ADAPTIVE CONTROL THEORY
SC617
ENDSEM PROJECT REPORT

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Task A. Simulate the motion of the spacecraft for $t \in [0, 60]$ sec in the absence of any control assuming any non zero initial condition on ω. Plot $\omega(t)$, $||\mathbf{q}_v||$ and q_0 with time. Do the trajectories remain bounded? Verify that the unit norm constraint on the quaternions is maintained.

Solution: In absence of any control input the dynamics are as follows:

$$\boldsymbol{q} = \begin{pmatrix} q_0 \\ \boldsymbol{q}_v \end{pmatrix} \tag{1}$$

$$\begin{pmatrix} \dot{q}_0 \\ \dot{\boldsymbol{q}}_v \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -\boldsymbol{q}_v^T \\ q_o \boldsymbol{I} + \boldsymbol{S}(\boldsymbol{q}_v) \end{pmatrix} \delta\boldsymbol{\omega} \tag{2}$$

$$\mathbf{J}\dot{\boldsymbol{\omega}} = \boldsymbol{\omega} \times \mathbf{J}\boldsymbol{\omega} + \boldsymbol{u} \tag{3}$$

Following are the plots generated for $t \in [0, 60]$sec with initial conditions given as $\omega = [1, 0.2, 0.3]$ and $q = [-0.9487, 0.1826, 0.1826, 0.1826]$. Clearly our states $\boldsymbol{\omega}(t)$ and $\boldsymbol{q}$ are bounded.

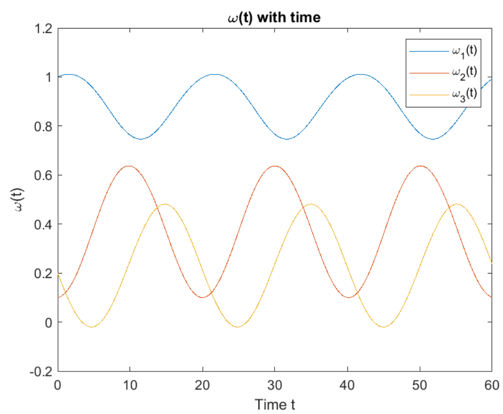

Figure 1: $\omega(t)$ vs t)

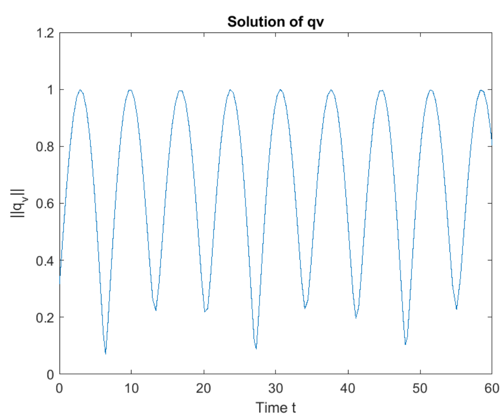

Figure 2: Norm of $||q_v||$ vs time

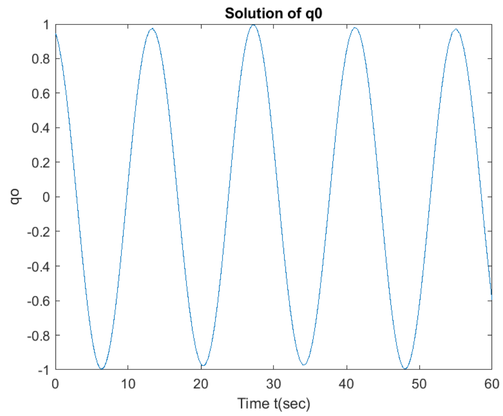

Figure 3: Norm of q_0 vs time

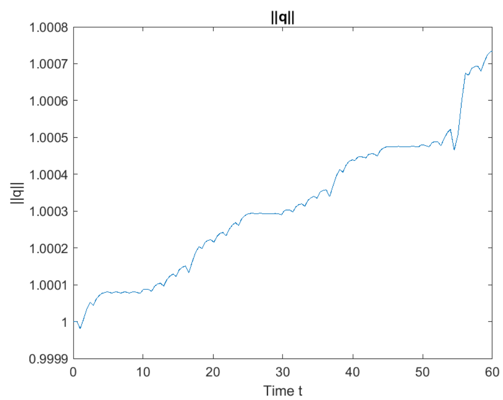

Figure 4: Norm of q vs time

We are having the $||\boldsymbol{q}||$ nearly 1 and the condition is satisfied.

Task B. Design and simulate a control law $u(t)$ for the spacecraft to achieve stabilization. Plot the state and the control signals. The gain parameters in your design must be chosen to ensure that the norm of the state vector $[\omega, \mathbf{q}_v]^T$ remains less than 10^{-3} for all $t \geq 20$.

Solution: Stabilization Problem
For the given dynamics,

$$\boldsymbol{q} = \begin{pmatrix} q_0 \\ \boldsymbol{q}_v \end{pmatrix}$$

$$\begin{pmatrix} \dot{q}_0 \\ \dot{\boldsymbol{q}}_v \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -\boldsymbol{q}_v^T \\ q_o \boldsymbol{I} + \boldsymbol{S}(\boldsymbol{q}_v) \end{pmatrix} \delta\boldsymbol{\omega}$$

$$\mathbf{J}\dot{\boldsymbol{\omega}} = \boldsymbol{\omega} \times \mathbf{J}\boldsymbol{\omega} + \boldsymbol{u}$$

Our objective is to achieve(as $t \to \infty$):

$$\boldsymbol{q} = \begin{pmatrix} \pm\mathbf{1} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix}$$

$$\boldsymbol{\omega} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix}$$

Choosing Lyapunov function as:

$$V = k_p[\boldsymbol{q}_v^T \boldsymbol{q}_v + (1 - q_o)^2] + \frac{1}{2}\boldsymbol{\omega}^T \mathbf{J}\boldsymbol{\omega}$$

$$\implies \dot{V} = k_p \boldsymbol{q}_v^T \boldsymbol{\omega} + \boldsymbol{\omega}^T \mathbf{J}\dot{\boldsymbol{\omega}}$$

$$\implies \dot{V} = k_p \boldsymbol{q}_v^T \boldsymbol{\omega} + \boldsymbol{\omega}^T(\boldsymbol{\omega} \times \mathbf{J}\boldsymbol{\omega} + \boldsymbol{u})$$

Choose $u = -k_p \boldsymbol{q}_v - k_w \boldsymbol{\omega} - \boldsymbol{W}\theta^*$; where $\boldsymbol{\theta}^* \in \mathbb{R}^{6\times 1}$ and $\boldsymbol{W} \in \mathbb{R}^{3\times 6}$. Define $\boldsymbol{W}\boldsymbol{\theta}^* := -\boldsymbol{\omega} \times \mathbf{J}\boldsymbol{\omega}$.
This choice of control input yeilds,

$$\dot{V} = -k_w ||\boldsymbol{\omega}||^2 \leq 0$$

As stated in problem statement, thus $\dot{V}$ is negative semi-definite and using signal chasing analysis and Barbalat's lemma, it can be shown that,

$$\boldsymbol{\omega} \to \mathbf{0}$$

$$\dot{\boldsymbol{\omega}} \to \mathbf{0}$$

$$\implies \boldsymbol{q}_v \to \mathbf{0}$$

Simulation Results: Following are the plots obtained on using above control input and initial conditions:

$$\boldsymbol{\omega}(t=0) = [1, 0.2, 0.3]$$
$$k_w = 20$$
$$k_p = 18$$
$$\boldsymbol{q}(t=0) = [-0.9487, 0.1826, 0.1826, 0.1826]$$
$$J = \begin{bmatrix} 20 & 1.2 & 0.9 \\ 1.2 & 17 & 1.4 \\ 0.9 & 1.4 & 15 \end{bmatrix}$$

