MAT110 Discrete Mathematics
Assignment-I

Total Marks: 20 Marks

Due Date: 14^{th} May 2024

Instructions:
1. The students with even numbered student identification number will solve even numbered questions.
2. The students with odd numbered student identification number will solve odd numbered questions.
3. Each student will have to solve 15 questions.

Question 1
Translate each of these quantifications into english and determine its truth value.

1. $\exists x \in \mathbb{R}(x^3 = 1)$.
2. $\forall x \in \mathbb{Z}(x^2 = \mathbb{Z})$.

Question 2
Find the truth set of each of these predicates where the domain is the set of integers.

1. $P(x) : x^2 < 3$.
2. $R(x) : 2x + 1 = 0$.

Question 3
Prove the De Morgan Law by showing that if A and B are sets, then $\overline{A \cup B} = \bar{A} \cap \bar{B}$.

1. by showing each side is a subset of the other side.
2. uisng a membership table.

Question 4
Let A and B be sets. Show that

1. $(A \cap B) \subseteq A$.
2. $A \cap (B - A) = \emptyset$.
3. $A \cup (B - A) = A \cup B$.

Question 5
Draw the Venn diagrams for each of these combinations of the sets A, B, and C.

1. $A \cap (B - C)$.
2. $(A \cap B) \cup (A \cap C)$.
3. $(A \cap \bar{B})) \cup (A \cap \bar{C})$.

Question 6
Show that $A \oplus B = (A \cup B) - (A \cap B)$.

Question 7
Find the domain and range of these functions.

1. the function that assigns to each pair of positive integers the maximum of these two integers.
2. the function that assigns to each positive integer the number of the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 that do not appear as decimal digits of the integer.
3. the function that assigns to a bit string the number of times the block 11 appears.
4. the function that assigns to a bit string the numerical position of the first 1 in the string and that assigns the value 0 to a bit string consisting of all 0s.

Question 8
Give an explicit formula for a function from the set of integers to the set of positive integers that is

1. one-to-one, but not onto.
2. onto, but not one-to-one.
3. one-to-one and onto.
4. neither one-to-one nor onto.

Question 9
Determine whether each of these functions is a bijection from $\mathbb{R}$ to $\mathbb{R}$.

1. $f(x) = 2x + 1$.
2. $f(x) = x^2 + 1$.
3. $f(x) = \frac{(x^2+1)}{(x^2+2)}$.

Question 10
Suppose that g is a function from A to B and f is a function from B to C.

1. Show that if both f and g are one-to-one functions, then $f \circ g$ is also one-to-one.
2. Show that if both f and g are onto functions, then $f \circ g$ is also onto.

Question 11
Draw graphs of each of these functions.

1. $f(x) = \lfloor x + \frac{1}{2} \rfloor$.
2. $f(x) = \lfloor 2x + 1 \rfloor$.
3. $f(x) = \lceil \frac{x}{3} \rceil$.
4. $f(x) = \lceil \frac{1}{x} \rceil$.
5. $f(x) = \lfloor 2x \rfloor \lceil \frac{x}{2} \rceil$.

Question 12
Show that if x is a real number, then $\lceil x \rceil - \lfloor x \rfloor = 1$ if x is not an integer and $\lceil x \rceil - \lfloor x \rfloor = 0$ if x is an integer.

Question 13
Prove that if n is an integer, then$\lfloor \frac{n}{2} \rfloor = \frac{n}{2}$ if n is even and $\frac{(n-1)}{2}$ if n is odd.

Question 14 Show that if A and B are sets, A is uncountable, and $A \subseteq B$, then B is uncountable.

Quesiton 15
Show that the union of a countable number of countable sets is countable.

Question 16
For each of these relations on the set $\{1, 2, 3, 4\}$, decide whether it is reflexive, whether it is symmetric, whether it is antisymmetric, and whether it is transitive.

1. $\{(2, 2), (2, 3), (2, 4), (3, 2), (3, 3), (3, 4)\}$
2. $\{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (4, 4)\}$
3. $\{(2, 4), (4, 2)\}$
4. $\{(1, 2), (2, 3), (3, 4)\}$
5. $\{(1, 1), (2, 2), (3, 3), (4, 4)\}$
6. $\{(1, 3), (1, 4), (2, 3), (2, 4), (3, 1), (3, 4)\}$

Question 17
Determine whether the relation $\mathbb{R}$ on the set of all real numbers is reflexive, symmetric, antisymmetric, and/or transitive, where $(x, y) \in \mathbb{R}$ if and only if

1. $x + y = 0$.
2. $x = \pm y$.
3. $x - y$ is a rational number.
4. $x = 2y$.
5. $xy \geq 0$.
6. $xy = 0$.
7. $x = 1$.
8. $x = 1$ or $y = 1$.

