7.1 Modeling with Differential Equations

CA #1

Calculus   Name: ______________________________

Write a differential equation that describes each relationship. If necessary, use k as the constant of proportionality.

1. The rate of change of Y with respect to w is directly proportional to the square of x.

2. The rate of change of S with respect to y is proportional to the square root of u and inversely proportional to v.

3. L is increasing with respect to x at a rate that is proportional to the cube root of m. The rate of change of L is 12 when $m = 5$.

4. The rate of change of U with respect to a is inversely proportional to the cube of v. The rate of change of U is -5 when $v = \frac{1}{2}$.

5. The height of a rocket is given by the function $h(t)$, where t is measured in seconds since the launch and h is measured in meters. The acceleration is proportional to the cube root of the time since the start of the launch. At 12 seconds, the acceleration is 3 meters per second per second.

6. A scientist is studying the relationship of two quantities A and B in an experiment. The scientist finds that the quantity of A decreases and the quantity of B increases. The scientist determines that the rate of change of the quantity of A with respect to the quantity of B is inversely proportional to the square of the quantity of B.

7. The number of packets, p, Mr. Sullivan completes for Pre-Calculus is increasing as he nears the end of the school year. The rate of change of p with respect to time t is inversely proportional to the natural log of t.

8. Mr. Brust is running down his street. His position is given by the function $p(t)$, where t is measured in minutes since the start of his run. His acceleration is inversely proportional to the cube of the time since the start of his run.

Answers to 7.1 CA #1

1. $\frac{dY}{dw} = kx^2$	2. $\frac{dS}{dy} = \frac{k\sqrt{u}}{v}$	3. $\frac{dL}{dx} = 7.0176\sqrt[3]{m}$	4. $\frac{dU}{da} = -\frac{0.625}{v^3}$
5. $\frac{d^2h}{dt^2} = 1.310\sqrt[3]{t}$	6. $\frac{dA}{dB} = \frac{k}{B^2}$	7. $\frac{dp}{dt} = \frac{k}{\ln t}$	8. $\frac{d^2p}{dt^2} = \frac{k}{t^3}$

7.2 Verifying Solutions for Differential Equations

Calculus Name: ______________________________ **CA #1**

For each differential equation, find the particular solution that passes through the given point.

1. $\frac{dy}{dx} = \frac{12}{3x-2} - \frac{1}{x^2}$; $(1, -3)$

2. $\frac{dy}{dx} = 10\sin(5x)$; $\left(\frac{\pi}{5}, 1\right)$

3. $\frac{dy}{dx} = 6e^{2x} - 4x$; $(0, -2)$

4. $\frac{d^2y}{dx^2} = \sin(2x)$ and $y'\left(\frac{\pi}{6}\right) = \frac{5}{4}$ and $y\left(\frac{\pi}{2}\right) = \pi$

5. $\frac{d^2y}{dx^2} = e^{2x} + 4x$ and $y'(0) = 2$ and $y(0) = \frac{3}{4}$

6. For what value of k, if any, will $y = ke^{-3x} + 8\sin(2x)$ be a solution to the differential equation $y'' + 4y = 26e^{-3x}$?

7. For what value of k, if any, will $y = k\cos(3x) - \sin(5x)$ be a solution to the differential equation $y'' + 25y = 8\cos(3x)$?

Answers to 7.2 CA #1

1. $y = 4\ln\lvert 3x-2\rvert + \frac{1}{x} - 4$	2. $y = -2\cos(5x) - 1$	3. $y = 3e^{2x} - 2x^2 - 5$

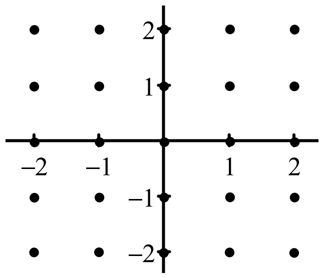
4. $y = -\frac{1}{4}\sin(2x) + \frac{3}{2}x + \frac{\pi}{4}$	5. $y = \frac{1}{4}e^{2x} + \frac{2}{3}x^3 + \frac{3}{2}x + \frac{1}{2}$	6. $k = 2$	7. $k = \frac{1}{2}$

7.3 Sketching Slope Fields

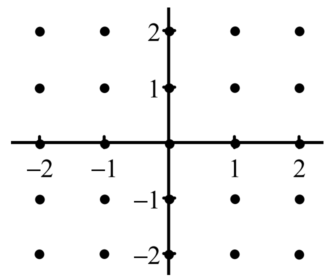
Calculus Name: ______________________________ **CA #1**

Draw a slope field for each of the following differential equations. Use each of the coordinate points shown in the graph.

