AP CALCULUS BC	YouTube Live Virtual Lessons	Mr. Bryan Passwater Mr. Anthony Record
Topic: All Units	**Free Response Question Stem Types** **Algebraic**	**Date:** April 29, 2020

Free Response Questions Stem Types: Algebraic

2020 FRQ Practice Problem BC1

BC1: Let g be the function defined by $g(x) = \dfrac{f(x)}{(2x)(2x-1)}$.

(**a**) If the slope of the line tangent to the graph of g at $x = \dfrac{1}{4}$ is -12, find the slope of the line tangent to the graph of f at $x = \dfrac{1}{4}$.

$$g'(x) = \frac{\left(4x^2 - 2x\right) f'(x) - (8x-2) f(x)}{\left(4x^2-2x\right)^2}$$

$$g'\left(\frac{1}{4}\right) = \frac{\left(4\left(\frac{1}{4}\right)^2 - 2\left(\frac{1}{4}\right)\right) f'\left(\frac{1}{4}\right) - \left(8\left(\frac{1}{4}\right)-2\right) f\left(\frac{1}{4}\right)}{\left(4\left(\frac{1}{4}\right)^2 - 2\left(\frac{1}{4}\right)\right)^2} = \frac{\left(\left(\frac{1}{4}\right)-\left(\frac{2}{4}\right)\right) f'\left(\frac{1}{4}\right) - (2-2) f\left(\frac{1}{4}\right)}{\left(\left(\frac{1}{4}\right)-\left(\frac{2}{4}\right)\right)^2}$$

$$= \frac{\left(-\frac{1}{4}\right) f'\left(\frac{1}{4}\right)}{\frac{1}{16}} = -4f'\left(\frac{1}{4}\right) = -12 \Rightarrow f'\left(\frac{1}{4}\right) = 3 \Rightarrow \text{ slope of the tangent line at } x = \frac{1}{4}$$

The problem has been restated.

BC1: Let g be the function defined by $g(x) = \dfrac{f(x)}{(2x)(2x-1)}$.

(**b**) When $f(x) = 1$, g can be written as $g(x) = \dfrac{1}{2x-1} - \dfrac{1}{2x}$. For $f(x) = 1$, determine

if $\sum_{n=1}^{\infty} a_n$ converges or diverges where $a_n = g(n)$.

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty}\left(\frac{1}{2n-1} - \frac{1}{2n}\right) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots \Rightarrow \text{alternating harmonic series } \Rightarrow \text{convergent}$$

(**c**) If the function g has a critical point at $x = 1$, find the x intercept the line tangent to $f(x)$ at $x = 1$.

$$g(x) = \frac{f(x)}{(2x)(2x-1)}$$

$$g'(1) = \frac{\left(4(1)^2 - 2(1)\right)f'(1) - \left(8(1) - 2\right)f(1)}{\left(4(1)^2 - 2(1)\right)^2} = \frac{(2)f'(1) - (6)f(1)}{4} = 0$$

$$(2)f'(1) - (6)f(1) = 0 \Rightarrow f'(1) = 3f(1)$$

$$T(x) = f(1) + f'(1)(x-1) \qquad x - \text{intercept} \Rightarrow T(x) = 0 = f(1) + f'(1)(x-1)$$

$$f(1) + f'(1)(x-1) = 0 \Rightarrow x - 1 = -\frac{f(1)}{f'(1)} \Rightarrow x = -\frac{f(1)}{f'(1)} + 1 = -\frac{f(1)}{3f(1)} + 1 = -\frac{1}{3} + 1 = \frac{2}{3}$$

The problem has been restated.

BC1: Let g be the function defined by $g(x) = \dfrac{f(x)}{(2x)(2x-1)}$.

(**d**) Consider the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ where $a_n = f(n)$ and $b_n = g(n)$. If $\lim_{n\to\infty}\left|\dfrac{b_{n+1}}{b_n}\right| = \dfrac{11}{7}$ use the ratio test to determine if the series $\sum_{n=1}^{\infty}\left(\dfrac{2}{3}\right)^n a_n$ converges or diverges.