Question 18
Let $R_1 = \{(1,2),(2,3),(3,4)\}$ and $R_2 = \{(1,1),(1,2),(2,1),(2,2),(2,3),(3,1),(3,2),(3,3),(3,4)\}$ be relations from $\{1,2,3\}$ to $\{1,2,3,4\}$. Find

1. $R_1 \cup R_2$
2. $R_1 \cap R_2$
3. $R_1 - R_2$
4. $R_2 - R_1$

Question 19
Let R be the relation $\{(1,2),(1,3),(2,3),(2,4),(3,1)\}$, and let S be the relation $\{(2,1),(3,1),(3,2),(4,2)\}$. Find $S \circ R$.

Question 20
Find the relations S_i^2 for $i = 1,2,3,4,5,6$ where

1. $S_1 = \{(a,b) \in \mathbb{Z}^2 | a > b\}$, the greater than relation.
2. $S_2 = \{(a,b) \in \mathbb{Z}^2 | a \geq b\}$, the greater than or equal to relation.
3. $S_3 = \{(a,b) \in \mathbb{Z}^2 | a < b\}$, the less than relation.
4. $S_4 = \{(a,b) \in \mathbb{Z}^2 | a \leq b\}$, the less than or equal to relation.
5. $S_5 = \{(a,b) \in \mathbb{Z}^2 | a = b\}$, the equal to relation.
6. $S_6 = \{(a,b) \in \mathbb{Z}^2 | a \neq b\}$, the unequal to relation.

Question 21
Determine whether each of these sets is finite, countably infinite, or uncountable. For those that are countably infinite, exhibit a one-to-one correspondence between the set of positive integers and that set.

1. the negative integers.
2. the odd negative integers.
3. integers divisible by 5 but not by 7.
4. the real numbers between 0 and $\frac{1}{2}$.
5. the positive integers less than 1,000,000,000.
6. the integers that are multiples of 7.
7. all bit strings not containing the bit 0.
8. all positive rational numbers that cannot be written with denominators less than 4.

Question 22

Determine whether each of these sets is finite, countably infinite, or uncountable. For those that are countably infinite, exhibit a one-to-one correspondence between the set of positive integers and that set.

1. the integers greater than 10.
2. the integers with absolute value less than 1,000,000.
3. the integers less than 100.
4. integers not divisible by 3.
5. integers divisible by 5 but not by 7.
6. the real numbers with decimal representations consisting of all 1s.
7. the real numbers not containing 0 in their decimal representation.
8. the real numbers containing only a finite number of 1s in their decimal representation.

Question 23

Suppose that Hilbert's Grand Hotel is fully occupied on the day the hotel expands to a second building which also contains a countably infinite number of rooms. Show that the current guests can be spread out to fill every room of the two buildings of the hotel.

Question 24

Show that a countably infinite number of guests arriving at Hilbert's fully occupied Grand Hotel can be given rooms without evicting any current guest.

Question 25

Use mathematical induction to prove summation formulae. Be sure to identify where you use the inductive hypothesis.

Let $P(n)$ be the statement that $1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$ for the positive integer n.

1. What is the statement $P(1)$?
2. Show that $P(1)$ is true, completing the basis step of a proof that $P(n)$ is true for all positive integers n.
3. What is the inductive hypothesis of a proof that $P(n)$ is true for all positive integers n?
4. What do you need to prove in the inductive step of a proof that $P(n)$ is true for all positive integers n?
5. Complete the inductive step of a proof that $P(n)$ is true for all positive integers n, identifying where you use the inductive hypothesis.
6. Explain why these steps show that this formula is true whenever n is a positive integer.

Question 26

Use mathematical induction to prove summation formulae. Be sure to identify where you use the inductive hypothesis.
Let $P(n)$ be the statement that $1^3 + 2^3 + \cdots + n^3 = (\frac{n(n+1)}{2})^2$ for the positive integer n.

1. What is the statement $P(1)$?
2. Show that $P(1)$ is true, completing the basis step of a proof that $P(n)$ is true for all positive integers n.
3. What is the inductive hypothesis of a proof that $P(n)$ is true for all positive integers n?
4. What do you need to prove in the inductive step of a proof that $P(n)$ is true for all positive integers n?
5. Complete the inductive step of a proof that $P(n)$ is true for all positive integers n, identifying where you use the inductive hypothesis.
6. Explain why these steps show that this formula is true whenever n is a positive integer.

Question 27

Use mathematical induction to prove summation formulae. Be sure to identify where you use the inductive hypothesis.
Prove that $1^2 + 3^2 + 5^2 + \cdots + (2n+1)^2 = \frac{(n+1)(2n+1)(2n+3)}{3}$ whenever n is a nonnegative integer.

Question 28

Use mathematical induction to prove summation formulae. Be sure to identify where you use the inductive hypothesis.

Prove that $1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! = (n+1)! - 1$ whenever n is a positive integer.

Question 29

Use mathematical induction to prove divisibility facts..
Prove that 2 divides $n^2 + n$ whenever n is a positive integer.

Question 30

Use mathematical induction to prove divisibility facts..
Prove that 3 divides $n^3 + 2n$ whenever n is a positive integer.