The Particle Swarm—Explosion, Stability, and Convergence in a Multidimensional Complex Space

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***Abstract*—The particle swarm is an algorithm for finding optimal regions of complex search spaces through the interaction of individuals in a population of particles. Even though the algorithm, which is based on a metaphor of social interaction, has been shown to perform well, researchers have not adequately explained how it works. Further, traditional versions of the algorithm have had some undesirable dynamical properties, notably the particles' velocities needed to be limited in order to control their trajectories. The present paper analyzes a particle's trajectory as it moves in discrete time (the algebraic view), then progresses to the view of it in continuous time (the analytical view). A five-dimensional depiction is developed, which describes the system completely. These analyses lead to a generalized model of the algorithm, containing a set of coefficients to control the system's convergence tendencies. Some results of the particle swarm optimizer, implementing modifications derived from the analysis, suggest methods for altering the original algorithm in ways that eliminate problems and increase the ability of the particle swarm to find optima of some well-studied test functions.**

***Index Terms*—Convergence, evolutionary computation, optimization, particle swarm, stability.**

I. Introduction

PARTICLE swarm adaptation has been shown to successfully optimize a wide range of continuous functions [1]–[5]. The algorithm, which is based on a metaphor of social interaction, searches a space by adjusting the trajectories of individual vectors, called "particles" as they are conceptualized as moving points in multidimensional space. The individual particles are drawn stochastically toward the positions of their own previous best performance and the best previous performance of their neighbors.

While empirical evidence has accumulated that the algorithm "works," e.g., it is a useful tool for optimization, there has thus far been little insight into *how* it works. The present analysis begins with a highly simplified deterministic version of the particle swarm in order to provide an understanding about how it searches the problem space [4], then continues on to analyze the full stochastic system. A generalized model is proposed, including methods for controlling the convergence properties of the particle system. Finally, some empirical results are given, showing the performance of various implementations of the algorithm on a suite of test functions.

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A. The Particle Swarm

A population of particles is initialized with random positions $\vec{x}_i$ and velocities $\vec{v}_i$ and a function f is evaluated, using the particle's positional coordinates as input values. Positions and velocities are adjusted and the function evaluated with the new coordinates at each time step. When a particle discovers a pattern that is better than any it has found previously, it stores the coordinates in a vector $\vec{p}_i$. The difference between $\vec{p}_i$ (the best point found by i so far) and the individual's current position is stochastically added to the current velocity, causing the trajectory to oscillate around that point. Further, each particle is defined within the context of a topological neighborhood comprising itself and some other particles in the population. The stochastically weighted difference between the neighborhood's best position $\vec{p}_g$ and the individual's current position is also added to its velocity, adjusting it for the next time step. These adjustments to the particle's movement through the space cause it to search around the two best positions.

The algorithm in pseudocode follows.

```
Intialize population
Do
    For i = 1 to Population Size
    if f(x_i) < f(p_i) then p_i = x_i
    p_g = min(p_neighbors)
        For d = 1 to Dimension
            v_id = v_id + φ1(p_id − x_id) + φ2(p_gd − x_id)
            v_id = sign(v_id) · min(abs(v_id), v_max)
            x_id = x_id + v_id
        Next d
    Next i
Until termination criterion is met
```

The variables φ_1 and φ_2 are random positive numbers, drawn from a uniform distribution and defined by an upper limit $\varphi_{\max}$, which is a parameter of the system. In this version, the term variable v_{id} is limited to the range $\pm V_{\max}$ for reasons that will be explained below. The values of the elements in $\vec{p}_g$ are determined by comparing the best performances of all the members of i's topological neighborhood, defined by indexes of some other population members and assigning the best performer's index to the variable g. Thus, $\vec{p}_g$ represents the best position found by any member of the neighborhood.

The random weighting of the control parameters in the algorithm results in a kind of explosion or a "drunkard's walk" as particles' velocities and positional coordinates careen toward infinity. The explosion has traditionally been contained through implementation of a $V_{\max}$ parameter, which limits step size or velocity. The current paper, however, demonstrates that the implementation of properly defined constriction coefficients can prevent explosion; further, these coefficients can induce particles to converge on local optima.

An important source of the swarm's search capability is the interactions among particles as they react to one another's findings. Analysis of interparticle effects is beyond the scope of this paper, which focuses on the trajectories of single particles.

B. Simplification of the System

We begin the analysis by stripping the algorithm down to a most simple form; we will add things back in later. The particle swarm formula adjusts the velocity $\vec{v}_i$ by adding two terms to it. The two terms are of the same form, i.e., $\varphi(\vec{p} - \vec{x}_i)$, where $\vec{p}$ is the best position found so far, by the individual particle in the first term, or by any neighbor in the second term. The formula can be shortened by redefining p_{id} as follows:

$$p_{id} \leftarrow \frac{\varphi_1 p_{id} + \varphi_2 p_{gd}}{\varphi_1 + \varphi_2}.$$

Thus, we can simplify our initial investigation by looking at the behavior of a particle whose velocity is adjusted by only one term

$$v_{id}(t+1) = v_{id}(t) + \varphi(p_{id} - x_{id}(t))$$

where $\varphi = \varphi_1 + \varphi_2$. This is algebraically identical to the standard two-term form.

When the particle swarm operates on an optimization problem, the value of $\vec{p}_i$ is constantly updated, as the system evolves toward an optimum. In order to further simplify the system and make it understandable, we set $\vec{p}_i$ to a constant value in the following analysis. The system will also be more understandable if we make φ a constant as well; where normally it is defined as a random number between zero and a constant upper limit, we will remove the stochastic component initially and reintroduce it in later sections. The effect of φ on the system is very important and much of the present paper is involved in analyzing its effect on the trajectory of a particle.

The system can be simplified even further by considering a one-dimensional (1-D) problem space and again further by reducing the population to one particle. Thus, we will begin by looking at a stripped-down particle by itself, e.g., a population of one 1-D deterministic particle, with a constant p.

Thus, we begin by considering the reduced system

$$\begin{cases} v(t+1) = v(t) + \varphi(p - x(t)) \\ x(t+1) = x(t) + v(t+1) \end{cases} \tag{1.1}$$

where p and φ are constants. No vector notation is necessary and there is no randomness.

In [4], Kennedy found that the simplified particle's trajectory is dependent on the value of the control parameter φ and recognized that randomness was responsible for the explosion of the system, although the mechanism that caused the explosion was not understood. Ozcan and Mohan [6], [7] further analyzed the system and concluded that the particle as seen in discrete time "surfs" on an underlying continuous foundation of sine waves.

The present paper analyzes the particle swarm as it moves in discrete time (the algebraic view), then progresses to the view of it in continuous time (the analytical view). A five-dimensional (5-D) depiction is developed, which completely describes the system. These analyses lead to a generalized model of the algorithm, containing a set of coefficients to control the system's convergence tendencies. When randomness is reintroduced to the full model with constriction coefficients, the deleterious effects of randomness are seen to be controlled. Some results of the particle swarm optimizer, using modifications derived from the analysis, are presented; these results suggest methods for altering the original algorithm in ways that eliminate some problems and increase the optimization power of the particle swarm.

II. Algebraic Point of View

The basic simplified dynamic system is defined by

$$\begin{cases} v_{t+1} = v_t + \varphi y_t \\ y_{t+1} = -v_t + (1-\varphi) y_t \end{cases} \tag{2.1}$$

where $y_t = p - x_t$.

Let

$$P_t = \begin{bmatrix} v_t \\ y_t \end{bmatrix}$$

be the current point in R^2 and

$$M = \begin{bmatrix} 1 & \varphi \\ -1 & 1-\varphi \end{bmatrix}$$

the matrix of the system. In this case, we have $P_{t+1} = MP_t$ and, more generally, $P_t = M^t P_0$. Thus, the system is defined completely by M.

The eigenvalues of M are

$$\begin{cases} e_1 = 1 - \dfrac{\varphi}{2} + \dfrac{\sqrt{\varphi^2 - 4\varphi}}{2} \\ e_2 = 1 - \dfrac{\varphi}{2} - \dfrac{\sqrt{\varphi^2 - 4\varphi}}{2} \end{cases}. \tag{2.2}$$

We can immediately see that the value $\varphi = 4$ is special. Below, we will see what this implies.

For $\varphi \neq 4$, we can define a matrix A so that

$$AMA^{-1} = L = \begin{bmatrix} e_1 & 0 \\ 0 & e_2 \end{bmatrix} \tag{2.3}$$

(note that A^{-1} does not exist when $\varphi = 4$).

For example, from the canonical form $A = \begin{bmatrix} a & 1 \\ c & 1 \end{bmatrix}$, we find

$$\begin{cases} a = \dfrac{\varphi + \sqrt{\varphi^2 - 4\varphi}}{2\varphi} \\ c = \dfrac{\varphi - \sqrt{\varphi^2 - 4\varphi}}{2\varphi} \end{cases}. \tag{2.4}$$

In order to simplify the formulas, we multiply by 2φ to produce a matrix A

$$A = \begin{bmatrix} \varphi + \sqrt{\varphi^2 - 4\varphi} & 2\varphi \\ \varphi - \sqrt{\varphi^2 - 4\varphi} & 2\varphi \end{bmatrix}. \tag{2.5}$$

TABLE I
SOME φ VALUES FOR WHICH THE SYSTEM IS CYCLIC

φ	*Cycle period*
3	3 (see Figure 1(a))
2	4
$\frac{5\pm\sqrt{5}}{2}$	5 (see Figure 1(b) for the sum and Figure 1(c) for the difference)
1, 3	6,3
1, 2, 3, $2\pm\sqrt{3}$	6, 4, 3, 12

So, if we define $Q_t = AP_t$, we can now write

$$\begin{aligned} P_{t+1} &= A^{-1}LAP_t \\ AP_{t+1} &= LAP_t \\ Q_{t+1} &= LQ_t \end{aligned} \tag{2.6}$$

and, finally, $Q_t = L^t Q_0$

However, L is a diagonal matrix, so we have simply

$$L^t = \begin{bmatrix} e_1^t & 0 \\ 0 & e_2^t \end{bmatrix}. \tag{2.7}$$

In particular, there is cyclic behavior in the system if and only if $Q_t = Q_0$ (or, more generally, if $Q_{t+k} = Q_t$). This just means that we have a system of two equations

$$\begin{cases} e_1^t = 1 \\ e_2^t = 1 \end{cases}. \tag{2.8}$$

A. Case $\varphi < 4$

For $0 < \varphi < 4$, the eigenvalues are complex and there is always at least one (real) solution for φ. More precisely, we can write

$$\begin{cases} e_1 = \cos(\theta) + i\sin(\theta) \\ e_2 = \cos(\theta) - i\sin(\theta) \end{cases} \tag{2.9}$$

with $\cos(\theta) = 1 - (\varphi/2)$ and $\sin(\theta) = \sqrt{4\varphi - \varphi^2}/2$. Then

$$\begin{cases} e_1^t = \cos(t\theta) + i\sin(t\theta) \\ e_2^t = \cos(t\theta) - i\sin(t\theta) \end{cases} \tag{2.10}$$

and cycles are given by any θ such that $\theta = (2k\pi)/t$.

