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# Particle swarm optimization almost surely finds local optima ☆

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ABSTRACT

Particle swarm optimization (PSO) is a popular nature-inspired meta-heuristic for solving continuous optimization problems. Although this technique is widely used, up to now only some partial aspects of the method have been formally investigated. In particular, while it is well-studied how to let the swarm converge to a single point in the search space, no general theoretical statements about this point or on the best position any particle has found have been known. For a very general class of objective functions, we provide for the first time results about the quality of the solution found. We show that a slightly adapted PSO almost surely finds a local optimum. To do so, we investigate the newly defined *potential* of the swarm. The potential drops when the swarm approaches the point of convergence, but increases if the swarm remains close to a point that is not a local optimum, meaning that the swarm charges potential and continues its movement.



## 1. Introduction

*Background* Particle swarm optimization (PSO), introduced by Kennedy and Eberhart [14,6], is a very popular meta-heuristic for solving continuous optimization problems. It is inspired by the social interaction of individuals living together in groups and supporting and cooperating with each other. Fields of very successful application are, among many others, Biomedical Image Processing [27], Geosciences [18], and Materials Science [21], to name just a few, where the continuous objective function on a multi-dimensional domain is not given in a closed form, but by a "black box." The popularity of the PSO framework in these scientific communities is due to the fact that it on the one hand can be realized and, if necessary, adapted to further needs easily, but on the other hand shows in experiments good performance results with respect to the quality of the obtained solution and the speed needed to obtain it. By adapting its parameters, users may in real-world applications easily and successfully control the swarm's behavior with respect to "exploration" ("searching where no one has searched before") and "exploitation" ("searching around a good position"). A thorough discussion of PSO can be found in [15].

To be precise, let an objective function $f : \mathbb{R}^D \to \mathbb{R}$ on a $D$-dimensional domain be given that (w.l.o.g.) has to be minimized. A population of *particles*, each consisting of a position (the candidate for a solution), a velocity and a local attractor, moves through the search space $\mathbb{R}^D$. The local attractor of a particle is the best position with respect to $f$ this particle has encountered so far. The population in motion is the *swarm*. In contrast to evolutionary algorithms, the individuals of the swarm cooperate by sharing information about the search space via the global attractor, which is the best position any particle has found so far. The particles move in time-discrete iterations. The movement of a particle is governed by

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so-called movement equations that depend on both the particle's velocity and its two attractors and on some additional fixed parameters (for details, see Section 2).

Although this method is widely used in real-world applications, there has unfortunately not been any formal analysis explaining more than only partial aspects of the algorithm. Theoretical analyses of the particles' trajectories can be found in, e.g., [4,19,20,30,7,17]. A discussion of runtime aspects is presented in [28]. For the case of a bounded high-dimensional search space, theoretical work about the initial behavior of the particle swarm is presented in [9,10]. Some guidelines for the choice of the fixed parameters that control the impact of the current velocity and the attractors on the updated velocity of a particle can be found in [25,12], where the authors prove that under certain conditions about the parameters the swarm provably converges. Analyses of the convergence time of a statistical interpretation of PSO can be found in [3,2]. However, mathematical properties of the limit, i.e., the quality of the solution, are unknown. Van den Bergh/Engelbrecht [26] substantially modify the movement equations, enabling the particles to count the number of times they improve the global attractor and use that information. Empirical evidence for the capability of their method to find local optima on common benchmarks is given.

Closest to our work, Lehre/Witt [16] modify the movement equations by adding in *every* step a small random perturbation to the velocity. Their detailed analysis for one specific, simple, one-dimensional objective function shows that their modified PSO finds this function's unique optimum. But it is not clear how to apply this special analysis to higher dimensional search spaces and larger classes of functions.

An overview of known theoretical results about PSO can be found in [29].

*New results* Up to now, every formal analysis providing results about the quality of the best search point obtained by the particle swarm has been restricted to a very special class of objective functions. In this paper, we provide the first *general* mathematical analysis of the quality of the global attractor when it is considered as a solution for objective functions from a very general class $\mathcal{F}$ of functions (see Definition 2 below) and therefore of the quality of the algorithm's return value. Informally, functions in $\mathcal{F}$ resemble (if restricted to 3-dimensional pictures) an (everted) island. On its almost arbitrary surface, the lowest point has to be found. The sea around the island is of no interest. We introduce the new approach of defining the *potential* $\Phi$ of the particle swarm that changes after every step. $\Phi$ covers two properties of the swarm: It tends to zero if the particles converge, but it increases if the whole swarm stays close to a search point that is no local minimum. In the latter case, the swarm charges potential and resumes its movement. As a consequence, we prove an emergent property of PSO for $\mathcal{F}$, namely that in the one-dimensional case the swarm almost surely (in the well-defined probabilistic sense) finds a local optimum. For the general $D$-dimensional case, we slightly modify the classical PSO and prove that this adapted swarm almost surely finds a local optimum. Necessity of such a modification is constituted in [24], where we report on significant experiments that show that even on the very simple sphere function the swarm does not necessarily converge towards the only local optimum at the center. Note that our analysis carries over to presumably all variants of PSO developed so far. One would expect such strong results to hold only for a very small class of objective functions. Indeed, we need some restrictions, but as it turns out the class $\mathcal{F}$ of objective functions for which our results hold is much more general than, e.g., the subset of the class of unimodal functions that is considered in [11] in the context of restricted $(1+1)$ evolutionary algorithms.

We conclude this paper by presenting experimental observations that confirm that our mathematical results on continuous processes and observations carry over to actual optimization on concrete "discrete" computers.

## 2. Analysis of the classical PSO algorithm

First we present the model we are going to use for our analysis of the PSO algorithm. The model describes the positions of the particles, the velocities and the global and local attractors as real-valued stochastic processes. Furthermore, we define in Definition 5 the potential of the swarm which depends on the state of the particles and will be a measure for their movement. A swarm with high potential is more likely to reach search points far away from the current global attractor, while a swarm with potential approaching 0 is converging. (The basic mathematical tools from probability theory we need for our analysis can be found in, e.g., [5].)

**Definition 1** *(Classical PSO Process).* A *swarm* $\mathcal{S}$ of $N$ particles moves through the $D$-dimensional search space $\mathbb{R}^D$. Let $f : \mathbb{R}^D \to \mathbb{R}$ be the objective function. For $\mathcal{S}$, we define the stochastic process $(S_t)_{t\in\mathbb{N}_0} = ((X_t, V_t, L_t, G_t))_{t\in\mathbb{N}_0} = ((X_0, V_0, L_0, G_0), (X_1, V_1, L_1, G_1), \ldots)$, consisting of

- $X_t = (X_t^{n,d})_{1\le n\le N, 1\le d\le D}$ ($d$-th coordinate of the *position* of particle $n$ after step $t$),
- $V_t = (V_t^{n,d})_{1\le n\le N, 1\le d\le D}$ ($d$-th coordinate of the *velocity* of particle $n$ after step $t$),
- $L_t = (L_t^{n,d})_{1\le n\le N, 1\le d\le D}$ ($d$-th coordinate of the *local attractor* of particle $n$ after step $t$),
- $G_t = (G_t^{n,d})_{1\le n\le N, 1\le d\le D}$, ($d$-th coordinate of the *global attractor* before the $t$-th step of particle $n$).

We will write $X_t^n$ for the vector $(X_t^{n,1}, \ldots, X_t^{n,D})$ (analogously, $V_t^n$, $L_t^n$, $G_t^n$) and $X_t^d$ for the vector $(X_t^{1,d}, \ldots, X_t^{N,d})$ (analogously, $V_t^d$, $L_t^d$, $G_t^d$). Furthermore $\tilde{G}_t^{n,d}$ denotes the $d$-th coordinate of the global attractor *after* the $t$-th step of particle $n$, i.e., $\tilde{G}_t^{n,d} =$

**Algorithm 1:** classical PSO.

**output**: $G \in \mathbb{R}^D$
**repeat**
  **for** $n = 1 \to N$ **do**
    **for** $d = 1 \to D$ **do**
      $V_d^n := \chi \cdot V_d^n + c_1 \cdot \mathrm{rand}() \cdot (L_d^n - X_d^n) + c_2 \cdot \mathrm{rand}() \cdot (G_d - X_d^n)$;
      $X_d^n := X_d^n + V_d^n$;
    **if** $f(X^n) \le f(L^n)$ **then** $L^n := X^n$;
    **if** $f(X^n) \le f(G)$ **then** $G := X^n$;
**until** *termination criterion met*;

$G_t^{n+1,d}$ if $n < N$, and $\tilde{G}_t^{N,d} = G_{t+1}^{1,d}$. With a given distribution for $(X_0, V_0)$ and the values $G_0^1 := \mathrm{argmin}_{1 \le n \le N}\{f(X_0^n)\}$ and $L_0 := X_0$, $S_{t+1} = (X_{t+1}, V_{t+1}, L_{t+1}, G_{t+1})$ is determined by the following recursive equations that are called the *movement equations*:

- $G_t^{n+1} = \mathrm{argmin}\{f(L_t^n), f(G_t^n)\}$ for $t \ge 0$, $1 \le n \le N-1$,
- $G_{t+1}^1 = \mathrm{argmin}\{f(L_t^N), f(G_t^N)\}$ for $t \ge 0$,
- $V_{t+1}^{n,d} = \chi \cdot V_t^{n,d} + c_1 \cdot r_t^{n,d} \cdot (L_t^{n,d} - X_t^{n,d}) + c_2 \cdot s_t^{n,d} \cdot (G_{t+1}^{n,d} - X_t^{n,d})$ for $t \ge 0$,
- $X_{t+1}^{n,d} = X_t^{n,d} + V_{t+1}^{n,d}$ for $t \ge 0$,
- $L_{t+1}^n = \mathrm{argmin}\{f(X_{t+1}^n), f(L_t^n)\}$.