1. $\frac{dy}{dx} = x - 2y$



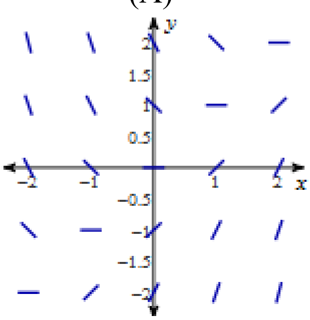
2. $\frac{dy}{dx} = -\frac{x}{y}$



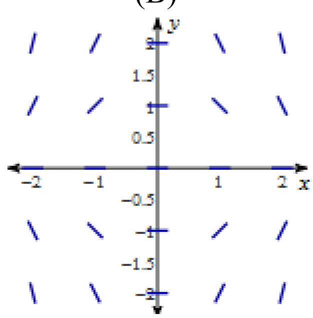
Match the differential equation with its slope field.

3. $\frac{dy}{dx} = xy$

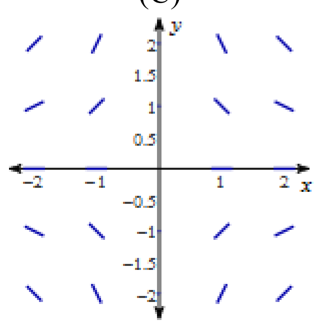
(A)



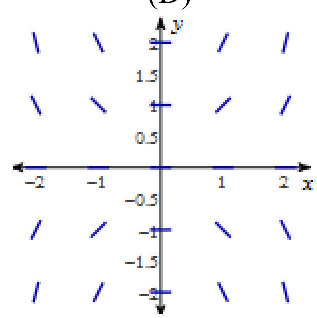
(B)



(C)

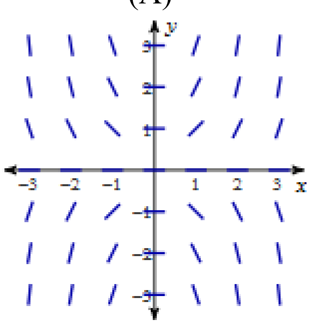


(D)

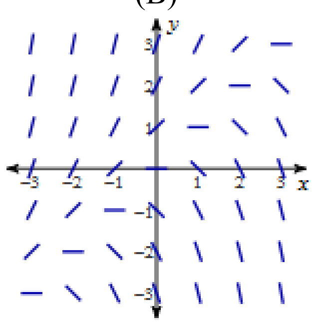


4. $\frac{dy}{dx} = x - y$

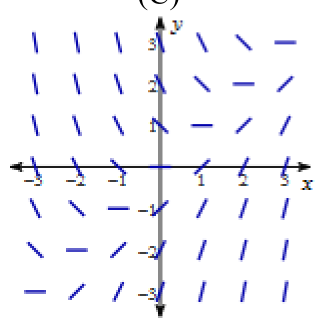
(A)



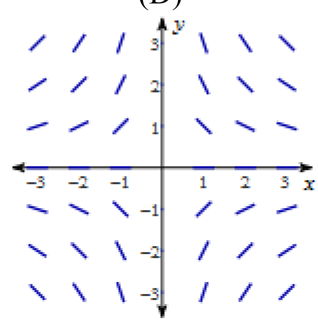
(B)



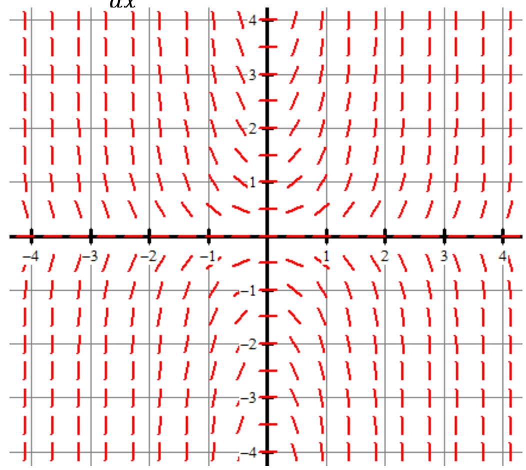
(C)



(D)



5. The figure below shows the slope field for the differential equation $\frac{dy}{dx} = 2xy$



Let f be the function that satisfies the given differential equation. Write an equation for the tangent line to the curve $y = f(x)$ through the point $(-2, 3)$.