So for each t, the solutions for φ are given by

$$\varphi = 2\left(1 - \cos\left(\frac{2k\pi}{t}\right)\right) \text{ for } k \in \{1, 2, \ldots, t-1\}. \tag{2.11}$$

Table I gives some nontrivial values of φ for which the system is cyclic.

Fig. 1(a)–(d) show the trajectories of a particle in phase space, for various values of φ. When φ takes on one of the values from Table I, the trajectory is cyclical, for any other value, the system is just quasi-cyclic, as in Fig. 1(d).

We can be a little bit more precise. Below, $\|\|$ is the 2-norm (the Euclidean one for a vector)

$$\begin{aligned} \|Q_t\| = \|AP_t\| &= \|Q_0\| \\ \left\|A^{-1}\right\| \|Q_0\| \geq \|P_t\| &\geq \frac{\|Q_0\|}{\|A\|}. \end{aligned} \tag{2.12}$$

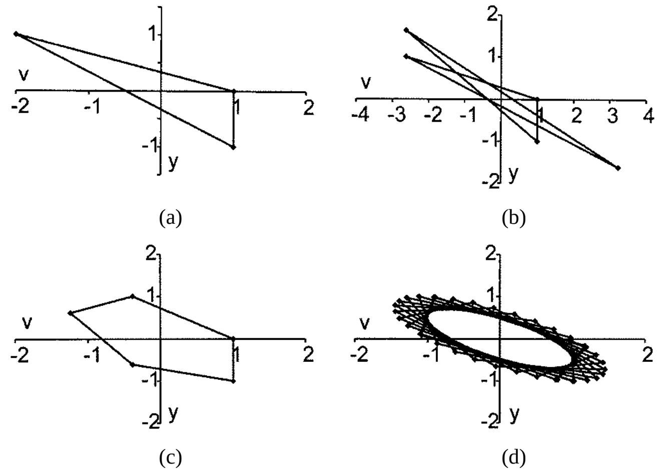


Fig. 1. (a) Cyclic trajectory of a nonrandom particle when $\varphi = 3$. (b) Cyclic trajectory of a nonrandom particle when $\varphi = (5+\sqrt{5})/2$. (c) Cyclic trajectory of a nonrandom particle when $\varphi = (5-\sqrt{5})/2$. (d) Particle's more typical quasi-cyclic behavior when φ does not satisfy (2.11). Here, $\varphi = 2.1$.

For example, for $v_0 = 0$ and $y_0 = 1$, we have

$$\begin{aligned} &\sqrt{2\varphi \max\left(\frac{1}{2}\left|\frac{3\varphi^2 - 4\varphi \pm \sqrt{5\varphi^4 - 8\varphi^3 + 16\varphi^2}}{2\varphi^4 - 8\varphi^3}\right|\right)} \\ &\geq \|P_t\| \\ &\geq \sqrt{\frac{2\varphi}{\max\left(\left|3\varphi^2 - 4\varphi \pm \sqrt{5\varphi^4 - 8\varphi^3 + 16\varphi^2}\right|\right)}}. \end{aligned} \tag{2.13}$$

B. Case $\varphi > 4$

If $\varphi > 4$, then e_1 and e_2 are real numbers (and $|e_1| \leq |e_2|$), so we have either:

1) $e_1 = e_2 = 1$ (for t even) which implies $\varphi = 0$, not consistent with the hypothesis $\varphi > 4$;
2) $e_1 = -e_2 = 1$ (or -1), which is impossible;
3) $e_1 = e_2 = -1$, that is to say $\varphi = 4$, not consistent with the hypothesis $\varphi > 4$.

So, and this is the point: there is no cyclic behavior for $\varphi > 4$ and, in fact, the distance from the point P_t to the center (0,0) is strictly monotonic increasing with t, which means that

$$\begin{aligned} Q_t &= AP_t \\ L^t Q_0 &= AP_t. \end{aligned} \tag{2.14}$$

So

$$\begin{aligned} \left\|L^t Q_0\right\| &\leq \|A\| \|P_t\| \\ \frac{\left\|L^t Q_0\right\|}{\|A\|} &\leq \|P_t\|. \end{aligned} \tag{2.15}$$

One can also write

$$\begin{aligned} P_t &= A^{-1}Q_t \\ \|P_t\| &\leq \left\|A^{-1}\right\| \|Q_t\| \\ \|P_t\| &\leq \left\|A^{-1}\right\| \left\|L^t Q_0\right\|. \end{aligned} \tag{2.16}$$

So, finally, $\|P_t\|$ increases like $\|L^t Q_0\|$.

In Section IV, this result is used to prevent the explosion of the system, which can occur when particle velocities increase without control.

C. Case $\varphi = 4$

In this situation

$$M = \begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix}.$$

In this particular case, the eigenvalues are both equal to -1 and there is just one family of eigenvectors, generated by

$$V = \begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

So, we have $MV = -V$.

Thus, if P_0 is an eigenvector, proportional to V (that is to say, if $v_0 + 2y_0 = 0$), there are just two symmetrical points, for

$$P_{t+1} = \pm \begin{bmatrix} 2y_0 \\ -y_0 \end{bmatrix} = -P_t. \tag{2.17}$$

In the case where P_0 is not an eigenvector, we can directly compute how $\|P_t\|$ decreases and/or increases.

Let us define $\Delta_t = \|P_{t+1}\|^2 - \|P_t\|$. By recurrence, the following form is derived:

$$\Delta_t = a_t v_0^2 + b_t v_0 y_0 + c_t y_0^2 \tag{2.18}$$

where a_t, b_t, c_t are integers so that $\Delta_t = 0$ for $v_0 + 2y_0 = 0$. The integers can be negative, zero, or positive.

Supposing for a particular t we have $\Delta_t > 0$, one can easily compute $\Delta_t = v_t^2 + 14 v_t y_t + 24 y_t^2$. This quantity is positive if and only if v_t is not between (or equal to) the roots $\{-2y_t, -12y_t\}$.

Now, if Δ_{t+1} is computed, then we have $\Delta_{t+1} = 11v_t^2 + 54 v_t y_t + 64 y_t^2$ and the roots are $\{-2y_t, -(32y_t)/11\}$. As $(32/11) < 12$, this result means that Δ_{t+1} is also positive. So, as soon as $\|P_t\|$ begins to increase, it does so infinitely, but it can decrease, at the beginning. The question to be answered next is, how long can it decrease before it begins increasing?

Now take the case of $\Delta_0 < 0$. This means that v_0 is between $-2y_0$ and $-12y_0$. For instance, in the case where $y_0 > 0$[1]

$$v_0 = -2y_0 - \varepsilon, \text{ with } \varepsilon \in]0, 10y_0[. \tag{2.19}$$

By recurrence, the following is derived:

$$\begin{aligned} \Delta_0 &= -10y_0\varepsilon + \varepsilon^2 \\ \Delta_1 &= -10y_0\varepsilon + 11\varepsilon^2 \\ \Delta_2 &= -10y_0\varepsilon + 21\varepsilon^2 \\ \Delta_{t+2} &= -\Delta_t + 2\Delta_{t+1} \\ &= -10y_0\varepsilon + k_{t+2}\varepsilon^2, \text{ with} \\ k_{t+2} &= -k_t + 2k_{t+1}. \end{aligned} \tag{2.20}$$

Finally

$$\Delta_t = -10y_0\varepsilon + (1 + 10t)\varepsilon^2 \tag{2.21}$$

[1]Note that the present paper uses the Bourbaki convention of representing open intervals with reversed brackets. Thus,]a,b[is equivalent to parenthetical notation (a,b).

as long as $(1 + 10t)\varepsilon^2 \leq 10y_0\varepsilon$, which means that $\|P_t\|$ decreases as long as

$$t \leq 1 + \text{Integer\_part}\left(\frac{-\frac{1}{10} + y_0}{\varepsilon}\right). \tag{2.22}$$

After that, $\|P_t\|$ increases.

The same analysis can be performed for $y_0 < 0$. In this case, $\varepsilon < 0$, as well, so the formula is the same. In fact, to be even more precise, if

$$\begin{aligned} \alpha &= -10y_0\varepsilon + \varepsilon^2 \\ \beta &= 10\varepsilon^2 \end{aligned}$$

then we have

$$\|P_t\| = t\sqrt{\left|\frac{\beta}{2} + \frac{\left(\alpha - \frac{\beta}{2}\right)}{t}\right| + \frac{\|P_0\|^2}{t^2}}. \tag{2.23}$$

Thus, it can be concluded that $\|P_t\|$ decreases/increases almost linearly when t is big enough. In particular, even if it begins to decrease, after that it tends to increase almost like $t\sqrt{5}\,|v_0 + 2y_0|$.

III. Analytic Point of View

A. Basic Explicit Representation

From the basic iterative (implicit) representation, the following is derived:

$$\begin{aligned} v(t+2) &= v(t+1) + \varphi y(t+1) \\ &= v(t+1) - \varphi v(t) + (1-\varphi)v(t+1) - v(t) \\ v(t+2) &+ (\varphi - 2)v(t+1) + v(t) = 0. \end{aligned} \tag{3.1}$$

Assuming a continuous process, this becomes a classical second-order differential equation

$$\frac{\partial^2 v}{\partial t^2} + \ln(e_1 e_2)\frac{\partial v}{\partial t} + \ln(e_1)\ln(e_2)v = 0 \tag{3.2}$$

where e_1 and e_2 are the roots of

$$\lambda^2 + (\varphi - 2)\lambda + 1 = 0. \tag{3.3}$$

As a result

$$\begin{cases} e_1 = 1 - \dfrac{\varphi}{2} + \dfrac{\sqrt{\varphi^2 - 4\varphi}}{2} \\ e_2 = 1 - \dfrac{\varphi}{2} - \dfrac{\sqrt{\varphi^2 - 4\varphi}}{2} \end{cases}. \tag{3.4}$$

The general solution is

$$v(t) = c_1 e_1^t + c_2 e_2^t. \tag{3.5}$$

A similar kind of expression for $y(t)$ is now produced, where

$$y(t) = \frac{1}{\varphi}\left(c_1 e_1^t (e_1 - 1) + c_2 e_2^t (e_2 - 1)\right). \tag{3.6}$$

The coefficients c_1 and c_2 depend on $v(0)$ and $y(0)$. If $e_1 \neq e_2$, we have

$$\begin{cases} c_1 = \dfrac{-\varphi y(0) - (1-e_2)v(0)}{e_2 - e_1} \\ c_2 = \dfrac{\varphi y(0) + (1-e_1)v(0)}{e_2 - e_1} \end{cases}. \tag{3.7}$$

In the case where $e_1 = e_2$ ($\varphi = 4$), (3.5) and (3.6) give

$$\begin{cases} v(0) = c_1 + c_2 \\ y(0) = -\dfrac{c_1 + c_2}{2} \end{cases} \tag{3.8}$$

so we must have

$$v(0) + 2y(0) = 0 \tag{3.9}$$

in order to prevent a discontinuity.