In case of a tie when applying argmin, the new value prevails, i.e., whenever a particle finds a search point with value equal to the one of its local attractor, this point becomes the new local attractor. If additionally the value is equal to the one of the global attractor, this one is also updated. Here, $\chi$, $c_1$ and $c_2$ are some positive constants called the *fixed parameters* of $\mathcal{S}$, and $r_t^{n,d}$, $s_t^{n,d}$ are uniformly distributed over $[0, 1]$ and all independent.

If after the $t$-th step the process is stopped, the *solution* found by $\mathcal{S}$ so far is $\tilde{G}_t^N$. Algorithm 1 gives a representation of the classical PSO algorithm in pseudo code.

Basically, this definition describes the common movement equations with two specifications: If a particle visits a point with the same objective value as its local attractor or the global attractor, then the attractor is updated to the new point. And the global attractor is updated after every step of a single particle, not only after every iteration of the whole swarm. Another common variant of PSO only updates the global attractor after every iteration of the whole swarm, however, due to our choice the information shared between the particles is as recent as possible.

PSO is designed to handle any objective function. But for the rest of this paper, we consider only objective functions from the set $\mathcal{F}$ defined below.

**Definition 2.** Let $f : \mathbb{R}^D \to \mathbb{R}$ be a function. $f \in \mathcal{F}$ iff

(i) there is a compact set $K \subset \mathbb{R}^D$ with positive Lebesgue measure, such that $P(X_0^n \in K) = 1$ for every $n$ and $\{x \in \mathbb{R}^D \mid f(x) \le \sup_K f\}$ (the island) is bounded;
(ii) $f$ is continuous and has a continuous derivative.

Restriction (i) states that there is a compact set $K$ such that for all $x \in K$, the set of all search points $y$ at least as good as $x$, i.e., all $y$ with $f(y) \le f(x)$, is a bounded set. Since the particles are initialized inside $K$ and since $f(G_t^n)$ is nonincreasing in $t$, (i) ensures that the possible area for the global attractor is limited if the positions of all particles are initialized inside of $K$ (being on any point of the island is better than being in the sea). If for example $\lim_{|x| \to \infty} f(x) = \infty$ or if $f$ has compact support and is negative on $K$, (i) is already satisfied. E.g., common benchmark functions like the sphere function $f(x) = \sum_{i=1}^D x_i^2$ or the Rosenbrock function $f(x) = \sum_{i=1}^{D-1}((1-x_i)^2 + 100 \cdot (x_{i+1} - x_i^2)^2)$ are in $\mathcal{F}$. On functions that violate (i), the swarm might move forever because either they do not necessarily have a local optimum like $f(x) = x$ or they have an optimum, but improvements can be made arbitrary far away from it, like, e.g. in the case of the function $f(x) = x^2/(x^4+1)$, where $x = 0$ is the only local and the global optimum, but if the particles are far away from 0, they tend to further increase the distance because $f$ converges to 0 as $|x|$ approaches $\infty$. Under such circumstances, convergence cannot be expected and it is necessary to restrict the function class in order to avoid this. However, (ii) might be the only true restriction.

From Definition 1 it follows that $(S_t)_{t \in \mathbb{N}}$ has the strong Markov property, i.e., the distribution of $S_{t+t_0}$ given all the previous values $(S_0, \ldots, S_{t_0})$ is the same as the distribution of $S_{t+t_0}$ given only $S_{t_0}$. This is clear because $S_{t+1}$ is formulated in terms of $S_t$ and some independent random variables $r_t^{n,d}$ and $s_t^{n,d}$. As a result from the theory of probability, for our discrete time space $\mathbb{N}_0$ the property still holds if one replaces the constant $t_0$ by an almost surely finite stopping time $\tau$. Another interesting property of this stochastic process that follows immediately from the movement equations is the following:

**Observation 1.** *Let $\mathcal{S}$ be a swarm and $((X_t, V_t, L_t, G_t))_{t\in\mathbb{N}_0}$ its corresponding stochastic process. Let $\mathcal{L}^k$ denote the k-dimensional Lebesgue measure, $\mathcal{L}[Y]$ the distribution of a random variable Y and "$\ll$" (just in this observation) absolute continuity between two distributions. Assuming $\mathcal{L}[(X_0, V_0)] \ll \mathcal{L}^{2\cdot N\cdot D}$, it follows $\mathcal{L}[X_t] \ll \mathcal{L}^{N\cdot D}$ for every $t \geq 0$. If $X_t^n \neq \tilde{G}_t^N$ for every n, then for every $t' > t$, $\mathcal{L}[X_{t'} \mid S_t] \ll \mathcal{L}^{N\cdot D}$ almost surely.*[1]

This observation follows from the structure of the movement equations, i.e., the entries of $X_t$ are weighted sums of the entries of $X_0$, which by assumption has probability 0 to hit any fixed Lebesgue null set, and values that are chosen uniformly and independently from certain intervals. If $X_t^n \neq \tilde{G}_t^N$ for every $n$, then for every $t' > t$, the entries of $X_{t'}$ are sums of constants depending on $S_t$ and a non-empty sum of values, chosen uniformly and independently from certain intervals.

If the positions and velocities of the initial population are distributed in some natural way, e.g., uniformly at random over $K$, Observation 1 states that the swarm has similar restrictions as a process consisting only of variables that are sampled u.a.r. in the sense that events with probability 0 in the latter case also have probability 0 in the first case. If the function contains no plateaus, we cannot expect the swarm to ever hit a local optimum since the probability to hit that particular point by just sampling is zero and so is the probability for hitting it with the swarm. Another consequence is that no point in $\mathbb{R}^D$ is visited more than once because that would not happen under uniform sampling since $\mathbb{R}^D$ is not enumerable. This implies that the well-studied equilibrium when every particle is at the global attractor and has velocity 0 is a state that the process may converge to but that will never be reached. Therefore we give a definition of the kind of convergent behavior that can be expected.

**Definition 3** *(Convergence).* Swarm $\mathcal{S}$ *converges* if there almost surely is a point $z$ such that the following two conditions hold:

1. $\lim_{t\to\infty} V_t = 0$ (the movement of the particles tends to zero),
2. $\lim_{t\to\infty} X_t^n = z$ for each $n \in \{1, \ldots, N\}$ (every particle moves towards $z$).

A consequence of the above conditions is that for every $n \in \{1, \ldots, N\}$ $\lim_{t\to\infty} G_t^n = z$ almost surely and $\lim_{t\to\infty} L_t^n = z$ almost surely. Although the convergence analysis in the literature [12] usually makes the assumption that at least the global attractor is constant forever, a prerequisite that because of Observation 1 we cannot assume, the generalization of the convergence proof from [12], showing that their results still hold under the weaker assumption of only the convergence of the attractors, is straight-forward.

We want to prove more, namely, that under the stated assumptions about $f$ the swarm is able to find a local minimum. Here, the notion of the potential of a swarm comes into play. Roughly speaking, as long as the swarm has potential high enough to overcome the distance to at least one local minimum, the probability to find it within a few steps is still positive. A problem occurs when the value of the potential is too low for the swarm to overcome the distance to the next optimum by only a small number of steps. In other words, if $f$ is monotonically decreasing in some direction and on an area that is large in comparison to the potential of the swarm, the particles must be able to "run down the hill," i.e., they must be able to surpass every distance as long as $f$ decreases. The following definition formally describes the situation of a swarm while it is "running down the hill" and will lead to the definition of the potential.

**Definition 4** *(Running particle swarm).* Let $d_0 \leq D$ be an arbitrary dimension. We call $\mathcal{S}$ positively running in direction $d_0$, if the following two properties hold: First, $G_t^{n,d_0} = \max_{1\leq i\leq N}\{X_{t'}^{i,d_0}\}$ for $t' = t$ if $i < n$ and $t' = t-1$ otherwise. Second, $L_t^n = X_t^n$ for every $n$. In other words, while the swarm is running, each particle updates its local attractor at every step and the global attractor is always the local attractor with greatest value in the $d_0$-th dimension. If the global attractor is always the position with smallest $d_0$-value instead, we call the swarm negatively running in direction $d_0$.

