10.4.1.1.2

EE24BTECH11019 - Dwarak A

Question:

Find the roots of the quadratic equation:

$$x^2 - 2x = (-2)(3 - x) \tag{0.1}$$

Solution:

Rearranging terms,

$$x^2 - 2x = 2x - 6 \tag{0.2}$$

$$x^2 - 4x + 6 = 0 \tag{0.3}$$

Theoretical solution (Quadratic formula):

The roots are,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{0.4}$$

$$= \frac{4 \pm \sqrt{16 - 24}}{2} \tag{0.5}$$

$$= 2 \pm \sqrt{2}i \tag{0.6}$$

Computational solution:

(1) Eigenvalues of Companion Matrix:

The roots of a polynomial equation $x^n + b_{n-1}x^{n-1} + \cdots + b_2x^2 + b_1x + b_0 = 0$ is given by finding eigenvalues of the companion matrix (C).

$$C = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -b_0 & -b_1 & -b_2 & \cdots & -b_{n-1} \end{pmatrix} \tag{0.7}$$

Here $b_0 = 6$, $b_1 = -4$

$$C = \begin{pmatrix} 0 & 1 \\ -6 & 4 \end{pmatrix} \tag{0.8}$$

We find the eigenvalues using the QR algorithm. The basic principle behind this algorithm is a similarity transform,

$$A' = X^{-1}AX \tag{0.9}$$

which does not alter the eigenvalues of the matrix A.
We use this to get the Schur Decomposition,

$$A = Q^{-1}UQ = Q^{*}UQ \tag{0.10}$$

where Q is a unitary matrix $\left(Q^{-1} = Q^{*}\right)$ and U is an upper triangular matrix whose diagonal entries are the eigenvalues of A.
To efficiently get the Schur Decomposition, we first householder reflections to reduce it to an upper hessenberg form.
A householder reflector matrix is of the form,

$$P = I - 2\mathbf{u}\mathbf{u}^{*} \tag{0.11}$$

Householder reflectors transforms any vector $\mathbf{x}$ to a multiple of $\mathbf{e_1}$,

$$P\mathbf{x} = \mathbf{x} - 2\mathbf{u}\left(\mathbf{u}^{*}\mathbf{x}\right) = \alpha\mathbf{e_1} \tag{0.12}$$

P is unitary, which implies that,

$$\|P\mathbf{x}\| = \|\mathbf{x}\| \tag{0.13}$$

$$\implies \alpha = \rho\|\mathbf{x}\| \tag{0.14}$$

$$\tag{0.15}$$

As $\mathbf{u}$ is unit norm,

$$\mathbf{u} = \frac{\mathbf{x} - \rho\|\mathbf{x}\|\mathbf{e_1}}{\|\mathbf{x} - \rho\|\mathbf{x}\|\mathbf{e_1}\|} = \frac{1}{\|\mathbf{x} - \rho\|\mathbf{x}\|\mathbf{e_1}\|}\begin{pmatrix} x_1 - \rho\|\mathbf{x}\| \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \tag{0.16}$$

Selection of ρ is flexible as long as $|\rho| = 1$. To ease out the process, we take $\rho = \frac{x_1}{|x_1|}$, $x_1 \neq 0$. If $x_1 = 0$, we take $\rho = 1$.
Householder reflector matrix (P_i) is given by,

$$P_i = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \end{bmatrix} = \begin{bmatrix} 1 & \mathbf{0}^{*} \\ \mathbf{0} & I_{n-i} - 2\mathbf{u_i}\mathbf{u_i}^{*} \end{bmatrix} \tag{0.17}$$

$$\begin{bmatrix} \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \end{bmatrix} \xrightarrow{P_1} \begin{bmatrix} \times & \times & \times & \times \\ \times & \times & \times & \times \\ 0 & \times & \times & \times \\ 0 & \times & \times & \times \end{bmatrix} \xrightarrow{P_2} \begin{bmatrix} \times & \times & \times & \times \\ \times & \times & \times & \times \\ 0 & \times & \times & \times \\ 0 & 0 & \times & \times \end{bmatrix} \tag{0.18}$$

Next step is to do Given's rotation to get the QR Decomposition.