Regarding the expressions e_1 and e_2, eigenvalues of the matrix M, as in Section II above, the same discussion about the sign of ($\varphi^2 - 4\varphi$) can be made, particularly about the (non) existence of cycles.

The above results provide a guideline for preventing the explosion of the system, for we can immediately see that it depends on whether we have

$$\max(|e_1|, |e_2|) > 1. \tag{3.10}$$

B. A Posteriori Proof

One can directly verify that $v(t)$ and $y(t)$ are, indeed, solutions of the initial system.

On one hand, from their expressions

$$\begin{cases} v(t+1) = e_1 c_1 e_1^t + e_2 c_2 e_2^t \\ y(t+1) = \frac{1}{\varphi}\Big(e_1 c_1 e_1^t (e_1 - 1) + e_2 c_2 e_2^t (e_2 - 1)\Big) \end{cases} \tag{3.11}$$

and on the other hand

$$\begin{aligned} v(t) + \varphi y(t) =& c_1 e_1^t + c_2 e_2^t + \big(c_1 e_1^t (e_1 - 1) \\ & + c_2 e_2^t (e_2 - 1)\big) \\ =& e_1 c_1 e_1^t + e_2 c_2 e_2^t \\ =& v(t+1) \end{aligned} \tag{3.12}$$

and also

$$\begin{aligned} -v(t) + (1-\varphi) y(t) =& - c_1 e_1^t - c_2 e_2^t \\ & + \frac{1}{\varphi}\left(c_1 e_1^t (e_1 - 1) + c_2 e_2^t (e_2 - 1)\right) \\ & - \left(c_1 e_1^t (e_1 - 1) + c_2 e_2^t (e_2 - 1)\right) \\ =& \frac{1}{\varphi}\Big[\big(e_1 c_1 e_1^t (e_1 - 1) + e_2 c_2 e_2^t (e_2 - 1)\big) \\ & - \big(e_1 c_1 e_1^t (e_1 - 1) + e_2 c_2 e_2^t (e_2 - 1)\big) \\ & - \varphi\, c_1 e_1^t - \varphi\, c_2 e_2^t \\ & + \big(c_1 e_1^t (e_1 - 1) + c_2 e_2^t (e_2 - 1)\big) \\ & - \varphi\left(c_1 e_1^t (e_1 - 1) + c_2 e_2^t (e_2 - 1)\right)\Big] \\ =& \frac{1}{\varphi}\Big[\big(e_1 c_1 e_1^t (e_1 - 1) + e_2 c_2 e_2^t (e_2 - 1)\big) \\ & - c_1 e_1^t \left(e_1^2 - (\varphi - 2) e_1 + 1\right) \\ & - c_2 e_2^t \big(e_2^2 - (\varphi - 2) e_2 + 1\big)\Big] \\ =& y(t+1). \end{aligned} \tag{3.13}$$

C. General Implicit and Explicit Representations

A more general implicit representation (IR) is produced by adding five coefficients $\{\alpha, \beta, \gamma, \delta, \eta\}$, which will allow us to identify how the coefficients can be chosen in order to ensure convergence. With these coefficients, the system becomes

$$\begin{aligned} &\begin{cases} v_{t+1} = \alpha v_t + \beta\varphi y_t \\ y_{t+1} = -\gamma v_t + (\delta - \eta\varphi) y_t \end{cases} \\ &\varphi \in R_+^* \\ &\forall t \in N, \{y_t, v_t\} \in R^2. \end{aligned} \tag{3.14}$$

The matrix of the system is now

$$M' = \begin{bmatrix} \alpha & \beta\varphi \\ -\gamma & \delta - \eta\varphi \end{bmatrix}.$$

Let e_1' and e_2' be its eigenvalues.

The (analytic) explicit representation (ER) becomes

$$\begin{aligned} &\begin{cases} v(t) = c_1 \left(e_1'\right)^t + c_2 \left(e_2'\right)^t \\ y(t) = \frac{1}{\beta\varphi}\Big(c_1 \left(e_1'\right)^t (e_1' - \alpha) + c_2 \left(e_2'\right)^t (e_2' - \alpha)\Big) \end{cases} \\ &\varphi \in R_+^* \\ &\forall t \in N, \{y(t), v(t)\} \in R^2 \end{aligned} \tag{3.15}$$

with

$$\begin{cases} c_1 = \dfrac{-\beta\varphi y(0) - (\alpha - e_2')v(0)}{e_2' - e_1'} \\ c_2 = \dfrac{\beta\varphi y(0) + (\alpha - e_1')v(0)}{e_2' - e_1'} \end{cases}. \tag{3.16}$$

Now the constriction coefficients (see Section IV for details) χ_1 and χ_2 are defined by

$$\begin{cases} e_1' = \chi_1 e_1 \\ e_2' = \chi_2 e_2 \end{cases} \tag{3.17}$$

with

$$\begin{cases} e_1 = 1 - \dfrac{\varphi}{2} + \dfrac{\sqrt{\varphi^2 - 4\varphi}}{2} \\ e_2 = 1 - \dfrac{\varphi}{2} - \dfrac{\sqrt{\varphi^2 - 4\varphi}}{2} \end{cases} \tag{3.18}$$

which are the eigenvalues of the basic system. By computing the eigenvalues directly and using (3.17), χ_1 and χ_2 are

$$\begin{cases} \chi_1 = \frac{\alpha + \delta - \eta\varphi + \sqrt{(\eta\varphi)^2 + 2\varphi(\alpha\eta - \delta\eta - 2\beta\gamma) + (\alpha - \delta)^2}}{2 - \varphi + \sqrt{\varphi^2 - 4\varphi}} \\ \chi_2 = \frac{\alpha + \delta - \eta\varphi - \sqrt{(\eta\varphi)^2 + 2\varphi(\alpha\eta - \delta\eta - 2\beta\gamma) + (\alpha - \delta)^2}}{2 - \varphi - \sqrt{\varphi^2 - 4\varphi}} \end{cases}. \tag{3.19}$$

The final complete ER can then be written from (3.15) and (3.16) by replacing e_1' and e_2', respectively, by $\chi_1 e_1$ and $\chi_2 e_2$ and then e_1, e_2, χ_1, χ_2 by their expressions, as seen in (3.18) and (3.19).

It is immediately worth noting an important difference between IR and ER. In the IR, t is always an integer and $v(t)$ and $y(t)$ are real numbers. In the ER, real numbers are obtained if and only if t is an integer; nothing, however, prevents the assignment of any real positive value to t, in which case $v(t)$ and $y(t)$ become true complex numbers. This fact will provide an elegant way of explaining the system's behavior, by conceptualizing it in a 5-D space, as discussed in Section IV.

Note 3.1: If χ_1 and χ_2 are to be real numbers for a given φ value, there must be some relations among the five real coefficients $\{\alpha,\beta,\gamma,\delta,\eta\}$. If the imaginary parts of χ_1 and χ_2 are set equal to zero, (3.20) is obtained, as shown at the bottom of the page, with

$$\begin{aligned} A &= \text{sign}\left(\varphi^2-4\varphi\right)\\ B &= \left|\varphi^2-4\varphi\right|\\ C &= 2-\varphi+\frac{1}{2}\sqrt{\left|\varphi^2-4\varphi\right|}\left(1+\text{sign}\left(\varphi^2-4\varphi\right)\right)\\ C' &= 2-\varphi-\frac{1}{2}\sqrt{\left|\varphi^2-4\varphi\right|}\left(1+\text{sign}\left(\varphi^2-4\varphi\right)\right)\\ D &= C^2+\frac{1}{4}\left|\varphi^2-4\varphi\right|\left(1-\text{sign}\left(\varphi^2-4\varphi\right)\right)^2\\ E &= (\eta\varphi)^2+2\varphi\left(\alpha\eta-\delta\eta-2\beta\gamma\right)+(\alpha-\delta)^2. \end{aligned} \tag{3.21}$$

The two equalities of (3.20) can be combined and simplified as follows:

$$\begin{cases} \sqrt{|E|}\left(1-\text{sign}\left(E\right)\right)\left(2-\varphi\right)-\left(\alpha+\delta-\eta\varphi\right)\sqrt{B}\left(1-A\right)=0\\ \sqrt{|E|}\sqrt{B}\;\text{sign}\left(E\right)\left(1+A\right)=0 \end{cases}. \tag{3.22}$$

The solutions are usually not completely independent of φ. In order to satisfy these equations, a set of possible conditions is

$$\begin{cases} E>0\\ A=-1\left(\Leftrightarrow \varphi<4\right).\\ \alpha+\delta-\eta\varphi=0 \end{cases} \tag{3.23}$$

However, these conditions are not necessary. For example, an interesting particular situation (studied below) exists where $\alpha=\beta=\gamma=\delta=\eta=\chi\in R_+^*$. In this case, $\chi_1=\chi_2=\chi$ for any φ value and (3.20) is always satisfied.

D. From ER to IR

The ER will be useful to find convergence conditions. Nevertheless, in practice, the iterative form obtained from (3.19) is very useful, as shown in (3.24) at the bottom of the page.

Although there are an infinity of solutions in terms of the five parameters $\{\alpha,\beta,\gamma,\delta,\eta\}$, it is interesting to identify some particular classes of solutions. This will be done in the next section.

E. Particular Classes of Solutions

1) Class 1 Model: The first model implementing the five-parameter generalization is defined by the following relations:

$$\begin{cases} \alpha=\delta\\ \beta\gamma=\eta^2 \end{cases}. \tag{3.25}$$

In this particular case, α and η are

$$\begin{cases} \alpha=\frac{1}{4}\left(2\left(\chi_1+\chi_2\right)+\left(\chi_1-\chi_2\right)\left(\sqrt{\varphi^2-4\varphi}+\varphi\frac{2-\varphi}{\sqrt{\varphi^2-4\varphi}}\right)\right)\\ \eta=\frac{1}{2}\left(\chi_1+\chi_2+\frac{2-\varphi}{\sqrt{\varphi^2-4\varphi}}\left(\chi_1-\chi_2\right)\right) \end{cases}. \tag{3.26}$$

An easy way to ensure real coefficients is to have $\chi_1=\chi_2=\chi\in R$. Under this additional condition, a class of solution is simply given by

$$\alpha=\beta=\gamma=\delta=\eta=\chi. \tag{3.27}$$

2) Class $1'$ Model: A related class of model is defined by the following relation:

$$\begin{cases} \alpha=\beta\\ \gamma=\delta=\eta \end{cases}. \tag{3.28}$$

The expressions in (3.29), shown at the bottom of the next page, for α and γ are derived from (3.24).