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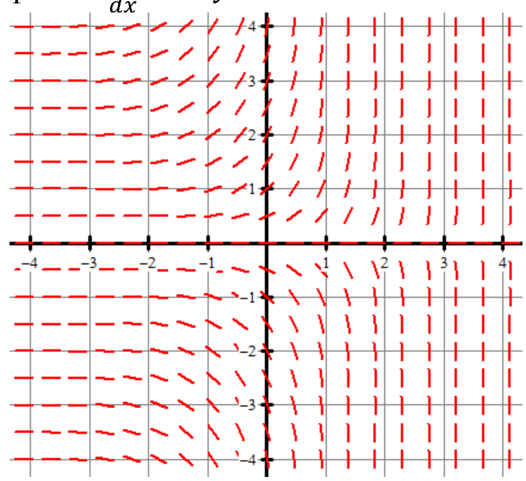
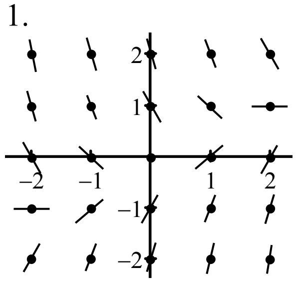
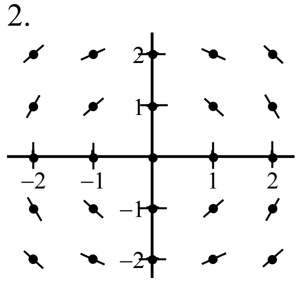
6. The figure below shows the slope field for the differential equation $\frac{dy}{dx} = e^x y$



Let f be the function that satisfies the given differential equation. Write an equation for the tangent line to the curve $y = f(x)$ through the point $(0, -3)$.

Answers to 7.3 CA #1

1. 	2. 	3. D	4. C
		5. $y - 3 = -12(x + 2)$	6. $y + 3 = -3x$

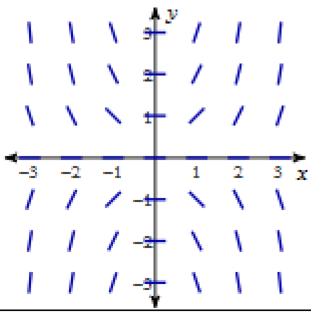
7.4 Reasoning Using Slope Fields

Calculus Name: ______________________________ **CA #1**

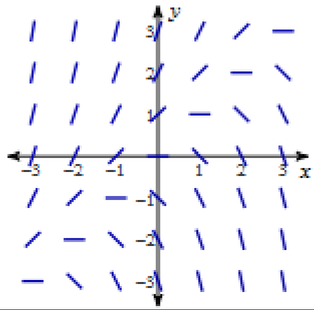
Match the slope field with the differential equation.

1. $\frac{dy}{dx} = x - y$

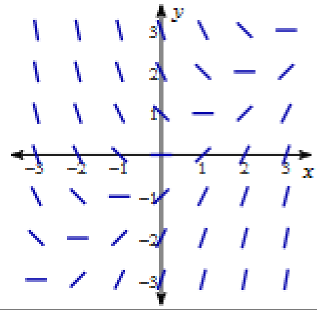
(A)



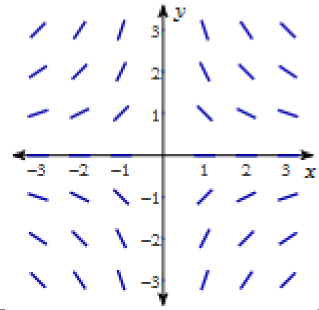
(B)



(C)



(D)



2.

