Optimization problems

SOLUTIONS

0.5 mark for variables

1 mark restriction function(s)

1 mark optimization function

1 mark reducing optimizing function to single variable

1 mark for derivative of optimizing function

0.5 mark for checking CP of form f' DNE

1 mark for checking CP of form f' = 0

1 mark for classification using one of the three methods (if applicable)

1 mark for solution (eg other dimensions, time, etc)

1. (8 points) Find the dimensions of the rectangle of maximum area with sides parallel to the axes that can be inscribed in the ellipse $4x^2 + y^2 = 16$. Give your answer as an exact value — no approximations.

Solution: Let (x, y) be the corner of the rectangle which is on the ellipse in the first quadrant. Then the area is $A = (2x)(2y) = 4xy$, with the restriction $4x^2 + y^2 = 16$. The latter gives $y = \sqrt{16 - 4x^2} = 2\sqrt{4 - x^2}$ since $y \geq 0$, so that the optimizing function is

$$A(x) = 4x \cdot 2\sqrt{4 - x^2} = 8x\sqrt{4 - x^2}$$

Differentiating,

$$\begin{aligned} A'(x) &= 8\sqrt{4 - x^2} + 8x \cdot \frac{-2x}{2\sqrt{4 - x^2}} \\ &= \frac{8(4 - x^2) - 8x^2}{\sqrt{4 - x^2}} = \frac{8(4 - 2x^2)}{\sqrt{4 - x^2}} = \frac{16(2 - x^2)}{\sqrt{4 - x^2}} \end{aligned}$$

Since $0 \leq x \leq 2$, this gives critical points $x = \sqrt{2}, 2$. We have

$$\begin{aligned} A(0) &= 0, \\ A(\sqrt{2}) &> 0, \\ A(2) &= 0, \end{aligned}$$

so $x = \sqrt{2}$ is the maximum. Thus the rectangle's dimensions are $2\sqrt{2}$ in the x-direction and $4\sqrt{2}$ in the y-direction.

2. (8 points) Alice is standing at the bank of a river which is 85 m wide. She sees Bob 250 m downstream, on the opposite shore, lying in distress. She can swim at 2 m/s and can run at 8 m/s, and plans to swim to the opposite shore at some point, then run the remaining distance to Bob. At what point should she swim to on the opposite shore in order to minimize the time it takes to reach Bob? (Give your answer in m, to 3 decimal places.)

Solution:

Let x be the distance downstream of the point directly opposite the river from Alice. Then she will swim $\sqrt{85^2 + x^2}$ m and run $250 - x$ meters. We wish to minimize

$$T(x) = \frac{250 - x}{8} + \frac{\sqrt{85^2 + x^2}}{2}$$

Differentiating,

$$T'(x) = -\frac{1}{8} + \frac{1}{2} \cdot \frac{x}{\sqrt{85^2 + x^2}}$$

T' always exists, so we solve $T' = 0$:

$$\begin{aligned}
\frac{1}{8} &= \frac{1}{2} \cdot \frac{x}{\sqrt{85^2 + x^2}} \\
\sqrt{85^2 + x^2} &= 4x \\
85^2 + x^2 &= 16x^2 \\
85^2 &= 15x^2 \\
x &= \frac{85}{\sqrt{15}} \approx 21.947
\end{aligned}$$

Clearly, $T'(x) < 0$ for $0 \leq x < \frac{85}{\sqrt{15}}$ and $T'(x) > 0$ for $x > \frac{85}{\sqrt{15}}$, so by the First Derivative Test for global extrema, $x = \frac{85}{\sqrt{15}}$ is a global minimum. Therefore, Alice should swim to a point 21.947 m downstream from the point directly opposite her.

3. (8 points) We want to construct a cylindrical can. Due to the manufacturing process, the side of the can costs 0.6 cents per cm^2 to produce and the top and bottom costs 1.3 cents per cm^2 to produce. If the can must have a volume of $540\,\text{cm}^3$, determine the dimensions of the can (to the nearest hundredth of a centimetre) that will minimize the cost of the can.

Solution: Let the radius and height of the can be r and h, respectively. We have $\pi r^2 h = 540$, and we wish to minimize

$$C = 1.3 \cdot 2\pi r^2 + 0.6 \cdot 2\pi r h$$

Since $\pi r^2 h = 540$, we have $h = \dfrac{540}{\pi r^2}$, the cost in cents can be expressed in terms of the radius:

$$\begin{aligned} C(r) &= 2.6\pi r^2 + 1.2 \cdot \frac{540}{r} \\ &= 2.6\pi r^2 + \frac{648}{r} \end{aligned}$$

where $r > 0$. Differentiating,

$$C'(r) = 5.2\pi r - \frac{648}{r^2} = \frac{5.2\pi r^3 - 648}{r^2}$$

C' does not exist at $r = 0$, but this is not in the domain. Solving $C'(r) = 0$, we get

$$\begin{aligned} 5.2\pi r^3 &= 648 \\ r &= \sqrt[3]{\frac{648}{5.2\pi}} \approx 3.4104 \end{aligned}$$

Notice that $C' < 0$ to the left of this point and $C' > 0$ to the right of this point, so by the First Derivative Test for Global Extrema, this is the minimum. Solving for the height,

$$h \approx 14.7785$$

To the nearest hundredth of a centimetre,

$$\begin{aligned} r &\approx 3.41\,\text{cm} \\ h &\approx 14.79\,\text{cm} \end{aligned}$$

Note that the method from section 4.2 fails since we do not have a closed and bounded interval for the domain. We can, however, use the Second Derivative Test for Global Extrema:

$$C'' = 5.2\pi + \frac{2 \cdot 648}{r^3}$$

Since $C''(3.4104) > 0$ and this is the only critical point, $r = 3.4104$ is the minimum.