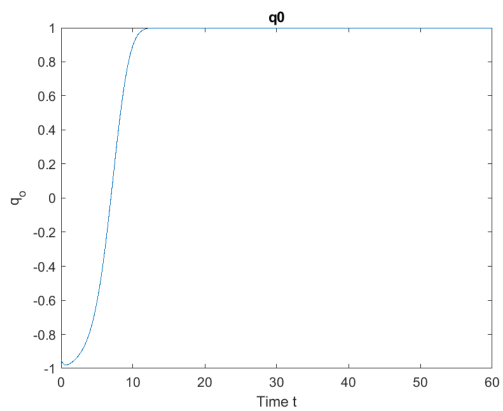

Figure 5: q_0 vs time

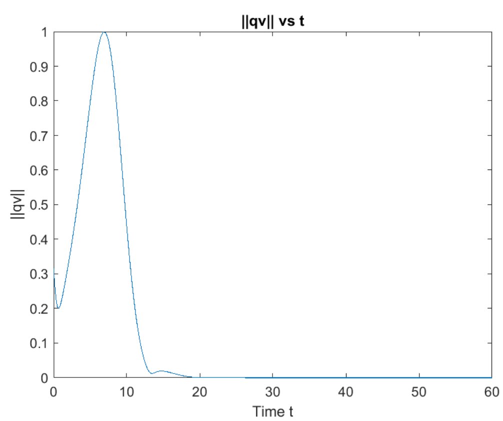

Figure 6: $||q_v||$ vs time

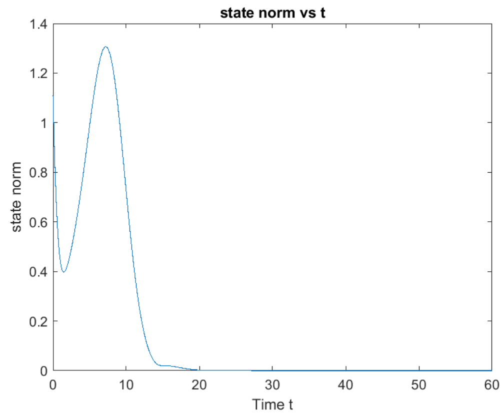

Figure 7: $||q, \omega||$ vs time

Here the gains have been tuned in such a way that we have achieved the requirement of having the $||[\boldsymbol{\omega}, \boldsymbol{q}]^T|| \leq 10^{-3}$ for $t \geq 20sec$. Here at $t = 20sec$ we are having $||[\boldsymbol{\omega}, \boldsymbol{q}]^T|| \approx 0.001$

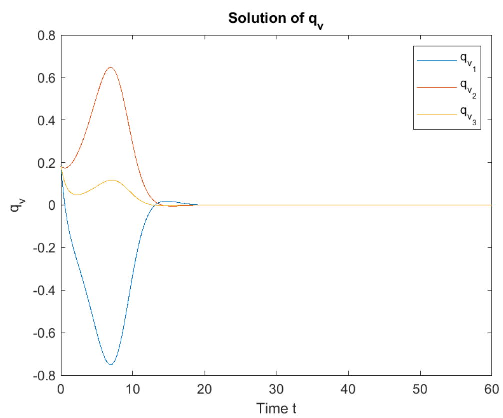

Figure 8: q_v vs t

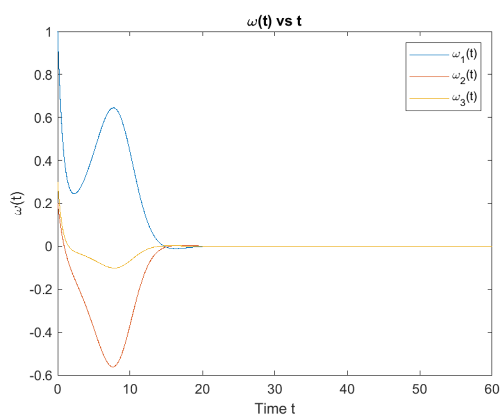

Figure 9: $\omega(t)$ vs t

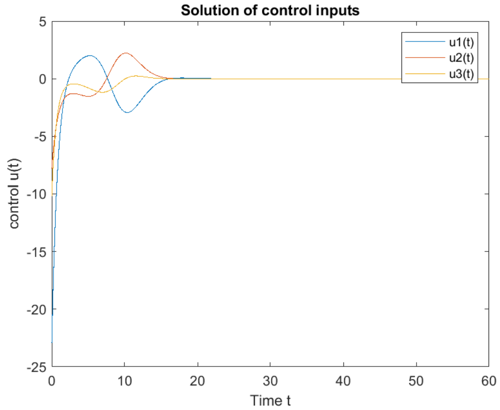

Figure 10: $u(t)$ vs t

Task C. Consider the trajectory tracking problem for the aforementioned spacecraft. The desired angular velocity trajectory in a desired reference frame is given by $\omega_d(t) = [0.3\cos(t)(1-e^{-0.01t^2}) + (0.08\pi + 0.006\sin(t))te^{-0.01t^2}, 0.3\cos(t)(1-e^{-0.01t^2}) + (0.08\pi + 0.006\sin(t))te^{-0.01t^2}, 1]^T$. The desired reference frame $\mathbf{q_d}(t)$ is obtained from $\omega_d(t)$ by solving the corresponding quaternion dynamics ($\dot{\mathbf{q}}_d(t) = 0.5E(\mathbf{q}_d)\omega_d$) with the following initial conditions $\mathbf{q}_{d,o}(t) = 0$ $\mathbf{q}_{d,v}(0) = [0, 0, 0]^T$. Assuming that the inertia matrix is perfectly defined, design and simulate a full state feedback attitude controller that tracks the desired trajectory. Plot the error quaternion, angular velocity and the control. Choose the gains so as to make sure that magnitude of control required on each axis is less than 5 Nm.

Solution: Tracking Problem
For the given dynamics,

$$\boldsymbol{q} = \begin{pmatrix} q_0 \\ \boldsymbol{q}_v \end{pmatrix}$$

$$\begin{pmatrix} \dot{q}_0 \\ \dot{\boldsymbol{q}}_v \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -\boldsymbol{q}_v^T \\ q_o\boldsymbol{I} + \boldsymbol{S}(\boldsymbol{q}_v) \end{pmatrix} \delta\boldsymbol{\omega}$$

$$\mathbf{J}\dot{\boldsymbol{\omega}} = \boldsymbol{\omega} \times \mathbf{J}\boldsymbol{\omega} + \boldsymbol{u}$$

The error dynamics is as;

$$\dot{\mathbf{s}} = \frac{1}{2}\mathbf{E}(\mathbf{s})\delta\boldsymbol{\omega} \tag{4}$$

$$\mathbf{J}\dot{\delta\boldsymbol{\omega}} = -\boldsymbol{\omega} \times \mathbf{J}\boldsymbol{\omega} + \boldsymbol{u} - \mathbf{J}\phi \tag{5}$$

where,

$$\boldsymbol{\phi} := \dot{\boldsymbol{R}}_s\boldsymbol{\omega}_d + \boldsymbol{R}_s\dot{\boldsymbol{\omega}}_d \tag{6}$$

$$\boldsymbol{s} = \begin{pmatrix} s_0 \\ \boldsymbol{s}_v \end{pmatrix} \tag{7}$$

$$\begin{pmatrix} \dot{s}_0 \\ \dot{\boldsymbol{s}}_v \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -\boldsymbol{s}_v^T \\ s_o\boldsymbol{I} + \boldsymbol{S}(\boldsymbol{s}_v) \end{pmatrix} \delta\boldsymbol{\omega} \tag{8}$$

Here $\mathbf{S(a)}$ defined as $\mathbf{S(a)} := \boldsymbol{a} \times \boldsymbol{b}$.
Our objective is to achieve(as $t \to \infty$):

$$\begin{pmatrix} \boldsymbol{s}_v \\ \delta\boldsymbol{\omega} \end{pmatrix} \to \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix} \tag{9}$$

Choosing Lyapunov function as:

$$\begin{aligned} V &= [\boldsymbol{s}_v^T\boldsymbol{s}_v + (1-s_o)^2] + \frac{1}{2}\delta\boldsymbol{\omega}^T\mathbf{J}\delta\boldsymbol{\omega} \\ \implies \dot{V} &= \boldsymbol{s}_v^T\delta\boldsymbol{\omega} + \delta\boldsymbol{\omega}^T\mathbf{J}\dot{\delta\boldsymbol{\omega}} \\ &= \boldsymbol{s}_v^T\delta\boldsymbol{\omega} + \delta\boldsymbol{\omega}^T[-\boldsymbol{\omega} \times \mathbf{J}\boldsymbol{\omega} + \boldsymbol{u} - \mathbf{J}\phi] \end{aligned}$$

Here define $\boldsymbol{W}\theta^* := -\boldsymbol{\omega} \times \mathbf{J}\boldsymbol{\omega} - \mathbf{J}\phi$, for some $\boldsymbol{W} \in \mathbb{R}^{3\times 6}$ since the terms are linear in J.
Choose $\boldsymbol{u} = -k_p\boldsymbol{s}_v - k_w\delta\boldsymbol{\omega} - \boldsymbol{W}\theta^*$.
This yields,

$$\dot{V} = -k_w||\delta\boldsymbol{\omega}||^2 \leq 0$$

As stated in problem statement, thus $\dot{V}$ is negative semi-definite and using signal chasing analysis and Barbalat's lemma, it can be shown that,

$$\delta\boldsymbol{\omega} \rightarrow \mathbf{0}$$
$$\delta\dot{\boldsymbol{\omega}} \rightarrow \mathbf{0}$$
$$\implies \boldsymbol{s}_v \rightarrow \mathbf{0}$$

