Mini-math Div 3/4: Friday, March 15, 2023 (20 minutes)

SOLUTIONS

1. (3 points) Determine whether the following series converges or diverges. If possible, write down the value of the series if it converges.

$$\sum_{n=0}^{\infty} \frac{n\sqrt{n} - n + 2}{\sqrt[3]{8n^7 + n^3 + 1}}$$

Solution: We compare with $\frac{n^{3/2}}{n^{7/3}} = \frac{1}{n^{5/6}}$:

$$\lim_{n\to\infty} \frac{n\sqrt{n} - n + 2}{\sqrt[3]{8n^7 + n^3 + 1}} \cdot \frac{n^{7/3}}{n^{3/2}} = \lim_{n\to\infty} \frac{1 - \frac{1}{n^{1/2}} + \frac{2}{n^{3/2}}}{\sqrt[3]{8 + \frac{1}{n^4} + \frac{1}{n^7}}} = \frac{1}{2}$$

Since $0 < 1/2 < \infty$, the Limit Comparison Test holds. Since $5/6 < 1$, $\sum \frac{1}{n^{5/6}}$ diverges by p-series, and hence the original series also diverges.

2. (3 points) Determine whether the following series converges or diverges. If possible, write down the value of the series if it converges.

$$\sum_{n=3}^{\infty} \frac{(n+1)2^n}{n!}$$

Solution:

$$\lim_{n\to\infty} \left| \frac{(n+2)2^{n+1}}{(n+1)! \cdot \frac{n!}{(n+1)2^n}} \right| = \lim_{n\to\infty} \frac{n+2}{n+1} \cdot \frac{2}{n+1} = 0 < 1$$

By the Ratio Test, the series converges.

3. (3 points) Determine whether the following series converges or diverges. If possible, write down the value of the series if it converges.

$$\sum_{n=0}^{\infty} \frac{n(n+1)^3}{2n^4+1}$$

Solution:

$$\lim_{n\to\infty} \frac{n(n+1)^3}{2n^4+1} = \frac{1}{2} \neq 0$$

so by the nth term test, the series diverges.

4. (3 points) Determine whether the following series converges or diverges. If possible, write down the value of the series if it converges.

$$\sum_{n=0}^{\infty} \frac{(-1)^n 5^n}{3^{2n+1}}$$

Solution: This is a geometric series with $r = -5/9$, and $|r| < 1$ gives convergence. Furthermore,

$$\sum_{n=0}^{\infty} \frac{(-1)^n 5^n}{3^{2n+1}} = \frac{\frac{1}{3}}{1+\frac{5}{9}} = \frac{3}{9+5} = \frac{3}{14}$$

5. (3 points) Determine whether the following series converges or diverges. If possible, write down the value of the series if it converges.

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$$

Solution: By inspection (or using the substitution $u = \ln x$),

$$\begin{aligned}\int_2^{\infty} \frac{1}{x(\ln x)^2}\,dx &= \lim_{b\to\infty} \int_2^{b} \frac{1}{x(\ln x)^2}\,dx = \lim_{b\to\infty} -\frac{1}{\ln x}\Big|_2^b \\ &= \lim_{b\to\infty} -\frac{1}{\ln b} + \frac{1}{\ln 2} = \frac{1}{\ln 2} < \infty\end{aligned}$$

By the integral test, $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ converges.