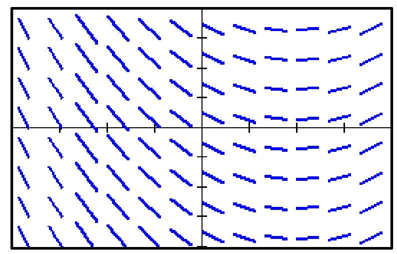
(A) $\frac{dy}{dx} = (x-2)^2$

(B) $\frac{dy}{dx} = 0.5x - 1$

(C) $\frac{dy}{dx} = x - y$

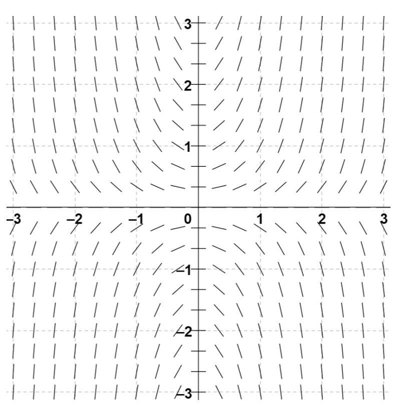
(D) $\frac{dy}{dx} = x + y$

(E) $\frac{dy}{dx} = 0.5y$



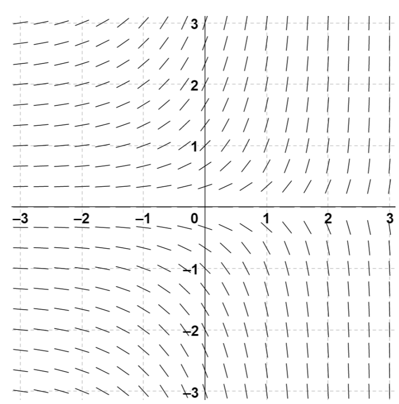
For each slope field, plot and label the points A and B and sketch the particular solution that passes through each of those points. (Two separate solutions for each slope field.)

3. $\frac{dy}{dx} = 2xy$



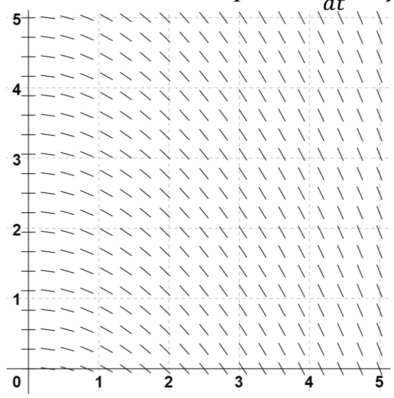
Point A: $(0, 1)$
Point B: $(2, -1)$

4. $\frac{dy}{dx} = e^x y$

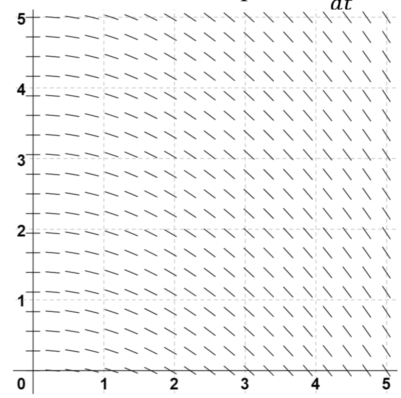


Point A: $(2, 1)$
Point B: $(0, -1)$

5. Let $f(t)$ be an increasing, differentiable function. Explain why the following slope field cannot represent the differential equation $\frac{dy}{dt} = f'(t)$

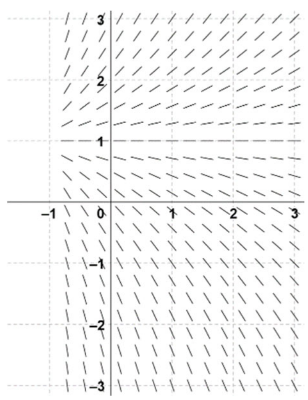


6. Explain why the following slope field cannot represent the differential equation $\frac{dy}{dt} = -0.3y$



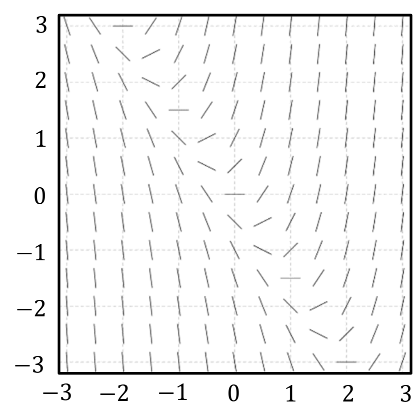
Consider the differential equation and its slope field. Describe all points in the xy-plane that match the given condition.