Note that while a particle swarm is positively running, every particle has positive velocity in direction $d_0$, in terms: $V_t^{n,d_0} > 0$. If the swarm is negatively running in direction $d_0$, then $V_t^{n,d_0} < 0$. Intuitively one may think of running as the behavior a swarm shows when it moves through an area that is monotone in one dimension and changes in any other dimension are insignificant.

For our analysis, we now define the potential of $\mathcal{S}$.

**Definition 5** *(Potential).* For $a > 0$, the potential of swarm $\mathcal{S}$ in dimension $d$ right before the $t$-th step of particle $n$ is $\Phi_t^{n,d}$ and its potential in dimension $d$ after the $t$-th step of particle $n$ is $\tilde{\Phi}_t^{n,d}$ with

$$\Phi_t^{n,d} := \sqrt{\sum_{n'=1}^{n-1}\left(a\left|V_t^{n',d}\right| + \left|G_t^{n,d} - X_t^{n',d}\right|\right) + \sum_{n'=n}^{N}\left(a\left|V_{t-1}^{n',d}\right| + \left|G_t^{n,d} - X_{t-1}^{n',d}\right|\right)},$$

[1] Let $(\Omega, \mathcal{A}, P)$ be a probability space. An event $E \in \mathcal{A}$ happens almost surely iff $P(E) = 1$.

$$\tilde{\Phi}_t^{n,d} := \sqrt{\sum_{n'=1}^{n}\left(a\left|V_t^{n',d}\right| + \left|\tilde{G}_t^{n,d} - X_t^{n',d}\right|\right) + \sum_{n'=n+1}^{N}\left(a\left|V_{t-1}^{n',d}\right| + \left|\tilde{G}_t^{n,d} - X_{t-1}^{n',d}\right|\right)}.$$

Basically, the potential $\Phi$ is an extension of the physical interpretation of the particle swarm model. If the particles move faster and get farther away from their global attractor, the potential increases. If the swarm converges, the potential tends towards 0. However, for technical reasons explained later, we need the additional parameter $a$ and the square root. Note that general tendencies towards 0 or $\infty$ are invariant under different choices of $a$.

**Example 1.** Consider a 1-dimensional particle swarm and the objective function $f(x) = -x$. Assume that the velocities of the particles are all positive. Then the swarm is positively running in direction 1 forever. It is obvious that the position with the greatest $x$-value leads to the smallest value of $f(x)$ and therefore becomes the global attractor. It remains to prove that the velocity of every particle stays positive. Given the old velocity $V_t^{n,1}$, the new velocity $V_{t+1}^{n,1}$ is a positive linear combination of the three components $V_t^{n,1}$, $G_t^{n,1} - X_t^{n,1}$ and $L_t^{n,1} - X_t^{n,1}$. The value for $V_t^{n,1}$ is positive by assumption, $G_t^{n,1} - X_t^{n,1}$ and $L_t^{n,1} - X_t^{n,1}$ are non-negative since $G_t^{n,1} \geq L_t^{n,1} \geq X_t^{n,1}$. Therefore, the velocity stays positive and the swarm will stay positively running forever. In that situation, a good behavior would be increasing (or at least non-decreasing) $\Phi$.

Informally speaking, if a swarm $\mathcal{S}$ has a too little potential $\Phi$ left to make it to the next local minimum, it is necessary that $\Phi$ increases after $\mathcal{S}$ has become running, and so $\Phi$ enables the swarm to overcome every distance. The following lemma is the central technical observation of our work and makes a statement about how to choose the parameters to make sure that a running swarm has an increasing potential.

**Lemma 1** *(Running to infinity lemma). For certain parameters $N$, $\chi$, $c_1$, $c_2$ and the swarm $\mathcal{S}$ positively (negatively, resp.) running in direction $d_0$, $V_t^{n,d_0} + X_t^{n,d_0}$ ($-V_t^{n,d_0} - X_t^{n,d_0}$, resp.) tends to $\infty$ for every $n$ almost surely. In particular, the swarm leaves every bounded set $B \subset \mathbb{R}^D$ almost surely.*

**Proof.** To ease up notation, we omit the upper index $d_0$ for the rest of this proof. Without loss of generality, assume that the swarm is positively running. The main idea is to show that on expectation the potential increases after every step of the particle swarm. More precisely, we want to show that $\mathrm{E}[1/\Phi_t^n] \xrightarrow{t\to\infty} 0$, which is equivalent to $1/\Phi_t^n \xrightarrow{t\to\infty} 0$ almost surely and therefore $\Phi_t^n \xrightarrow{t\to\infty} \infty$ almost surely. Our strategy to prove the convergence of $\mathrm{E}[1/\Phi_t^n]$ is to prove that for $\Phi_t^n \neq 0$ the bound

$$\mathrm{E}\left[\Phi_t^n/\Phi_{t+1}^n \;\middle|\; S_t\right] \leq q \tag{1}$$

holds for some fixed $q < 1$ almost surely (due to Observation 1, the case $\Phi_t^n = 0$ for some $t$ and some $n$ has probability 0).

Next, we bound $\mathrm{E}[\Phi_t^n/\Phi_{t+1}^n \mid S_t]$ for our concrete choice of potential from Definition 5. We know that there are better choices for the definition of a "potential" leading to larger areas of parameters that match our requirements, but for our existence proof this one is sufficient.

We need to determine the values for $N$, $\chi$, $c_1$ and $c_2$, for which this potential fulfills Eq. (1) for a $q < 1$. In other words, during one iteration of all particles, we want the reciprocal of the potential to decrease on expectation by at least a factor of $q$. Inserting the definition of $\Phi$ and applying the fact that the swarm is positively running and therefore $X_{t-1}^n \leq G_t^1$ and $V_{t-1}^n \geq 0$ for every $n$, we obtain:

$$\begin{aligned}
\frac{\Phi_t^1}{\Phi_{t+1}^1} &= \sqrt{\frac{\sum_{n'=1}^N (a\cdot|V_{t-1}^{n'}| + |G_t^1 - X_{t-1}^{n'}|)}{\sum_{n'=1}^N (a\cdot|V_t^{n'}| + |G_{t+1}^1 - X_t^{n'}|)}} \\
&= \sqrt{\frac{\sum_{n'=1}^N (a\cdot V_{t-1}^{n'} + G_t^1 - X_{t-1}^{n'})}{\sum_{n'=1}^N (a\cdot V_t^{n'} + G_{t+1}^1 - X_t^{n'})}} \\
&= \sqrt{\frac{\sum_{n'=1}^N (a\cdot V_{t-1}^{n'} + G_t^1 - X_{t-1}^{n'})}{\sum_{n'=1}^N (a\cdot V_t^{n'} + N\cdot(\tilde{G}_t^{n'} - G_t^{n'}) + G_t^1 - X_t^{n'})}} \\
&= \sqrt{\frac{1}{\sum_{n'=1}^N \frac{w_{n'}}{x_{n'}}}} \leq \sum_{n'=1}^N w_{n'}\cdot\sqrt{x_{n'}},
\end{aligned}$$

where $w_{n'} := (a\cdot V_{t-1}^{n'} + G_t^1 - X_{t-1}^{n'})/(\sum_{k=1}^N (a\cdot V_{t-1}^k + G_t^1 - X_{t-1}^k))$ and $x_{n'} := (a\cdot V_{t-1}^{n'} + G_t^1 - X_{t-1}^{n'})/(a\cdot V_t^{n'} + N\cdot(\tilde{G}_t^{n'} - G_t^{n'}) + G_t^1 - X_t^{n'})$. The last inequality follows from the generalized weighted mean inequality between the $(-1)$-mean and the $(1/2)$-mean with weights $w_{n'}$. Note that the $w_{n'}$ sum up to 1 and that they only depend on $S_t$. Therefore, it follows

$$\mathrm{E}\left[\frac{\Phi_t^1}{\Phi_{t+1}^1}\;\middle|\;S_t\right] \le \mathrm{E}\left[\sum_{n'=1}^{N} w_{n'}\cdot\sqrt{x_{n'}}\;\middle|\;S_t\right]$$
$$= \sum_{n'=1}^{N} w_{n'}\cdot \mathrm{E}[\sqrt{x_{n'}}\mid S_t] \le \max_{n'=1\dots N} \mathrm{E}[\sqrt{x_{n'}}\mid S_t]$$

