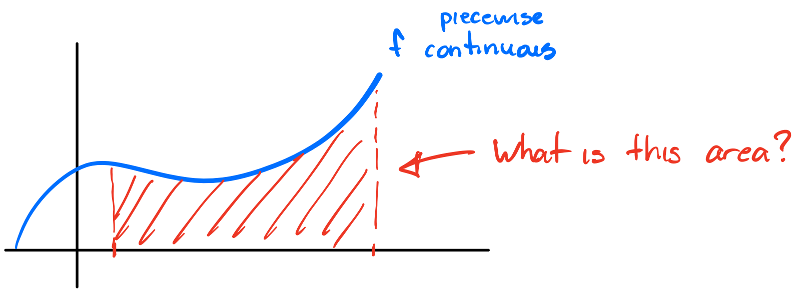
Integration



Analogy: exponents What is 2^0? 2^{-3}? $2^{1/2}$? 2^{π}?

$$\underbrace{\left(1 \quad \frac{1}{2^3} \quad \sqrt{2} \quad * \right)}_{\text{can't prove any of these}}$$

We can define exponents when the power is a positive integer:

$$a^b = \underbrace{a \times a \times \dots \times a}_{b \text{ copies of } a} \qquad \text{if } b \in \mathbb{Z}^+$$

We <u>define</u>

$$a^0 = 1 \qquad (\text{for } a \neq 0)$$

$$a^{-b} = \frac{a}{b}$$

$$a^{1/b} = \sqrt[b]{a}$$

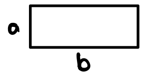
$$\left(\begin{array}{l} a^b \cdot a^c = a^{b+c} \\ (a^b)^c = a^{bc} \end{array} \qquad b,c \in \mathbb{Z}^+ \right)$$

extend powers to more numbers so that it makes sense

In a similar fashion, we extend our definition of area beyond that of a rectangle in a way that makes sense.

We begin with:

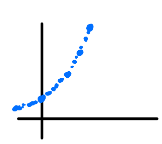
Area of a rectangle = length × width



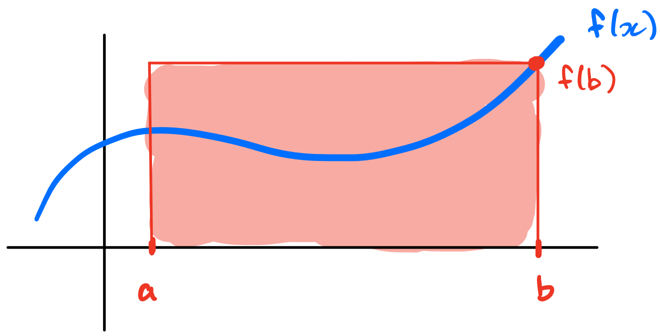
← Area = ab

What about a curve?

Analogy: $2^{\pi} = \lim_{\substack{x \to \pi, \\ x \in \mathbb{Q}}} 2^x$



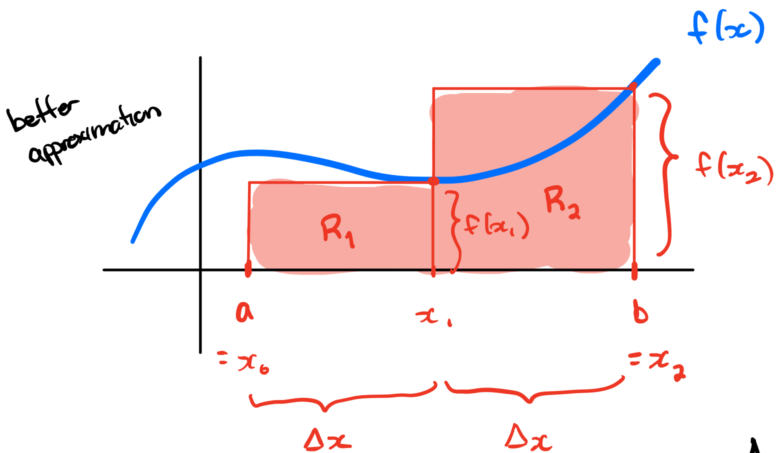
We will approximate using rectangles.