Let $c_n = \left(\dfrac{2}{3}\right)^n a_n = \left(\dfrac{2}{3}\right)^n f(n) \qquad b_n = \dfrac{f(n)}{(2n)(2n-1)}$

$$\lim_{n\to\infty}\left|\frac{b_{n+1}}{b_n}\right| = \lim_{n\to\infty}\left|\frac{f(n+1)}{(2n+2)(2n+1)}\cdot\frac{(2n)(2n-1)}{f(n)}\right| = \lim_{n\to\infty}\left|\frac{f(n+1)}{f(n)}\right| = \frac{11}{7}$$

$$\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right| = \lim_{n\to\infty}\left|\frac{\left(\frac{2}{3}\right)^{n+1} f(n+1)}{\left(\frac{2}{3}\right)^n f(n)}\right| = \lim_{n\to\infty}\left|\frac{2}{3}\,\frac{f(n+1)}{f(n)}\right| = \left(\frac{2}{3}\right)\left(\frac{11}{7}\right) = \frac{22}{21}$$

$\sum_{n=1}^{\infty}\left(\dfrac{2}{3}\right)^n a_n$ diverges by the ratio test because $\lim_{n\to\infty}\left|\dfrac{c_{n+1}}{c_n}\right| = \dfrac{22}{21} > 1$

(**e**) If $\int g(x)dx = \dfrac{1}{2} ln|(2x)(2x-1)| + C$, find an expression for $f(x)$.

$$\int g(x)dx = \frac{1}{2}\left[\ln(2x) + \ln(2x-1)\right] + C \qquad \frac{d}{dx}\int g(x)dx = \frac{d}{dx}\left[\frac{1}{2}\left[\ln(2x) + \ln(2x-1)\right] + C\right]$$

$$g(x) = \frac{1}{2}\left[\frac{2}{2x} + \frac{2}{2x-1}\right] = \left[\frac{1}{2x} + \frac{1}{2x-1}\right] \qquad \frac{f(x)}{(2x)(2x-1)} = \left[\frac{1}{2x} + \frac{1}{2x-1}\right]$$

$$f(x) = \left[\frac{1}{2x} + \frac{1}{2x-1}\right]\left[(2x)(2x-1)\right] = (2x-1) + 2x = \boxed{4x-1}$$

Free Response Questions Stem Types: Algebraic
2020 FRQ Practice Problem BC2

BC2: Let f be the piecewise defined function defined by $f(x) = \begin{cases} ax+b, & x<0 \\ 4e^{-2x}, & x \geq 0 \end{cases}$ where a and b are constants.

(**a**) Find the values for a and b such that $f(x)$ is differentiable at $x = 0$.

Must have continuity $\Rightarrow \lim\limits_{x\to 0^-} f(x) = a(0)+b = b \qquad \lim\limits_{x\to 0^+} f(x) = 4e^{-2(0)} = 4 \Rightarrow \boxed{b=4}$

$\lim\limits_{x\to 0^-} f'(x) = a \qquad \lim\limits_{x\to 0^+} f'(x) = \lim\limits_{x\to 0^+}\left(4e^{-2x}(-2)\right) = -8 \Rightarrow \boxed{a=-8}$

(**b**) Let p be a function such that $f(x) = x^2 - \int_0^x p(t)dt$. Find any nonzero value(s) of x where p has a critical point.

$f'(x) = 2x - p(x) \Rightarrow p(x) = 2x - f'(x) = 2x - \left(-8e^{-2x}\right) = 2x + 8e^{-2x}$

$f''(x) = 2 - p'(x) \Rightarrow \quad p'(x) = 2 - f''(x) = 2 - \left(16e^{-2x}\right) = 2 - 16e^{-2x}$

critical values $= p'(x) = 0 \Rightarrow 2 - 16e^{-2x} = 0 \Rightarrow e^{-2x} = \frac{1}{8} \Rightarrow -2x = \ln\left(\frac{1}{8}\right) \Rightarrow x = -\frac{1}{2}\ln\left(\frac{1}{8}\right) = \frac{1}{2}\ln 8$

(**c**) Let $k(x) = f(f(x))$ where $a = b = 2$. Find $k'\left(-\frac{1}{2}\right)$.

$x = -\frac{1}{2} < 0 \Rightarrow f(x) = 2x+2 \qquad x = 1 \geq 0 \Rightarrow f(x) = 4e^{-2x} \Rightarrow f'(x) = -8e^{-2x}$

$k'(x) = f'(f(x))(f'(x)) \Rightarrow k'\left(-\frac{1}{2}\right) = f'\left(f\left(-\frac{1}{2}\right)\right)\left(f'\left(-\frac{1}{2}\right)\right) = f'(1)(2) = 2\left(-8e^{-2}\right) = -16e^{-2}$

(**d**) If $\lim\limits_{x\to -1} \frac{f(x)}{1-x^2} = 3$, find the values of a and b.