It remains to show, that $\mathrm{E}[\sqrt{x_{n'}}\mid S_t] \le q$ for every $n'$ and a $q<1$. Applying the definition of $x_{n'}$ and the movement equations, we obtain

$$\begin{aligned}
x_{n'} &= \frac{a\cdot V_{t-1}^{n'} + G_t^1 - X_{t-1}^{n'}}{a\cdot V_t^{n'} + N\cdot(\tilde{G}_t^{n'} - G_t^{n'}) + G_t^1 - X_t^{n'}}\\
&= \left[a\cdot V_{t-1}^{n'} + G_t^1 - X_{t-1}^{n'}\right]\\
&\quad\times \big[a\cdot\big(\chi\cdot V_{t-1}^{n'} + c_2\cdot s_{t-1}^{n'}\cdot\big(G_t^{n'} - X_{t-1}^{n'}\big)\big) + \ldots + N\cdot\max\big\{0, X_{t-1}^{n'} + \chi\cdot V_{t-1}^{n'}\\
&\quad + c_2\cdot s_{t-1}^{n'}\cdot\big(G_t^{n'} - X_{t-1}^{n'}\big) - G_t^{n'}\big\} + \ldots + G_t^1 - X_{t-1}^{n'} - \chi\cdot V_{t-1}^{n'} - c_2\cdot s_{t-1}^{n'}\cdot\big(G_t^{n'} - X_{t-1}^{n'}\big)\big]^{-1}\\
&= \left[a\cdot V_{t-1}^{n'} + G_t^1 - X_{t-1}^{n'}\right]\times\big[(a-1)\cdot\big(\chi\cdot V_{t-1}^{n'} + c_2\cdot s_{t-1}^{n'}\cdot\big(G_t^{n'} - X_{t-1}^{n'}\big)\big) + G_t^1 - X_{t-1}^{n'} + \ldots\\
&\quad + N\cdot\max\big\{0, X_{t-1}^{n'} + \chi\cdot V_{t-1}^{n'} + c_2\cdot s_{t-1}^{n'}\cdot\big(G_t^{n'} - X_{t-1}^{n'}\big) - G_t^{n'}\big\}\big]^{-1}
\end{aligned}$$

There are two distinct cases. In the first case, the position of particle $n'$ before its step is the global attractor, in terms $G_t^1 = X_{t-1}^{n'}$. Then its move is deterministic and its new position will be the new global attractor. In this case, we obtain:

$$\mathrm{E}[\sqrt{x_{n'}}\mid S_t] \le \sqrt{\frac{a\cdot V_{t-1}^{n'}}{(a-1)\cdot\chi\cdot V_{t-1}^{n'} + N\cdot\chi\cdot V_{t-1}^{n'}}} = \sqrt{\frac{a}{(a-1)\cdot\chi + N\cdot\chi}},$$

which is less than $q$ if and only if $a < (N-1)\cdot\frac{\chi\cdot q^2}{1-\chi\cdot q^2}$.

The second case when $G_{t+1}^1 > X_t^1$ requires more exhaustive but still straight-forward calculations. We need to bound the expression

$$\sup_{v>0,\, g\ge d>0}\int_0^1\sqrt{\frac{av+d}{(a-1)(\chi v + c_2 sg) + N\max\{0, \chi v + c_2 sg - g\} + d}}\,ds$$
$$= \sup_{v'>0,\, g'>1}\int_0^1\sqrt{\frac{av'+1}{(a-1)(\chi v' + c_2 sg') + N\max\{0, \chi v' + c_2 sg' - g'\} + 1}}\,ds$$

The calculation of the integral can be done explicitly and finding the values for $v'$ and $g'$ maximizing it for given $a$, $\chi$, $c_2$ and $N$ using standard techniques from analysis is straight-forward. Obviously, the greater the number of particles is, the smaller is the value of the integral, so we calculated the minimal number of necessary particles ensuring that the integral is less than 1 for three common parameter choices obtained from the literature. For the choice of $\chi = 0.729$, $c_2 = 1.49$ recommended in [4], we set $a := 2.3543$ and obtain for $N = 2$ an upper bound of $q \le 0.9812$. For the choice $\chi = 0.729$ and $c_2 = 1.3\cdot\chi$ [1], one requires at least $N = 3$ particles and the choice of $a := 5.1298$ leads to $q \le 0.9964$. Finally, for the choice $\chi = 0.6$, $c_2 = 1.7$ as proposed in [25], we obtain for $N = 3$ and $a := 2.5847$ a value of $q \le 0.9693$. □

In Fig. 1, one can see the borderlines between choices for $c_2$ and $\chi$ that satisfy the conditions of Lemma 1 and those that do not. The parameters that satisfy both Lemma 1 and convergence requirements will be referred to as good parameters. Note that in [16], the authors have shown the existence of "bad" parameters that allow stagnation on arbitrary search points with positive probability. They have proven that for $N = 1$, all choices of parameters are bad, so a swarm with good parameters needs at least two particles. From here on we assume the parameters to be good. Lemma 1 says that, given the parameters are good, a swarm that moves into the right direction can overcome every distance and increase its potential, no matter how small it was in the beginning. In other words: The equilibrium when all attractors and particles are on the same point and every velocity is zero is not stable because arbitrary small changes of an attractor, a position or a velocity can be sufficient to lead the swarm far away from this equilibrium, as long as there is a direction with decreasing value of the fitness function.

**Theorem 1.** *If $D = 1$, then every accumulation point of $(G_t^n)_{n=1,\ldots,N;t\in\mathbb{N}}$ is a local minimum of $f$ almost surely.*

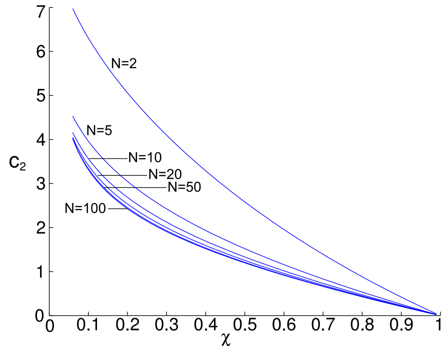



**Fig. 1.** Borders between the too low values for $c_2$ and $\chi$ and the ones large enough to satisfy the requirements in Lemma 1 for some swarm sizes $N$.

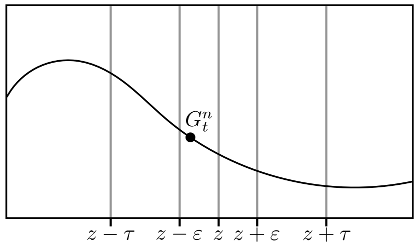



**Fig. 2.** Fitness function $f$ monotonic on $B_\tau(z)$, global attractor in $\varepsilon$-neighborhood of $z$.

**Proof.** Assume for contradiction, that there is an accumulation point $z \in \mathbb{R}$ such that (w.l.o.g.) $f$ is monotonically decreasing on $B_\tau(z) = (z - \tau, z + \tau)$ for some $\tau > 0$. Since $z$ is an accumulation point, for every $\varepsilon > 0$ $G_t^n$ is inside the neighborhood $B_\varepsilon(z) = (z - \varepsilon, z + \varepsilon)$ of $z$ infinitely often. Fig. 2 gives a visualization of the described situation. Note that $G_t^n$ entering $[z, z + \tau]$ for any $t$ and any $n$ violates the assumption of $z$ being an accumulation point. That is because the global attractor does not accept worsenings, so for any $\varepsilon$ with $\varepsilon < |z - G_t^n|$, $B_\varepsilon(z)$ will not be entered by the global attractor anymore.

Now we consider two cases: The first case is that there is at least a second accumulation point (that might even be a local optimum). This second accumulation point must yield the same function value as $z$ and can therefore not be inside $B_\tau(z)$. Consequently, infinitely often, a particle has a distance of at least $\tau$ between its local and its global attractor. From this situation, the probability for hitting $[z, z + \tau]$ within the next few steps is positive and the probability for never hitting $[z, z + \tau]$ would be 0. We outline a sequence that moves the particle into $[z, z + \tau]$. Several cases could occur: There are essentially three possible orders of the particle, its local attractor and $z$, depending on which of this three points is between the other two. Furthermore, the velocity could be close to 0 or large and it could point in two different directions. Since the calculations showing that hitting $[z, z + \tau]$ has a positive probability from each of the situations are straight-forward, we will just do this exemplary. Consider the case where $z$ is located between the particles position $X_t^n$ and its local attractor $L_t^n$ and has a velocity $V_t^n$ with $|V_t^n| \le (c_1 \cdot |L_t^n - X_t^n| - |z - X_t^n| - \tau)/\chi$ pointing away from $L_t^n$. Then, any choice of $r_t^n \in I$ with

$$I = \left[\frac{|z - X_t^n| + \tau/4 - \chi \cdot |V_t^n|}{c_1 \cdot |L_t^n - X_t^n|}, \frac{|z - X_t^n| + 3/4 \cdot \tau - \chi \cdot |V_t^n|}{c_1 \cdot |L_t^n - X_t^n|}\right] \cap [0, 1]$$

leads to $X_{t+1}^n \in [z, z + \tau]$. Since

$$\left|z - X_t^n\right| + \tau/4 - \chi \cdot \left|V_t^n\right| \le \left|z - X_t^n\right| + \tau/4 \le c_1 \cdot \left|L_t^n - X_t^n\right| - 3/4 \cdot \tau$$

and

$$\left|z - X_t^n\right| + 3/4 \cdot \tau - \chi \cdot \left|V_t^n\right| \ge c_1 \cdot \left|L_t^n - X_t^n\right| - \tau/4,$$

we have $|I| \ge \min\{1, \frac{\tau}{4 \cdot c_1 \cdot |L_t^n - X_t^n|}\}$. If $|V_t^n|$ is larger than assumed above, one requires at most two additional steps in order to decrease it sufficiently. Similar calculations for all the other cases show that there is indeed always a positive probability to hit $[z, z + \tau]$ within a constant number of iterations. Additionally, since $|L_t^n - X_t^n|$ is stochastically bounded due to our choice of admissible fitness functions, the probability cannot converge to 0. It follows that the probability to never hit $[z, z + \tau]$ in case of a second accumulation point is 0.

