Mini-math Div 3/4: Friday, January 14, 2022 (20 minutes)

SOLUTIONS

1. Consider the continuous function whose derivative is $f'(x) = \dfrac{(x+4)^5(x-3)^2\left(x-\frac{1}{7}\right)}{(x-1)^{1/3}}$

   (a) (3 points) Find the interval(s) on which the original function f is increasing.

   Solution: The critical points are $-4, 1/7, 1, 3$.
   On each subinterval, the derivative has the following sign:

		-4		$1/7$		1		3	
$(x+4)^5$	$-$	|	$+$	|	$+$	|	$+$	|	$+$
$x-1/7$	$-$	|	$-$	|	$+$	|	$+$	|	$+$
$(x-1)^{1/3}$	$-$	|	$-$	|	$-$	|	$+$	|	$+$
$(x-3)^2$	$+$	|	$+$	|	$+$	|	$+$	|	$+$
$f'(x)$	$-$	|	$+$	|	$-$	|	$+$	|	$+$

   Then $f(x)$ is increasing on $(-4, 1/7)$, $(1, 3)$, or $(3, \infty)$. (Note: for the AP exam, you must state your justification such as "because $f' > 0$ on those intervals")

   (b) (2 points) Find and classify the local extrema of f.

   Solution: By part (a) and the First Derivative Test, $x = -4$ and $x = 1$ are local minima and $x = 1/7$ is a local maximum. (Note: for the AP exam, you must state your justification such as "f' changes from negative to positive at -4, so the First Derivative Test tells us that $x = -4$ is a local minimum")

2. (3 points) Find the global maximum and minimum of $f(x) = 2x^3 - 9x^2 - 10$ on $[1, 4]$.

Solution: The derivative is given by

$$f'(x) = 6x^2 - 18x = 6x(x-3)$$

which has critical points $x = 0, 3$, since f' exists everywhere (Note: you **must** check where f' DNE and make it clear you have considered it). Only 3 is in the domain of consideration. We compute

$$\begin{aligned} f(1) &= 2(1)^3 - 9(1)^2 - 10 = -17, \\ f(3) &= 2(3)^3 - 9(3)^2 - 10 = -37, \\ f(4) &= 2(4)^3 - 9(4)^2 - 10 = -26 \end{aligned}$$

so f has a global maximum at $x = 1$ (with value -17) and a global minimum at $x = 3$ (with value -37).

3. (3 points) Consider the function

$$f(x) = \frac{3}{5}x^5 + 4x^4 + 8x^3 + 12x + 10.$$

Find the interval(s) on which f is concave down.

Solution: Differentiating twice,

$$f'(x) = 3x^4 + 16x^3 + 24x^2 + 12,$$
$$f''(x) = 12x^3 + 48x^2 + 48x$$

We find the critical points: $f''(x)$ always exists, and

$$0 = 12x^3 + 48x^2 + 48x = 12x(x^2 + 4x + 4) = 12x(x+2)^2$$

so the critical points are $x = -2, 0$.

On each subinterval, the second derivative has the following sign:

		-2		0	
$(x+2)^2$	$+$	|	$+$	|	$+$
x	$-$	|	$-$	|	$+$
$f''(x)$	$-$	|	$-$	|	$+$

Then $f(x)$ is concave down on $(-\infty, -2)$ and $(-2, 0)$. (Note: for the AP exam, you must state your justification such as "because $f'' < 0$ on those intervals")

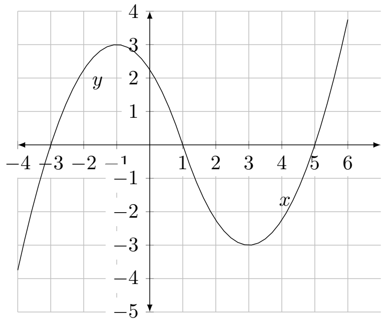
4. (2 points) Suppose there is a function $f(x)$ such that $f'(x) = 0$ if and only if $x = 0, 1$, and whose **second derivative** is given by $f''(x) = \dfrac{x^3 + x^2 - x}{(x+1)^2}$. What would the Second Derivative Test tell you about the critical points $x = 0$ and $x = 1$?

Solution: We calculate:

$$f''(0) = 0,$$
$$f''(1) = \frac{1}{2^2} > 0$$

so the Second Derivative Test is inconclusive for $x = 0$ and tells us that $x = 1$ is a local minimum.

5. (2 points) Assume f is a continuous function such that the following is a graph of f'. Find the points of inflection of f.



Solution: We want the concavity to change signs, so that f' has a local extremum. Therefore, there are points of inflection at $x = -1$ and $x = 3$.

6. (1 point) (AP) Which of the following functions does not satisfy the conditions of the Mean Value Theorem on the interval specified?

(A) $f(x) = \dfrac{1}{x}$ on $[1, 4]$

(B) $f(x) = \sqrt{x}$ on $[0, 2]$

(C) $f(x) = \sqrt[3]{x}$ on $[-1, 1]$

(D) $f(x) = x^2 - 1$ on $[-2, 2]$

Solution: (C) $\sqrt[3]{x}$ is not differentiable at $0 \in [-1, 1]$.

7. (1 point) (AP) Given the curve $y3^y = \sin x$, for what value of y, if any, does the derivative of y with respect to x not exist?

Solution: Differentiating,

$$\begin{aligned} 3^y \frac{dy}{dx} + y3^y \ln 3 \frac{dy}{dx} &= \cos x \\ \frac{dy}{dx} &= \frac{\cos x}{3^y(1 + y \ln 3)} \end{aligned}$$

Then $\frac{dy}{dx}$ is undefined at $y = -1/\ln 3$.