The tracking objective is achieved with the control law $\boldsymbol{u} = -k_p\boldsymbol{s}_v - k_w\delta\boldsymbol{\omega} - \boldsymbol{W}\theta^*$ and following are the plots obtained on using above control input and following initial conditions:

$$\begin{aligned}
\boldsymbol{\omega}(t=0) &= [0.41381, -0.2795, 0.867] \\
\boldsymbol{\omega}_d(t=0) &= [0, 0, 1] \\
k_w &= 29.1 \\
k_p &= 30 \\
\boldsymbol{q}(t=0) &= [-0.9487, 0.1826, 0.1826, 0.1826] \\
\boldsymbol{q}_d(t=0) &= [1, 0, 0, 0] \\
J &= \begin{bmatrix} 20 & 1.2 & 0.9 \\ 1.2 & 17 & 1.4 \\ 0.9 & 1.4 & 15 \end{bmatrix}
\end{aligned}$$

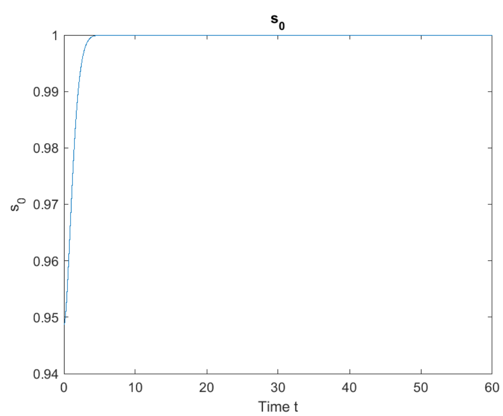

Figure 11: s_0 vs time

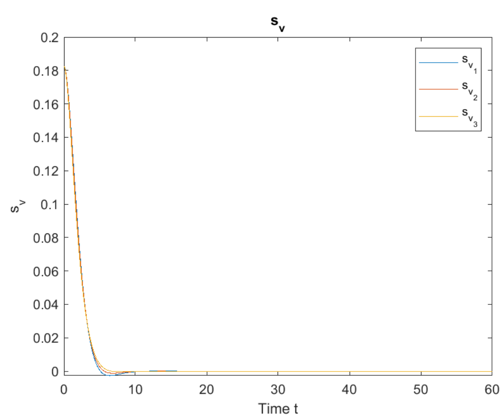

Figure 12: s_v vs t

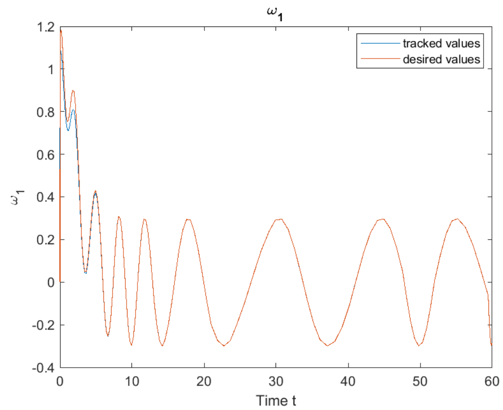

Figure 13: $\omega_{d1}(t)$ and $\omega_1(t)$ vs t

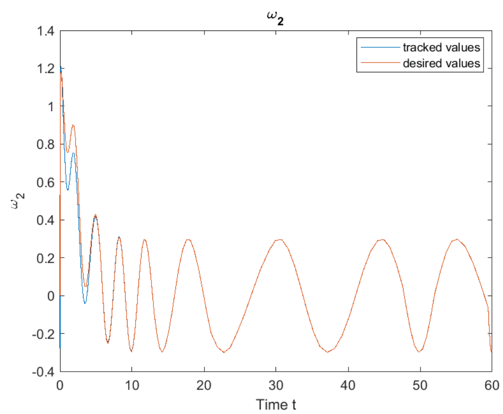

Figure 14: $\omega_{d2}(t)$ and $\omega_2(t)$ vs t

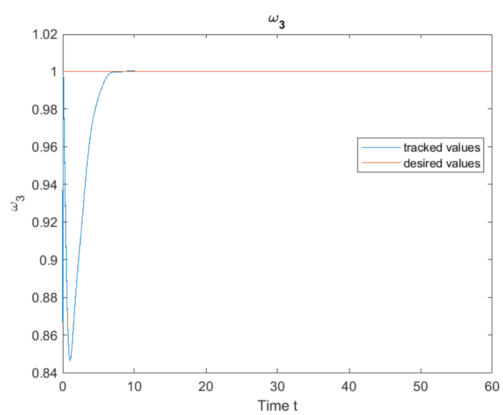

Figure 15: $\omega_{d3}(t)$ and $\omega_3(t)$ vs t

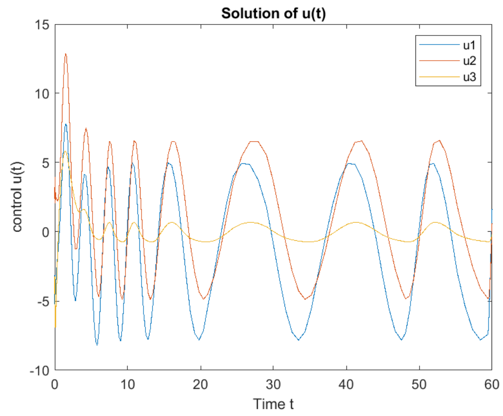

Figure 16: $u(t)$ vs t)

The gains have been tuned to make the control inputs less than 5 Nm. But since the initial values of quaternion is large enough so the maximum I achieved by tuning the control gains is nearly 8 Nm.

Task D. Consider the spacecraft tracking error dynamics (9)-(10). Derive a control law $\mathbf{u}(t)$ to guarantee perfect attitude and angular velocity tracking using an integrator backstepping type construction proposed for linear double integrators for the known inertia case. Provide complete stability analysis along with the control design

Solution: We have the error dynamics as:

$$\dot{\mathbf{s}} = \frac{1}{2}\mathbf{E}(\mathbf{s})\delta\boldsymbol{\omega} \tag{10}$$

$$\mathbf{J}\dot{\delta\boldsymbol{\omega}} = -\boldsymbol{\omega} \times \mathbf{J}\boldsymbol{\omega} + \boldsymbol{u} - \mathbf{J}\phi \tag{11}$$

where,

$$\boldsymbol{\phi} := \dot{\boldsymbol{R}}_s\boldsymbol{\omega}_d + \boldsymbol{R}_s\dot{\boldsymbol{\omega}}_d \tag{12}$$

$$\boldsymbol{s} = \begin{pmatrix} s_0 \\ \boldsymbol{s}_v \end{pmatrix} \tag{13}$$

$$\begin{pmatrix} \dot{s}_0 \\ \dot{\boldsymbol{s}}_v \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -\boldsymbol{s}_v^T \\ s_o\boldsymbol{I} + \boldsymbol{S}(\boldsymbol{s}_v) \end{pmatrix}\delta\boldsymbol{\omega} \tag{14}$$

Here $\mathbf{S(a)}$ defined as $\mathbf{S(a)} := \boldsymbol{a} \times \boldsymbol{b}$.
Our objective is to achieve(as $t \to \infty$):

$$\begin{pmatrix} \boldsymbol{s}_v \\ \delta\boldsymbol{\omega} \end{pmatrix} \to \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix} \tag{15}$$

<u>Integrator backstepping construction</u>: Assume $\delta\boldsymbol{\omega}$ to be control and choose:

$$\delta\boldsymbol{\omega} = \delta\boldsymbol{\omega}_d := -k_p\boldsymbol{s}_v \tag{16}$$

with $k_p > 0$ and $\delta\boldsymbol{\omega}_d$ is desired one.
Therefore we had,

$$\dot{\delta\boldsymbol{\omega}}_d = -k_p\dot{\boldsymbol{s}}_v \tag{17}$$

$$= \frac{-k_p}{2}[s_o\delta\boldsymbol{\omega} + \boldsymbol{s}_v \times \delta\boldsymbol{\omega}] \tag{18}$$

