

Continuous Additive Functions

Definition 1. A function f defined on the real numbers is said *additive* if $f(x + y) = f(x) + f(y)$ for all real numbers x and y .

- (1) Give an example of an additive function and show that it is additive.
- (2) Give an example of a function which is not additive and show that it is not additive.
- (3) Suppose f is an additive function and m is a real number. Define $g(x) = f(x) - mx$. Prove that g is an additive function.
- (4) Suppose f is an additive function. Define $g(x) = f(x) - mx$ where $m = f(1)$. Prove $g(x + 1) = g(x)$ for all real numbers x .

Definition 2. A function f is *bounded on* $[a, b]$ if there is a real number M such that $|f(x)| \leq M$ for all x in $[a, b]$. We could say f is *bounded on* $[a, b]$ *by* M .

A function f is *bounded* (or for emphasis, *bounded everywhere*) if there is a real number M such that $|f(x)| \leq M$ for all real numbers x . We could say f is *bounded by* M (or for emphasis, *bounded everywhere by* M).

- (5) Suppose f is an additive function that is bounded on $[0, 1]$. Define $g(x) = f(x) - mx$ where $m = f(1)$. Prove that g is bounded everywhere.
- (6) Let g be an additive function which is bounded by M . Prove that if there is a real number a such that $g(a) \neq 0$, then there must be a real number b such that $|g(b)| > M$ (a contradiction). What can you conclude about g ? (Hint: consider $g(2a)$ and $g(3a)$...)
- (7) Prove that if f is additive and continuous, there is a real number m such that $f(x) = mx$ for all x . (Hint: use the previous parts!)