7. $\frac{dy}{dx} = \frac{y-1}{\sqrt{x+1}}$



When is $\frac{dy}{dx}$ positive?

8. $\frac{dy}{dx} = 3x + 2y$

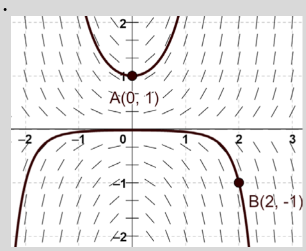


When does $\frac{dy}{dx} = -2$?

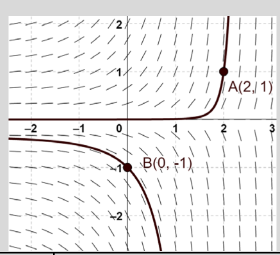
Answers to 7.4 CA #1

1. C
2. B

3.



4.



5. $\frac{dy}{dt} > 0$ when $y > 0$, but the slope field shows line segments with nonpositive slope.

6. Possible answer: When $y = 0, \frac{dy}{dt} = 0$. However, in the slope field, the slopes of the line segments for $y = 0$ are nonzero.

7. All points where $y > 1$.

8. All points that fall on the line $y = -\frac{3}{2}x - 1$

7.5 Euler's Method

CA #1

Calculus Name: ______________________________

1. The table below gives the values of f', the derivative of f. If $f(1.4) = 3$, what is the approximation to $f(2.6)$ obtained by using Euler's method with 3 steps of equal size?

$\boldsymbol{x}$	1	1.4	1.8	2.2	2.6
$\boldsymbol{f'(x)}$	.1	.3	.5	.8	1.2

2. The table below gives the values of f', the derivative of f. If $f(0) = 7$, what is the approximation to $f(1)$ obtained by using Euler's method with 2 steps of equal size?

$\boldsymbol{x}$	0	0.5	1.0
$\boldsymbol{f'(x)}$	$-.5$	$-.3$	$-.1$

3. Let $y = f(x)$ be the solution to the differential equation $\frac{dy}{dx} = x + y$ with initial condition $f(0) = 3$. What is the approximation for $f(0.5)$ obtained using Euler's method with 2 steps of equal length, starting at $x = 0$?

4. Let $y = f(x)$ be the solution to the differential equation $\frac{dy}{dx} = \frac{1}{x}$ with initial condition $f(1) = 2$. What is the approximation for $f(1.4)$ obtained using Euler's method with 4 steps of equal length, starting at $x = 1$?

5. Let $h(x) = \int_0^x \sqrt{1 + 4t^2}\, dt$. Use Euler's method, starting at $x = 0$ with two steps of equal size, to approximate $h(3)$.

Answers to 7.5 CA #1

1. $f(2.6) \approx 3.64$	2. $f(1.0) \approx 6.6$	3. $f(0.5) \approx 4.75$	4. $f(1.4) \approx 2.351$	5. $h(3) \approx 6.243$

7.6 Separation of Variables (General Solutions)

CA #1

Calculus Name: _______________________________

Find the general solution of each differential equation.

1. $\frac{dy}{dx} = 3e^{x-y}$

2. $\frac{dy}{dx} = \frac{x^2}{2y}$

3. $\frac{dy}{dx} = y\cos x$

4. $\frac{dy}{dx} = 6x(y-1)$

5. $\frac{dy}{dx} = \frac{2x+1}{10y}$

6. $\frac{dy}{dx} = 4xy^2$

Answers to 7.6 CA #1

1. $y = \ln(3e^x + C)$	2. $y = \pm\sqrt{\frac{1}{3}x^3 + C}$	3. $y = Ce^{\sin x}$
4. $y = Ce^{3x^2} + 1$	5. $y = \pm\sqrt{\frac{1}{5}x^2 + \frac{1}{5}x + C}$	6. $y = -\frac{1}{2x^2+C}$

7.7 Separation of Variables (Particular Solutions)

Calculus Name: ________________________________

CA #1

For each differential equation, find the solution that passes through the given initial condition.