Using 1 rectangle:

choose height = $f(b)$

$$A \approx f(b)(b-a)$$



Using 2 rectangles:

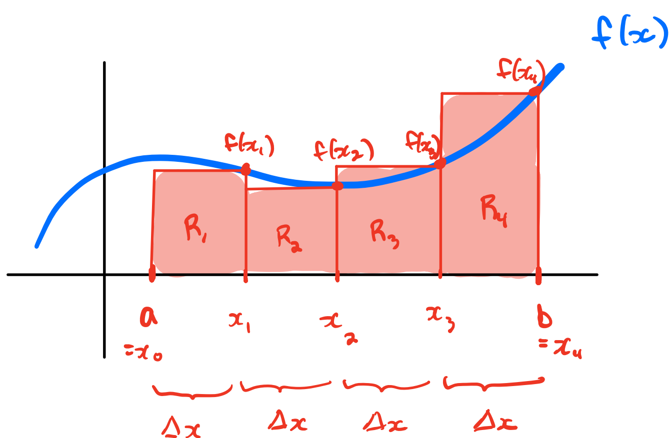
$$\Delta x = \frac{b-a}{2}$$

$$x_1 = a + \Delta x$$

On $[x_0, x_1]$, use height $f(x_1)$

$[x_1, x_2]$, use height $f(x_2)$

$$A \approx R_1 + R_2$$
$$= f(x_1)\Delta x + f(x_2)\Delta x$$



Using 4 rectangles:

$$\Delta x = \frac{b-a}{4}$$

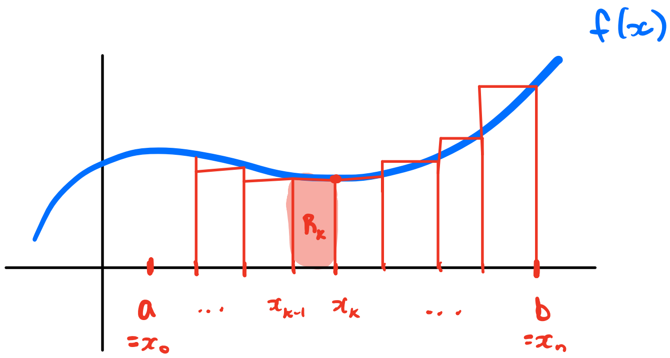
$$x_k = a + k\Delta x$$

We partition $[a,b]$ into 4 subintervals:

$$[x_{k-1}, x_k] \text{ for } k = 1,2,3,4$$

On each subinterval, approx. the area with R_k of height $f(x_k)$

$$A \approx R_1 + R_2 + R_3 + R_4 = \sum_{k=1}^{4} R_k = \sum_{k=1}^{4} f(x_k)\Delta x$$



In general, use n rectangles:

$$\Delta x = \frac{b-a}{n}$$

$$x_k = a + k\Delta x$$

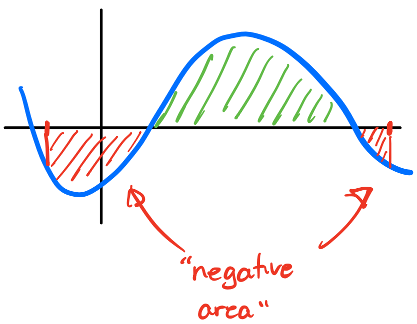
On $[x_{k-1}, x_k]$, use a rectangle of height $f(x_k)$.

$$A \approx \sum_{k=1}^{n} f(x_k)\Delta x$$ Right-hand Riemann Sum

As n increases, the approximation gets better.

$$\boxed{A = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\Delta x}$$ definition of area

This formula still works if f takes on negative values, but we get the signed area:



<u>Note</u>: Instead of using $f(x_k)$ as the height of R_k on $[x_{k-1}, x_k]$, we can use $x_k^* \in [x_{k-1}, x_k]$ and then

$$A = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x$$

i.e. the choice of which x_k^* to use does not affect the area.