The Givens rotation matrix $G(i, j, c, s)$ is defined by

$$G(i, j, c, s) = \begin{bmatrix} 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & c & \cdots & s & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & -\overline{s} & \cdots & \overline{c} & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 \end{bmatrix} \tag{0.19}$$

where $|c|^2 + |s|^2 = 1$, and G is a unitary matrix.
Say we take a vector $\mathbf{x}$, and $\mathbf{y} = G(i, j, c, s)\,\mathbf{x}$, then

$$y_k = \begin{cases} cx_i + sx_j, & k = i \\ -\overline{s}x_i + \overline{c}x_j, & k = j \\ x_k, & k \neq i, j \end{cases} \tag{0.20}$$

For y_j to be zero, we set

$$c = \frac{\overline{x_i}}{\sqrt{|x_i|^2 + \left|x_j\right|^2}} = c_{ij} \tag{0.21}$$

$$s = \frac{\overline{x_j}}{\sqrt{|x_i|^2 + \left|x_j\right|^2}} = s_{ij} \tag{0.22}$$

Using this Givens rotation matrix, we zero out elements of subdiagonal in the hessenberg matrix H.

$$H = \begin{bmatrix} \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \\ 0 & 0 & \times & \times & \times \\ 0 & 0 & 0 & \times & \times \end{bmatrix} \xrightarrow{G(1,2,c_{12},s_{12})} \begin{bmatrix} \times & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \\ 0 & 0 & \times & \times & \times \\ 0 & 0 & 0 & \times & \times \end{bmatrix}$$
$$\xrightarrow{G(2,3,c_{23},s_{23})} \begin{bmatrix} \times & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \\ 0 & 0 & \times & \times & \times \\ 0 & 0 & \times & \times & \times \\ 0 & 0 & 0 & \times & \times \end{bmatrix} \xrightarrow{G(3,4,c_{34},s_{34})} \begin{bmatrix} \times & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \\ 0 & 0 & \times & \times & \times \\ 0 & 0 & 0 & \times & \times \\ 0 & 0 & 0 & \times & \times \end{bmatrix}$$
$$\xrightarrow{G(4,5,c_{45},s_{45})} \begin{bmatrix} \times & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \\ 0 & 0 & \times & \times & \times \\ 0 & 0 & 0 & \times & \times \\ 0 & 0 & 0 & 0 & \times \end{bmatrix} = R \tag{0.23}$$

where R is upper triangular. For the given companion matrix,

$$c_{11} = \frac{0}{\sqrt{0^2+1^2}} = 0 \tag{0.24}$$

$$s_{11} = \frac{1}{\sqrt{0^2+1^2}} = 1 \tag{0.25}$$

Let $G_k = G\left(k, k+1, c_{k,k+1}, s_{k,k+1}\right)$, then we deduce that

$$G_4G_3G_2G_1H = R \tag{0.26}$$

$$H = G_1^*G_2^*G_3^*G_4^*R \tag{0.27}$$

$$H = QR, \text{ where } Q = G_1^*G_2^*G_3^*G_4^* \tag{0.28}$$

Using this QR algorithm, we get the following update equation,

$$A_k = Q_kR_k \tag{0.29}$$

$$A_{k+1} = R_kQ_k \tag{0.30}$$

$$= \left(G_n \dots G_2G_1\right) A_k \left(G_1^*G_2^* \dots G_n^*\right) \tag{0.31}$$

Running the eigenvalue code for our companion matrix we get,

$$x_1 = 2.0 + 1.4142135623730971j \tag{0.32}$$

$$x_2 = 2.0 + -1.4142135623730971j \tag{0.33}$$

(2) Newton-Raphson iterative method:

$$f(x) = x^2 - 4x + 6 \tag{0.34}$$

$$f'(x) = 2x - 4 \tag{0.35}$$

Difference equation,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{0.36}$$

$$x_{n+1} = x_n - \frac{x_n^2 - 4x_n + 6}{2x_n - 4} \tag{0.37}$$

$$x_{n+1} = \frac{x_n}{2} - 1 + \frac{1}{x_n - 2} \tag{0.38}$$

Picking two initial guesses,

$$x_0 = 1 + i \text{ converges to } 2.0 + 1.4142135623730954i \tag{0.39}$$

$$x_0 = -1 - i \text{ converges to } 2.0000000000000733 + -1.4142135623729934i \tag{0.40}$$