If $z$ is the only accumulation point, the attractors converge towards $z$. As a consequence of the results in [12], this implies that the whole swarm converges to $z$. That is the point were Lemma 1 becomes useful. Since $f$ is monotonic on

**Algorithm 2:** modified PSO.

**output**: $G \in \mathbb{R}^D$
**repeat**
  **for** $n = 1 \to N$ **do**
    **for** $d = 1 \to D$ **do**
      **if** $\forall n' \in \{1, \ldots, N\} : |V^{n',d}| + |G^d - X^{n',d}| < \delta$ **then**
        $V^{n,d} := (2 \cdot \mathrm{rand}() - 1) \cdot \delta$;
      **else**
        $V^{n,d} := \chi \cdot V^{n,d} + c_1 \cdot \mathrm{rand}() \cdot (L^{n,d} - X^{n,d})$
          $+ c_2 \cdot \mathrm{rand}() \cdot (G^d - X^{n,d})$;
      $X^{n,d} := X^{n,d} + V^{n,d}$;
    **if** $f(X^n) \le f(L^n)$ **then** $L^n := X^n$;
    **if** $f(X^n) \le f(G)$ **then** $G := X^n$;
**until** *termination criterion met*;

$B_\tau(z)$, the local and the global attractor are always greater or equal to the current position of the particle. Therefore the velocities will with probability 1 all become positive after a finite number of iterations and stay positive. It follows that each particle will exceed its local attractor almost surely after a finite number of iterations. At that time, the swarm becomes positively running at least until the first particle surpasses a local minimum and therefore leaves $B_\tau(z)$. With Lemma 1, this will happen after a finite number of iterations almost surely, a contradiction to the convergence of the swarm towards $z$.

So, $z$ is no accumulation point of $G_t^n$. $\square$

**Corollary 1.** *If $D = 1$, then $f(G_t^n)$ converges towards the value of a local minimum. Particularly, if no two local minima have the same value, then $G_t^n$ converges towards a local minimum. If the swarm converges towards a point $z \in \mathbb{R}$, then $z$ is a local minimum.*

**Proof.** The first statement follows directly from Theorem 1. From Definition 2 it follows that the sequence of the global attractors over the time is bounded and therefore has at least one accumulation point. If there is more than one accumulation point, then $f$ has the same value on each of them because $f$ is continuous. Due to Theorem 1 every accumulation point is a local minimum, so if there are no two local minima with the same value, there is only one accumulation point that therefore is the limit of $G_t^n$. That proves the second statement. The third statement again is a direct consequence of Theorem 1 because convergence of the swarm implies convergence of $G_t^n$. $\square$

## 3. Modified PSO algorithm

Now the question arises how much of the results from the 1-dimensional PSO can be transferred to the general case. Unfortunately, the stated result is not true in a $D$-dimensional situation with $D > 1$. The main problem is the following: Assume that the whole swarm is close to a point that allows improvements neither in positive nor in negative changes of the first coordinate. Furthermore let the swarm have high potential in the first and low potential in any other dimension. Then an improvement of the global attractor is still possible and will indeed happen infinitely often, but it is very unlikely and between two updates are many iterations without an update. The reason is that any improvement in some of the dimensions $2, \ldots, D$ is voided by the much larger worsening in dimension 1. In the meantime, the swarm tends to converge and therefore continuously looses potential and never gets running. A small and simple modification of the PSO algorithm avoids that problem by enabling the swarm to rebalance the potentials in the different dimensions:

**Definition 6** *(Modified PSO).* For some arbitrary small but fixed $\delta > 0$, we define the modified PSO via the same equations as the classic PSO in Definition 1, only modifying the third equation to

$$V_{t+1}^{n,d} = \begin{cases} (2 \cdot r_t^{n,d} - 1) \cdot \delta, \\ \quad \text{if } \forall n' \in \{1, \ldots, N\} : |V_t^{n',d}| + |G_{t+1}^{n',d} - X_t^{n',d}| < \delta, \\ \chi \cdot V_t^{n,d} + c_1 \cdot r_t^{n,d} \cdot (L_t^{n,d} - X_t^{n,d}) + c_2 \cdot s_t^{n,d} \cdot (G_{t+1}^{n,d} - X_t^{n,d}), \\ \quad \text{otherwise.} \end{cases}$$

Whenever the first case applies, we call the step *forced*.

An algorithmic overview over our modified PSO is given in Algorithm 2. In words: As soon as in one dimension the sum of the velocity and the distance between the position and the global attractor are below the bound of $\delta$ for every single particle, the updated velocity of this dimension is drawn u.a.r. from the interval $[-\delta, \delta]$. Note the similarity between this condition and the definition of the potential. Indeed, we could have used the condition $\Phi_{t+1}^{n,d} < \delta$ (with some fixed $a$) instead, but we decided to keep the modification as simple and independent from the terms occurring in the analysis as

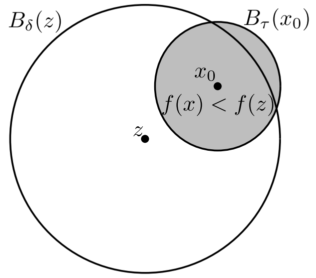



**Fig. 3.** Every point $x_0$ with $f(x_0) < f(z)$ has a neighborhood $B_\tau(x_0)$, such that $f(x) < f(z)$ for $x \in B_\tau(x_0)$.

possible. Now the potential can no longer converge to 0 while staying unbalanced because if it decreases below a certain bound, we randomly assign a value to the velocity which on expectation has an absolute value of $\delta/2$.

This modified PSO is similar to the Noisy PSO proposed by Lehre and Witt in [16] where they generally add a random perturbation drawn u.a.r. from $[-\delta/2, \delta/2]$ for some small $\delta$ and prove that their swarm is able to find a local optimum. However, their analysis is restricted to one specific 1-dimensional fitness function.

In case of our modified PSO, we consider the change from the classic PSO as comparatively simple. The main difference to previous approaches (e.g., [26]) is that the PSO uses the modification not as its engine. Rather, we will see that the number of forced steps is small and if the swarm is not already within a $\delta$-neighborhood of a local optimum, after some forced steps the potential increases and the swarm switches back to classical steps, a behavior we also observed experimentally (see Section 4.5)

Note that we sacrifice the convergence of the swarm in order to increase the quality of the solution, since the potential cannot approach 0 anymore. Instead, we can only expect the global attractor to converge.

**Theorem 2.** *Using the modified PSO algorithm, every accumulation point of $G_t^n$ is a local minimum of $f$ almost surely.*

**Proof.** Assume, for contradiction, that there is some accumulation point $z$ of $G_t^n$ that is no local minimum. Then, in any neighborhood of $z$ and therefore in particular in $B_\delta(z)$, there is a point $x_0 \in B_\delta(z)$ with $f(x_0) < f(z)$. Since $f$ is continuous, $x_0$ has some neighborhood $B_\tau(x_0)$, such that $f(x) < f(z)$ for every $x \in B_\tau(x_0)$. Fig. 3 gives an overview over the situation.

The set $B_\tau(x_0)$ plays the role of the interval $(z, z+\tau)$ from the proof of Theorem 1. Now we investigate what happens when $G_t^n$ enters $B_\varepsilon(z)$. This will for each $\varepsilon > 0$ happen infinitely often because $z$ is an accumulation point. The modification of the PSO algorithm enables us to construct a sequence of steps leading a particle into $B_\tau(x_0)$. In principle, the sequence can be obtained by using the sequence from the proof of Theorem 1 for every single dimension. However, that may result in a particle being at the desired positions in two distinct dimensions at two different points in time, so we construct the sequence basically by a simple composition of modified sequences from Theorem 1 for every single dimension in which the steps are not forced. The modification of the sequences ensuring that they all have the same length is straight-forward. For forced dimensions the probability for hitting $B_\tau(x_0)$ within the next step is obviously positive and so is the probability for obtaining a velocity suitable for ensuring that the next step will be forced in case the other dimensions are not at the end of their sequences. Note that due to the modification, we do not need to split cases on whether there is a second accumulation point or not. $\square$

This result is not surprising because in the modified PSO random perturbations occur when the swarm tends to converge and it is easy to see that small random perturbations can optimize any continuous function (but with a very poor runtime). Note that the proof of Theorem 2 does neither make use of $f$ having a continuous derivative nor of Lemma 1. To supplement this result, we need to prove a statement about how often the modification actually applies. It is obvious that for $\delta$ chosen too large, every step of the particles could be forced. The case of $\delta$ being small with respect to the structure of the function is the interesting one. On the other hand, if the distance of a particle and a local optimum is smaller than $\delta$, presumably all upcoming steps will be forced because there is no room for further improvements. But one can show that, given the swarm is sufficiently far away from the closest local optimum, the forced steps only balance the potentials between the different dimensions and enable the swarm to become running. In particular, consider the following situation: Let for some dimension $d_0$ and some $c \gg 1$ be $\frac{\partial f}{\partial d_0} < 0$ on a $(c \cdot \delta)$-neighborhood of the current global attractor and let the swarm have low potential, i.e., every particle has in every dimension potential of order $\delta$. Instead of only being driven by the random perturbation, we would like the swarm to become running in direction $d_0$, increasing the potential in that direction, so the velocity updates can be done according to the classical case again.

