \int \frac{1}{(t-3)^2+1}\, dt = \arctan(t-3) + C$$

Problem 11.

$$\int_{-1}^{1} \frac{1}{x^3}\, dx$$

Solution: This is an improper integral of type II - you cannot use FTC II or that this is an odd function over a symmetric interval. Instead, the discontinuity is at 0, so

$$\int_{-1}^{1} \frac{1}{x^3}\, dx = \int_{-1}^{0} \frac{1}{x^3}\, dx + \int_{0}^{1} \frac{1}{x^3}\, dx$$

Considering the second integral,

$$\int_0^1 \frac{1}{x^3}\, dx = \lim_{c\to 0^+} \int_c^1 \frac{1}{x^3}\, dx = \lim_{c\to 0^+} -\frac{1}{2x^2}\Big|_c^1$$
$$= \lim_{c\to 0^+}\left(\frac{1}{2c^2} - \frac{1}{2}\right) = \infty$$

Therefore, the integral diverges.

Problem 12.

$$\int \frac{8x^2 - 2x + 3}{2x - 1}\,dx$$

Solution:

$$\begin{array}{r|l} & 4x + 1 \\ \hline 2x - 1 \,) & 8x^2 - 2x + 3 \\ & -8x^2 + 4x \\ \hline & 2x + 3 \\ & -2x + 1 \\ \hline & 4 \end{array}$$

$$\begin{aligned} \int \frac{8x^2 - 2x + 3}{2x - 1}\,dx &= \int \left(4x + 1 + \frac{4}{2x - 1}\right)\,dx \\ &= 2x^2 + x + 2\ln|2x - 1| + C \end{aligned}$$

Problem 13.

$$\int x^2 e^{2x}\,dx$$

Solution: Use $f = x^2, g' = e^{2x}$, so $f' = 2x, g = \frac{1}{2}e^{2x}$. By IBP,

$$\int x^2 e^{2x}\,dx = \frac{1}{2}x^2 e^{2x} - \int x e^{2x}\,dx$$

Use $f = x, g' = e^{2x}$, so $f' = 1, g = \frac{1}{2}e^{2x}$. By IBP again,

$$\begin{aligned} \int x^2 e^{2x}\,dx &= \frac{1}{2}x^2 e^{2x} - \frac{1}{2}x e^{2x} + \frac{1}{2}\int e^{2x}\,dx \\ &= \frac{1}{2}x^2 e^{2x} - \frac{1}{2}x e^{2x} + \frac{1}{4}e^{2x} + C \end{aligned}$$

Problem 14.

$$\int_e^{e^2} \frac{1}{x\ln x}\,dx$$

Solution: Method 1: Using $u = \ln x$, $du = \frac{1}{x}\,dx$, $e \mapsto 1$, $e^2 \mapsto 2$, so

$$\int_e^{e^2} \frac{1}{x\ln x}\,dx = \int_1^2 \frac{1}{u}\,du = \ln|u|\Big|_1^2 = \ln 2$$

Method 2: Integrating directly,

$$\int_e^{e^2} \frac{1}{x\ln x}\,dx = \ln\ln x\Big|_e^{e^2} = \ln\ln e^2 - \ln\ln e = \ln 2$$

Problem 15.

$$\int_0^4 \sqrt{3x+4}\,dx$$

Solution:

Method 1: Using $u = 3x+4$, $du = 3\,dx$, $0 \mapsto 4$, $4 \mapsto 16$, so

$$\begin{aligned}\int_0^4 \sqrt{3x+4}\,dx &= \int_4^{16} \frac{1}{3}\sqrt{u}\,du = \frac{2}{9}u^{3/2}\Big|_4^{16} \\ &= \frac{2}{9}(16^{3/2} - 4^{3/2}) = \frac{2}{9}(4^3 - 2^3) = \frac{112}{9}\end{aligned}$$

Method 2: Integrating directly,

$$\begin{aligned}\int_0^4 \sqrt{3x+4}\,dx &= \frac{2}{9}\sqrt{3x+4}\Big|_0^4 = \frac{2}{9}(16^{3/2} - 4^{3/2}) \\ &= \frac{2}{9}(4^3 - 2^3) = \frac{112}{9}\end{aligned}$$

Problem 16.