Define $\mathbf{J}\boldsymbol{\xi} := \mathbf{J}\delta\boldsymbol{\omega} - \mathbf{J}\delta\boldsymbol{\omega}_d$
Now,

$$\mathbf{J}\dot{\boldsymbol{\xi}} = \mathbf{J}\dot{\delta\boldsymbol{\omega}} - \mathbf{J}\dot{\delta\boldsymbol{\omega}}_d \tag{19}$$

$$= (-\boldsymbol{\omega} \times \mathbf{J}\boldsymbol{\omega} + \boldsymbol{u} - \mathbf{J}\phi) + \frac{k_p}{2}\mathbf{J}[s_o\delta\boldsymbol{\omega} + \boldsymbol{s}_v \times \delta\boldsymbol{\omega}] \tag{20}$$

$$= (-\boldsymbol{\omega} \times \mathbf{J}\boldsymbol{\omega} - \mathbf{J}\phi + \frac{k_p}{2}\mathbf{J}[s_o\delta\boldsymbol{\omega} + \boldsymbol{s}_v \times \delta\boldsymbol{\omega}]) + \boldsymbol{u} \tag{21}$$

$$= \boldsymbol{W}\boldsymbol{\theta}^* + \boldsymbol{u} \tag{22}$$

where $\mathbf{W} \in \mathbb{R}^{3\times 6}$. Notice that $\mathbf{J}$ is symmetric and hence has only 6 distinct entries, which we denote as $\boldsymbol{\theta}^* \in \mathbb{R}^{6\times 1}$. Define $\boldsymbol{W}\boldsymbol{\theta}^* := -\boldsymbol{\omega} \times \mathbf{J}\boldsymbol{\omega} - \mathbf{J}\phi + \frac{k_p}{2}\mathbf{J}[s_o\delta\boldsymbol{\omega} + \boldsymbol{s}_v \times \delta\boldsymbol{\omega}]$. Define Lyapunov function V as:

$$V = [\boldsymbol{s}_v^T\boldsymbol{s}_v + (1 - s_o)^2] + \frac{1}{2}\boldsymbol{\xi}^T\mathbf{J}\boldsymbol{\xi} \tag{23}$$

$$\implies \dot{V} = \boldsymbol{s}_v^T\delta\boldsymbol{\omega} + \boldsymbol{\xi}^T\mathbf{J}\dot{\boldsymbol{\xi}} \tag{24}$$

$$= \boldsymbol{s}_v^T(\boldsymbol{\xi} - k_p\boldsymbol{s}_v) + \boldsymbol{\xi}^T(\boldsymbol{W}\boldsymbol{\theta}^* + \boldsymbol{u}) \tag{25}$$

$$= -k_p||\boldsymbol{s}_v||^2 + \boldsymbol{\xi}^T(\boldsymbol{W}\boldsymbol{\theta}^* + \boldsymbol{u} + \boldsymbol{s}_v) \tag{26}$$

Now choose, $\boldsymbol{u} = -\boldsymbol{W}\boldsymbol{\theta}^* - k_\xi\boldsymbol{\xi}$, we get

$$\dot{V} = -k_p||\boldsymbol{s}_v||^2 - k_\xi||\boldsymbol{\xi}||^2 + \boldsymbol{\xi}^T\boldsymbol{s}_v \tag{27}$$

$$\implies \dot{V} = -(k_p - \frac{1}{2})||\boldsymbol{s}_v||^2 - (k_\xi - \frac{1}{2})||\boldsymbol{\xi}||^2 \tag{28}$$

Using property $ab \leq \frac{a^2+b^2}{2}$ we can see that $\dot{V} < 0$ for $k_p > \frac{1}{2}$ and $k_\xi > \frac{1}{2}$. Therefore V is serving as strict Lyapunov function.

Stability Analysis:
We will use following Lemma:

Lemma 1. Consider a scalar valued function, $f(t) : \mathbb{R} \to \mathbb{R}$. Suppose $f(t)$ is bounded from below and non-increasing. Then, $f(t)$ has a finite limit as $t \to \infty$.

Lemma 2. Barbalat's Lemma:Consider $f(.) : \mathbb{R} \to \mathbb{R}^n$(scalar or vector valued) such that the signal is integrable, i.e.

$$\lim_{t\to\infty} \int_0^t f(\sigma)d\sigma$$

exists and is finite. Further, suppose $f(t)$ is uniformly continuous, then

$$\lim_{t\to\infty} f(t) = 0$$

Note: In case of vector-valued $f(.)$, the integral has to be finite component-wise.

Claim: As $t \to \infty$, $\boldsymbol{s}_v \to 0$ and $\boldsymbol{\xi} \to 0$.
We will prove this using Signal Chasing Analysis-
Proof:

1 $V(t)$ is bounded - Since $V(t)$ is lower bounded ($V(t) \geq 0$) and non-increasing ($\dot{V}(t) < 0$ [for $k_p > \frac{1}{2}$ and $k_\xi > \frac{1}{2}$]), using Lemma 1,

$$V_\infty := \lim_{t\to\infty} V(t) < \infty$$

where $V(t) : \mathbb{R} \to \mathbb{R}$

2 $\boldsymbol{s}_v$, $\boldsymbol{\xi}$ are bounded component-wise-
Now,

$$V(t) \leq V(0)$$

$$\implies V \text{ is bounded}$$

$$\implies ||s_v||^2 \text{ and } ||\xi||^2 \text{ are bounded}$$

$$\implies \boldsymbol{s}_v \text{ and } \boldsymbol{\xi} \text{ are bounded component wise}$$

$$\implies \text{component wise } \boldsymbol{s}_v \text{ and } \boldsymbol{\xi} \in \mathcal{L}_\infty$$

Since $\boldsymbol{s}_v \in \mathcal{L}_\infty \implies s_0 \in \mathcal{L}_\infty$ as $s_0 = \sqrt{1 - ||\boldsymbol{s}_v||^2}$

3 $\boldsymbol{s}_v$, $\boldsymbol{\xi} \in \mathcal{L}_2$ component-wise-

$$\int_0^\infty \dot{V}(t)dt = -(k_p - \frac{1}{2})\int_o^\infty ||\boldsymbol{s}_v(t)||^2 dt - (k_\xi - \frac{1}{2})\int_0^\infty ||\boldsymbol{\xi}(t)||^2 dt$$
$$\implies V_\infty - V(0) = -(k_p - \frac{1}{2})\int_o^\infty ||\boldsymbol{s}_v(t)||^2 dt - (k_\xi - \frac{1}{2})\int_0^\infty ||\boldsymbol{\xi}(t)||^2 dt$$
$$\implies V_\infty - V(0) = -(k_p - \frac{1}{2})\int_o^\infty \sum_{i=1}^{3} ||\boldsymbol{s}_{vi}(t)||^2 dt - (k_\xi - \frac{1}{2})\int_0^\infty \sum_{n=1}^{3} ||\boldsymbol{\xi}_i(t)||^2 dt$$

Now for $k_p > \frac{1}{2}$ and $k_\xi > \frac{1}{2}$:LHS is finite and therefore each term on RHS must also be finite as both integral are positive. So component wise, we know,

$$||\boldsymbol{s}_{vi}||_2 = \sqrt{\int_0^\infty \boldsymbol{s}_{vi}(t)^2}$$

$$\text{and } ||\boldsymbol{\xi}_i||_2 = \sqrt{\int_0^\infty \boldsymbol{\xi}_i(t)^2}$$

So,

$$\boldsymbol{s}_v \in \mathcal{L}_2 \text{ and } \boldsymbol{\xi} \in \mathcal{L}_2 \text{ component wise}$$
$$\implies s_0 \in \mathcal{L}_2$$

4 $\dot{s_0} \in \mathcal{L}_\infty$ and $\dot{\boldsymbol{s}_v}$, $\dot{\boldsymbol{\xi}} \in \mathcal{L}_\infty$ component-wise-

Since, from dynamics:

$$\dot{s_0} = \frac{-1}{2}\boldsymbol{s}_v^T\delta\boldsymbol{\omega}$$
$$\dot{\boldsymbol{s}_v} = \frac{1}{2}(s_o\delta\boldsymbol{\omega} + \boldsymbol{s}_v \times \delta\boldsymbol{\omega})$$
$$\mathbf{J}\dot{\boldsymbol{\xi}} = -k_{\boldsymbol{\xi}}\boldsymbol{\xi} = -k_{\boldsymbol{\xi}}(\delta\boldsymbol{\omega} - \delta\boldsymbol{\omega}_d)$$

We know that $\delta\boldsymbol{\omega}, \boldsymbol{s}_v, s_0$ and $\delta\boldsymbol{\omega}_d$ all are bounded, so $\dot{s_0}$, $\dot{\boldsymbol{s}_v}$, $\dot{\boldsymbol{\xi}} \in \mathcal{L}_\infty$ component-wise

5 Using Barbalat's Lemma-$\boldsymbol{s}_v \to 0$, $\boldsymbol{\xi} \to 0$ as $t \to \infty$ -

Here,

- s_o, $\boldsymbol{s}_v, \boldsymbol{\xi} \in \mathcal{L}_\infty \cap \mathcal{L}_2$
- $\dot{s_0}$, $\dot{\boldsymbol{s}_v}$, $\dot{\boldsymbol{\xi}} \in \mathcal{L}_\infty$