**Theorem 3.** *In the situation described above, the probability for the swarm to become running within a constant number of iterations is positive and independent of $\delta$.*

**Proof.** We will explicitly describe a possible sequence of iterations enabling the swarm to become running. First, the particles decrease their distance to the global attractor in every single dimension to at most $\delta \cdot \varepsilon/2$ with $\varepsilon \ll 1$ and a velocity of absolute value less than $\delta \cdot (1-\varepsilon/2)$, such that the local attractor is updated for all particles except the one whose local attractor is equal to the global attractor. If the current global attractor $G_t^n$ is no local maximum, this can be done because every local attractor has a function value worse than the global attractor and since $f$ is continuous, so the function values of $f$ approach $f(G_t^n)$ when $x$ approaches $G_t^n$. The case of $G_t^n$ being a local maximum has probability 0. Then the next step of each particle is forced. In the next iteration, the velocity of every particle gets smaller than $\delta \cdot \varepsilon/2$ in each dimension except $d_0$. In dimension $d_0$, one particle obtains velocity greater than $\delta \cdot (1+\varepsilon)/2$, such that it gets to a search point that is in dimension $d_0$ more than $\delta/2$ and in any other dimension at most $\varepsilon \cdot \delta$ away from the previous global attractor. For $\varepsilon$ sufficiently small, this particle will update the global attractor since $f$ has a positive partial derivative in dimension $d_0$. Every other particle obtains in $d_0$ a velocity less than $-\delta \cdot (1+\varepsilon)/2$, making sure that its new position and the new global attractor after that step differ by more than $\delta$. So the next step will not be forced and the potentials have order $\sqrt{\delta}$ in dimension $d_0$ and only $\sqrt{\delta \cdot \varepsilon}$ in every other dimension. So for $\varepsilon$ sufficiently small with respect to the function $f$, the swarm will become running and therefore the steps will actually become unforced. □

The behavior of the modified PSO is the same as of the classic PSO, except that due to the modification the particles can overcome “corners,” i.e., if the global attractor stagnates because the potential of wrong dimensions is too high and the potential of dimensions where the function value can be decreased is too low, the modification helps to balance the potentials of the different dimensions. The “blind” algorithm that just randomly checks a point around the previous best solution with range $\delta$ would of course find a local minimum but the running time would tend towards $\infty$ if $\delta$ approaches 0.

## 4. Experimental results

In this section, we describe the experiments we performed to confirm our theoretical results. When nothing else is mentioned, we use the parameters $\chi = 0.729$, $c_1 = c_2 = 1.49$ determined in [4] and the choice $a = 1$ for the potential (see Definition 5). All experiments were performed using MATLAB version 8.2.0.701 (R2013b).

### *4.1. Potential gain*

First we examine the behavior of a 1-dimensional PSO with respect to the potential. If the swarm is close to a local optimum and there is no second local optimum within range, the attractors converge and it is well-known that with appropriate choices for the parameters of the PSO, convergence of the attractors implies convergence of the whole swarm. Such parameter selection guidelines can be found, e.g., in [12].

On the other hand, if the swarm is far away from the next local optimum and the function is monotone on an area that is large compared to the current potential of the swarm, the preferred behavior of the swarm is to increase the potential and move in the direction that yields the improvement until a local optimum is surpassed and the monotonicity of the function changes. In [16], the authors show that there are non-trivial choices of parameters for which the swarm converges even on a monotone function. In particular, if $N = 1$, every parameter choice either allows convergence to an arbitrary point in the search space, or it generally prevents the one-particle-swarm from converging, even if the global attractor is already at the global optimum.

We ran the particle swarm algorithm on a monotone function to measure the development of the potential over time. For our experiment, we chose the 1-dimensional function $f(x) = -x$ as objective function wanting the swarm always “running down the hill.” Note that this choice is not a restriction, since the particles compare points only qualitatively and the behavior is exactly the same on any monotone decreasing function: the new attractors are the points with greater $x$-coordinate. Therefore, we used only one function in our experiment. The parameters for the movement equations are common choices obtained from the literature. We let the particles make 1000 iterations and stored the potential (with $a$ set to 1) at every iteration. We made a total of 1000 experiments for each set of parameters and calculated both average and standard deviation. The averages are stated in Fig. 4, the standard deviations are of the same order and therefore omitted.

As stated in the proof of Lemma 1, the increase of the potential of a running particle swarm is expected to be exponential in the number of steps the swarm makes. The cases (a), (c) and (e) are covered by our analysis and show the expected behavior as an exponential increase of the potential. Cases (b) and (d) are not covered by our analysis, here the number of particles was below the bound that is proven to be sufficient. However, case (d) still works, even though the increase of the potential is much smaller than in the provably good cases (a), (c) and (e). Only in case (b) where only two particles are involved, we see the potential decreasing exponentially because the number of particles is presumably too small. In this case, the swarm will eventually stop, i.e., stagnate. But we also see in case (c) that using one additional particle and not changing the remaining parameters, the swarms keeps its motion.

In all cases, for the small swarm size of $\geq 3$, the common parameter choices avoid the problem mentioned in [16].

### *4.2. Imbalanced potentials*

In the $D$-dimensional case, the situation is more complicated as now the relations between distinct dimensions become important. A new problem arising is the following: Assume that the whole swarm is close to a point $x \in \mathbb{R}^D$ such that every

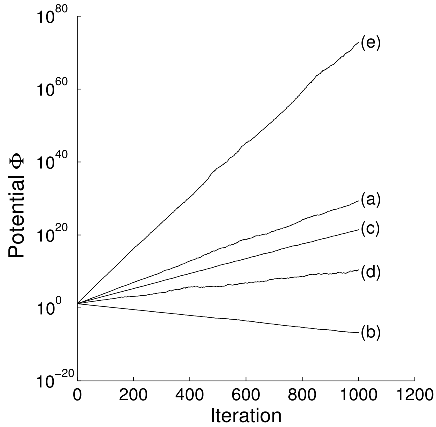
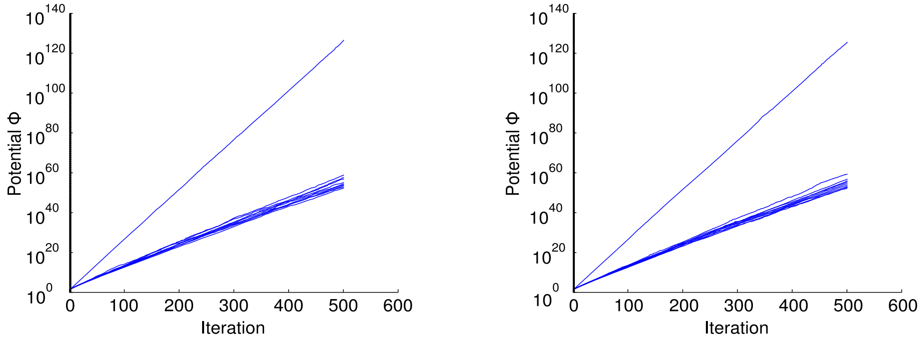



**Fig. 4.** (a) $\chi = 0.729$, $c_1 = c_2 = 1.49$, $N = 2$ [4] (b) $\chi = 0.729$, $c_1 = 2.8 \cdot \chi$, $c_2 = 1.3 \cdot \chi$, $N = 2$ [1] (c) $\chi = 0.729$, $c_1 = 2.8 \cdot \chi$, $c_2 = 1.3 \cdot \chi$, $N = 3$ [1] (d) $\chi = 0.6$, $c_1 = c_2 = 1.7$, $N = 2$ [25] (e) $\chi = 0.6$, $c_1 = c_2 = 1.7$, $N = 3$ [25].




(a) Fitness function $f$ (b) Fitness function $g$

**Fig. 5.** Growth of potential when processing (a) $f(\vec{x}) = -\sum_{i=1}^{D} x_i$, (b) $g(\vec{x}) = -\sum_{i=1}^{D} i \cdot x_i$.

change of the first coordinate leads to a significantly worse value of the objective function, while in the other dimensions there is still room for improvements. Furthermore let the swarm have high potential in the first and low potential in any other dimension. Then an improvement of the global attractor is still possible, but it is very unlikely and between two updates are many steps without an update. The reason is that any improvement in some of the dimensions $2, \dots, D$ is voided by the much larger worsening in dimension 1. It follows that the attractors stay constant for long times between two updates and so the swarm tends to converge and therefore looses potential. As long as the global attractor stays constant, the situation is symmetric in every dimension, so while converging, the imbalance is still maintained.