$$\int \frac{x+1}{x(2x+1)}\,dx$$

Solution: If $\dfrac{x+1}{x(2x+1)} = \dfrac{A}{x} + \dfrac{B}{2x+1}$, the Heaviside cover-up method gives $A = \frac{0+1}{2(0)+1} = 1$ and $B = \frac{-\frac{1}{2}+1}{-\frac{1}{2}} = -1$, so

$$\begin{aligned}\int \frac{x+1}{x(2x+1)}\,dx &= \int \left(\frac{1}{x} + \frac{-1}{2x+1}\right)\,dx \\ &= \ln|x| - \frac{1}{2}\ln|2x+1| + C\end{aligned}$$

Problem 17.

$$\int_0^2 \frac{1}{\sqrt{2-x}}\,dx$$

Solution: The upper limit is a discontinuity, so

$$\begin{aligned}\int_0^2 \frac{1}{\sqrt{2-x}}\,dx &= \lim_{b\to 2^-} \int_0^b \frac{1}{\sqrt{2-x}}\,dx = \lim_{b\to 2^-} (-2\sqrt{2-x})\Big|_0^b \\ &= \lim_{b\to 2^-} (-2\sqrt{2-b} + 2\sqrt{2-0})) = 2\sqrt{2}\end{aligned}$$

Problems 18-30 are still good for review

Problem 18.

$$\int_0^3 \sqrt{9-r^2}\,dr$$

Solution: The function is a quarter-circle of radius 3, so

$$\int_0^3 \sqrt{9-r^2}\,dr = \frac{9\pi}{4}$$

Problem 19.

$$\int_{-2}^2 (x^2+1)(x-\sin x)\,dx$$

Solution: The integrand is odd and the interval is symmetric, so

$$\int_{-2}^2 (x^2+1)(x-\sin x)\,dx = 0$$

Problem 20.

$$\int_1^4 \frac{2+1/\sqrt{x}}{x+\sqrt{x}}\,dx$$

Solution: This is logarithmic differentiation:

$$\begin{aligned}\int_1^4 \frac{2+1/\sqrt{x}}{x+\sqrt{x}}\,dx &= 2\ln|x+\sqrt{x}|\Big|_1^4 \\ &= 2\ln|4+2| - 2\ln|1+1| \\ &= 2\ln 3 = \ln 9\end{aligned}$$

Problem 21.

$$\int x(1-x)^{2025}\,dx$$

Solution: Let $u = 1-x$, so that $du = -dx$.

$$\begin{aligned}\int x(1-x)^{2025}\,dx &= -\int (1-u)u^{2025}\,du \\ &= \int (u^{2026}-u^{2025})\,du \\ &= \frac{u^{2027}}{2027} - \frac{u^{2026}}{2026} + C \\ &= \frac{(1-x)^{2027}}{2027} - \frac{(1-x)^{2026}}{2026} + C\end{aligned}$$

Note: Wolfram Alpha cannot do this integral.

Problem 22.

$$\int_0^1 \frac{x}{1+x^4}\,dx$$

Solution:

$$\begin{aligned}\int_0^1 \frac{x}{1+x^4}\,dx &= \frac{1}{2}\arctan(x^2)\Big|_0^1 \\ &= \frac{1}{2}(\arctan 1 - \arctan 0) = \frac{\pi}{8}\end{aligned}$$

Problem 23.

$$\int \frac{1}{2025^x}\,dx$$

Solution:

$$\int \frac{1}{2025^x}\,dx = \int 2025^{-x}\,dx = -\frac{2025^{-x}}{\ln 2025} + C$$

Problem 24.

$$\int_1^\infty \frac{1}{x(x^2+1)}\,dx$$

Solution: Using partial fraction decomposition or inspection,

$$\begin{aligned}\int \frac{1}{x(x^2+1)}\,dx &= \int \left(\frac{1}{x} - \frac{x}{x^2+1}\right)\,dx \\ &= \ln|x| - \frac{1}{2}\ln|x^2+1| + C \\ &= \ln\left|\frac{x}{\sqrt{x^2+1}}\right| + C\end{aligned}$$

Then

$$\begin{aligned}\int_1^\infty \frac{1}{x(x^2+1)}\,dx &= \lim_{b\to\infty} \ln\left|\frac{x}{\sqrt{x^2+1}}\right|\Bigg|_1^b \\ &= \lim_{b\to\infty}\left(\ln\left|\frac{1}{\sqrt{1+\frac{1}{b^2}}}\right| - \ln\left|\frac{1}{\sqrt{2}}\right|\right) \\ &= -\ln 2^{-1/2} = \frac{\ln 2}{2}\end{aligned}$$