So directly by Corollary of **Lemma 2** we have $s_0 \to 0$, $\boldsymbol{s}_v \to 0$, $\boldsymbol{\xi} \to 0$ as $t \to \infty$

6 $\dot{s_0}$, $\dot{\boldsymbol{s}_v}$, and $\dot{\boldsymbol{\xi}}$ are integrable

We are having s_0, $\boldsymbol{s}_v$, $\boldsymbol{\xi} \in \mathcal{L}_\infty$, so we are having all $\dot{s_0}$, $\dot{\boldsymbol{s}_v}$, and $\dot{\boldsymbol{\xi}}$ integrable

7 $\ddot{s_0}$, $\ddot{\boldsymbol{s}_v}$, and $\ddot{\boldsymbol{\xi}} \in \mathcal{L}_\infty$

We are having these as following dynamic equations:

$$\ddot{s_0} = \frac{-1}{2}[\dot{\boldsymbol{s}_v}^T\delta\boldsymbol{\omega} + \boldsymbol{s}_v^T\dot{\delta\boldsymbol{\omega}}]$$
$$\ddot{\boldsymbol{s}_v} = \frac{1}{2}[\dot{s_o}\delta\boldsymbol{\omega} + s_o\dot{\delta\boldsymbol{\omega}} + \dot{\boldsymbol{s}_v} \times \delta\boldsymbol{\omega} + \boldsymbol{s}_v \times \dot{\delta\boldsymbol{\omega}}]$$
$$\mathbf{J}\ddot{\boldsymbol{\xi}} = -k_{\boldsymbol{\xi}}\dot{\boldsymbol{\xi}} = -k_{\boldsymbol{\xi}}(\dot{\delta\boldsymbol{\omega}} - \dot{\delta\boldsymbol{\omega}}_d)$$

We know that $\delta\boldsymbol{\omega}, \boldsymbol{s}_v, s_0, \delta\boldsymbol{\omega}_d, \dot{\delta\boldsymbol{\omega}}, \dot{\boldsymbol{s}_v}, \dot{s_0}$ and $\dot{\delta\boldsymbol{\omega}}_d$ all are bounded, so $\ddot{s_0}$, $\ddot{\boldsymbol{s}_v}$, and $\ddot{\boldsymbol{\xi}} \in \mathcal{L}_\infty$

8 Using Barbalat's Lemma-$\dot{s_0} \to 0$, $\dot{\boldsymbol{s}_v} \to 0$, $\dot{\boldsymbol{\xi}} \to 0$ as $t \to \infty$ -

As here,

- $\dot{s_0}$, $\dot{\boldsymbol{s}_v}$, $\dot{\boldsymbol{\xi}} \in \mathcal{L}_\infty \cap \mathcal{L}_2$
- $\ddot{s_0}$, $\ddot{\boldsymbol{s}_v}$, $\ddot{\boldsymbol{\xi}} \in \mathcal{L}_\infty$

So directly by Corollary of **Lemma 2** we have $s_0 \dot{\to} 0$, $\boldsymbol{s}_v \dot{\to} 0$, $\dot{\boldsymbol{\xi}} \to 0$ as $t \to 0$.

Task E. Design and simulate an adaptive attitude tracking controller for the above case when the inertia matrix is unknown but a symmetric positive definite constant matrix. Assume starting inertial values at a 30% off-set from the true value. Provide plots of error quaternion, angular velocity, control and the parameter error. Do the parameters converge to their true values ? What is the persistence condition? Bonus points if you can alter desired trajectory to ensure convergence.

Solution: We had from know $\mathbf{J}$ dynamics our Lyapunov function as:

$$V = [\boldsymbol{s}_v^T \boldsymbol{s}_v + (1 - s_o)^2] + \frac{1}{2}\boldsymbol{\xi}^T \mathbf{J} \boldsymbol{\xi}$$

Adaptive Integrator backstepping construction: $\mathbf{J}$ is symmetric and hence has only 6 distinct entries, which we denote as $\boldsymbol{\theta}^* \in \mathbb{R}^{6\times1}$. Let $\hat{\boldsymbol{\theta}}$ denote the parameter estimate of $\boldsymbol{\theta}^*$ and define parameter error as:

$$\tilde{\boldsymbol{\theta}} = \boldsymbol{\theta}^* - \hat{\boldsymbol{\theta}} \tag{29}$$

Choose candidate Lyapunov function given by:

$$V = [\boldsymbol{s}_v^T \boldsymbol{s}_v + (1 - s_o)^2] + \frac{1}{2}\boldsymbol{\xi}^T \mathbf{J} \boldsymbol{\xi} + \frac{1}{2\gamma}\tilde{\boldsymbol{\theta}}^T \tilde{\boldsymbol{\theta}} \tag{30}$$

$$\tag{31}$$

where $\gamma \in \mathbb{R}, \gamma > 0$ is positive adaptive gain. Note, $\dot{\tilde{\boldsymbol{\theta}}} = -\dot{\hat{\boldsymbol{\theta}}}$ as we are following the fundamental assumption of adaptive control of constant unknown parameter estimation i.e. $\boldsymbol{\theta}^*$ is constant. Now taking time derivative of Lyapunov candidate function:

$$\dot{V} = \boldsymbol{s}_v^T \delta\boldsymbol{\omega} + \boldsymbol{\xi}^T \mathbf{J} \dot{\boldsymbol{\xi}} - \frac{1}{\gamma}\tilde{\boldsymbol{\theta}}^T \dot{\hat{\boldsymbol{\theta}}} \tag{32}$$

Substituting $\delta\boldsymbol{\omega}$ and $\mathbf{J}\dot{\boldsymbol{\xi}}$ from (16) and (22) in (32), we get

$$\implies \dot{V} = \boldsymbol{s}_v^T(\boldsymbol{\xi} - k_p \boldsymbol{s}_v) + \boldsymbol{\xi}^T(\boldsymbol{W}\boldsymbol{\theta}^* + \boldsymbol{u}) - \frac{1}{\gamma}\tilde{\boldsymbol{\theta}}^T \dot{\hat{\boldsymbol{\theta}}} \tag{33}$$

$$\tag{34}$$

Now choose control

$$\boldsymbol{u}(t) = -\boldsymbol{W}\hat{\boldsymbol{\theta}} - k_\xi \boldsymbol{\xi} \tag{35}$$

$$\implies \dot{V} = -k_p||\boldsymbol{s}_v||^2 - k_\xi||\boldsymbol{\xi}||^2 + \boldsymbol{\xi}^T \boldsymbol{s}_v + \boldsymbol{\xi}^T \boldsymbol{W}\tilde{\boldsymbol{\theta}} - \frac{1}{\gamma}\tilde{\boldsymbol{\theta}}^T \dot{\hat{\boldsymbol{\theta}}} \tag{36}$$

Since $\tilde{\boldsymbol{\theta}}^T \dot{\hat{\boldsymbol{\theta}}}$ is a scalar quantity,

$$\tilde{\boldsymbol{\theta}}^T \dot{\hat{\boldsymbol{\theta}}} = \dot{\hat{\boldsymbol{\theta}}}^T \tilde{\boldsymbol{\theta}} \tag{37}$$

$$\implies \dot{V} = -k_p||\boldsymbol{s}_v||^2 - k_\xi||\boldsymbol{\xi}||^2 + \boldsymbol{\xi}^T \boldsymbol{s}_v + [\boldsymbol{\xi}^T \boldsymbol{W} - \frac{1}{\gamma}\dot{\hat{\boldsymbol{\theta}}}^T]\tilde{\boldsymbol{\theta}} \tag{38}$$

Now we can choose the adaption law such that we can eliminate the $\tilde{\boldsymbol{\theta}}$ term from the $\dot{V}$ expression

$$\dot{\hat{\boldsymbol{\theta}}}^T = \gamma \boldsymbol{\xi}^T \boldsymbol{W} \tag{39}$$
$$\implies \dot{\hat{\boldsymbol{\theta}}} = \gamma \boldsymbol{W}^T \boldsymbol{\xi} \tag{40}$$
$$\implies \dot{V} = -k_p||\boldsymbol{s}_v||^2 - k_\xi||\boldsymbol{\xi}||^2 + \boldsymbol{\xi}^T \boldsymbol{s}_v \tag{41}$$
$$\implies \dot{V} = -(k_p - \frac{1}{2})||\boldsymbol{s}_v||^2 - (k_\xi - \frac{1}{2})||\boldsymbol{\xi}||^2 \tag{42}$$

