Mini-math Div 3/4: Friday, March 4, 2022 (12 minutes)

SOLUTIONS

1. (2 points) (AP)

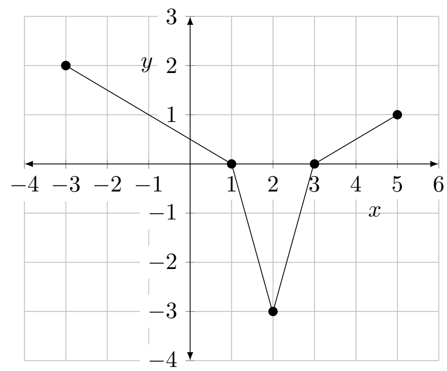
   The graph of the piecewise linear function f is shown in the figure to the right. What is the average value of f over $[-3, 5]$?

   A. -1

   B. $-1/8$

   C. $1/4$

   D. 2



Solution:

$$f_{avg} = \frac{\int_{-3}^{5} f\,dx}{5-(-3)} = \frac{4-3+1}{8} = \frac{1}{4}$$

C is correct.

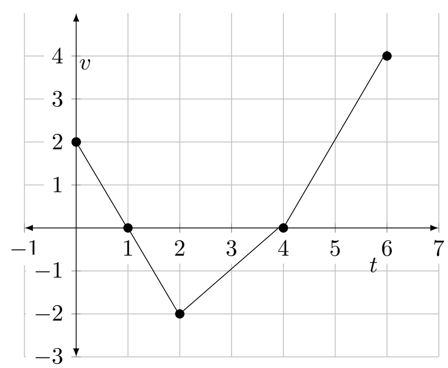
2. (2 points) The graph of the velocity of a function is the piecewise linear function shown in the figure to the right. The initial position of the particle at time $t = 0$ is $x = 1$. What is the total distance the particle travels from $t = 0$ to $t = 6$?

   A. 2

   B. 3

   C. 8

   D. 9



Solution:

$$\int_0^6 |v(t)|\,dt = 1 + 3 + 4 = 8$$

C is correct.

3. (2 points) The acceleration of a particle is modelled by $a(t) = 2t + 3$ for $t \geq 0$. At $t = 0$, the velocity of the particle is -2. What is the change in displacement of the particle from $t = 0$ to $t = 3$?

A. 9 B. 16 **C. 16.5** D. 22.5

Solution:

$$v(t) = \int a(t)\, dt = \int (2t + 3)\, dt = t^2 + 3t + C$$

Since $v(0) = -2$, we know $C = -2$. Then the change in displacement is

$$\Delta x = \int_0^3 v(t)\, dt = \int_0^3 \left(t^2 + 3t - 2\right)\, dt = \left(\frac{1}{3}t^3 + \frac{3}{2}t^2 - 2t\right)\Big|_0^3 = 9 + \frac{27}{2} - 6 = 16.5$$

C is correct.

4. (2 points) (AP) Suppose f is a differentiable function. Which of the following statements are true:

(I) The average value of the derivative of f over $[a, b]$ is the same as the average rate of change of f over $[a, b]$.

(II) There exists a $c \in [a, b]$ for which $f(c)$ equals the average value of f over $[a, b]$.

A. (I) only B. (II) only **C. Both (I) and (II)** D. Neither (I) nor (II)

Solution: By FTC II,

$$\frac{\int_a^b f'(x)\, dx}{b - a} = \frac{f(b) - f(a)}{b - a}$$

so (I) is true.

Since f is continuous and $[a, b]$ is a closed and bounded interval, the Extreme Value Theorem tells us that there are m and M for which $m \leq f(x) \leq M$ for all $x \in [a, b]$. Then

$$\frac{\int_a^b m\, dx}{b - a} \leq \frac{\int_a^b f(x)\, dx}{b - a} \leq \frac{\int_a^b M\, dx}{b - a}$$
$$m \leq f_{avg} \leq M$$

By the Intermediate Value Theorem, there exists $c \in [a, b]$ such that $f(c) = f_{avg}$, so (II) is true.

C is correct.

5. (2 points) Water is leaking out of a tub at a rate modelled by $r(t) = \dfrac{1}{t^2+1}\text{cm}^3/\text{min}$, where t is in minutes. If the initial volume of the tub is 160 000 cm^3, ehich of the following represents the volume of the tub at time t?

   A. $\dfrac{1}{t^2+1}$

   B. $160000 + \displaystyle\int_0^t r(t)\,dt$

   C. $160000 - \displaystyle\int_0^t r(t)\,dt$

   D. $160000 - \dfrac{1}{t^2+1}$

Solution: By FTC II,

$$-\int_0^t r(t)\,dt = V(t) - V(0)$$

$$V(t) = 160000 - \int_0^t r(t)\,dt$$

C is correct.

6. (2 points) Find the area of the bounded region in the first quadrant below both $y = x^2$ and $y = 2 - x$ and above the x-axis.

   A. 2/3 **B. 5/6** C. 7/6 D. 3

Solution: Integrating with respect to y,

$$A = \int_0^1 [(2-y) - \sqrt{y}]\,dy = \left(2y - \frac{y^2}{2} - \frac{2}{3}y^{3/2}\right)\Big|_0^1 = 2 - \frac{1}{2} - \frac{2}{3} = \frac{5}{6}$$

OR Integrating with respect to x (with 2 regions),

$$A = \int_0^1 x^2\,dx + \int_1^2 (2-x)\,dx = \frac{x^3}{3}\Big|_0^1 + \left(2x - \frac{x^2}{2}\right)\Big|_1^2 = \frac{1}{3} + 4 - 2 - 2 + \frac{1}{2} = \frac{5}{6}$$

B is correct.