Problem 25.

$$\int \frac{2^x + 3^x}{5^x}\, dx$$

Solution: Simplify first,

$$\begin{aligned}
\int \frac{2^x + 3^x}{5^x}\, dx &= \int \left(\left(\frac{2}{5}\right)^x + \left(\frac{3}{5}\right)\right)^x dx \\
&= \frac{(2/5)^x}{\ln(2/5)} + \frac{(3/5)^x}{\ln(3/5)} + C
\end{aligned}$$

Problem 26.

$$\int \frac{1}{(x^2-1)^2}\, dx$$

Solution: By partial fractions,

$$\begin{aligned}
&\int \frac{1}{(x^2-1)^2}\, dx \\
&= \int \frac{1}{(x+1)^2(x-1)^2}\, dx \\
&= \int \left(\frac{1/4}{x+1} + \frac{1/4}{(x+1)^2} + \frac{-1/4}{x-1} + \frac{1/4}{(x-1)^2}\right) dx \\
&= \frac{1}{4}\left(\ln|x+1| - \frac{1}{x+1} - \ln|x-1| - \frac{1}{x-1}\right) + C \\
&= \frac{1}{4}\left(-\frac{2x}{x^2-1} + \ln|x+1| - \ln|x-1|\right) + C \\
&= \frac{1}{4}\left(-\frac{2x}{x^2-1} + \ln\left|\frac{x+1}{x-1}\right|\right) + C
\end{aligned}$$

Any of the last three or their distributed forms are acceptable.

Problem 27.

$$\int_1^e \ln(x^5)\, dx$$

Solution: Using integration by parts,

$$\begin{aligned}
\int_1^e \ln(x^5)\, dx &= 5\int_1^e \ln x\, dx \\
&= 5(x\ln x - x)\Big|_1^e \\
&= 5(e - e - 0 + 1) = 5
\end{aligned}$$

Problem 28.

$$\int_{-4}^{0} \sqrt{-4x - x^2}\, dx$$

Solution: Completing the square, $-4x - x^2 = 4 - (x+2)^2$. Then the integral is the area under the circle of radius 2 centred at $(-2, 0)$ and above the x-axis, so

$$\int_{-4}^{0} \sqrt{-4x - x^2}\, dx = \frac{2^2\pi}{2} = 2\pi$$

Alternatively, upon completing the square use $x + 2 = 2\sin\theta$, $-4 \mapsto -\pi/2$, $4 \mapsto \pi/2$, so

$$\begin{aligned}
\int_{-4}^{0} \sqrt{-4x - x^2}\, dx &= \int_{-\pi/2}^{\pi/2} \sqrt{4 - 4\sin^2\theta} \cdot 2\cos\theta\, d\theta \\
&= \int_{-\pi/2}^{\pi/2} 4\cos^2\theta\, d\theta \\
&= 2\int_{-\pi/2}^{\pi/2} (1 + \cos 2\theta)\, d\theta \\
&= (2\theta + \sin 2\theta)\Big|_{-\pi/2}^{\pi/2} = 2\pi
\end{aligned}$$

Problem 29.

$$\int \frac{1}{1 + e^{-x}}\, dx$$

Solution: This is a logarithmic integral:

$$\int \frac{1}{1 + e^{-x}}\, dx = \int \frac{e^x}{e^x + 1}\, dx = \ln|e^x + 1| + C$$

Problem 30.

$$\int_{-2}^{2} (x + 3)\sqrt{4 - x^2}\, dx$$

Solution: Breaking this integral into two parts,

$$\begin{aligned}
&\int_{-2}^{2} (x + 3)\sqrt{4 - x^2}\, dx \\
&= \int_{-2}^{2} x\sqrt{4 - x^2}\, dx + 3\int_{-2}^{2} \sqrt{4 - x^2}\, dx
\end{aligned}$$

The first integral is of an odd integrand over a symmetric interval, so gives 0. The second integral is a semi-circle of radius 2. Then

$$\int_{-2}^{2} (x + 3)\sqrt{4 - x^2}\, dx = 3 \cdot \frac{1}{2}\pi(2^2) = 6\pi$$