Using property $ab \leq \frac{a^2+b^2}{2}$ we can see that $\dot{V} \leq 0$ for $k_p > \frac{1}{2}$ and $k_\xi > \frac{1}{2}$. Here $\dot{V}$ is semi-negative definite as we are not having all states appearing in $\dot{V}$.
So our Persistence condition for the system is,

$$\dot{\hat{\boldsymbol{\theta}}} = \gamma \boldsymbol{W}^T \boldsymbol{\xi}$$

Following are the simulated results for the adaptive controller for $t \in [0, 60]$ sec with initial conditions given as:

$$\begin{aligned}
\omega &= [0.8, 0.2, 0.4] \\
\hat{J}_0 &= \begin{bmatrix} 14 & 0.84 & 0.63 \\ 0.84 & 11.9 & 0.98 \\ 0.63 & 0.98 & 10.5 \end{bmatrix} \\
k_\xi &= 28 \\
k_p &= 30 \\
\gamma &= 1 \\
q &= [-0.9487, 0.1826, 0.1826, 0.1826]
\end{aligned}$$

and all the desired trajectory conditions from the given information of trajectory:

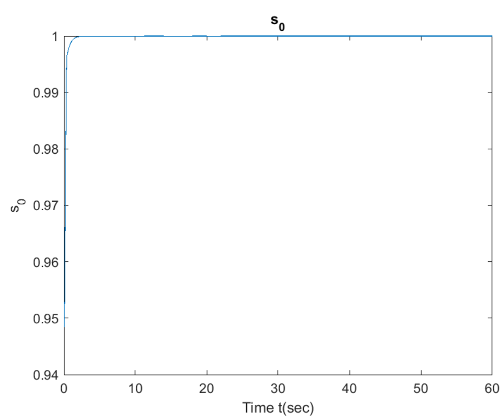

Figure 17: s_0 vs time

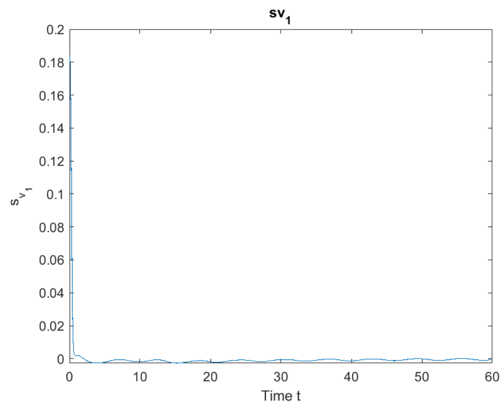

Figure 18: s_{v_1} vs time

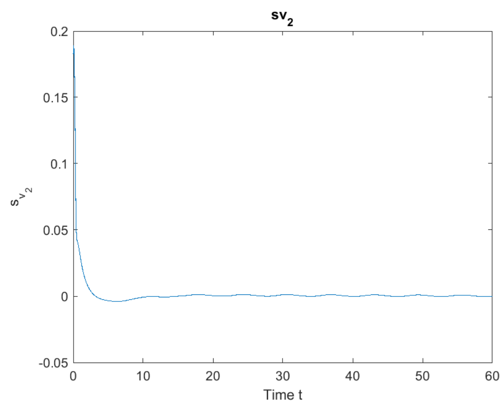

Figure 19: s_{v_2} vs time

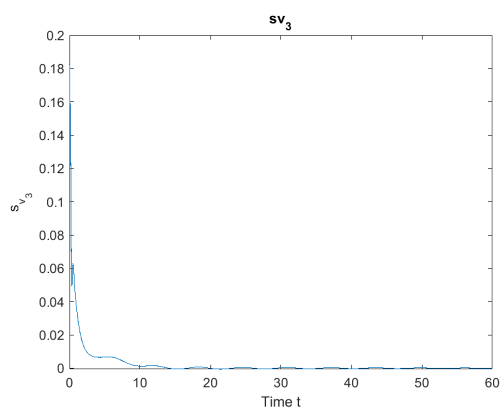

Figure 20: s_{v_3} vs time

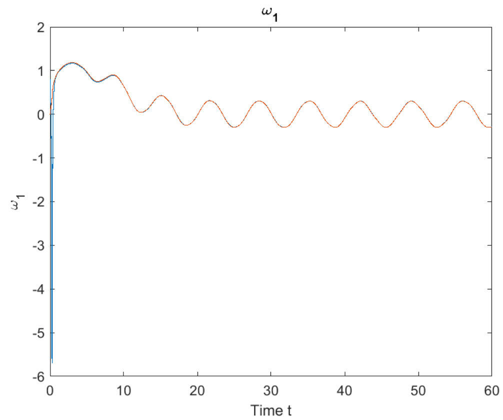

Figure 21: $\omega_1(t)$ vs t)

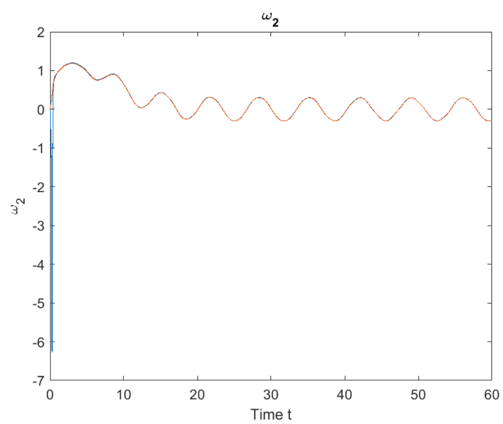

Figure 22: $\omega_2(t)$ vs t)

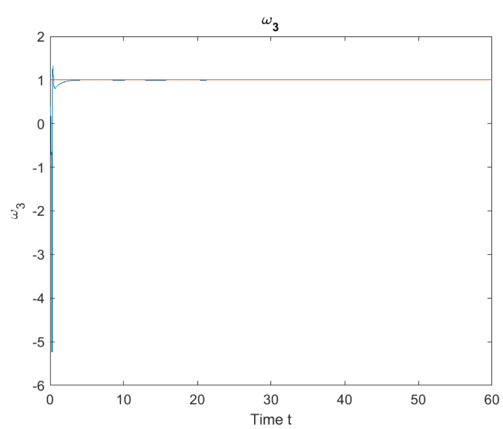

Figure 23: $\omega_3(t)$ vs t)

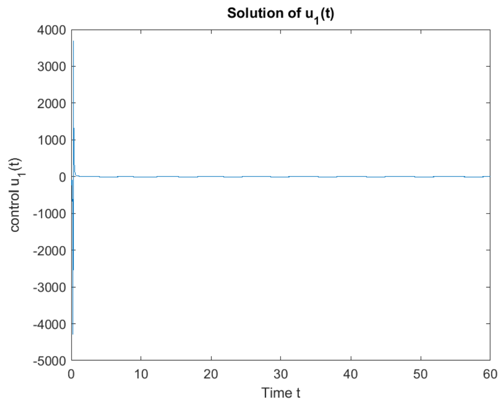

Figure 24: $u_1(t)$ vs t)

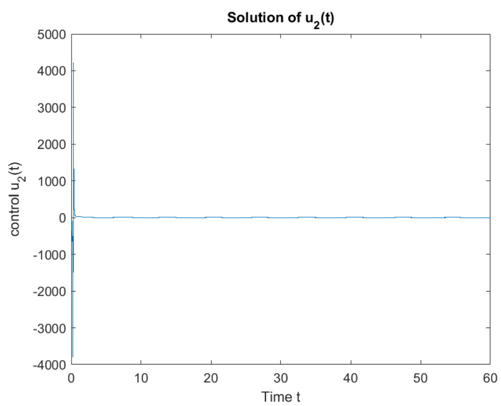

Figure 25: $u_2(t)$ vs t)

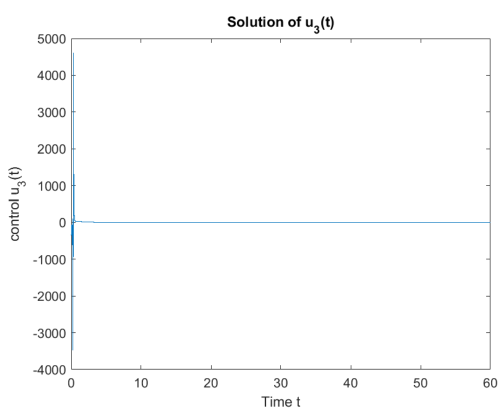

Figure 26: $u_3(t)$ vs t)