If the condition $\chi_1=\chi_2=\chi$ is added, then

$$\begin{cases} \alpha=(2-\varphi)\,\chi+\varphi-1\\ \gamma=\chi \text{ or } \gamma=\frac{\chi}{\varphi-1} \end{cases}. \tag{3.30}$$

Without this condition, one can choose a value for γ, for example, $\gamma=1$ and a corresponding α value (χ_1,χ_2), which give a convergent system.

$$\begin{cases} \sqrt{|E|}\left(1-\text{sign}(E)\right)C-\left(\alpha+\delta-\eta\varphi+\frac{1}{2}\sqrt{|E|}(1+\text{sign}(E))\right)\sqrt{B}(1-A)=0\\ \sqrt{|E|}\left(1-\text{sign}(E)\right)C'-\left(\alpha+\delta-\eta\varphi-\frac{1}{2}\sqrt{|E|}(1+\text{sign}(E))\right)\sqrt{B}(1-A)=0 \end{cases} \tag{3.20}$$

$$\begin{cases} 2\left(\alpha+\delta-\eta\varphi\right)=\left(\chi_1+\chi_2\right)\left(2-\varphi\right)+\left(\chi_1-\chi_2\right)\sqrt{\varphi^2-4\varphi}\\ 2\sqrt{\left(\eta\varphi\right)^2+2\varphi\left(\alpha\eta-\delta\eta-2\beta\gamma\right)+\left(\alpha-\delta\right)^2}=\left(\chi_1+\chi_2\right)\sqrt{\varphi^2-4\varphi}+\left(\chi_1-\chi_2\right)\left(2-\varphi\right) \end{cases} \tag{3.24}$$

3) Class $1''$ Model: A second model related to the Class 1 formula is defined by

$$\alpha = \beta = \gamma = \eta \tag{3.31}$$

$$\alpha = \frac{2\delta + (\chi_1 + \chi_2)(\varphi - 2) - (\chi_1 - \chi_2)\sqrt{\varphi^2 - 4\varphi}}{2(\varphi - 1)}. \tag{3.32}$$

For historical reasons and for its simplicity, the case $\delta = 1$ has been well studied. See Section IV-C for further discussion.

4) Class 2 Model: A second class of models is defined by the relations

$$\begin{cases} \alpha = \beta = 2\delta \\ \eta = 2\gamma \end{cases}. \tag{3.33}$$

Under these constraints, it is clear that

$$\begin{cases} 2(3\delta - 2\gamma\varphi) = (\chi_1 + \chi_2)(2 - \varphi) + (\chi_1 - \chi_2)\sqrt{\varphi^2 - 4\varphi} \\ 2\,|2\gamma\varphi - \delta| = (\chi_1 + \chi_2)\sqrt{\varphi^2 - 4\varphi} + (\chi_1 - \chi_2)(2 - \varphi) \end{cases} \tag{3.34}$$

which gives us γ and δ, respectively.

Again, an easy way to obtain real coefficients for every φ value is to have $\chi_1 = \chi_2 = \chi$. In this case

$$\begin{cases} 3\delta - 2\gamma\varphi = \chi(2 - \varphi) \\ |2\gamma\varphi - \delta| = \chi\sqrt{\varphi^2 - 4\varphi} \end{cases}. \tag{3.35}$$

In the case where $2\gamma\varphi \geq \delta$, the following is obtained:

$$\begin{cases} \delta = \chi\frac{2 - \varphi + \sqrt{\varphi^2 - 4\varphi}}{2} = \chi e_1 \\ \gamma = \chi\frac{2 - \varphi + 3\sqrt{\varphi^2 - 4\varphi}}{4\varphi} \end{cases}. \tag{3.36}$$

From the standpoint of convergence, it is interesting to note that we have the following.

1) For the Class 1 models, with the condition $\chi_1 = \chi_2 = \chi$

$$\begin{cases} |e_1'| = \chi|e_1| \\ |e_2'| = \chi|e_2| \end{cases}. \tag{3.37}$$

2) For the Class $1'$ models, with the conditions $\chi_1 = \chi_2 = \chi$ and $\varphi \leq 2$

$$\begin{cases} |e_1'| = \left|\chi\left(1 - \frac{\varphi}{2}\right) + \frac{\sqrt{\chi^2(4 - 4\varphi + \varphi^2) + 4\chi(\varphi - 2) + 4(\varphi - 1)}}{2}\right| \leq \chi|e_1| = \chi \\ |e_2'| = \left|\chi\left(1 - \frac{\varphi}{2}\right) - \frac{\sqrt{\chi^2(4 - 4\varphi + \varphi^2) + 4\chi(\varphi - 2) + 4(\varphi - 1)}}{2}\right| \leq \chi|e_2| = \chi \end{cases}. \tag{3.38}$$

3) For the the Class 2 models, see (3.39) at the bottom of the page, with $\Delta = \sqrt{\varphi^2 - 4\varphi}$.

This means that we will just have to choose $\chi < 1/|e_2|$, $\chi < 1$, and $\varphi \leq 2$, $\chi < 1/|e_{2,\text{class } 2}|$, respectively, to have a convergent system. This will be discussed further in Section IV.

F. Removing the Discontinuity

Depending on the parameters $\{\alpha, \beta, \gamma, \delta, \eta\}$ the system may have a discontinuity in φ due to the presence of the term $\sqrt{(\eta\varphi)^2 - 4\beta\gamma\varphi + (\alpha - \delta)^2 + 2\eta\varphi(\alpha - \delta)}$ in the eigenvalues.

Thus, in order to have a completely continuous system, the values for $\{\alpha, \beta, \gamma, \delta, \eta\}$ must be chosen such that

$$\begin{cases} \{\alpha, \beta, \gamma, \delta, \eta\} \in R^5 \\ \forall\varphi \in R^+, (\eta\varphi)^2 - 4\beta\gamma\varphi + (\alpha - \delta)^2 + 2\eta\varphi(\alpha - \delta) \geq 0 \end{cases}. \tag{3.40}$$

By computing the discriminant, the last condition is found to be equivalent to

$$\beta\gamma(-\beta\gamma + \eta(\alpha - \delta)) > 0. \tag{3.41}$$

In order to be "physically plausible," the parameters $\{\alpha, \beta, \gamma, \delta, \eta\}$ must be positive. So, the condition becomes

$$\beta\gamma < \eta(\alpha - \delta). \tag{3.42}$$

The set of conditions taken together specify a volume in R^5 for the admissible values of the parameters.

G. Removing the Imaginary Part

When the condition specified in (3.42) is met, the trajectory is usually still partly in a complex space whenever one of the eigenvalues is negative, due to the fact that $(-1)^t$ is a complex

$$\begin{cases} \alpha = \frac{(\chi_1 + \chi_2)(2 - \varphi) + (\chi_1 - \chi_2)\sqrt{\varphi^2 - 4\varphi}}{2} + \varphi - 1 \\ \gamma = \frac{1}{4(\varphi - 1)}\left(\begin{array}{l} (\chi_1 + \chi_2)(\varphi - 2) - (\chi_1 - \chi_2)\sqrt{\varphi^2 - 4\varphi} \\ \mp\sqrt{2\chi_1^2\left(\varphi^2 - 4\varphi + 2 - \varphi\sqrt{\varphi^2 - 4\varphi}\right) + 2\chi_2^2\left(\varphi^2 - 4\varphi + 2 + \varphi\sqrt{\varphi^2 - 4\varphi}\right) + 8\chi_1\chi_2(2\varphi - 1)} \end{array} \right) \end{cases} \tag{3.29}$$

$$\begin{cases} |e_1'| = \chi\left|\frac{3}{2} - \frac{3}{4}\varphi + \frac{3}{4}\Delta - \frac{1}{2}\varphi^2 + \frac{1}{4}\varphi^3 - \frac{3}{4}\varphi^2\Delta + \frac{1}{4}\left|2 - \varphi - 2\varphi^2 - 2\varphi^3 + \Delta - 3\varphi^2\Delta\right|\right| = \chi\left|e_{1,\text{class } 2}\right| \\ |e_2'| = \chi\left|\frac{3}{2} - \frac{3}{4}\varphi + \frac{3}{4}\Delta - \frac{1}{2}\varphi^2 + \frac{1}{4}\varphi^3 - \frac{3}{4}\varphi^2\Delta - \frac{1}{4}\left|2 - \varphi - 2\varphi^2 - 2\varphi^3 + \Delta - 3\varphi^2\Delta\right|\right| = \chi\left|e_{2,\text{class } 2}\right| \end{cases} \tag{3.39}$$

number when t is not an integer. In order to prevent this, we must find some stronger conditions in order to maintain positive eigenvalues.

Since

$$\begin{cases} e_1' > 0 \\ e_2' > 0 \end{cases} \Leftrightarrow \begin{cases} e_1' + e_2' > 0 \\ e_1' e_2' > 0 \end{cases} \tag{3.43}$$

the following conditions can be used to ensure positive eigenvalues:

$$\begin{cases} \alpha(\delta - \eta\varphi) + \gamma\beta\varphi > 0 \\ \alpha + \delta - \eta\varphi > 0 \end{cases}. \tag{3.44}$$

Note 3.2: From an algebraic point of view, the conditions described in (3.43) can be written as

$$\begin{cases} \det(M) > 0 \\ \mathrm{trace}(M) > 0 \end{cases}. \tag{3.45}$$

Now, these conditions depend on φ. Nevertheless, if the maximum φ value is known, they can be rewritten as

$$\begin{cases} \frac{\alpha\delta}{\alpha\eta - \gamma\beta} > \varphi_{\max} \\ \frac{\alpha+\delta}{\eta} > \varphi_{\max} \end{cases}. \tag{3.46}$$

Under these conditions, all system variables are real numbers in conjunction with the conditions in (3.42) and (3.44), the parameters can be selected so that the system is completely continuous *and* real.

