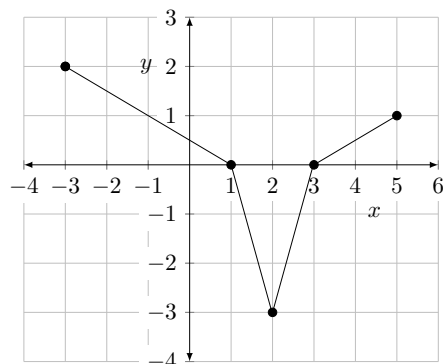


SOLUTIONS

1. (2 points) The graph of the piecewise linear function f is shown in the figure to the right. What is the average value of f over $[-3, 5]$?

- A. -1
- B. $-1/8$
- C. 0
- D. $1/4$**
- E. 2



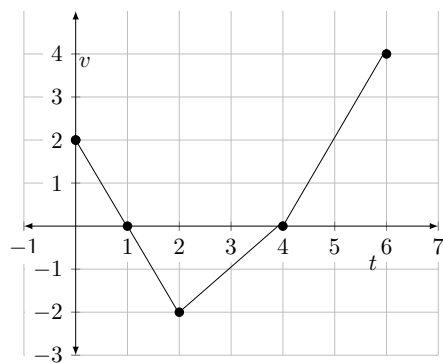
Solution:

$$f_{avg} = \frac{\int_{-3}^5 f \, dx}{5 - (-3)} = \frac{4 - 3 + 1}{8} = \frac{1}{4}$$

D is correct.

2. (2 points) The graph of the velocity of a function is the piecewise linear function shown in the figure to the right. The initial position of the particle at time $t = 0$ is $x = 1$. What is the total distance the particle travels from $t = 0$ to $t = 6$?

- A. 2
- B. 3
- C. 4
- D. 8**
- E. 9



Solution:

$$\int_0^6 |v(t)| \, dt = 1 + 3 + 4 = 8$$

D is correct.

3. (2 points) The acceleration of a particle is modelled by $a(t) = 2t + 3$ for $t \geq 0$. At $t = 0$, the velocity of the particle is -2 and its position is 2.5 . What is the change in displacement of the particle from $t = 0$ to $t = 3$?
- A. 9 B. 16 **C. 16.5** D. 19 E. 22.5

Solution:

$$v(t) = \int a(t) dt = \int (2t + 3) dt = t^2 + 3t + C$$

Since $v(0) = -2$, we know $C = -2$. Then the change in displacement is

$$\Delta x = \int_0^3 v(t) dt = \int_0^3 (t^2 + 3t - 2) dt = \left(\frac{1}{3}t^3 + \frac{3}{2}t^2 - 2t \right) \Big|_0^3 = 9 + \frac{27}{2} - 6 = 16.5$$

C is correct.

4. (2 points) Suppose f is a differentiable function. Which of the following statements are true:
- (I) The average value of the derivative of f over $[a, b]$ is the same as the average rate of change of f over $[a, b]$.
- (II) There exists a $c \in [a, b]$ for which $f(c)$ equals the average value of f over $[a, b]$.
- A. (I) only B. (II) only **C. Both (I) and (II)** D. Neither (I) nor (II)
- E. The truth of both statements depend on the specific choice of f

Solution: By FTC II,

$$\frac{\int_a^b f'(x) dx}{b-a} = \frac{f(b) - f(a)}{b-a}$$

so (I) is true.

Since f is continuous and $[a, b]$ is a closed and bounded interval, the Extreme Value Theorem tells us that there are m and M for which $m \leq f(x) \leq M$ for all $x \in [a, b]$. Then

$$\frac{\int_a^b m dx}{b-a} \leq \frac{\int_a^b f(x) dx}{b-a} \leq \frac{\int_a^b M dx}{b-a}$$

$$m \leq f_{avg} \leq M$$

By the Intermediate Value Theorem, there exists $c \in [a, b]$ such that $f(c) = f_{avg}$, so (II) is true.

C is correct.

5. (2 points) Water is leaking out of a tub at a rate modelled by $r(t) = \frac{1}{t^2 + 1} \text{cm}^3/\text{min}$, where t is in minutes. If the initial volume of the tub is $160\,000 \text{ cm}^3$, which of the following represents the volume of the tub at time t ?

A. $160000 + \int_0^t r(x) dx$

B. $160000 - \int_0^t r(x) dx$

C. $160000 - \frac{1}{t^2 + 1}$

D. $160000 + \frac{r(t)}{t^2 + 1}$

E. $\frac{1}{t^2 + 1}$

Solution: By FTC II,

$$-\int_0^t r(t) dt = V(t) - V(0)$$

$$V(t) = 160000 - \int_0^t r(t) dt$$

B is correct.

6. (2 points) Find the area of the bounded region in the first quadrant below both $y = x^2$ and $y = 2 - x$ and above the x -axis.

A. $2/3$

B. **$5/6$**

C. 1

D. $7/6$

E. 3

Solution: Integrating with respect to y ,

$$A = \int_0^1 [(2 - y) - \sqrt{y}] dy = \left(2y - \frac{y^2}{2} - \frac{2}{3}y^{3/2} \right) \Big|_0^1 = 2 - \frac{1}{2} - \frac{2}{3} = \frac{5}{6}$$

OR Integrating with respect to x (with 2 regions),

$$A = \int_0^1 x^2 dx + \int_1^2 (2 - x) dx = \frac{x^3}{3} \Big|_0^1 + \left(2x - \frac{x^2}{2} \right) \Big|_1^2 = \frac{1}{3} + 4 - 2 - 2 + \frac{1}{2} = \frac{5}{6}$$

B is correct.

7. (4 points) Write an integral (or integrals) to calculate the area of the finite region(s) bounded by the given curves.

$$x + y = 1, \quad 2x - y = -1, \quad 4x - y = 4$$

Solution: Finding the intersections of the curves:

$y = 1 - x$ and $y = 2x + 1$ intersect at $x = 0$

$y = 1 - x$ and $y = 4x - 4$ intersect at $x = 1$

$y = 2x + 1$ and $y = 4x - 4$ intersect at $x = 5/2$

On $[0, 1]$, $2x + 1 \geq 1 - x$ and on $[1, 5/2]$, $2x + 1 \geq 4x - 4$, so the area is

$$\begin{aligned} & \int_0^1 [(2x + 1) - (1 - x)] dx + \int_1^{5/2} [(2x + 1) - (4x - 4)] dx \\ &= \int_0^1 3x dx + \int_1^{5/2} (5 - 2x) dx = \frac{3}{2}x^2 \Big|_0^1 + (5x - x^2) \Big|_1^{5/2} \\ &= \frac{3}{2} + \left(\left(5 \cdot \frac{5}{2} - \frac{25}{4} \right) - (5 - 1) \right) \\ &= \frac{3}{2} + \frac{25}{4} - 4 = \frac{31 - 16}{4} = \frac{15}{4} \end{aligned}$$

OR

In terms of y ,

$$\int_0^1 [(y + 4)/4 - (1 - y)] dy + \int_1^6 [(y + 4)/4 - (y - 1)/2] dy = \frac{5}{8} + \frac{25}{8} = \frac{15}{4}$$