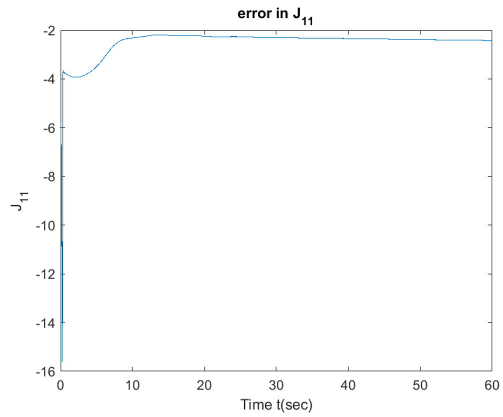

Figure 27: Error in $J_{11}(t)$ vs t

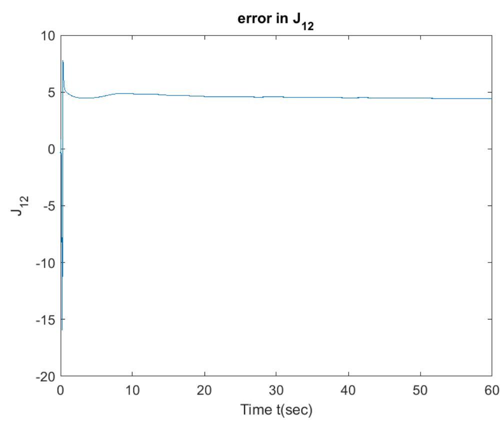

Figure 28: Error in $J_{12}(t)$ vs t

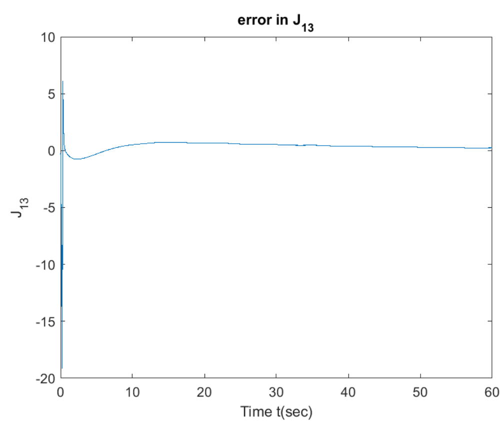

Figure 29: Error in $J_{13}(t)$ vs t

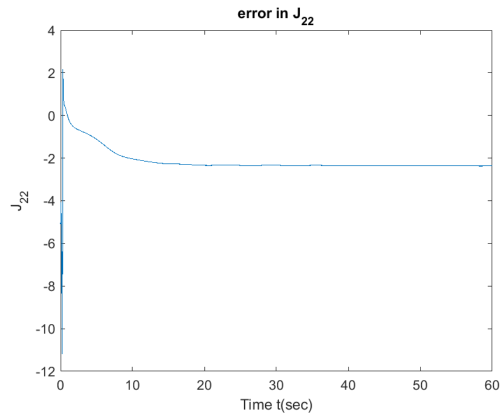

Figure 30: Error in $J_{22}(t)$ vs t

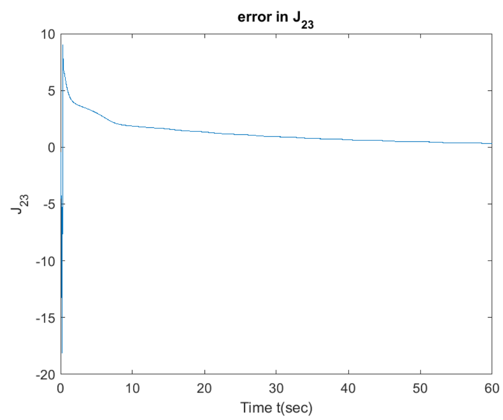

Figure 31: Error in $J_{23}(t)$ vs t

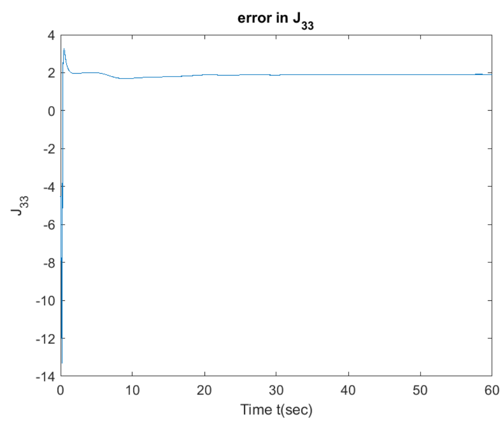

Figure 32: Error in $J_{33}(t)$ vs t

We can see that parameters are not converging to their real values but after $t \geq 20sec$ we have steady constant error in Parameters estimate.

Task F. Consider the alternate form of (9)-(10) below,

$$\dot{\mathbf{s}} = \frac{1}{2}\mathbf{E}(\mathbf{s})\delta\boldsymbol{\omega}$$
$$\mathbf{J}\dot{\delta\boldsymbol{\omega}} = \boldsymbol{\omega} \times \mathbf{J}\boldsymbol{\omega} + \boldsymbol{\nu} - \mathbf{J}\phi$$
$$\dot{\boldsymbol{\nu}} = u$$

where $\boldsymbol{\nu} \in \mathbb{R}^3$ is the control. Design an adaptive backstepping based law using the tuning function method? Provide simultaions with same data as in part (c) and assuming starting inertial values at a 30% off-set from the true value.

Solution: For the above dynamics we will first claim an Adaptive Control Lyapunov Function(**ACLF**) for the original system dynamics without augmenting the integrator term. So for following system dynamics:

$$\dot{\mathbf{s}} = \frac{1}{2}\mathbf{E}(\mathbf{s})\delta\boldsymbol{\omega} \tag{43}$$
$$\mathbf{J}\dot{\delta\boldsymbol{\omega}} = \boldsymbol{\omega} \times \mathbf{J}\boldsymbol{\omega} + \boldsymbol{u} - \mathbf{J}\phi \tag{44}$$

Tuning Function Method: Assume $\delta\boldsymbol{\omega}$ to be control and choose:

$$\delta\boldsymbol{\omega} = \delta\boldsymbol{\omega}_d := -k_p\boldsymbol{s}_v \tag{45}$$

with $k_p > 0$ and $\delta\boldsymbol{\omega}_d$ is desired one.
Therefore we had,

$$\dot{\delta\boldsymbol{\omega}}_d = -k_p\dot{\boldsymbol{s}}_v \tag{46}$$
$$= \frac{-k_p}{2}[s_o\delta\boldsymbol{\omega} + \boldsymbol{s}_v \times \delta\boldsymbol{\omega}] \tag{47}$$

Define $\mathbf{J}\boldsymbol{\xi} := \mathbf{J}\delta\boldsymbol{\omega} - \mathbf{J}\delta\boldsymbol{\omega}_d$
Now,

$$\mathbf{J}\dot{\boldsymbol{\xi}} = \mathbf{J}\dot{\delta\boldsymbol{\omega}} - \mathbf{J}\dot{\delta\boldsymbol{\omega}}_d \tag{48}$$
$$= (-\boldsymbol{\omega} \times \mathbf{J}\boldsymbol{\omega} + \boldsymbol{u} - \mathbf{J}\phi) + \frac{k_p}{2}\mathbf{J}[s_o\delta\boldsymbol{\omega} + \boldsymbol{s}_v \times \delta\boldsymbol{\omega}] \tag{49}$$
$$= (-\boldsymbol{\omega} \times \mathbf{J}\boldsymbol{\omega} - \mathbf{J}\phi + \frac{k_p}{2}\mathbf{J}[s_o\delta\boldsymbol{\omega} + \boldsymbol{s}_v \times \delta\boldsymbol{\omega}]) + \boldsymbol{u} \tag{50}$$
$$= \boldsymbol{W}\boldsymbol{\theta}^* + \boldsymbol{u} \tag{51}$$

where $\mathbf{W} \in \mathbb{R}^{3\times6}$. Notice that $\mathbf{J}$ is symmetric and hence has only 6 distinct entries, which we denote as $\boldsymbol{\theta}^* \in \mathbb{R}^{6\times1}$. Define $\boldsymbol{W}\boldsymbol{\theta}^* := -\boldsymbol{\omega} \times \mathbf{J}\boldsymbol{\omega} - \mathbf{J}\phi + \frac{k_p}{2}\mathbf{J}[s_o\delta\boldsymbol{\omega} + \boldsymbol{s}_v \times \delta\boldsymbol{\omega}]$.
Now combine (43) and (51) to have combined dynamics of the system as:

$$\begin{bmatrix} \dot{s} \\ \mathbf{J}\dot{\boldsymbol{\xi}} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\mathbf{E}(\mathbf{s})\delta\boldsymbol{\omega} \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{W} \end{bmatrix}\boldsymbol{\theta}^* + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\boldsymbol{u}$$

Claim:$\boldsymbol{V}_a$ be the Adaptive Control Lyapunov function for the above dynamics

$$\boldsymbol{V}_a = [\boldsymbol{s}_v^T\boldsymbol{s}_v + (1 - s_o)^2] + \frac{1}{2}\boldsymbol{\xi}^T\mathbf{J}\boldsymbol{\xi} \tag{52}$$