H. Example

As an example, suppose that $\alpha = \beta = 1$ and $\delta = \eta$. Now the conditions become

$$\begin{cases} \delta < \dfrac{1}{\varphi_{\max}} \\ \dfrac{\delta(\varphi_{\max} - 1)}{\varphi_{\max}} < \gamma < \delta(1 - \delta) \end{cases}. \tag{3.47}$$

For example, when

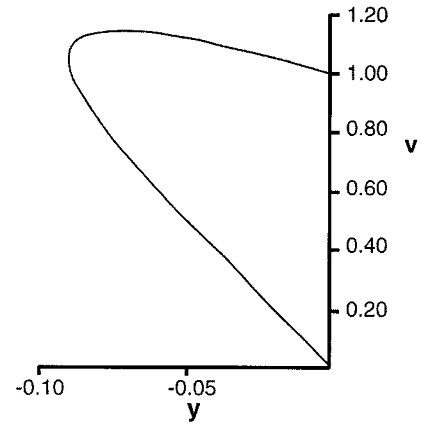
$$\begin{cases} \varphi_{\max} = 10 \\ y_0 = 0, v_0 = 1 \\ \alpha = \beta = 1 \\ \gamma = \frac{1}{2}\left(\frac{\delta(\varphi_{\max} - 1}{\varphi_{\max}} + \delta(1 - \delta)\right) = 0.089\,15 \\ \delta = \eta = \frac{0.99}{\varphi_{\max}} = 0.099 \end{cases} \tag{3.48}$$

the system converges quite quickly after about 25 time steps and at each time step the values of y and v are almost the same over a large range of φ values. Fig. 2(a) shows an example of convergence ($v \geq 0$ and $y \geq 0$) for a continuous real-valued system with $\varphi = 4$.

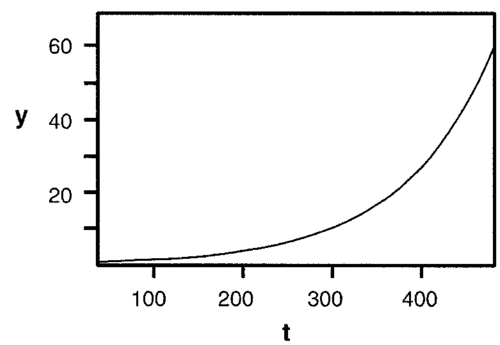
I. Reality and Convergence

The quick convergence seen in the above example suggests an interesting question. Does reality—using real-valued variables—imply convergence? In other words, does the following hold for real-valued system parameters:

$$\begin{cases} \frac{\alpha\delta}{\alpha\eta - \gamma\beta} > \varphi_{\max} \\ \frac{\alpha+\delta}{\eta} > \varphi_{\max} \end{cases} \Rightarrow \begin{cases} |e_1'| < 1 \\ |e_2'| < 1 \end{cases}. \tag{3.49}$$



(a)



(b)

Fig. 2. (a) Convergent trajectory in phase space of a particle when $\alpha = \beta = 1$ and $\delta = \eta$, where $\varphi = 4$. Both velocity v and y, the difference between the previous best p, and the current position x converge to 0.0. (b) y increases over time, even when the parameters are real and not complex.

The answer is no. It can be demonstrated that convergence is not always guaranteed for real-valued variables. For example, given the following parameterization:

$$\begin{cases} \varphi_{\max} = 10 \\ y_0 = 0, v_0 = 1 \\ \alpha = \beta = 1.1 \\ \gamma = 0.089\,149\,5 \\ \delta = \eta = 0.099 \end{cases} \tag{3.50}$$

the relations are

$$\begin{cases} \dfrac{\alpha\delta}{\alpha\eta - \gamma\beta} = 10.05 > \varphi_{\max} \\ \dfrac{\alpha+\delta}{\eta} = 12.11 > \varphi_{\max} \end{cases} \tag{3.51}$$

which will produce system divergence when $\varphi = 0.1$ (for instance), since $|e_1'| = 1.09 > 1$. This is seen in Fig. 2(b)

IV. Convergence and Space of States

From the general ER, we find the criterion of convergence

$$\begin{cases} |e_1'| < 1 \\ |e_2'| > 1 \end{cases} \tag{4.1}$$

where $v(t)$ and $y(t)$ are usually true complex numbers. Thus, the whole system can be represented in a 5-D space $(\mathrm{Re}(y), \mathrm{Im}(y), \mathrm{Re}(v), \mathrm{Im}(v), \varphi)$.

In this section, we study some examples of the most simple class of constricted cases: the ones with just one *constriction coefficient*. These will allow us to devise methods for controlling the behavior of the swarm in ways that are desirable for optimization.

A. Constriction for Model Type 1

Model Type 1 is described as follows:

$$\begin{cases} v(t+1) = \chi\left(v(t) + \varphi y(t)\right) \\ y(t+1) = -\chi\left(v(t) + (1-\varphi) y(t)\right) \end{cases}. \tag{4.2}$$

We have seen that the convergence criterion is satisfied when $\chi < \min(1/|e_1|, 1/|e_2|)$. Since $|e_1| \le |e_2|$, the constriction coefficient below is produced

$$\chi = \frac{\kappa}{|e_2|}, \kappa \in \left]0,1\right[. \tag{4.3}$$

B. Constriction for Model Type $1'$

Just as a constriction coefficient was found for the Type 1 model, the following IR (with χ instead of α) is used for Type $1'$:

$$\begin{cases} v(t+1) = \chi(v(t) + \varphi y(t)) \\ y(t+1) = -v(t) + (1-\varphi) y(t) \end{cases}. \tag{4.4}$$

The coefficient becomes

$$\chi = \frac{\kappa}{|e_2|}, \kappa \in]0,1[, \text{ for } \varphi \in]0,2[. \tag{4.5}$$

However, as seen above, this formula is *a priori* valid only when $\varphi < 2$, so it is interesting to find another constriction coefficient that has desirable convergence properties. We have here

$$e_2' = \frac{\chi + 1 - \varphi}{2} - \frac{\sqrt{(\chi - 1)^2 + \varphi^2 - 2\varphi - 2\chi\varphi}}{2}. \tag{4.6}$$

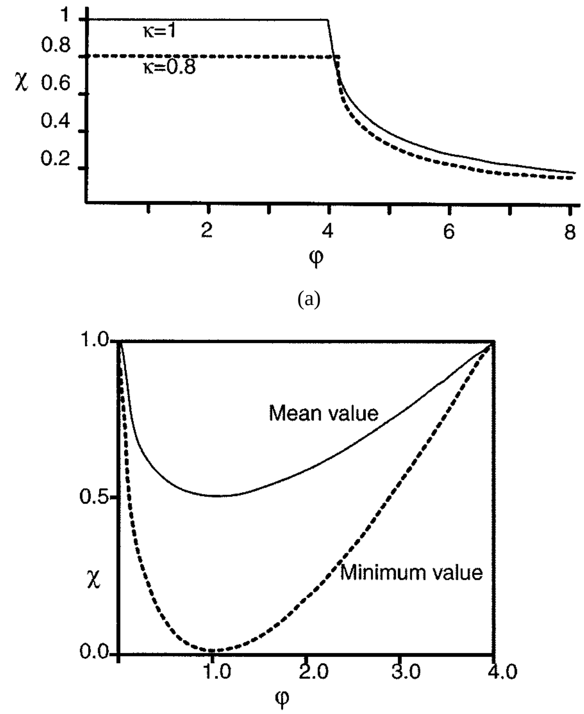
The expression under the square root is negative for $\chi \in]1 + \varphi - 2\sqrt{\varphi}, 1 + \varphi + 2\sqrt{\varphi}[$. In this case, the eigenvalue is a true complex number and $|e_2'| = \sqrt{\chi}$. Thus, if $1+\varphi-2\sqrt{\varphi} < 1$, that is to say, if $\varphi < 4$, a χ needs to be selected such that $\chi \in]1 + \varphi - 2\sqrt{\varphi}, 1[$ in order to satisfy the convergence criterion. So, for example, define χ as

$$\chi = \frac{2 + \varphi - 2\sqrt{\varphi}}{2}, \text{ for } \varphi \in]0,4[. \tag{4.7}$$

Now, can another formula for greater φ values be found? The answer is no. For in this case, e_2' is a real number and its absolute value is:

1) strictly decreasing on $\alpha \in \left]0, 1 + \varphi - 2\sqrt{\varphi}\right]$ and the minimal value is $\sqrt{\varphi} - 1$ (greater than 1);
2) strictly decreasing on $\alpha \in \left[1 + \varphi + 2\sqrt{\varphi}, \infty\right[$, with a limit of 1.

For simplicity, the formula can be the same as for Type 1, not only for $\varphi < 2$, but also for $\varphi < 4$. This is, indeed, also possible, but then κ cannot be too small, depending on φ. More precisely, the constraint $\kappa > \left(1 + \varphi - 2\sqrt{\varphi}\right)|e_2|$ must be satisfied. However, as for $\varphi < 4$, we have $|e_2| = 1$, which means that the curves in Fig. 3(a) and (b) can then be interpreted as the mean and minimally acceptable χ values for sure convergence. For example, for $\varphi = 3$, the constraint $\kappa > 0.536$ must hold, but there is no such restriction on $\kappa \in \left]0,1\right[$ if $\varphi = 1$.



(a)

(b)

Fig. 3. (a) Type 1 constriction coefficient χ as a function of φ and κ. It drops below κ only when $\varphi > 4.0$. (b) Type $1'$ coefficient is less than 1.0 when $\varphi < 4.0$. These coefficients identify the conditions for convergence of the particle system.

Note 4.1: The above analysis is for $\varphi =$ constant. If φ is random, it is nevertheless possible to have convergence, even with a small constriction coefficient, when at least one φ value is strictly inside the interval of variation.

C. Constriction Type $1''$

Referring to the Class $1''$ model, in the particular case where $\delta = 1$, we use the following IR (with χ instead of α)

$$\begin{cases} v(t+1) = \chi(v(t) + \varphi y(t)) \\ y(t+1) = -\chi v(t) + (1 - \chi\varphi) y(t) \end{cases}. \tag{4.8}$$

In fact, this system is hardly different from the classical particle swarm as described in the Section I

$$\begin{cases} v(t+1) = \chi(v(t) + \varphi(p - x(t))) \\ x(t+1) = v(t+1) + x(t) \end{cases} \tag{4.9}$$

so it may be interesting to detail how, in practice, the constriction coefficient is found and its convergence properties proven.

