Mini-math Div 3/4: Friday, March 14, 2022 (15 minutes)

SOLUTIONS

1. Find the area of the finite region(s) bounded by the given curves.

   (a) (4 points) $y = e^x, y = 1$ from $x = -1$ to $x = 1$.

Solution: The curves $y = e^x$ and $y = 1$ intersect when $e^x = 1$, so $x = 0$. On $[-1, 0]$, $1 \geq e^x$, and on $[0, 1]$, $e^x \geq 1$. Then the area between the curves is

$$\begin{aligned}\int_{-1}^{1} |e^x - 1|\, dx &= \int_{-1}^{0} (1 - e^x)\, dx + \int_{0}^{1} (e^x - 1)\, dx \\ &= (x - e^x)\Big|_{-1}^{0} + (e^x - x)\Big|_{0}^{1} \\ &= (-1 + 1 + e^{-1}) + (e - 1 - 1) \\ &= e + \frac{1}{e} - 2\end{aligned}$$

   (b) (4 points) $x + y = 1, 2x - y = -1, 4x - y = 4$

Solution: Finding the intersections of the curves:
$y = 1 - x$ and $y = 2x + 1$ intersect at $x = 0$
$y = 1 - x$ and $y = 4x - 4$ intersect at $x = 1$
$y = 2x + 1$ and $y = 4x - 4$ intersect at $x = 5/2$
On $[0, 1]$, $2x + 1 \geq 1 - x$ and on $[1, 5/2]$, $2x + 1 \geq 4x - 4$, so the area is

$$\begin{aligned}&\int_{0}^{1} [(2x + 1) - (1 - x)]\, dx + \int_{1}^{5/2} [(2x + 1) - (4x - 4)]\, dx \\ &= \int_{0}^{1} 3x\, dx + \int_{1}^{5/2} (5 - 2x)\, dx = \frac{3}{2}x^2\Big|_{0}^{1} + (5x - x^2)\Big|_{1}^{5/2} \\ &= \frac{3}{2} + \left(\left(5 \cdot \frac{5}{2} - \frac{25}{4}\right) - (5 - 1)\right) \\ &= \frac{3}{2} + \frac{25}{4} - 4 = \frac{31 - 16}{4} = \frac{15}{4}\end{aligned}$$

2. (AP) In this question, you do not need to simplify your answer. Find an integral (but do not evaluate) that represents the volume of the solid whose base is the region R bounded by $y = x/2$ and $y = \sqrt{2x}$, if:

(a) (2 points) cross-sections perpendicular to the x-axis are rectangles whose heights are twice their base.

Solution:

$$\int_0^8 2(\sqrt{2x} - x/2)^2 \, dx$$

(b) (2 points) cross-sections perpendicular to the x-axis are right isosceles triangles whose hypotenuse lies on the base.

Solution:

$$\int_0^8 \frac{1}{4}(\sqrt{2x} - x/2)^2 \, dx$$

(c) (2 points) cross-sections perpendicular to the y-axis are semi-circles.

Solution:

$$\int_0^4 \frac{\pi}{2}\left(\frac{2y - y^2/2}{2}\right)^2 dy = \int_0^4 \frac{\pi}{8}\left(2y - y^2/2\right)^2 dy$$

(d) (2 points) cross-sections perpendicular to the y-axis are right isosceles triangles whose hypotenuse does not lie on the base.

Solution:

$$\int_0^4 \frac{1}{2}\left(2y - y^2/2\right)^2 dy$$

3. (AP) In this question, you do not need to simplify your answer. Find an integral (but do not evaluate) that represents the volume of the solid of revolution if we revolve the region R bounded by $y = x/2$ and $y = \sqrt{2x}$:

(a) (2 points) about the x-axis.

Solution:

$$\pi \int_0^8 \left[\sqrt{2x}^2 - (x/2)^2\right] \, dx = \pi \int_0^8 \left(2x - \frac{x^2}{4}\right) \, dx$$

(b) (2 points) about the y-axis.

Solution:

$$\pi \int_0^4 \left[(2y)^2 - (y^2/2)^2\right] \, dx$$

(c) (2 points) about the line $y = -1$.

Solution:

$$\pi \int_0^8 \left[(\sqrt{2x} + 1)^2 - (x/2 + 1)^2\right] \, dx$$

(d) (2 points) about the line $x = 10$.

Solution:

$$\pi \int_0^4 \left[(10 - y^2/2)^2 - (10 - 2y)^2\right] \, dx$$