Solving Linear Equations

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Problem Statement

Solve the following pair of linear equations:

$$\frac{3x}{2} - \frac{5y}{3} = -2 \tag{1}$$

$$\frac{x}{3} + \frac{y}{2} = \frac{13}{6} \tag{2}$$

Matrix Representation

Let

$$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} \tag{3}$$

Expressing the system in matrix form:

$$\begin{pmatrix} \frac{3}{2} & \frac{-5}{3} \\ \frac{1}{3} & \frac{1}{2} \end{pmatrix} \mathbf{x} = \begin{pmatrix} -2 \\ \frac{13}{6} \end{pmatrix} \tag{4}$$

$$\begin{pmatrix} 9 & -10 \\ 2 & 3 \end{pmatrix} \mathbf{x} = \begin{pmatrix} -12 \\ 13 \end{pmatrix} \tag{5}$$

$$A\mathbf{x} = \mathbf{b} \tag{6}$$

LU Decomposition

Any non-singular matrix A can be expressed as a product of a lower triangular matrix L and an upper triangular matrix U, such that

$$A = LU \tag{7}$$

$$\implies LU\mathbf{x} = \mathbf{b} \tag{8}$$

U is determined by row reducing A using a pivot:

$$\begin{pmatrix} 9 & -10 \\ 2 & 3 \end{pmatrix} \xrightarrow{R_2 \to R_2 - \frac{2}{9} R_1} \begin{pmatrix} 9 & -10 \\ 0 & \frac{47}{9} \end{pmatrix} \tag{9}$$

Thus,

$$U = \begin{pmatrix} 9 & -10 \\ 0 & \frac{47}{9} \end{pmatrix} \tag{10}$$

LU Decomposition

Let

$$L = \begin{pmatrix} 1 & 0 \\ l & 1 \end{pmatrix} \tag{11}$$

l is the multiplier used to zero out a_{21} in A:

$$L = \begin{pmatrix} 1 & 0 \\ \frac{2}{9} & 1 \end{pmatrix} \tag{12}$$

Two-Step Solution Process

We can get the solution to our problem by the two-step process:

$$Ly = \mathbf{b} \quad (13)$$

$$U\mathbf{x} = \mathbf{y} \quad (14)$$

Forward Substitution

To solve $L\mathbf{y} = \mathbf{b}$, use forward substitution:

$$\begin{pmatrix} 1 & 0 \\ \frac{2}{9} & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} -12 \\ 13 \end{pmatrix} \tag{15}$$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} -12 \\ \frac{47}{3} \end{pmatrix} \tag{16}$$

Back Substitution

To solve $U\mathbf{x} = \mathbf{y}$, use back substitution:

$$\begin{pmatrix} 9 & -10 \\ 0 & \frac{47}{9} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -12 \\ \frac{47}{3} \end{pmatrix} \tag{17}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \tag{18}$$

Update Equations

This LU decomposition could also be computationally found using Doolittle's algorithm. The update equations are given by:

$$U_{ij} = \begin{cases} A_{ij} & i = 0 \\ A_{ij} - \sum_{k=0}^{i-1} L_{ik} U_{kj} & i > 0 \end{cases} \tag{19}$$

$$L_{ij} = \begin{cases} \frac{A_{ij}}{U_{jj}} & j = 0, U_{jj} \neq 0 \\ \frac{A_{ij} - \sum_{k=0}^{j-1} L_{ik} U_{kj}}{U_{jj}} & j > 0 \end{cases} \tag{20}$$

Plot

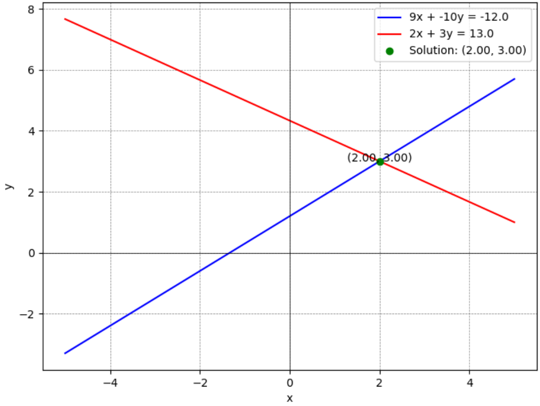

Figure: Graphical solution of the linear equations