Step 1) *Matrix of the System*

We have immediately

$$M = \begin{bmatrix} \chi & \chi\varphi \\ -\chi & 1 - \chi\varphi \end{bmatrix}. \tag{4.10}$$

Step 2) *Eigenvalues*

They are the two solutions for the equation

$$Z^2 - \text{trace}(M)\, Z + \text{determinant}(M) = 0 \tag{4.11}$$

or

$$Z^2 - (\chi + 1 - \chi\varphi)\,Z + \chi = 0. \tag{4.12}$$

Thus

$$\begin{cases} e'_1 = \frac{\chi+1-\chi\varphi+\sqrt{\Delta}}{2} \\ e'_2 = \frac{\chi+1-\chi\varphi-\sqrt{\Delta}}{2} \end{cases} \tag{4.13}$$

with

$$\begin{aligned}\Delta =& \text{trace}\,(M)^2 - 4\text{determinant}\,(M) \\ =& \chi^2\left(\varphi^2 - 4\varphi + 2\varphi\left(1 - \frac{1}{\chi}\right)\right. \\ & \left. + \left(1 - \frac{1}{\chi}\right)^2\right).\end{aligned} \tag{4.14}$$

Step 3) *Complex and Real Areas on* φ

The discriminant Δ is negative for the φ values in $]1 + ((1/\chi) - (2/\sqrt{\chi})), 1 + (1/\chi) + (2/\sqrt{\chi})[$. In this area, the eigenvalues are true complex numbers and their absolute value (i.e., module) is simply $\sqrt{\chi}$.

Step 4) *Extension of the Complex Region and Constriction Coefficient*

In the complex region, according to the convergence criterion, $\chi < 1$ in order to get convergence. So the idea is to find a constriction coefficient depending on φ so that the eigenvalues are true complex numbers for a large field of φ values. In this case, the common absolute value of the eigenvalues is

$$\begin{cases} \sqrt{\frac{2\kappa}{\varphi-2+\sqrt{\varphi^2-4\varphi}}}, & \text{for } \varphi > 4 \\ \text{else } \sqrt{\kappa} \end{cases} \tag{4.15}$$

which is smaller than one for all φ values as soon as κ is itself smaller than one.

This is generally the most difficult step and sometimes needs some intuition. Three pieces of information help us here:

1) the determinant of the matrix is equal to χ;
2) this is the same as in Constriction Type 1;
3) we know from the algebraic point of view the system is (eventually) convergent like M^T.

So it appears very probable that the same constriction coefficient used for Type 1 will work. First, we try

$$\chi = \frac{\kappa}{|e_2|}, \kappa \in]0, 1[\tag{4.16}$$

that is to say

$$\begin{cases} \frac{2\kappa}{\varphi-2+\sqrt{\varphi^2-4\varphi}}, & \text{for } \varphi > 4 \\ \text{else } \kappa \end{cases}. \tag{4.17}$$

It is easy to see that Δ is negative only between $\varphi_{\min}$ and $\varphi_{\max}$, depending on κ. The general algebraic form of $\varphi_{\max}$ is quite complicated (polynomial in κ^6 with some coefficients being roots of an equation in κ^4) so it is much easier to compute it indirectly for some κ values. If $\varphi_{\min}$ is smaller than four, then $\chi = \kappa$ and by solving $\Delta = 0$ we find that $\varphi_{\min} = (\kappa^2 + \kappa - 2\kappa^{3/2})/\kappa^2$. This relation is valid as soon as $\kappa \geq 1/9$. Fig. 4 shows how the discriminant depends on φ, for two κ values. It is negative between the φ values given in Table II.

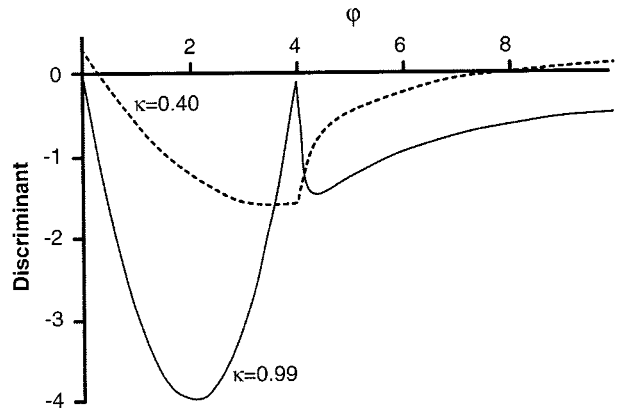


Fig. 4. Discriminant remains negative within some bounds of φ, depending on the value of κ, ensuring that the particle system will eventually converge.

TABLE II
VALUES OF φ BETWEEN WHICH THE DISCRIMINANT IS NEGATIVE, FOR TWO SELECTED VALUES OF κ

κ	$\varphi_{\min}$	$\varphi_{\max}$
0.4	0.3377	8.07
0.99	0.000025	39799.76

D. Moderate Constriction

While it is desirable for the particle's trajectory to converge, by relaxing the constriction the particle is allowed to oscillate through the problem space initially, searching for improvement. Therefore, it is desirable to constrict the system moderately, preventing explosion while still allowing for exploration. To demonstrate how to produce moderate constriction, the following ER is used:

$$\begin{cases} v(t) = c_1 e_1^t + c_2\,(\chi e_2)^t \\ y(t) = \frac{1}{\varphi}\,(c_1 e_1^t (e_1 - 1) + c_2 (\chi e_2)^t (\chi e_2 - 1)) \end{cases}$$
$$\chi = \frac{\kappa}{|e_2|}, \kappa \in]0, 1[\tag{4.18}$$

that is to say

$$\begin{cases} \chi_1 = 1 \\ \chi_2 = \chi \end{cases}.$$

From the relations between ER and IR, (4.19) is obtained, as shown at the bottom of the next page.

There is still an infinity of possibilities for selecting the parameters $\alpha \cdots \eta$. In other words, there are many different IRs that produce the same explicit one. For example

$$\begin{cases} \alpha = \frac{\varphi+2\chi-\chi\varphi}{2} + \frac{\sqrt{\varphi^2-4\varphi}}{2}\,(1-\chi) \\ \beta = -\frac{1}{2\varphi}\Big(\varphi - 3\chi\varphi - \varphi^2 + \chi\varphi^2 + \sqrt{\varphi^2-4\varphi}\,(1 + \chi\varphi - \chi - \varphi)\Big) \\ \gamma = \delta = \eta = 1 \end{cases} \tag{4.20}$$

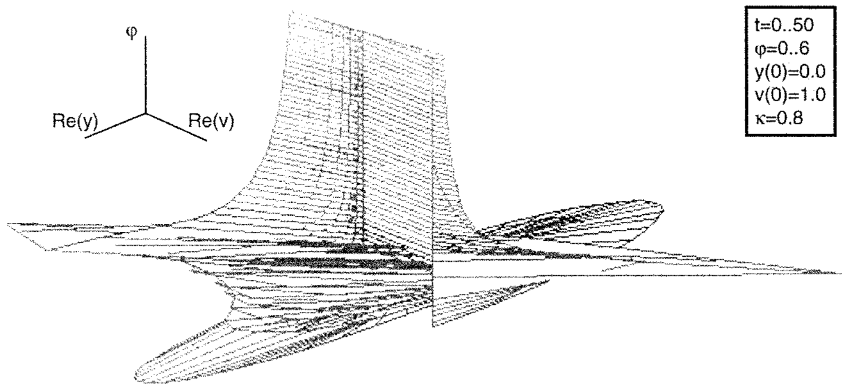


Fig. 5. Real parts of y and v, varying φ over 50 units of time, for a range of φ values.

or

$$\begin{cases} \alpha=\beta=1 \\ \gamma=\frac{\varphi(1+\chi)-\sqrt{\varphi^2-4\varphi}(1-\chi)}{2\varphi} \\ \delta=\eta=\frac{\varphi+\chi(\varphi-2)-\sqrt{\varphi^2-4\varphi}(1-\chi)}{2(\varphi-1)} \end{cases}. \quad (4.21)$$

From a mathematical point of view, this case is richer than the previous ones. There is no more explosion, but there is not always convergence either. This system is "stabilized" in the sense that the representative point in the state space tends to move along an attractor which is not always reduced to a single point as in classical convergence.

E. Attractors and Convergence

Fig. 5 shows a three-dimensional representation of the real restriction $(\text{Re}(y), \text{Re}(v), \varphi)$ of a particle moving in the 5-D space. Fig. 6(a)–(c) show the "real" restrictions $(\text{Re}(y), \text{Re}(v), \varphi)$ of the particles that are typically studied. We can clearly see the three cases:

1) "spiral" easy convergence toward a nontrivial attractor for $\varphi < 4$ [see Fig. 6(a)];
2) difficult convergence for $\varphi \approx 4$ [see Fig. 6(b)];
3) quick almost linear convergence for $\varphi > 4$ [see Fig. 6(c)].

Nevertheless, it is interesting to have a look at the true system, including the complex dimensions. Fig. 6(d)–(f) shows some other sections of the whole surface in R^5.

Note 4.2: There is a discontinuity, for the radius is equal to zero for $\varphi > 4$ (see Fig. 7).

Thus, what seems to be an "oscillation" in the real space is in fact a continuous spiralic movement in a complex space. More importantly, the attractor is very easy to define: it is the "circle" $c_1 e_1^t$ [center (0,0) and radius ρ]. When $\varphi < 4$, $\rho = |c_1 e_1|$ and when $\varphi > 4$, then $\rho = 0$ ($\lim_{t\to\infty} |c_1 e_1^t|$ with $|e_1| < 1$), for the constriction coefficient χ has been precisely chosen so that the part $c_2 (\chi e_2)^t$ of $v(t)$ tends to zero. This provides an intuitive way to transform this stabilization into a true convergence.

