$\sqrt{2}$ is irrational

Suppose $\sqrt{2} \in \mathbb{Q}$. Then $\sqrt{2} = \dfrac{a}{b}$ for $a, b \in \mathbb{Z}, b \neq 0$, with $\gcd(a, b) = 1$. Then

$$\sqrt{2}b = a$$
$$2b^2 = a^2 \qquad\qquad \text{Then } 2 \mid a^2 \implies 2 \mid a \implies a = 2c, c \in \mathbb{Z}$$
$$2b^2 = (2c)^2 = 4c^2$$
$$b^2 = 2c^2 \qquad\qquad \text{Then } 2 \mid b^2 \implies 2 \mid b$$

But then $2 \mid \gcd(a, b)$, a contradiction to $\gcd(a, b) = 1$. Therefore, $\sqrt{2} \notin \mathbb{Q}$.