$\lim\limits_{x\to -1}(ax+b) = -a+b \qquad \lim\limits_{x\to -1}\left(1-x^2\right) = 0$

Because $\lim\limits_{x\to -1} \frac{f(x)}{1-x^2} = 3$, we know $b - a = 0$ and l'Hospital's Rule must have been used

since we have an indeterminant form $\frac{0}{0} \Rightarrow a = b$

$\lim\limits_{x\to -1} \frac{f(x)}{1-x^2} = \underbrace{\lim\limits_{x\to -1} \frac{a}{-2x}}_{\text{l'Hospital's Rule}} = \frac{a}{2} = 3 \Rightarrow \boxed{a = 6 = b}$

The problem has been restated.

BC2: Let f be the piecewise defined function defined by $f(x) = \begin{cases} ax+b, & x<0 \\ 4e^{-2x}, & x \geq 0 \end{cases}$ where a and b are constants.

(**e**) Let $h(x) = e^{f(x)}$. The 2nd degree Taylor polynomial for $h(x)$ centered at $x = -3$ is given by $P_2(x) = 1 - 2(x+3) + 2(x+3)^2$. Find a and b.

$h(-3) = e^{f(-3)} = 1 \Rightarrow e^{-3a+b} = 1 \Rightarrow -3a+b=0$

$h'(x) = e^{f(x)}(a) \Rightarrow h'(-3) = e^{f(-3)}(a) = e^{0}(a) = -2 \Rightarrow \boxed{a=-2}$

$-3(-2)+b=0 \Rightarrow (6)+b=0 \Rightarrow \boxed{b=-6}$

Free Response Questions Stem Types: Algebraic
2020 FRQ Practice Problem BC3

BC3: Let g be the function defined by $g(x) = \dfrac{f(x)}{(2x)(2x-1)}$.

(a) Let $h(x) = \begin{cases} g(x), & x < \dfrac{1}{2} \\ 3e^{2x-1}, & x \geq \dfrac{1}{2} \end{cases}$. If h is continuous at $x = \dfrac{1}{2}$, write an equation of the line tangent to $f(x)$ at $x = \dfrac{1}{2}$.

$$f(x) = (2x)(2x-1)g(x) = (4x^2 - 2x)g(x) \qquad f\left(\frac{1}{2}\right) = \left(4\left(\frac{1}{2}\right)^2 - 2\left(\frac{1}{2}\right)\right)g\left(\frac{1}{2}\right) = 0$$

$$f'(x) = (8x-2)g(x) + (4x^2 - 2x)g'(x) \qquad f'\left(\frac{1}{2}\right) = (2)g\left(\frac{1}{2}\right) + (0)g'\left(\frac{1}{2}\right) = (2)g\left(\frac{1}{2}\right)$$

$$h \text{ is continuous at } x = \frac{1}{2} \Rightarrow \lim_{x\to 1/2^-} g(x) = \lim_{x\to 1/2^+} 3e^{2x-1} = 3e^0 = 3 \quad f'\left(\frac{1}{2}\right) = (2)g\left(\frac{1}{2}\right) = (2)(3)$$

$$T(x) = f\left(\frac{1}{2}\right) + f'\left(\frac{1}{2}\right)\left(x - \frac{1}{2}\right) = (0) + 6\left(x - \frac{1}{2}\right) = 6\left(x - \frac{1}{2}\right)$$

(b) If $\lim\limits_{x\to 0} g(x) = -5$ and $f(x) = a\sin(\pi x) + b$, find a and b.

$$g(x) = \frac{a\sin(\pi x) + b}{(2x)(2x-1)} \qquad \lim_{x\to 0}\left[a\sin(\pi x) + b\right] = b \qquad \lim_{x\to 0}\left[(2x)(2x-1)\right] = 0$$

Since $\lim\limits_{x\to 0} g(x) = -5$ must be the result of using l'Hospital's Rule on the indeterminant form $\dfrac{0}{0}$

which means $\lim\limits_{x\to 0}\left[a\sin(\pi x) + b\right] = \boxed{b = 0}$.

$$\lim_{x\to 0} g(x) = \underbrace{\lim_{x\to 0} \frac{a\pi\cos(\pi x)}{8x-2}}_{\text{l'Hospital's Rule}} = \frac{a\pi\cos(\pi(0))}{8(0)-2} = \frac{a\pi}{-2} = -5 \Rightarrow \boxed{a = \frac{10}{\pi}}$$

The problem has been restated.