First, we want to examine if and how such imbalances arise. Assume that the fitness function is (on some area) monotone in every dimension. One of our main observations is that indeed in such a situation the swarm tends to pick one dimension and favor it over all the others. As a consequence, the movement of the swarm becomes more and more parallel to one of the axes.

We used the fitness-functions $f(\vec{x}) = -\sum_{i=1}^{D} x_i$ and $g(\vec{x}) = -\sum_{i=1}^{D} i \cdot x_i$ which are both monotonically decreasing in every dimension and set $D$ to 10. Initially, we distribute the particles randomly over $[-100; 100]^D$ and the velocities over $[-50; 50]^D$ and let the swarm make 500 iterations. The swarm size $N$ was 10 and the parameters were set to $\chi = 0.729$, $c_1 = c_2 = 1.49$ as found in [4]. After each iteration, we calculated the potential for each dimension. We made 1000 runs and for each run, the dimensions were sorted according to the final value of $\Phi$, i.e., we switched the numbers of the dimensions such that after the last iteration dimension 1 always had the highest potential, dimension 2 the second highest and so on. We calculated the mean of the potentials over the 1000 runs for each of the sorted dimensions. The results are stated in Fig. 5. One can see that the dimension with the greatest potential has for both functions a value far higher than the others, while the other dimensions do not show such a significant difference between each other.

An explanation for this behavior is the following: Assume that at some time, one dimension $d_0$ has more potential than the others. Further assume that the advance is great enough such that for some number of steps the particle with the largest value in dimension $d_0$ is the one that determines the global attractor. In a companion paper, we call a swarm in this situation “running”. Since randomness is involved and this situation has a positive probability to occur, it will actually occur after sufficiently many iterations. Then, each update of the global attractor increases the potential in $d_0$ considerably because it increases the distance of every single particle to the global attractor except for the one particle that updated it.

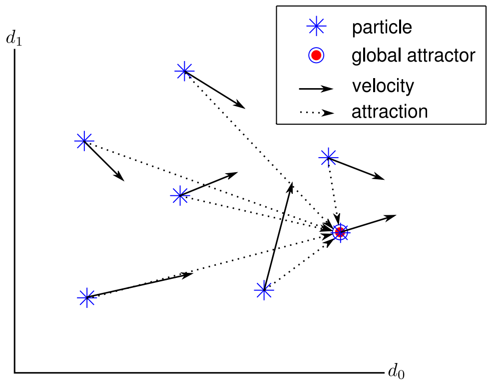



**Fig. 6.** Particles running in direction $d_0$. In dimension $d_0$, the differences between the coordinate of the particle and the global attractor is on average higher than in dimension $d_1$. The velocities of dimension $d_0$ point in the direction of the global attractor.

In any other dimension $d \neq d_0$, the situation is different. Here, the decision which particle updates the global attractor is stochastically independent of the value $x_d$ in dimension $d$. In other words: If one looks only on the dimension $d$, the global attractor is chosen uniformly at random from the set of all particles. As a consequence, after some iterations, the $d_0$-th coordinate of the velocity becomes positive for every particle, so the attraction towards the global attractor always goes into the same direction as the velocity, while in the remaining dimensions, the velocities may as well point away from the global attractor, meaning that the particles will be slowed down by the force of attraction. An overview over the situation is given in Fig. 6.

So, roughly speaking, most of the time the global attractor is somewhere in the middle of the different $x_d$ values, giving less potential increase then in dimension $d_0$ where it has a border position. That means that the balanced situation is not stable in a sense that after the imbalance has reached a certain critical value, it will grow unbounded.

If at some point no more improvements can be made in dimension $d_0$, the swarm is in the situation described above where it starts to converge while the convergence is, other than the acceleration phase, balanced. That means after the same time the potential of every dimension is decreased by approximately the same factor, so dimension $d_0$ has still far more potential than any other dimension and the swarm stays blind for possible improvements in dimensions other than $d_0$.

### *4.3. Imbalanced convergence*

To supplement the results about the behavior of the PSO in that "artificial" setting, we ran it on two well-known benchmark functions to show that the problems described above really occurs on actual instances. Since the described scenario may happen with positive but, depending on the situation, small probability, we choose the number of particles $N$ compared to the number of dimensions $D$ small in order to be able to view the phenomenon in a preferably pure condition. Table 1 lists our results on the sphere function with optimal solution $z^* = (0, \dots, 0)$, where we distributed the particles randomly over $[-100; 100]^D$ and the velocities over $[-50, 50]^D$, and another common benchmark, the Rosenbrock function with optimal solution $z^* = (1, \dots, 1)$ (found in [22]), where the initial population was randomly distributed over $[-5; 10]^D$ and the initial velocity over $[-2.5, 5]^D$. We repeated each experiment 1000 times and calculated the means. Additionally we calculated for each repetition the dimension with the minimal and the one with the maximal value for the potential $\Phi$ after the last iteration (see columns $\Phi$) and the difference between the global attractor and the optimal solution in the dimension with the lowest and highest remaining potential, resp. In the 5-dimensional case, the potential reached 0 due to double precision in every dimension, so there is no single dimension with highest or lowest potential. However, the function value obtained at the point where the particles converged to is still far away from the optimum. In case of the 50-dimensional sphere function, one can see that the dimension with the highest value for $\Phi$ usually is much closer to its optimal value than the dimension with the lower value. This confirms the concerns about the classical PSO we stated in the beginning of Section 3. Since the Rosenbrock function is non-separable, we do not obtain the same relationship between the remaining potentials and the distance to the optimum. However, what we can see is that the remaining potentials are much smaller than the distance to the optimum.

We repeated the experiment in the same setting as before, but using the modified PSO. The results can be seen in Table 2. It turns out that the modified PSO algorithm actually leads to a significantly better solution than the unmodified one. In particular, on the sphere function the values obtained by the modified algorithm are of order $\delta^2$, which means that the swarm was already closer than $\delta$ to the optimum. Fig. 7 shows the fitness value of the global attractor at each time during two particular runs. One can see that as long as the fitness value is larger than $\delta^2$, the swarm sometimes stagnates and does not improve until it finds another promising direction and accelerates again. On the Rosenbrock function, even the values obtained by modified algorithm are still far away from the optimum. So, some of the runs were not converged up to an error of $\delta$ when they were stopped.

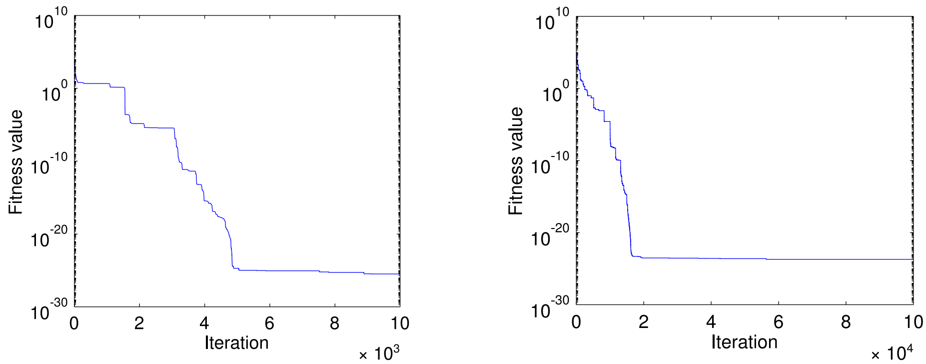
**Table 1**
Imbalanced potentials.

| Function | | Sphere | | Rosenbrock | |
|---|---|---|---|---|---|
| $D$ | | 5 | 50 | 5 | 50 |
| $N$ | | 2 | 8 | 2 | 8 |
| $t_{\max}$ | | 10 000 | 100 000 | 10 000 | 100 000 |
| Value | | 247.83 | 26.2706 | $4.19 \cdot 10^6$ | $4.1256 \cdot 10^5$ |
| min. $\Phi$ | $\Phi$ | 0[a] | 0[a] | 0[a] | $1.0320 \cdot 10^{-5}$ |
| | dist. opt. | – | 0.9188 | – | 0.3235 |
| max. $\Phi$ | $\Phi$ | 0[a] | $2.2778 \cdot 10^{-4}$ | 0[a] | 0.1789 |
| | dist. opt. | – | $2.5449 \cdot 10^{-8}$ | – | 104.2775 |

[a] Due to double precision.