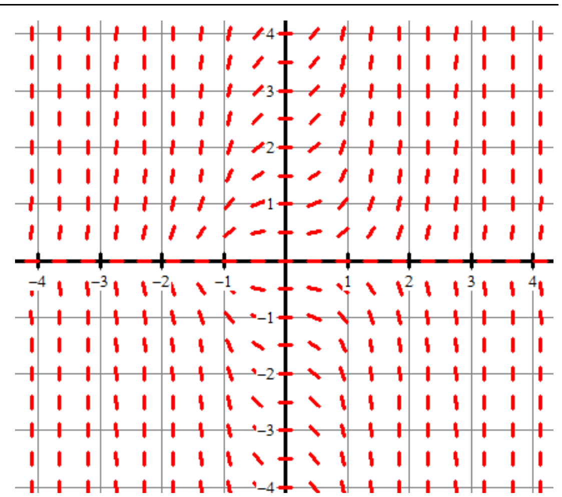
1. $\frac{dy}{dx} = e^{x+y}$ with initial condition $y(0) = -\ln 3$

2. $\frac{dy}{dx} = y \sec^2 x$ and $y = 2$ when $x = 0$.

3. $\frac{dy}{dx} = \frac{x^3-2}{y}$ with initial condition $y(2) = -4$

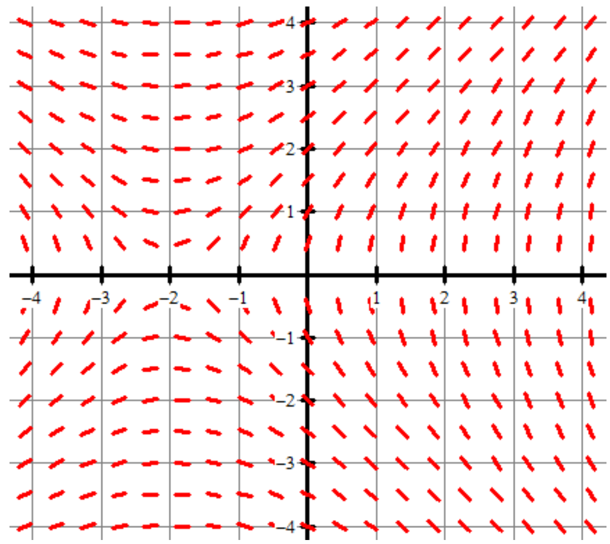
4. $\frac{dy}{dx} = 2x^2 y$ and $y = 1$ when $x = 3$.

5. The slope field of $\frac{dy}{dx} = 2x^2 y$ from question #4 is shown below. Draw the particular solution $y = f(x)$ when $f(3) = 1$ that you found in question #4 on the slope field.

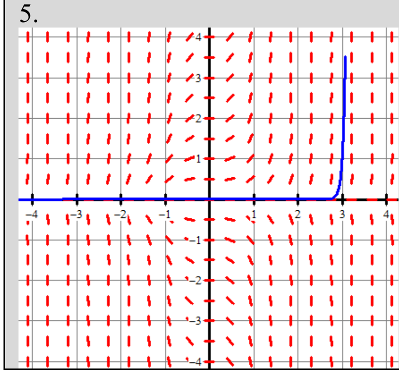
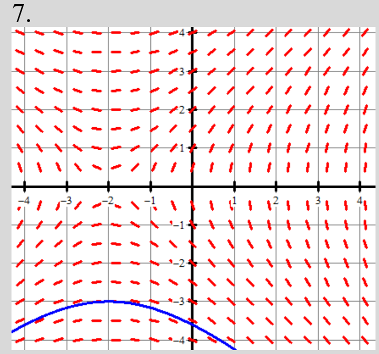


6. Solve the differential equation $\frac{dy}{dx} = \frac{x+2}{y}$ for the particular solution $y = f(x)$ when $f(-2) = -3$.

