Mini-math AP Calculus BC: Friday, February 17, 2022 (12 minutes)

SOLUTIONS

1. (2 points) What is the interval of convergence of the following series?

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x+1)^n}{n^{1/2}2^{2n}}$$

Solution: By the Ratio Test,

$$\lim_{n\to\infty}\left|\frac{(-1)^{n+1+1}(x+1)^{n+1}}{(n+1)^{1/2}2^{2(n+1)}}\cdot\frac{n^{1/2}2^{2n}}{(-1)^{n+1}(x+1)^n}\right| = \lim_{n\to\infty}\left|\frac{x+1}{4}\right| < 1 \quad \Leftrightarrow |x+1| < 4$$

When $x+1=4$, we get $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}4^n}{n^{1/2}2^{2n}} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^{1/2}}$ which converges by AST.

When $x+1=-4$, we get $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(-4)^n}{n^{1/2}2^{2n}} = \sum_{n=1}^{\infty}\frac{-1}{n^{1/2}}$ which diverges by p-series.

Therefore, the interval of convergence is $-5 < x \leq 3$.

2. (2 points) Evaluate

$$\frac{2^{-2}}{0!} - \frac{2^{-1}}{1!} + \frac{2^0}{2!} - \frac{2^1}{3!} + \frac{2^2}{4!} - \frac{2^3}{5!} + \cdots + \frac{(-1)^n 2^{n-2}}{n!} + \cdots$$

Solution:

$$\sum_{n=0}^{\infty}\frac{(-1)^n 2^{n-2}}{n!} = 2^{-2}\sum_{n=0}^{\infty}\frac{(-2)^n}{n!} = \frac{1}{4e^2}$$

3. (2 points) Give the first three non-zero terms of the Maclaurin series for the function

$$f(x) = (x^2 + 1)\sin x$$

Solution:

$$\begin{aligned}
(x^2+1)\sin x &= (x^2+1)\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right) \\
&= x^2\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right) + \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right) \\
&= \left(x^3 - \frac{x^5}{6} + \cdots\right) + \left(x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots\right) \\
&= x + \frac{5}{6}x^3 - \frac{19}{120}x^5 + \cdots
\end{aligned}$$

So the first three non-zero terms are $x + \frac{5}{6}x^3 - \frac{19}{120}x^5$.

4. (2 points) f is a function with $f(0) = 3$ and $f'(x) = e^{x^2}$. Find the first four non-zero terms of the Maclaurin series for f.

Solution: The Maclaurin series for e^{x^2} is $\sum_{n=0}^{\infty} \frac{x^{2n}}{n!}$, so

$$\int f'\,dx = \int \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}\,dx = \sum_{n=0}^{\infty} x^{2n+1}(2n+1)n! + C$$

Since $f(0) = 3$, $C = 3$ and so the first three non-zero terms are $3 + x + \frac{1}{3}x^3 + \frac{1}{10}x^5$