

Solving Linear Equations

Dwarak A - EE24BTECH11019

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Problem Statement

Solve the following pair of linear equations:

$$\frac{3x}{2} - \frac{5y}{3} = -2 \quad (1)$$

$$\frac{x}{3} + \frac{y}{2} = \frac{13}{6} \quad (2)$$

Matrix Representation

Let

$$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} \quad (3)$$

Expressing the system in matrix form:

$$\begin{pmatrix} \frac{3}{2} & \frac{-5}{3} \\ \frac{1}{3} & \frac{1}{2} \end{pmatrix} \mathbf{x} = \begin{pmatrix} -2 \\ \frac{13}{6} \end{pmatrix} \quad (4)$$

$$\begin{pmatrix} 9 & -10 \\ 2 & 3 \end{pmatrix} \mathbf{x} = \begin{pmatrix} -12 \\ 13 \end{pmatrix} \quad (5)$$

$$A\mathbf{x} = \mathbf{b} \quad (6)$$

LU Decomposition

Any non-singular matrix A can be expressed as a product of a lower triangular matrix L and an upper triangular matrix U , such that

$$A = LU \quad (7)$$

$$\implies LU\mathbf{x} = \mathbf{b} \quad (8)$$

U is determined by row reducing A using a pivot:

$$\begin{pmatrix} 9 & -10 \\ 2 & 3 \end{pmatrix} \xrightarrow{R_2 \rightarrow R_2 - \frac{2}{9}R_1} \begin{pmatrix} 9 & -10 \\ 0 & \frac{47}{9} \end{pmatrix} \quad (9)$$

Thus,

$$U = \begin{pmatrix} 9 & -10 \\ 0 & \frac{47}{9} \end{pmatrix} \quad (10)$$

LU Decomposition

Let

$$L = \begin{pmatrix} 1 & 0 \\ l & 1 \end{pmatrix} \quad (11)$$

l is the multiplier used to zero out a_{21} in A :

$$L = \begin{pmatrix} 1 & 0 \\ \frac{2}{9} & 1 \end{pmatrix} \quad (12)$$

Update Equations

This LU decomposition could also be computationally found using Doolittle's algorithm. The update equations are given by:

$$U_{ij} = \begin{cases} A_{ij} & i = 0 \\ A_{ij} - \sum_{k=0}^{i-1} L_{ik} U_{kj} & i > 0 \end{cases} \quad (13)$$

$$L_{ij} = \begin{cases} \frac{A_{ij}}{U_{jj}} & j = 0, U_{jj} \neq 0 \\ \frac{A_{ij} - \sum_{k=0}^{j-1} L_{ik} U_{kj}}{U_{jj}} & j > 0 \end{cases} \quad (14)$$

Two-Step Solution Process

We can get the solution to our problem by the two-step process:

$$L\mathbf{y} = \mathbf{b} \quad (15)$$

$$U\mathbf{x} = \mathbf{y} \quad (16)$$

Forward Substitution

To solve $L\mathbf{y} = \mathbf{b}$, use forward substitution:

$$\begin{pmatrix} 1 & 0 \\ \frac{2}{9} & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} -12 \\ 13 \end{pmatrix} \quad (17)$$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} -12 \\ \frac{47}{3} \end{pmatrix} \quad (18)$$

Back Substitution

To solve $U\mathbf{x} = \mathbf{y}$, use back substitution:

$$\begin{pmatrix} 9 & -10 \\ 0 & \frac{47}{9} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -12 \\ \frac{47}{3} \end{pmatrix} \quad (19)$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \quad (20)$$

Plot

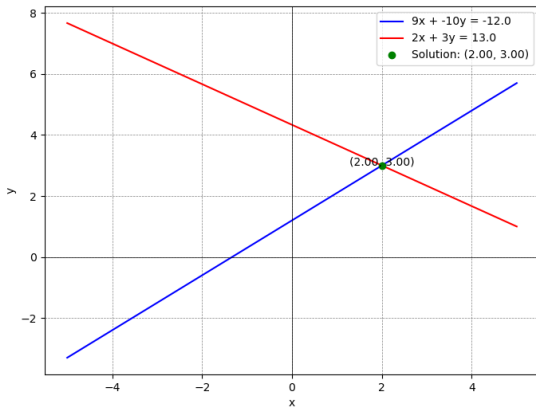


Figure: Graphical solution of the linear equations