Answer Key

Phrases for Rates of Change

Match each phrase in the first column of the table with the corresponding term or set of units in the second column. Each letter will be used only once. Indicate your answers in the Answer Table that follows.

1. instantaneous rate of change of position
2. possible units for acceleration
3. units for instantaneous rate of change of volume, $V(t)$, where V is measured in gallons and t is measured in hours
4. possible units for position
5. instantaneous rate of change of velocity
6. possible units for velocity
7. units for instantaneous rate of change of $R(t)$, R is measured in gallons per hour and t is measured in hours

A. velocity

B. acceleration

C. gallons per hour per hour

D. meters

E. meters per second

F. meters per second per second

G. gallons per hour

Answer Table:

1	2	3	4	5	6	7
A	F	G	D	B	E	C

Instantaneous vs Average Rate of Change

Part I: Decide whether the phrase used in each of the following problems corresponds to an instantaneous rate of change or an average rate of change. Indicate your decision by placing a check mark in the appropriate box.

1. Over the time interval from $t = 2$ to $t = 5$
   - ☐ instantaneous rate of change
   - ☑ average rate of change

2. After three seconds have elapsed
   - ☑ instantaneous rate of change
   - ☐ average rate of change

3. At noon
   - ☑ instantaneous rate of change
   - ☐ average rate of change

4. During the first five minutes
   - ☐ instantaneous rate of change
   - ☑ average rate of change

5. Over the twelve hour time period
   - ☐ instantaneous rate of change
   - ☑ average rate of change

6. When $t = 7$ hours
   - ☑ instantaneous rate of change
   - ☐ average rate of change

Part II: Complete the last column in the following table with the correct units for the derivative of the function given in the first column. The first row is completed as an example.

Function	Units for independent variable	Units for dependent variable	Units for derivative
$C(t)$	t is measured in hours	C is measured in dollars	$C'(t)$ is measured in dollars per hour
$A(t)$	t is measured in seconds	A is measured in cm^2	$A'(t)$ is measured in cm^2/s
$W(V)$	V is measured in cubic meters	W is measured in kilograms	$W'(V)$ is measured in kg/m^3
$M(v)$	v is measured in mi/hr	M is measured in mi/gal	$M'(v)$ is measured in mi/gal per mi/hr or hr/gal

Verbal Descriptions of Derivatives

Complete each of the following verbal descriptions by filling in the blanks with the information that is missing. Be sure to include appropriate units. The first one is completed for you as an example.

Example:
$V(t)$ gives the volume of water in a tank in liters after t minutes.

$V(3) = 20$ means that after $\boxed{\text{3 minutes}}$, there are $\boxed{\text{20 liters}}$ of water in the tank.

$V'(3) = -2$ means that $\boxed{\text{after 3 minutes}}$, the volume of water in the tank is $\boxed{\text{decreasing}}$ at a rate of $\boxed{\text{2 liters per minute}}$.

1. $A(t)$ gives the area of the surface of an oil slick, in square kilometers (km^2) after t hours.

$A(24) = 17$ means that after $\boxed{\text{24 hours}}$, the area of the surface of the oil slick is $\boxed{17\text{ km}^2}$.

$A'(5) = 4$ means that when $\boxed{\text{5 hours}}$ have elapsed, the area of the surface of the oil slick is $\boxed{\text{increasing}}$ at a rate of $\boxed{4\text{ km}^2/\text{hr}}$.

2. $C(x)$ gives the cost in dollars of digging a hole x feet deep.

$C(20) = 140$ means that it costs $\boxed{\$140}$ to dig a hole that is $\boxed{\text{20 feet}}$ deep. $C'(20) = 5$ means that when the hole is $\boxed{\text{20 feet deep}}$, the cost of digging is $\boxed{\text{increasing}}$ at a rate of $\boxed{\$5\text{ / foot}}$.

3. $a(t)$ gives the acceleration in cm/sec^2 after t seconds.

$a(2) = -6$ means that after $\boxed{\text{2 seconds}}$, the acceleration is $\boxed{-6\text{ cm/s}^2}$.