7. The slope field of $\frac{dy}{dx} = \frac{x+2}{y}$ from question #6 is shown below. Draw the particular solution $y = f(x)$ when $f(-2) = -3$ that you found in question #6 on the slope field.



Answers to 7.7 CA #1

<table>
<tr><td>1. $y = -\ln(-e^x + 4)$</td><td>2. $y = 2e^{\tan x}$</td><td>3. $y = -\sqrt{\frac{1}{2}x^4 - 4x + 16}$</td><td>4. $y = e^{\frac{2}{3}x^3 - 18}$</td></tr>
<tr><td>5. </td><td colspan="2">6. $y = -\sqrt{x^2 + 4x + 13}$</td><td>7. </td></tr>
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7.8 Exponential Models with Differential Equations

Calculus Name: ______________________________ **CA #1**

Find the particular solution $y = f(t)$ for each differential equation.

1. $\frac{dy}{dt} = 106y$ and $y = -15$ when $x = 0$, then $y =$	2. $\frac{dy}{dx} = -0.3y$ and $y = 41$ when $x = 0$, then $y =$	3. $\frac{dy}{dt} = 51y$ and $y = -0.5$ when $x = 0$, then $y =$

For each problem, use your understanding of exponential models and differential equations.

4. A dose of 75 milligrams of a drug is administered to a patient. The amount of the drug, in milligrams, in the person's bloodstream at time t, in hours, is given by $A(t)$. The rate at which the drug leaves the bloodstream can be modeled by the differential equation $\frac{dA}{dt} = -0.09A$. Write an expression for $A(t)$.

5. A population y grows according to the equation $\frac{dy}{dt} = ky$, where k is a constant and t is measured in years. If the population doubles every 4 years, then what is the value of k?

6. A population y grows according to the equation $\frac{dy}{dt} = ky$, where k is a constant and t is measured in years. If the population doubles every 28 years, then what is the value of k?

7. During a certain epidemic, the number of people that are infected at any time increases at a rate proportional to the number of people that are infected at that time. If 5,000 people are infected when the epidemic is first discovered, and 6,000 people are infected 3 days later, how many people are infected 20 days after the epidemic is first discovered?

Answers to 7.8 CA #1

1. $y = -15e^{106t}$	2. $y = 41e^{-0.3t}$	3. $y = -0.5e^{51t}$	4. $y = 75e^{-0.09t}$
5. $k \approx 0.173$	6. $k \approx 0.0247$	7. 16,859 people	

7.9 Logistic Models

Calculus Name: ______________________________ **CA #1**

1. A populations rate of growth is modeled by the logistic differential equation $\frac{dP}{dt} = \frac{1}{1000}P(600 - P)$, where t is in days and $P(0) = 60$. What is the greatest rate of change for this population?

2. Using the logistic differential equation $\frac{dP}{dt} = \frac{1}{5}P - \frac{1}{2000}P^2$, identify the carrying capacity.

3. A rate of change $\frac{dP}{dt}$ of a population is modeled by a logistic differential equation. If $\lim_{t\to\infty} P(t) = 1000$ and the rate of change of the population is 100 when the population size is 50, which of the following differential equations describe the situation?

   A. $\frac{dP}{dt} = 50P\left(1 - \frac{P}{1000}\right)$

   B. $\frac{dP}{dt} = 100P\left(1 - \frac{P}{1000}\right)$

   C. $\frac{dP}{dt} = \frac{19}{40}P\left(1 - \frac{P}{1000}\right)$

   D. $\frac{dP}{dt} = \frac{40}{19}P\left(1 - \frac{P}{1000}\right)$

4. A rate of change for a population is modeled by the differential equation $\frac{dP}{dt} = \frac{1}{5}P(40 - P)$. What is the population when the rate of change is the greatest?

5. Let k be a positive constant. Which of the following is a logistic differential equation?

A. $\frac{dy}{dt} = kt + C$

B. $\frac{dy}{dt} = ky$

C. $\frac{dy}{dt} = kt(2 - t)$

D. $\frac{dy}{dt} = ky(2 - y)$

Answers to 7.9 CA #1

1. 90/day	2. 400	3. D	4. 20	5. D