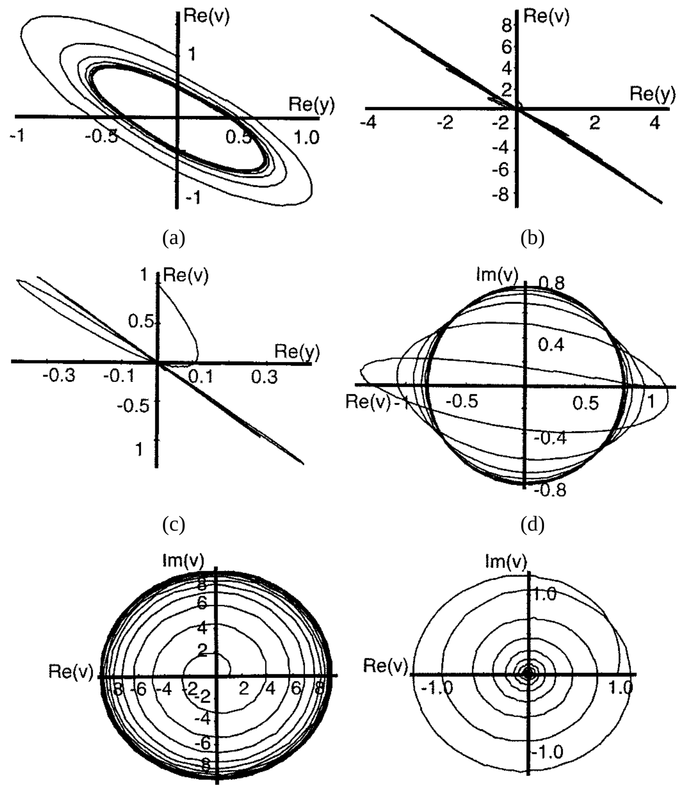
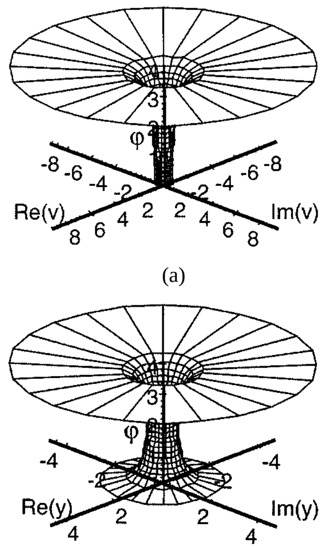


Fig. 6. Trajectories of a particle in phase space with three different values of φ. (a) (c) and (e) Real parts of the velocity v and position relative to the previous best y. (b) (d) and (f) Real and imaginary parts of v. (a) and (d) show the attractor for a particle with $\varphi = 2.5$. Particle tends to orbit, rather than converging to 0.0. (b) and (e) show the same views with $\varphi = 3.99$. (c) and (f) depict the "easy" convergence toward 0.0 of a constricted particle with $\varphi = 6.0$. Particle oscillates with quickly decaying amplitude toward a point in the phase space (and the search space).

$$\begin{cases} 2(\alpha+\delta-\eta\varphi) = (1+\chi)(2-\varphi) + (1-\chi)\sqrt{\varphi^2-4\varphi} \\ 2\sqrt{(\eta\varphi)^2 + 2\varphi(\alpha\eta-\delta\eta-2\beta\gamma) + (\alpha-\delta)^2} = (1+\chi)\sqrt{\varphi^2-4\varphi} + (1-\chi)(2-\varphi) \end{cases} \quad (4.19)$$



(a)

(b)

Fig. 7. "Trumpet" global attractor when $\varphi < 4$. Axis $(\mathrm{Re}(v), \mathrm{Im}(v), \varphi), \kappa = 8$. (a) Effect on φ of the real and imaginary parts of v. (b) Effects of the real and imaginary parts of y.

We just have to use a second coefficient in order to reduce the attractor, in the case $\varphi < 4$, so that

$$e_1 \to \chi' e_1, \chi' \le \frac{\kappa'}{|e_1|}, \kappa' \in \left]0, 1\right[. \tag{4.22}$$

The models studied here have only one constriction coefficient. If one sets $\chi' = \chi$, the Type 1 constriction is produced, but now, we understand better *why* it works.

V. Generalization of the Particle-Swarm System

Thus far, the focus has been on a special version of the particle swarm system, a system reduced to scalars, collapsed terms and nonprobabilistic behavior. The analytic findings can easily be generalized to the more usual case where φ is random and two vector terms are added to the velocity. In this section the results are generalized back to the original system as defined by

$$\begin{cases} v(t+1) = v(t) + \varphi_1(p_1 - x(t)) + \varphi_2(p_2 - x(t)) \\ x(t+1) = v(t+1) + x(t) \end{cases} \tag{5.1}$$

Now φ, p, and $y(t)$ are defined to be

$$\begin{aligned} \varphi &= \varphi_1 + \varphi_2 \\ p &= \frac{\varphi_1 p_1 + \varphi_2 p_2}{\varphi_1 + \varphi_2} \\ y(t) &= p - x(t) \end{aligned} \tag{5.2}$$

to obtain exactly the original nonrandom system described in Section I.

For instance, if there is a cycle for $\varphi = \varphi_c$, then there is an infinity of cycles for the values $\{\varphi_1, \varphi_2\}$ so that $\varphi_1 + \varphi_2 = \varphi_c$.

Upon computing the constriction coefficient, the following form is obtained:

$$\begin{aligned} \chi &= \frac{\kappa}{|e_2|} = \frac{\kappa}{\left|1 - \frac{\varphi}{2} - \frac{\sqrt{\varphi(\varphi-4)}}{2}\right|} \\ &= \frac{2\kappa}{\left|2 - \varphi - \sqrt{\varphi(\varphi - 4)}\right|} \\ &= \frac{2\kappa}{\left|2 - \varphi_1 - \varphi_2 - \sqrt{(\varphi_1 + \varphi_2)(\varphi_1 + \varphi_2 - 4)}\right|}, \\ &\quad \text{if } (\varphi_1 + \varphi_2) > 4 \\ &= \kappa \text{ else} \\ \kappa &\in \left]0, 1\right[. \end{aligned} \tag{5.3}$$

Coming back to the (v, x) system, v and x are

$$\begin{cases} v(t+1) = v(t) + \varphi_1(p_1 - x(t)) + \varphi_2(p_2 - x(t)) \\ x(t+1) = \chi v(t+1) + \chi x(t) + (1-\chi) \frac{\varphi_1 p_1 + \varphi_2 p_2}{\varphi_1 + \varphi_2} \end{cases}. \tag{5.4}$$

The use of the constriction coefficient can be viewed as a recommendation to the particle to "take smaller steps." The convergence is toward the point $(v = 0, x = (\varphi_1 p_1 + \varphi_2 p_2)/(\varphi_1 + \varphi_2))$. Remember v is in fact the velocity of the particle, so it will indeed be equal to zero in a convergence point.[2] Example

$$\begin{cases} v_0 = 1, x_0 = 4.5 \\ p_1 = 3, p_2 = 4 \\ \varphi_{\max,1} = 0.1, \varphi_{\max,2} = 5 \end{cases}$$

φ_1 and φ_2 are uniform random variables between 0 and $\varphi_{\max,1}$ and $\varphi_{\max,2}$ respectively. This example is shown in Fig. 8.

VI. Running the Particle Swarm With Constriction Coefficients

As a result of the above analysis, the particle swarm algorithm can be conceived of in such a way that the system's explosion can be controlled, without resorting to the definition of any arbitrary or problem-specific parameters. Not only can explosion be prevented, but the model can be parameterized in such a way that the particle system consistently converges on local optima. (Except for a special class of functions, convergence on global optima cannot be proven.)

The particle swarm algorithm can now be extended to include many types of constriction coefficients. The most general modification of the algorithm for minimization is presented in the following pseudocode.

```
Assign κ, φ_max
Calculate χ, α, β, γ, δ, η
Initialize population: random x_i, v_i
Do
   For i = 1 to population size
```

[2]Convergence implies velocity $= 0$, but the convergent point is not necessarily the one we want, particularly if the system is *too* constricted. We hope to show in a later paper how to cope with this problem, by defining the optimal parameters.

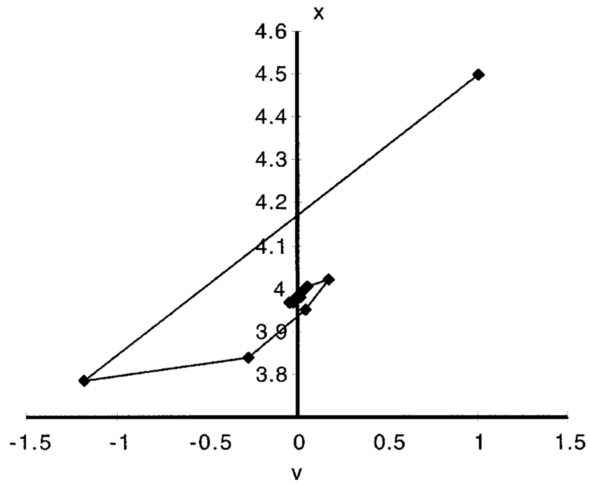


Fig. 8. Example of the trajectory of a particle with the "original" formula containing two $\varphi(p-x)$ terms, where φ is the upper limit of a uniform random variable. As can be seen, velocity v converges to 0.0 and the particle's position x converges on the previous best point p.

    if $f(x_i) < f(p_i)$ then $p_i = x_i$
        For $d = 1$ to dimension
            $\varphi 1 = \text{rand}() \times (\varphi_{\max}/2)$
            $\varphi 2 = \text{rand}() \times (\varphi_{\max}/2)$
            $\varphi = \varphi 1 + \varphi 2$
            $p = ((\varphi 1^* p_{id}) + (\varphi 2^* p_{gd}))/\varphi$
            $x = x_{id}$
            $v = v_{id}$
            $v_{id} = \alpha^* v + \beta^* \varphi^* (p - x)$
            $x_{id} = p + \gamma^* v - (\delta - (\eta^* \varphi))^* (p - x)$
        Next d
    Next i
Until termination criterion is met.

In this generalized version of the algorithm, the user selects the version and chooses values for κ and φ that are consistent with it. Then the two eigenvalues are computed and the greater one is taken. This operation can be performed as follows.

discrim $= ((\eta\varphi)^2 - 4\beta\gamma + (\alpha - \delta)^2 + 2\eta\varphi(\alpha - \delta))/4$
a $= (\alpha + \delta - \mu\varphi)/2$
if $(\text{discrim} > 0)$ then
    neprim1 $= \text{abs}(a + \sqrt{\text{discrim}})$
    neprim2 $= \text{abs}(a - \sqrt{\text{discrim}})$
else
    neprim1 $= \sqrt{a^2 + \text{abs}(\text{discrim})}$
    neprim2 $=$ neprim1
$\max(\text{eig.}) = \max(\text{neprim1}, \ \text{neprim2})$

These steps are taken only once in each program and, thus, do not slow it down. For the versions tested in this paper, the constriction coefficient is calculated simply as $\chi = \kappa/\max(\text{eig.})$. For instance, the Type 1 version is defined by the rules $\alpha = \beta = \gamma = \delta = \eta = \chi$.

The generalized description allows the user to control the degree of convergence by setting κ to various values. For instance, in the Type 1$''$ version, $\kappa = 1.0$ results in slow convergence, meaning that the space is thoroughly searched before the population collapses into a point.

In fact, the Type 1$''$ constriction particle swarm can be programmed as a very simple modification to the standard version presented in Section I. The constriction coefficient χ is calculated as shown in (4.15)

$$\varphi = \begin{cases} \sqrt{\frac{2\kappa}{\varphi - 2 + \sqrt{\varphi^2 - 4\varphi}}}, \text{ for } \varphi > 4 \\ \text{else} \sqrt{\kappa} \end{cases}.$$

The coefficient is then applied to the right side of the velocity adjustment.