$a'(7) = -3$ means that after $\boxed{\text{7 seconds}}$, the acceleration is $\boxed{\text{decreasing at a rate of } 3\text{ cm/s}^2 \text{ per second}}$.

4. $P(t)$ gives the population of population, in billions of people, of the world t years after 1950. $P(40) = 5.3$ means that in the year 1990,

the population of the world was 5.3 billion people

$P'(60) = 0.078$ means that

in the year 2010, the world population was increasing at a rate of 0.078 billion people per year

5. $R(t)$ gives the rate, in pints per second, that oil is flowing into a tank after t seconds.

$R(10) = 3$ means that

after 10 seconds, the rate that oil is flowing into the tank is 3 pints/s.

$R'(6) = -0.2$ means that

after 6 seconds, the rate that the oil is flowing into the tank is decreasing at a rate of 0.2 pints/s per second.

Check your understanding

The function $V(t)$ gives the volume in liters of water in a tank after t minutes. Read each of the following interpretations and then write the appropriate first or second derivative statement equivalent to it. The first one is completed as an example. (Note: In some cases, the units have been purposely left out in order to make it less obvious which derivative it corresponds to. However, whenever *you* are asked to write in words the meaning of a derivative statement, remember to always include units for both the independent and dependent variables.)

Example: After 3 minutes, the volume of water in the tank is decreasing at a rate of 8 liters per minute.
Derivative statement: $V'(3) = -8$

1. When 7 minutes have passed, the instantaneous rate of change of water in the tank is 12 liters per minute.

Derivative statement: $V'(7)=12$

2. After 2 minutes, the rate at which the volume of water is decreasing is 5.

Derivative statement: $V'(2)=-5$

3. When $t = 10$, the rate at which the volume of water is increasing is 4.

Derivative statement: $V'(10) = 4$

Additional Learning Resources: Rates of Change in Applied Contexts

Activity 1: Read then answer the questions that follow.

The instantaneous rate of change of a function f with respect to its independent variable x at the instant when $x = a$ is determined by computing $f'(a)$.

It is always important to read each and every word in a math problem. Mathematicians tend not to throw in lots of extra words, so usually, every word is important.

Read the two questions carefully, and then do the following for each:

a) Decide which of the two calculations indicated is the appropriate one to answer the question. Place a check mark in the appropriate box.

b) Verify both calculations, using your calculator. Remember that you can use the numerical derivative capability on a graphing calculator to compute a derivative of a function at a point.

c) Provide correct units for *both* calculations.

1. Water is flowing into a tank so that the volume of water in the tank, in liters, after t seconds is given by the function $V(t) = 74 - 9e^{-0.3t}$. At what rate is the amount of water in the tank increasing after 15 seconds?

☐ $V(15) = 73.900$ units: ______

☑ $V'(15) = 0.030$ units: liters/s

2. Oil is being siphoned out of a tank at a rate of $S(t) = 5 - \sqrt{t}$ gallons per hour where t is measured in hours and $0 \le t \le 5$. At what rate is oil being siphoned out of the tank after 2.4 hours have elapsed?

☑ $S(2.4) = 3.451$ units: gal/hr

☐ $S'(2.4) = -.323$ units: ______

Additional Learning Resources: Rates of Change in Applied Contexts

Activity 2 ~ Read then answer the questions that follow:

Almost certainly, somewhere on the AP Calculus Exam, you will be asked to state in words what a given derivative statement means in the context of the problem. Being able to do so succinctly and correctly is an important skill.

The key understanding is that the derivative gives the instantaneous rate of change of the given quantity at the indicated point in time. Ideally, you should use an action verb that indicates the direction of change.

Here is an example of what this might look like:

Prompt: The differentiable function $V(t)$ gives the volume of water, measured in gallons, in a tank at time t, where t is measured in minutes. Explain the meaning of $V'(5) = -6$.

☑ **Acceptable explanations:**

After 5 minutes, the volume of water in the tank is decreasing at a rate of 6 gallons per minute.

