Mini-math AP Calculus BC: Friday, April 1, 2022 (15 minutes)

SOLUTIONS

1. Consider the initial value problem $\dfrac{dy}{dx} = 2x + y$ and $y(1) = 2$.

   (a) (2 points) Find an approximation of $y(1.2)$ using Euler's Method with two equal steps.

   Solution:

   $y(1.1) \approx y(1) + y'(1)(1.1 - 1) = 2 + (2(1) + 2)(0.1) = 2.4,$
   $y(1.2) \approx y(1.1) + y'(1.1)(1.2 - 1.1) \approx 2.4 + (2(1.1) + 2.4)(1.2 - 1.1) = 2.4 + (4.6)(0.1) = 2.86$

   (b) (2 points) Is your estimate in part (a) an overestimate or an underestimate?

   Solution: At $(1, 2)$,

   $$\frac{d^2y}{dx^2} = 2 + \frac{dy}{dx} = 2 + 2x + y > 0$$

   so the function is concave up. Therefore, the estimate will be an underestimate. (Aside: the actual value is about 2.92842. If we used 10 equal steps, we would get the approximation 2.91397)

2. The number of squirrels in a park at time t is modelled by the function $y = F(t)$ that satisfies the logistic differential equation $\frac{dy}{dt} = \frac{1}{2000}y(1500 - y)$, where t is measured in weeks. The number of squirrels in the park at time $t = 0$ is $F(0) = b$, where b is a positive constant.

(a) (i) (1 point) If $b = 300$, what is the largest rate of increase in the number of squirrels in the park?

Solution: Since $300 < 1500/2 = 750$, F grows most rapidly when it is half the carrying capacity, 750. Then

$$\left.\frac{dy}{dt}\right|_{F=750} = \frac{750}{2000}(1500 - 750) = \frac{1125}{4} = 281.25$$

(ii) (1 point) If $b = 1000$, what is the largest rate of increase in the number of squirrels in the park?

Solution: For $F > 750$, $\frac{dy}{dt} > 0$ and $\frac{d^2y}{dt^2} < 0$, so F grows most rapidly when $F = 1000$.

$$\left.\frac{dy}{dt}\right|_{F=1000} = \frac{1000}{2000}(1500 - 1000) = 250$$

(b) (2 points) If $b = 150$, find $\lim_{t\to\infty} F(t)$ and interpret the meaning of this limit in the context of the problem.

Solution: The carrying capacity is 1500, so $\lim_{t\to\infty} F(t) = 1500$.

This means in the long term, the population of squirrels in the park will tend to 1500.

(c) (4 points) Find the function $F(t)$ if $b = 500$. For reference, the differential equation is $\frac{dy}{dt} = \frac{1}{2000}y(1500 - y)$.

Solution:

$$\int \frac{dy}{y(1500-y)} = \int \frac{dt}{2000}$$

$$\int \left(\frac{1/1500}{y} + \frac{1/1500}{1500-y}\right) = \int \frac{dt}{2000}$$

$$\ln\left|\frac{y}{1500-y}\right| = (\ln|y| - \ln|1500-y|) = \frac{1500}{2000}x + C = \frac{3}{4}x + C$$

$$\frac{y}{1500-y} = Ce^{3x/4}$$

Using the initial condition,

$$\frac{1}{2} = \frac{500}{1500-500} = C$$

so

$$\frac{2y}{1500-y} = e^{3x/4}$$

$$2y = 1500e^{3x/4} - e^{3x/4}y$$

$$y = \frac{1500e^{3x/4}}{e^{3x/4}+2}$$