Calculate χ
Initialize population
Do
    For $i = 1$ to Population Size
    if $f(\vec{x}_i) < f(\vec{p}_i)$ then $\vec{p}_i = \vec{x}_i$
    $\vec{p}_g = \min(\vec{p}_{\text{neighbors}})$
        For $d = 1$ to Dimension
            $v_{id} = \chi(v_{id} + \varphi_1 (p_{id} - x_{id}) + \varphi_2 (p_{gd} - x_{id}))$
            $x_{id} = x_{id} + v_{id}$
        Next d
    Next i
Until termination criterion is met.

Note that the algorithm now requires no explicit limit $V_{\max}$. The constriction coefficient makes it unnecessary. In [8], Eberhart and Shi recommended, based on their experiments, that a liberal $V_{\max}$, for instance, one that is equal to the dynamic range of the variable, be used in conjunction with the Type 1$''$ constriction coefficient. Though this extra parameter may enhance performance, the algorithm will still run to convergence even if it is omitted.

VII. Empirical Results

Several types of particle swarms were used to optimize a set of unconstrained real-valued benchmark functions, namely, several of De Jong's functions [9], Schaffer's f6, and the Griewank, Rosenbrock, and Rastrigin functions. A population of 20 particles was run for 20 trials per function, with the best performance evaluation recorded after 2000 iterations. Some results from Angeline's [1] runs using an evolutionary algorithm are shown for comparison.

Though these functions are commonly used as benchmark functions for comparing algorithms, different versions have appeared in the literature. The formulas used here for De Jong's f1, f2, f4 (without noise), f5, and Rastrigin functions are taken from [10]. Schaffer's f6 function is taken from [11]. Note that earlier editions give a somewhat different formula. The Griewank function given here is the one used in the First International Contest on Evolutionary Optimization held at ICEC 96 and the 30-dimensional generalized Rosenbrock function is taken from [1]. Functions are given in Table III.

TABLE III
FUNCTIONS USED TO TEST THE EFFECTS OF THE CONSTRICTION COEFFICIENTS

Sphere function (De Jong's f1)	$f_1(x) = \sum_{i=1}^{n} x_i^2$
Rosenbrock variant (De Jong's f2)	$f_2(x) = 100(x_1^2 - x_2)^2 + (1 - x_1)^2$
De Jong's f4 – no noise	$f_4(x) = \sum_{i=1}^{n} i \cdot x_i^4$
Foxholes (De Jong's f5)	$f_5(x) = 1/0.002 + \sum_{j=1}^{25} \frac{1}{j + \sum_{i=1}^{2} (x_i - a_{ij})^6}$
Shaffer's f6	$f_6(x) = 0.5 + \frac{(\sin\sqrt{x^2+y^2}) - 0.5}{(1.0 + 0.001(x^2+y^2))^2}$
Griewank function	$f_7(x) = \frac{1}{4000} \sum_{i=1}^{n} (x_i - 100)^2 - \prod_{i=1}^{n} \cos\left((x_i - 100)/\sqrt{i}\right) + 1$
Rosenbrock function	$f_9(x) = \sum_{i=1}^{n} (100(x_{i+1} - x^2)^2 + (x_i - 1)^2)$
Rastrigin function	$f_{10}(x) = \sum_{i=1}^{n} \left[x_i^2 - 10\cos(2\pi x_i) + 10\right]$

TABLE IV
FUNCTION PARAMETERS FOR THE TEST PROBLEMS

Function	Dimension	Initial Range
1	30	±20
2	2	±50
4	30	±20
5	2	±50
Shaffer's f6	2	±100
Griewank	30	±300
Ackley	30	±32
Rastrigin	30	±5.12
Rosenbrock	30	±10

A. Algorithm Variations Used

Three variations of the generalized particle swarm were used on the problem suite.

Type 1: The first version applied the constriction coefficient to all terms of the formula

$$\alpha = \beta = \gamma = \delta = \eta = \chi$$

using $\kappa = 0.8$.

Type 1″: The second version tested was a simple constriction, which was not designed to converge, but not to explode, either, as was assigned a value of 1.0. The model was defined as

$$\alpha = \beta = \chi$$
$$\gamma = \delta = \eta = 1.0.$$

Experimental Version: The third version tested was more experimental in nature. The constriction coefficient χ was initially defined as $\kappa/\max(e_1, e_2)$. If $\chi > 1$, then it was multiplied by 0.9 iteratively. Once a satisfactory value was found, the following model was implemented:

$$\alpha = \beta = 1$$
$$\gamma = \chi$$
$$\delta = \eta = \chi^2.$$

As in the first version, a "generic" value of $\kappa = 0.8$ was used. Table IV displays the problem-specific parameters implemented in the experimental trials.

B. Results

Table V compares various constricted particle swarms' performance to that of the traditional $V_{\max}$ particle swarm and evolutionary optimization (EO) results reported by [1]. All particle swarm populations comprised 20 individuals.

Functions were implemented in 30 dimensions except for f2, f5, and f6, which are given for two dimensions. In all cases except f5, the globally optimal function result is 0.0. For f5, the best known result is 0.998004. The limit of the control parameter φ was set to 4.1 for the constricted versions and 4.0 for the $V_{\max}$ versions of the particle swarm. The column labeled "E&S" was programmed according to the recommendations of [8]. This condition included both Type 1″ constriction and $V_{\max}$, with $V_{\max}$ set to the range of the initial domain for the function. Function results were saved with six decimal places of precision.

As can be seen, the Type 1″ and Type 1 constricted versions outperformed the $V_{\max}$ versions in almost every case; the experimental version was sometimes better, sometimes not. Further, the Type 1″ and Type 1 constricted particle swarms performed better than the comparison evolutionary method on three of the four functions. With some caution, we can at least consider the performances to be comparable.

Eberhart and Shi's suggestion to hedge the search by retaining $V_{\max}$ with Type 1″ constriction does seem to result in good performance on all functions. It is the best on all except the Rosenbrock function, where performance was still respectable. An analysis of variance was performed comparing the "E&S" version with Type 1″, standardizing data within functions. It was found that the algorithm had a significant main effect $F(1, 342) = 12.02, p < 0.0006$, but that there was a significant interaction of algorithm with function $F(8{,}342) = 3.68$, $p < 0.0004$, suggesting that the gain may not be robust across all problems. These results support those of [8].

Any comparison with Angeline's evolutionary method should be considered cautiously. The comparison is offered only as a *prima facie* standard by which to assess performances on these functions after this number of iterations. There are numerous versions of the functions reported in the literature

TABLE V
EMPIRICAL RESULTS

Function	$V_{max}=2$	$V_{max}=4$	Type 1"	Type 1	Exp. Version	E&S	Angeline
1	15.577775	59.301901	0	0	0	0	9.8808
2	0.000500	0.0013263	0	0	0	0	
4	271.107996	4349.137512	0	0	0	0	
5	2.874299	3.564808	0.998004	0.998004	3.507922	0.998004	
Shaffer's f6	0.000464	0.000247	0.001459	0.002915	29.173010	0.000155	
Griewank	0.562339	0.968623	0.003944	0.008614	0.038923	0.002095	0.4033
Ackley	4.287476	6.623447	0.204988	0.150886	7.135213	0.104323	
Rastrigin	223.834812	299.771716	82.95618	81.689550	63.222601	57.194136	46.4689
Rosenbrock	2770.882599	37111.70703	50.193877	39.118488	47.753953	50.798139	1610.359

Mean best evaluations at the end of 2 000 iterations for various versions of particle swarm and Angeline's evolutionary algorithm [1].

and it is extremely likely that features of the implementation are responsible for some variance in the observed results. The comparison though does allow the reader to confirm that constricted particle swarms are comparable in performance to at least one evolutionary algorithm on these test functions.

As has long been noted, the $V_{\max}$ particle swarm succeeds at finding optimal regions of the search space, but has no feature that enables it to converge on optima (e.g., [1]). The constriction techniques reported in this paper solve this problem, they do force convergence. The data clearly indicate an increase in the ability of the algorithm to find optimal points in the search space for these problems as a result.

No algorithmic parameters were adjusted for any of the particle swarm trials. Parameters such as $V_{\max}$, φ, population size, etc., were held constant across functions. Further, it should be emphasized that the population size of 20 is considerably smaller than what is usually seen in evolutionary methods, resulting in fewer function evaluations and consequently faster clock time in order to achieve a similar result. For instance, Angeline's results cited for comparison are based on populations of 250.

VIII. CONCLUSION

This paper explores how the particle swarm algorithm works from the inside, i.e., from the individual particle's point of view. How a particle searches a complex problem space is analyzed and improvements to the original algorithm based on this analysis are proposed and tested. Specifically, the application of constriction coefficients allows control over the dynamical characteristics of the particle swarm, including its exploration versus exploitation propensities.

Though the pseudocode in Section VI may look different from previous particle swarm programs, it is essentially the same algorithm rearranged to enable the judicious application of analytically chosen coefficients. The actual implementation may be as simple as computing one constant coefficient and using it to weight one term in the formula. The Type $1''$ method, in fact, requires only the addition of a single coefficient, calculated once at the start of the program, with almost no increase in time or memory resources.

In the current analysis, the sine waves identified by Ozcan and Mohan [6], [7] turn out to be the real parts of the 5-D attractor. In complex number space, e.g., in continuous time, the particle is seen to spiral toward an attractor, which turns out to be quite simple in form: a circle. The real-number section by which this is observed when time is treated discretely is a sine wave.

The 5-D perspective summarizes the behavior of a particle completely and permits the development of methods for controlling the explosion that results from randomness in the system. Coefficients can be applied to various parts of the formula in order to guarantee convergence, while encouraging exploration. Several kinds of coefficient adjustments are suggested in the present paper, but we have barely scratched the surface and plenty of experiments should be prompted by these findings. Simple modifications based on the present analysis resulted in an optimizer which appears, from these preliminary results, to be able to find the minima of some extremely complex benchmark functions. These modifications can guarantee convergence, which the traditional $V_{\max}$ particle swarm does not. In fact, the present analysis suggests that no problem-specific parameters may need to be specified.

We remind the reader that the real strength of the particle swarm derives from the interactions among particles as they search the space collaboratively. The second term added to the velocity is derived from the successes of others, it is considered a "social influence" term; when this effect is removed from the algorithm, performance is abysmal [3]. Effectively, the variable $\vec{p}_g$ keeps moving, as neighbors find better and better points in the search space and its weighting relative to $\vec{p}_i$ varies randomly with each iteration. As a particle swarm population searches over time, individuals are drawn toward one another's successes, with the usual result being clustering of individuals in optimal regions of the space. The analysis of the social-influence aspect of the algorithm is a topic for a future paper.

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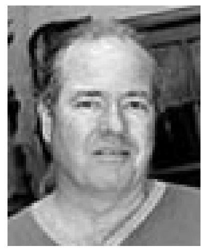

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