BC3: Let g be the function defined by $g(x) = \dfrac{f(x)}{(2x)(2x-1)}$.

(c) Find $\int_1^4 g(x)dx$ when $f(x) = 4x + 3$.

$$\frac{4x+3}{(2x)(2x-1)} = \frac{A}{2x} + \frac{B}{2x-1}$$

$$\int_1^4 g(x)dx = \int_1^4 \frac{4x+3}{(2x)(2x-1)}dx$$

$$4x+3 = A(2x-1) + B(2x)$$

$$x = 0 \Rightarrow 3 = A(-1) \Rightarrow A = -3$$

$$x = \frac{1}{2} \Rightarrow 5 = B$$

$$= \int_1^4 \left[\frac{-3}{2x} + \frac{5}{2x-1}\right]dx = \left[-\frac{3}{2}\ln|2x| + \frac{5}{2}\ln|2x-1|\right]_1^4 = \left[-\frac{3}{2}\ln|8| + \frac{5}{2}\ln|7|\right] - \left[-\frac{3}{2}\ln|2| + \frac{5}{2}\ln|1|\right]$$

$$= -\frac{3}{2}\ln|8| + \frac{5}{2}\ln|7| + \frac{3}{2}\ln|2|$$

Free Response Questions Stem Types: Algebraic
2020 FRQ Practice Problem BC4

BC4: Let f be the piecewise defined function defined by $f(x) = \begin{cases} ax+b, & x<0 \\ 4e^{-2x}, & x \geq 0 \end{cases}$ where a and b are constants.

(**a**) Let $g(x) = \dfrac{f(x)}{2x}$. Find $g'(1)$.

$$g'(x) = \frac{(2x)f'(x) - 2f(x)}{(2x)^2} \Rightarrow g'(1) = \frac{(2)f'(1) - 2f(1)}{(2)^2} = \frac{f'(1) - f(1)}{2}$$

$$= \frac{(-8e^{-2}) - (4e^{-2})}{2} = -6e^{-2}$$

(**b**) Let $a = 2$ and $b = 0$, find the average value of $f(x)$ over the interval $[-1, 1]$.

$$A = \frac{1}{2}\int_{-1}^{1} f(x)\,dx = \frac{1}{2}\left[\int_{-1}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx\right] = \frac{1}{2}\left[\int_{-1}^{0} 2x\,dx + \int_{0}^{1} 4e^{-2x}\,dx\right]$$

$$= \frac{1}{2}\left[\left[x^2\right]_{-1}^{0} + \left[-2e^{-2x}\right]_{0}^{1}\right] = \frac{1}{2}\left[\left[0 - (-1)^2\right] + \left[-2e^{-2} - (-2)\right]\right] = \frac{1}{2} - e^{-2}$$

(**c**) Let $a = b > 0$, find the values of a and b such that $\int_{-1}^{0} f(x)dx = \int_{0}^{\infty} f(x)dx$.

$$\int_{-1}^{0} f(x)\,dx = \int_{-1}^{0} (ax+b)\,dx = \left[\frac{a}{2}x^2 + ax\right]_{-1}^{0} = \left[\frac{a}{2}(0)^2 + (0)a\right] - \left[\frac{a}{2}(-1)^2 + (a)(-1)\right] = \frac{a}{2}$$

$$\lim_{b\to\infty}\int_{0}^{b} 4e^{-2x}\,dx = \lim_{b\to\infty}\left[-2e^{-2x}\right]_{0}^{b} = \lim_{b\to\infty}\left[(-2e^{-2b}) - (-2e^{-2(0)})\right] = \lim_{b\to\infty}\left[(-2e^{-2b}) + 2\right] = (0) + 2 = 2$$

$$\frac{a}{2} = 2 \Rightarrow a = b = 4$$

(**d**) Let $a_n = f(n)$. Find $\sum_{n=0}^{\infty} a_n$.

$$\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} (4e^{-2n}) = \sum_{n=0}^{\infty} 4\left(\frac{1}{e^2}\right)^n \Rightarrow \text{geometric series } r = \frac{1}{e^2}, a = 4 \Rightarrow \sum_{n=0}^{\infty} a_n = \frac{4}{1 - \frac{1}{e^2}} = \frac{4e^2}{e^2 - 1}$$