12.6.5.2.3

EE24BTECH11019 - Dwarak A

Question:

Solve the following pair of linear equations,

$$\frac{3x}{2} - \frac{5y}{3} = -2 \tag{0.1}$$

$$\frac{x}{3} + \frac{y}{2} = \frac{13}{6} \tag{0.2}$$

Solution:

Let

$$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} \tag{0.3}$$

Expressing the system in matrix form,

$$\begin{pmatrix} \frac{3}{2} & \frac{-5}{3} \\ \frac{1}{3} & \frac{1}{2} \end{pmatrix} \mathbf{x} = \begin{pmatrix} -2 \\ \frac{13}{6} \end{pmatrix} \tag{0.4}$$

$$\begin{pmatrix} 9 & -10 \\ 2 & 3 \end{pmatrix} \mathbf{x} = \begin{pmatrix} -12 \\ 13 \end{pmatrix} \tag{0.5}$$

$$A\mathbf{x} = \mathbf{b} \tag{0.6}$$

Any non-singular matrix A can be expressed as a product of a lower triangular matrix L and an upper triangular matrix U, such that

$$A = LU \tag{0.7}$$

$$\implies LU\mathbf{x} = \mathbf{b} \tag{0.8}$$

U is determined by row reducing A using a pivot,

$$\begin{pmatrix} 9 & -10 \\ 2 & 3 \end{pmatrix} \xrightarrow{R_2 \to R_2 - \frac{2}{9}R_1} \begin{pmatrix} 9 & -10 \\ 0 & \frac{47}{9} \end{pmatrix} \tag{0.9}$$

Thus

$$U = \begin{pmatrix} 9 & -10 \\ 0 & \frac{47}{9} \end{pmatrix} \tag{0.10}$$

Let

$$L = \begin{pmatrix} 1 & 0 \\ l & 1 \end{pmatrix} \tag{0.11}$$

l is the multiplier used to zero out a_{21} in A.

$$L = \begin{pmatrix} 1 & 0 \\ \frac{2}{9} & 1 \end{pmatrix} \tag{0.12}$$

This LU decomposition could also be computationally found using Doolittle's algorithm. The update equation is given by,

$$U_{ij} = \begin{cases} A_{ij} & i = 0 \\ A_{ij} - \sum_{k=0}^{i-1} L_{ik} U_{kj} & i > 0 \end{cases} \tag{0.13}$$

$$L_{ij} = \begin{cases} \frac{A_{ij}}{U_{jj}} & j = 0, U_{jj} \neq 0 \\ \frac{A_{ij} - \sum_{k=0}^{j-1} L_{ik} U_{kj}}{U_{jj}} & j > 0 \end{cases} \tag{0.14}$$

$$\tag{0.15}$$

Now,

$$A = \begin{pmatrix} 9 & -10 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{2}{9} & 1 \end{pmatrix} \begin{pmatrix} 9 & -10 \\ 0 & \frac{47}{9} \end{pmatrix} \tag{0.16}$$

Now we can get the solution to our problem by the two step process,

$$L\mathbf{y} = \mathbf{b} \tag{0.17}$$

$$U\mathbf{x} = \mathbf{y} \tag{0.18}$$

Using forward substitution to solve the first equation,

$$\begin{pmatrix} 1 & 0 \\ \frac{2}{9} & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} -12 \\ 13 \end{pmatrix} \tag{0.19}$$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} -12 \\ \frac{47}{3} \end{pmatrix} \tag{0.20}$$

Now using back-substitution for the second equation,

$$\begin{pmatrix} 9 & -10 \\ 0 & \frac{47}{9} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -12 \\ \frac{47}{3} \end{pmatrix} \tag{0.21}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \tag{0.22}$$

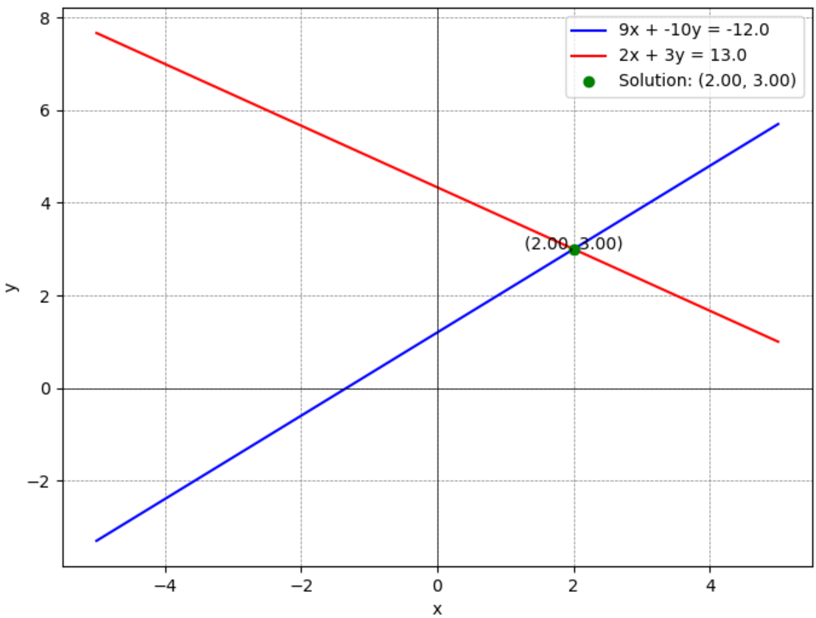

Fig. 0.1: Plot of local maximum and minimum