2022-05-18 Approximate integration

There are times when we want to find a definite integral but cannot use an antiderivative.

eg $\int_0^1 e^{-x^2}\,dx$ (shows up in statistics)

or we don't even know the integrand

eg given $f(x)$ at $x = 0, 1, 5, 7, 10$ (shows up often in sciences)

<u>Recall:</u> $\displaystyle\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$

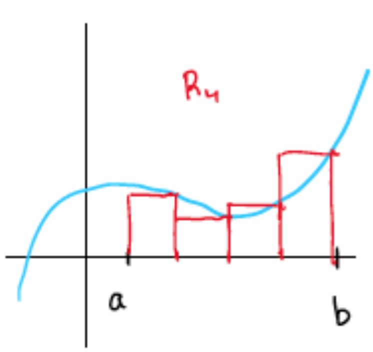
where $x_i^* \in [x_{i-1}, x_i]$

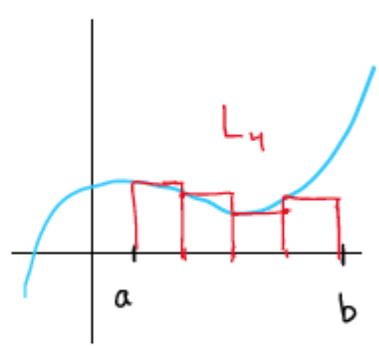
If we choose $x_i^* = x_i$ to be the right endpoint,

$$\int_a^b f(x)\,dx \approx R_n = \sum_{i=1}^{n} f(x_i)\,\Delta x$$

If we choose $x_i^* = x_{i-1}$ to be the left endpoint,

$$\int_a^b f(x)\,dx \approx L_n = \sum_{i=1}^{n} f(x_{i-1})\,\Delta x$$

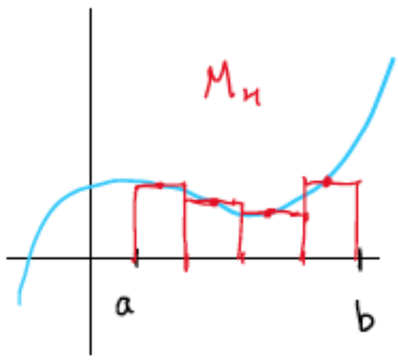




If we choose $x_i^* = \frac{x_{i-1}+x_i}{2}$ to be the midpoint,

(or $\bar{x}_i$)

$$\int_a^b f(x)\,dx \approx M_n = \sum_{i=1}^{n} f\left(\frac{x_{i-1}+x_i}{2}\right)\Delta x$$

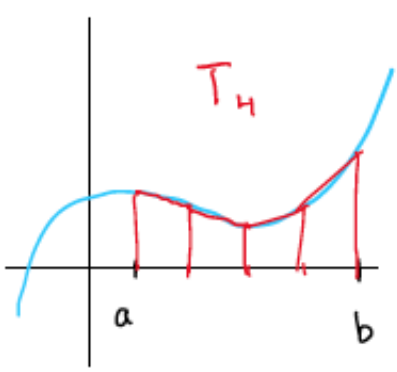


(other options: Lower Riemann Sum : x_i^* is the min of f on $[x_{i-1}, x_i]$

Upper Riemann Sum : x_i^* is the max of f on $[x_{i-1}, x_i]$)

Also, we can approximate by taking the average of L_n and R_n: this gives Trapezoidal Rule

$$\int_a^b f(x)\,dx \approx T_n = \sum_{i=1}^{n} \frac{f(x_{i-1}) + f(x_i)}{2}\,\Delta x$$



Try using these rules to appoximate $\int_1^2 x^2 dx$

$n = 4, 8, 16, 32$

$$E_L = \int_a^b f(x)dx - L_n$$

$$E_R = \int_a^b f(x)dx - R_n$$

$$E_M = \int_a^b f(x)dx - M_n$$

$$E_T = \int_a^b f(x)dx - T_n$$

n	L_n	R_n	M_n	T_n
4	1.96875	2.71875	2.32813	2.34375
8	2.14844	2.52344	2.33203	2.33594
16	2.24023	2.42773	2.33301	2.33398
32	2.28662	2.38037	2.33325	2.33350

n	E_L	E_R	E_M	E_T
4	0.36458333	-0.3854167	0.005203333	-0.0104167
8	0.18489333	-0.1901067	0.001303333	-0.0026067
16	0.09310333	-0.0943967	0.000323333	-0.0006467
32	0.04671333	-0.0470367	0.000083333	-0.0001667

Obs: 1) Better approximation as n increases.

2) E_L and E_R are opposite in sign

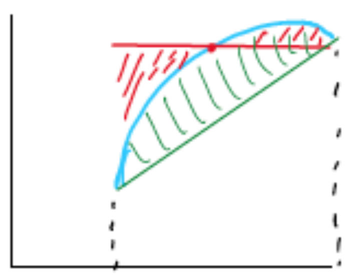
3) E_L, E_R appear to decrease by a factor of 2 when n doubles.

4) Midpoint and Trapezoid rules are much better than left/right sums,

5) E_M, E_T are opposite in sign

6) E_M, E_T appear to decrease by a factor of 4 when n doubles.

7) $|E_L| \approx |E_R|$, $|E_M| = \frac{1}{2}|E_T|$



Error bounds

$$|E_L|, |E_R| \leq \frac{K_1(b-a)^2}{2n}, \qquad |f'(x)| \leq K_1 \text{ on } [a,b]$$

$$|E_M| \leq \frac{K_2(b-a)^3}{24n^2}$$

$$|f''(x)| \leq K_2 \text{ on } [a,b]$$

$$|E_T| \leq \frac{K_2(b-a)^3}{12n^2}$$

Simpson's Rule

2 motivations:

1) $E_L = -E_R$ usually
$|E_L| \approx |E_R|$ usually $\Big\}$ try $\dfrac{L_n + R_n}{2}$ (this is T_n, which is much better)

$E_M = -E_T$ usually
$2|E_M| = |E_T|$ usually $\Big\}$ try $\dfrac{2M_n + T_n}{3} = S_{2n}$

2) Left, Right, and Midpoint Rules are an attempt at using a constant to approx. f on each subinterval (i.e. deg 0 poly)

Trap. Rule is an appox using a linear function (i.e. deg 1 poly - using 2 points to interpolate)

Simpson's Rule is an approx using a quadratic (i.e. deg 2 poly - using 3 points to interpolate)

(n even)

$$S_n = \frac{\Delta x}{3}\Big[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \dots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\Big]$$

$$|E_S| \le \frac{K_4 (b-a)^5}{180\, n^4}, \qquad |f^{(4)}(x)| \le K_4 \text{ on } [a,b].$$

We can go further: Newton-Cotes formulas

4-pt: Simpson's $3/8$ Rule

5-pt: Boole's Rule

$\vdots$