$x_k^* = x_k$ Right Riemann Sum

$x_k^* = x_{k-1}$ Left Riemann Sum

$x_k^* = \frac{x_{k-1}+x_k}{2}$ Midpoint Rule

Terminology: The definite integral of $f(x)$ between $x=a$ and $x=b$ is given by

upper limit of integration

integrating variable

Note: x is a dummy variable

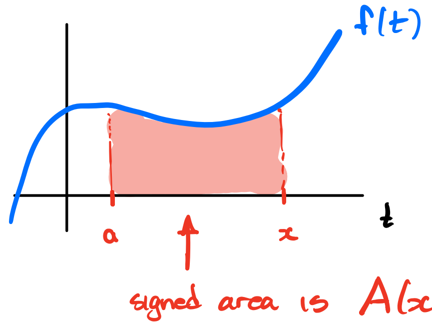
$$\int_a^b \underbrace{f(x)}_{\text{integrand}}\, dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\,\Delta x$$

lower limit of integration

integrand

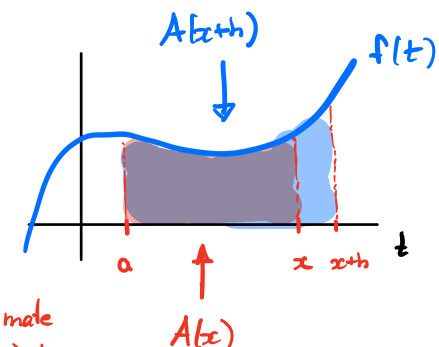
i.e. the definite integral of f between a and b is the signed area under f between a and b.

Consider $A(x) = \int_a^x f(t)\,dt$ ("area so far" function, accumulation function)



Q: What is $A'(x)$?

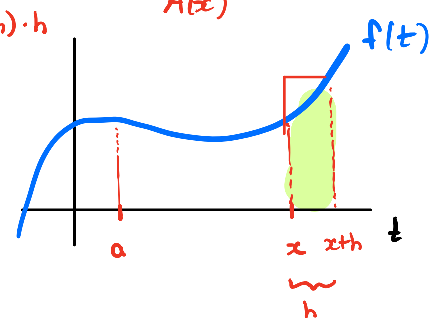
$$A'(x) = \lim_{h\to 0} \frac{A(x+h) - A(x)}{h}$$



$$= \lim_{h\to 0} \frac{\int_x^{x+h} f(t)\,dt}{h}$$

can approximate using $f(x+h)\cdot h$

$$= \lim_{h\to 0} \frac{f(x+h)\cdot \cancel{h}}{\cancel{h}}$$



$$= f(x)$$ if f is continuous at x

We have proved:

$\boxed{\underline{\text{Fundamental Theorem of Calculus I:}} \qquad \frac{d}{dx}\int_a^x f(t)\,dt = f(x)}$ (FTC I)

Challenge: Use Chain Rule to prove

$$\frac{d}{dx}\int_{g(x)}^{h(x)} f(t)\,dt = f(h(x))\cdot h'(x) - f(g(x))\cdot g'(x)$$

<u>Defin</u>: If $F'(x) = f(x)$, then we call $F(x)$ an antiderivative of $f(x)$.

eg $f(x) = x^2$ What are some antiderivatives of f?

$\frac{1}{3}x^3$, $\frac{1}{3}x^3+1$, $\frac{1}{3}x^3+2$, $\frac{1}{3}x^3+C$

Are there any others? <u>No</u>.

<u>Thm</u>: If F and G are antideriv. of f, then

$$F(x) = G(x) + C \quad \forall x$$

By FTC I, $A(x) = \int_a^x f(t)\,dt$ is an antideriv. of $f(x)$.

Suppose $F(x)$ is any antideriv. of f. Then $A(x) = F(x) + C$. (*)

$$\int_a^b f(x)dx = \int_a^b f(t)dt = A(b)$$

Note: $A(a) = 0$

$$= A(b) - A(a) = (F(b) + \cancel{C}) - (F(a) + \cancel{C}) \quad \text{by } (*)$$

$$= F(b) - F(a)$$

<u>Not'n</u>: $F(x)\Big|_a^b = F(b) - F(a)$

Fundamental Theorem of Calculus II

(FTC II)

If $F(x)$ is an antideriv. of $f(x)$, then

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b$$

eg $\displaystyle\int_0^2 x^2\,dx = \frac{1}{3}x^3\Big|_0^2$

$$= \frac{1}{3}(2^3 - 0^3)$$

$$= \frac{8}{3}$$