Or

At the instant when 5 minutes have elapsed, the amount of water in the tank is going down at a rate of 6 gallons per minute.

Or

The amount of water in the tank is decreasing 6 gallons per minute at time $t = 5$ minutes. *[The phrase "at a rate of" is not absolutely necessary since the units indicate that the statement reflects a rate of change. However, the following explanation signals more clearly that you understand what a derivative means.]*

Or

The amount of water in the tank is decreasing at a rate of 6 gallons per minute at time $t = 5$ minutes.

× **Explanations that are too vague or awkward phrasing:**

After 5 minutes, the amount of water in the tank is changing at a rate of 6 gallons per minute. *[Which direction?]*

Or

After 5 minutes, the instantaneous rate of change of water in the tank is -6 gallons per minute. *[Awkward – we don't really speak with negative numbers except for temperature when it is extremely cold!]*

× **Explanations that are wrong:**

After 5 minutes, the rate of change of water in the tank is decreasing at 6 gallons per minute. *[This is a statement about the rate of change of the rate of change of volume (with incorrect units), or the second derivative of volume.]*

Or

At time 5 minutes, during the next minute, the volume of water in the tank will decrease by 6 gallons. *[This is only an approximation of what the statement means and does not clearly indicate an instantaneous rate of change. Certainly the units for the -6 are not correctly indicated.]*

One way to reach an acceptable explanation is to start by thinking about how to describe a statement involving the original function, using the verb "is."

Then for the derivative statement, following the word "is," insert the phrase "increasing at a rate of" or "decreasing at a rate of," use the absolute value of the derivative, and change the units by inserting the word "per" and the unit of measure for the independent variable.

For our example, we might say:

- $V(5) = 60$ means that after 5 minutes, the volume of water in the tank is 60 gallons.
- $V'(5) = -6$ means that after 5 minutes, the volume of water in the tank is **decreasing at a rate of** 6 gallons **per minute.**

Practice writing acceptable verbal descriptions for the following prompts.

Prompt 1: The differentiable function $C(x)$, where C is measured in dollars, gives the cost of digging a hole x feet deep. Explain the meaning of $C'(20) = 55$.

When the hole is 20 feet deep, the cost of digging the hole is increasing at a rate of $55/foot

Prompt 2: The differentiable function $h(t)$ gives the height of a rocket, in meters, t seconds after launch. Explain the meaning of the statement $h'(20) = 510$.

After 20 seconds, the rocket is rising at a rate of 510 m/s.

Prompt 3: The differentiable function $P(t)$ gives the number of bacteria in a Petri dish after t minutes of observation. Explain the meaning of the statement $P'(12) = 720$.

After 12 minutes of observation, the number of bacteria in the Petri dish is increasing at a rate of 720 bacteria per minute.

Additional Learning Resources:
Rates of Change in Applied Contexts

Activity 3 ~ Read then answer the questions that follow:

When giving a verbal description of what a derivative statement means, we need to include the meaning of the independent variable. It is important that the words we use evoke an instantaneous rate of change rather than an average rate of change. An instantaneous rate of change is change at an *instant*, as the word suggests. An average rate of change is change over an *interval*.

Suppose we need to explain the meaning of $f'(4)$, where t is measured in seconds.

Each of the following phrases is acceptable to indicate an instant of time:

After 4 seconds
When 4 seconds have elapsed
At time 4 seconds

Each of the following phrases is unacceptable as it indicates an interval of time:

During the first 4 seconds
During the 4th second
From $t = 0$ to $t = 4$ seconds
Over the 4 seconds

Suppose $D(x)$ is a differentiable function giving the depth D, measured in feet, of a river, x feet downstream from the dock at point A. We are asked to explain the meaning of the statement $D'(100) = 2$.

1. Write one or more acceptable phrases using the input value 100 that signify an instantaneous rate of change:

   At a distance of 100 feet downstream from the dock.

2. Write one or more phrases using the input value 100 that would be unacceptable because they would signify an average rate of change rather than an instantaneous one:

   Over the first 100 feet downstream from the dock.