Optimization exercises

1. A printer needs to make a poster that will have a total area of 19,440 cm^2 with blank margins of 6 cm on the left and right, and 10 cm on the top and bottom. To the nearest cm, what dimensions will give the largest printed area?

Solution: Let the dimensions of the poster be w and h. We have $wh = 19440$, and wish to optimize

$$A = (w - 12)(h - 20)$$

Notice $w \geq 12$ and $h \geq 20$. Using $h = 19440/w$ (which gives $w \leq 1620$), we have

$$\begin{aligned} A &= (w - 12)\left(\frac{19440}{w} - 20\right) \\ &= 19680 - 20w - \frac{233280}{w} \end{aligned}$$

Differentiating,

$$A'(w) = -20 + \frac{233280}{w^2}$$

A' does not exist at $w = 0$, which is not in the domain. Solving $A'(w) = 0$, we get

$$\begin{aligned} 0 &= -20 + \frac{233280}{w^2} \\ w &= \sqrt{\frac{233280}{20}} = \sqrt{11664} = 108 \end{aligned}$$

Clearly $A'(w) > 0$ for $12 \leq w < 108$ and $A'(w) < 0$ for $w > 108$, so by the First Derivative Test for Global Extrema, $w = 108$ cm is the maximum. The corresponding height is $h = 19440/108 = 180$ cm.

Note that the method from section 4.2 works as well: the domain for w is $[12, 1620]$, and

$$A(12) = 0 = A(1620)$$

Since $A(108) > 0$, $w = 108$ is the maximum. We can also use the Second Derivative Test for Global Extrema:

$$A'' = -\frac{466560}{w^3}$$

Since $A''(108) < 0$ and this is the only critical point, $w = 108$ is the maximum.

2. A boat leaves a port at 4:00 PM and travels due north at a speed of 40 km/h. Another boat has been heading due west at 20 km/h and reaches the same port at 5:00 PM. At what time were the two boats closest together to the nearest minute?

Solution: Let y be the distance from the first boat to the port and x be the distance from the second boat to the port. Then the distance between the boats is given by $d = \sqrt{x^2 + y^2}$. To minimize d, it suffices to minimize $d^2 = x^2 + y^2$. We have $y = 40t$ and $x = 20 - 20t$, so we wish to minimize

$$\begin{aligned} f(t) &= (40t)^2 + (20 - 20t)^2 \\ &= 2000t^2 - 2 \cdot 400t + 20^2 \end{aligned}$$

Differentiating,

$$f'(t) = 2 \cdot 2000t - 2 \cdot 400$$

This has no points at which f' does not exist, and solving $f'(t) = 0$ yields

$$\begin{aligned} 0 &= 2 \cdot 2000t - 2 \cdot 400 \\ t &= \frac{400}{2000} = \frac{1}{5} \end{aligned}$$

Clearly $f'(t) < 0$ for $t < 1/5$ and $f'(t) > 0$ for $t > 1/5$, so by the First Derivative Test for Global Extrema, $t = 1/5$ is the minimum. This corresponds to the time 4:12 PM.

Note that the method from section 4.2 fails since we do not have a closed and bounded interval for the domain. We can, however, use the Second Derivative Test for Global Extrema:

$$f'' = 8000$$

Since $f''(1/5) > 0$ and this is the only critical point, $t = 1/5$ is the minimum.

3. We want to construct an open-topped box in the shape of a rectangular prism whose base is 60% longer than it is wide. The material used to build the sides costs 0.6 cents per cm^2 and the material used to build the base costs 1 cent per cm^2. If the box must have a volume of $2000\,\text{cm}^3$, determine the dimensions of the box (to the nearest tenth of a centimetre) that will minimize the cost of the box.

Solution: Let the dimensions of the box be L, W, H. We have $L = 1.6W$, $LWH = 1.6W^2H = 2000$, and we wish to minimize

$$\begin{aligned} C &= 1 \cdot LW + 0.6 \cdot 2(L + W)H \\ &= 1.6W^2 + 1.2 \cdot 2.6W^2H \end{aligned}$$

Since $1.6W^2H = 2000$, we have $H = \dfrac{2000}{1.6W^2} = \dfrac{1250}{W^2}$, the cost in cents can be expressed in terms of width:

$$\begin{aligned} C &= 1.6W^2 + 1.2 \cdot 2.6W^2 \cdot \frac{1250}{W^2} \\ &= 1.6W^2 + \frac{3900}{W} \end{aligned}$$

where $W > 0$. Differentiating,

$$C' = 3.2W - \frac{3900}{W^2} = \frac{3.2W^3 - 3900}{W^3}$$

C' does not exist at $W = 0$, but this is not in the domain. Solving $C'(W) = 0$, we get

$$\begin{aligned} 0 &= 3.2W^3 - 3900 \\ W &= \sqrt[3]{\frac{3900}{3.2}} = \sqrt[3]{1218.75} \end{aligned}$$

Notice that $C' < 0$ to the left of this point and $C' > 0$ to the right of this point, so by the First Derivative Test for Global Extrema, $W = \sqrt[3]{1218.75}$ is the minimum. Solving for the other dimensions,

$$\begin{aligned} L &= 1.6\sqrt[3]{1218.75} \\ H &= \frac{1250}{\sqrt[3]{1218.75}^2} \end{aligned}$$

To the nearest tenth of a centimetre,

$$\begin{aligned} W &\approx 10.7\,\text{cm} \\ L &\approx 17.1\,\text{cm} \\ H &\approx 11.0\,\text{cm} \end{aligned}$$

Note that the method from section 4.2 fails since we do not have a closed and bounded interval for the domain. We can, however, use the Second Derivative Test for Global Extrema:

$$C'' = 3.2 + \frac{7800}{W^3}$$

Since $C''(\sqrt[3]{1218.75}) > 0$ and this is the only critical point, $W = \sqrt[3]{1218.75}$ is the minimum.