**Table 2**
Comparison between the classic and the modified PSO algorithm.

| Function | $D$ | $N$ | $t_{\max}$ | $\delta$ | Value |
|---|---|---|---|---|---|
| Sphere | 5 | 2 | 10 000 | $10^{-12}$ | $1.91 \cdot 10^{-26}$ |
| Sphere | 5 | 2 | 10 000 | – | 247.83 |
| Sphere | 50 | 8 | 100 000 | $10^{-12}$ | $2.1402 \cdot 10^{-24}$ |
| Sphere | 50 | 8 | 100 000 | – | 26.27 |
| Rosenbrock | 5 | 2 | 10 000 | $10^{-12}$ | $2.67 \cdot 10^5$ |
| Rosenbrock | 5 | 2 | 10 000 | – | $4.19 \cdot 10^6$ |
| Rosenbrock | 50 | 8 | 100 000 | $10^{-12}$ | 220.66 |
| Rosenbrock | 50 | 8 | 100 000 | – | $4.13 \cdot 10^5$ |




(a) $D = 5$, $N = 2$ (b) $D = 50$, $N = 8$

**Fig. 7.** Curve of the fitness value of the global attractor when processing the sphere function with the modified PSO.

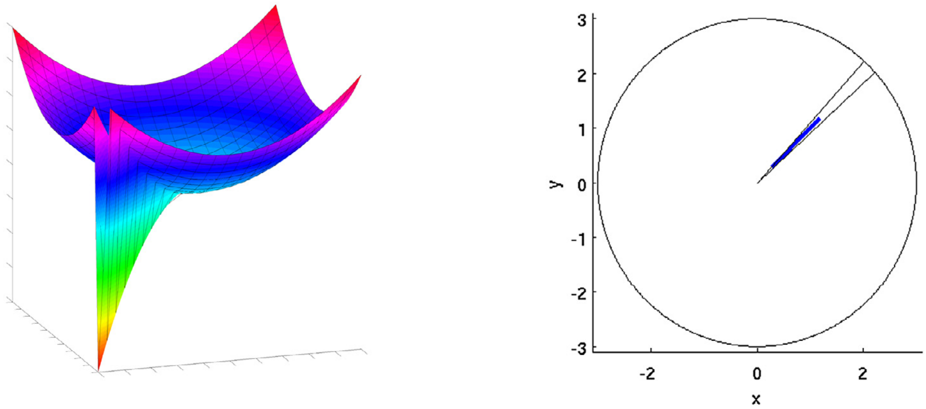
### 4.4. *Differentiability*

As we said in the introduction, the only substantial restriction for the objective function $f$ is that $f$ must have a continuous first derivative. In the following we will provide an example showing what can happen when $f$ is only continuous. Fix some $b > 1$ and define the $D$-dimensional function $f$ as follows:

$$f(\vec{x}) = \begin{cases} \sum_{i=1}^{n} x_i^2, & \exists i, j : x_i \geq b \cdot x_j \vee x_j \geq b \cdot x_i \\ \dfrac{\sum_{i=1}^{n} x_i^2}{b-1} \cdot (2 \max_{i \neq j}\{\frac{x_i}{x_j}\} - b - 1), & \text{otherwise} \end{cases}$$

In Fig. 8a one can see a plot of $f$. For $y$ not between $x/b$ and $x \cdot b$, this function behaves like the well-known sphere-function. For $x = y$, $f(x, y) = -2 \cdot x^2$ and from $y = x/b$ ($y = x \cdot b$) to $y = x$, the function falls into a valley. One can easily see that this function is continuous but has no derivative. The construction of a continuous function which behaves like $f$ on a bounded set and tends to infinity for $|x| + |y| \to \infty$ is straightforward. Therefore, the particles must be able to pass through the valley.

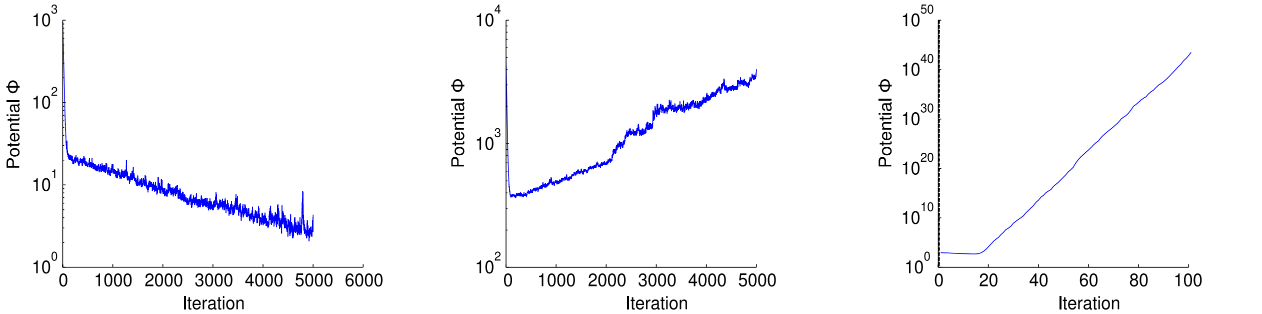
We initialized the particles uniformly at random over $[-100; 100]^D$ (except for the first particle that was initialized at $(1, \ldots, 1)$ such that the swarm could see the direction where the improvements are possible) and the velocities over $[-50; 50]^D$, with the value $D = 3$. We let the swarm do 1000 runs with 5000 iterations each. The potential of the dimension with the highest potential after the last iteration was determined and the mean and standard deviation of the respective dimensions were calculated over the 1000 repetitions. This was done for two different swarm sizes, namely $N = 10$ and




(a) Continuous, not differentiable function $f$

(b) Particles making almost only forced steps through the valley

**Fig. 8.** (a) Function $f$, (b) behavior of the particles on $f$.




(a) $f$, $b = 1.1$, 10 particles

(b) $f$, $b = 1.1$, 50 particles

(c) $f_{\mathrm{rot}}$, $b = 1.1$. 10 particles

**Fig. 9.** Behavior of the particles on functions $f$ and $f_{rot}$.

$N = 50$. We repeated the experiment with 10 particles and only 100 iterations, using the function $f_{rot}$, which is obtained by first rotating the input vector and then applying $f$ such that the valley now leads the particles along the $x_1$-axis. Formally speaking, the rotation maps the vector $(\sqrt{N}, 0, \ldots, 0)$ to $(1, 1, \ldots, 1)$ and keeps every vector that is orthogonal to this two invariant. The results of the three experiments can be seen in Fig. 9. In all three cases, for about the first 20 iterations, the swarm behaves like on the sphere function and reduces its potential. Then, it discovers the valley and tries to move through it. However, in the unrotated case with 10 particles (Fig. 9a), the swarm fails to accelerate and instead, it converges towards a non-optimal point. With much more effort, the swarm consisting of 50 particles (Fig. 9b) is able to accelerate, but the acceleration rate and therefore the speed are comparatively poor. Finally, Fig. 9c shows how the swarm handles the rotated version much better than the original function $f$ before. Here, after only 100 iterations, the potential increased to a value of about $10^{45}$. The reason for this large difference between the behavior on $f$ and on $f_{rot}$ is the capability of the swarm to favor one direction only if this direction is parallel to one of the axes.

In particular, this experiment also confirms the results in [8], namely that PSO is not invariant under rotations of the search space.

### 4.5. Impact of the modification

To make sure that the modification does not fully take over, we plotted the forced points with $\delta = 10^{-7}$ and the 2-dimensional sphere function as objective function in Fig. 10. As can be seen in the figure, the particles get forced near $(-2 \cdot 10^{-5}, 0)$ but their movement does not stay forced. Instead, the swarm becomes running again until the particles approached the optimum at $(0, 0)$. This implies that for sufficiently smooth functions, the modification does not take over, replacing the PSO by some random search routine. Instead, the modification just helps to overcome “corners”. As soon as there is a direction parallel to an axis with decreasing function value, the swarm becomes “running” again and the unmodified movement equations apply.

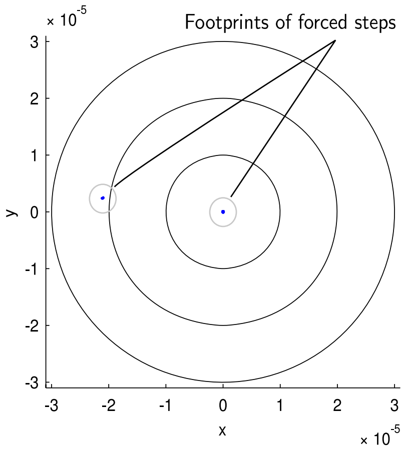



**Fig. 10.** Behavior of the modified PSO on the sphere function.

## 5. Conclusion

This paper focuses on the capability of a particle swarm to find a local minimum. The PSO algorithm is analyzed under this aspect and it is pointed out why the swarm might not always find a local minimum, namely the swarm gets stuck if the differences of the potentials between the dimensions are too large. A suggestion to modify the algorithm by randomly assigning a small velocity when the potential of a particle falls below a certain bound is suggested. It is proven that this modification together with some new parameter selection guidelines enables the swarm to find a local minimum for a large class of objective functions. Additionally, it is shown that the modification does not take over the swarm, it just corrects the direction before the classic movement equations are applied again.

The theoretical work is supported by experiments showing not only the superiority of the modified algorithm but also the described phenomena from the theoretical sections.

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