If V_a is ACLF for the above dynamics then it would be CLF for the modified system with $\Gamma = \Gamma^T > 0$ where $\Gamma \in \mathbb{R}^{6\times6}$.i.e..

$$\dot{V}_a := \begin{bmatrix} \frac{\partial V_a}{\partial s} & \frac{\partial V_a}{\partial \boldsymbol{\xi}} \end{bmatrix} \begin{bmatrix} \frac{1}{2}\mathbf{E}(\mathbf{s})\delta\boldsymbol{\omega} \\ \mathbf{J}^{-1}[\mathbf{W}(\boldsymbol{\theta}^* + \Gamma(\frac{\partial V_a}{\partial \boldsymbol{\theta}^*})^T) + \boldsymbol{u}] \end{bmatrix} \leq -\boldsymbol{M}(\mathbf{s}, \boldsymbol{\xi}, \boldsymbol{\theta}^*)$$

where $\boldsymbol{M}(\mathbf{s}, \boldsymbol{\xi}, \boldsymbol{\theta}^*)$is positive definite in $\mathbf{s}$ and $\boldsymbol{\xi}$.
Proof: Opening the above expression:

$$\dot{V}_a = \boldsymbol{s}_v^T \delta\boldsymbol{\omega} + \boldsymbol{\xi}^T \mathbf{J}[\mathbf{J}^{-1}[\mathbf{W}(\boldsymbol{\theta}^* + \Gamma(\frac{\partial V_a}{\partial \boldsymbol{\theta}^*})^T) + \boldsymbol{u}]] \tag{53}$$

$$= \boldsymbol{s}_v^T \delta\boldsymbol{\omega} + \boldsymbol{\xi}^T [\mathbf{W}(\boldsymbol{\theta}^* + \Gamma(\frac{\partial V_a}{\partial \boldsymbol{\theta}^*})^T) + \boldsymbol{u}] \tag{54}$$

$$= \boldsymbol{s}_v^T [\boldsymbol{\xi} - k_p \boldsymbol{s}_v] + \boldsymbol{\xi}^T [\mathbf{W}(\boldsymbol{\theta}^* + \Gamma(\frac{\partial V_a}{\partial \boldsymbol{\theta}^*})^T) + \boldsymbol{u}] \tag{55}$$

$$= -k_p \boldsymbol{s}_v^T \boldsymbol{s}_v + \boldsymbol{\xi}^T [\boldsymbol{s}_v + \mathbf{W}(\boldsymbol{\theta}^* + \Gamma(\frac{\partial V_a}{\partial \boldsymbol{\theta}^*})^T) + \boldsymbol{u}] \tag{56}$$

Now choose $\boldsymbol{u} := \boldsymbol{\alpha}(\mathbf{s}, \boldsymbol{\xi}, \boldsymbol{\theta}^*) = -k_{\boldsymbol{\xi}}\boldsymbol{\xi} - \mathbf{W}\boldsymbol{\theta}^* - \Gamma(\frac{\partial V_a}{\partial \boldsymbol{\theta}^*})^T$
This yields:

$$\begin{aligned} \dot{V}_a &= -k_p \boldsymbol{s}_v^T \boldsymbol{s}_v - k_{\boldsymbol{\xi}} \boldsymbol{\xi}^T \boldsymbol{\xi} \\ &= -k_p ||\boldsymbol{s}_v||^2 - k_{\xi} ||\boldsymbol{\xi}||^2 < 0 \end{aligned}$$

$\mathbb{Q}.\mathbb{E}.\mathbb{D}$
Now using directly the formulation of augmented systems that is Global adaptive asymptotic stabilizability of an integrator.
Our integrator dynamics is as follows:

$$\begin{bmatrix} \dot{s} \\ \mathbf{J}\dot{\boldsymbol{\xi}} \\ \dot{\boldsymbol{\nu}} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\mathbf{E}(\mathbf{s})\delta\boldsymbol{\omega} \\ \boldsymbol{\nu} \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{W} \\ \mathbf{0} \end{bmatrix} \boldsymbol{\theta}^* + \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ 1 \end{bmatrix} \boldsymbol{u}$$

Now we will have V_1(defined below) as ACLF for above integrator dynamics:

$$V_1(\mathbf{s}, \boldsymbol{\xi}, \boldsymbol{\nu}, \boldsymbol{\theta}^*) = V_a(\mathbf{s}, \boldsymbol{\xi}, \boldsymbol{\theta}^*) + \frac{1}{2}(\boldsymbol{\nu} - \boldsymbol{\alpha}(\mathbf{s}, \boldsymbol{\xi}, \boldsymbol{\theta}^*))^2$$

Define $\mathbf{z} = \boldsymbol{\nu} - \boldsymbol{\alpha}(\mathbf{s}, \boldsymbol{\xi}, \boldsymbol{\theta}^*)$ Which means CLF for modified system:

$$\dot{V}_1 := \begin{bmatrix} \frac{\partial V_1}{\partial s} & \frac{\partial V_1}{\partial \boldsymbol{\xi}} & \frac{\partial V_1}{\partial \boldsymbol{\nu}} \end{bmatrix} \begin{bmatrix} \frac{1}{2}\mathbf{E}(\mathbf{s})\delta\boldsymbol{\omega} \\ \mathbf{J}^{-1}[\mathbf{W}(\boldsymbol{\theta}^* + \Gamma(\frac{\partial V_a}{\partial \boldsymbol{\theta}^*})^T) + \boldsymbol{\nu}] \\ \mathbf{u} \end{bmatrix}$$

Now we are having;

$$\frac{\partial V_1}{\partial s} = \frac{\partial V_a}{\partial s} - \mathbf{z}\frac{\partial \alpha}{\partial s}$$

$$\frac{\partial V_1}{\partial \boldsymbol{\xi}} = \frac{\partial V_a}{\partial \boldsymbol{\xi}} - \mathbf{z}\frac{\partial \alpha}{\partial \boldsymbol{\xi}}$$

$$\frac{\partial V_1}{\partial \boldsymbol{\nu}} = \mathbf{z}$$

$$\frac{\partial \alpha}{\partial s} = 0$$

$$\frac{\partial \alpha}{\partial \boldsymbol{\xi}} = -k_{\boldsymbol{\xi}} - \Gamma \frac{\partial((\frac{\partial V_a}{\partial \boldsymbol{\theta}^*})^T)}{\partial \boldsymbol{\xi}}$$

So putting all that in there and opening the $\dot{V}_1$;

$$\dot{V}_1 = (\frac{\partial V_a}{\partial s} - \mathbf{z}\frac{\partial \alpha}{\partial s})[\frac{1}{2}\mathbf{E}(\mathbf{s})\delta\boldsymbol{\omega}] + (\frac{\partial V_a}{\partial \boldsymbol{\xi}} - \mathbf{z}\frac{\partial \alpha}{\partial \boldsymbol{\xi}})[\mathbf{J}^{-1}[\mathbf{W}(\boldsymbol{\theta}^* + \Gamma(\frac{\partial V_a}{\partial \boldsymbol{\theta}^*})^T) + \boldsymbol{\nu}]] + \mathbf{zu}$$

$$\implies \dot{V}_1 = \{-k_p \boldsymbol{s}_v^T \boldsymbol{s}_v + \boldsymbol{\xi}^T[\boldsymbol{s}_v + \mathbf{W}(\boldsymbol{\theta}^* + \Gamma(\frac{\partial V_a}{\partial \boldsymbol{\theta}^*})^T) + \boldsymbol{\nu}]\} - \mathbf{z}\{\frac{\partial \alpha}{\partial s}(\frac{1}{2}\mathbf{E}(\mathbf{s})\delta\boldsymbol{\omega}) + \frac{\partial \alpha}{\partial \boldsymbol{\xi}}(\mathbf{J}^{-1}[\mathbf{W}(\boldsymbol{\theta}^* + \Gamma(\frac{\partial V_a}{\partial \boldsymbol{\theta}^*})^T) + \boldsymbol{\nu}]) - \mathbf{u}\}$$

$$\implies \dot{V}_1 = \{-k_p \boldsymbol{s}_v^T \boldsymbol{s}_v + \boldsymbol{\xi}^T[\boldsymbol{s}_v + \mathbf{W}(\boldsymbol{\theta}^* + \Gamma(\frac{\partial V_a}{\partial \boldsymbol{\theta}^*})^T) + \boldsymbol{\nu}]\} - \mathbf{z}\{\frac{\partial \alpha}{\partial \boldsymbol{\xi}}(\mathbf{J}^{-1}[\mathbf{W}(\boldsymbol{\theta}^* + \Gamma(\frac{\partial V_a}{\partial \boldsymbol{\theta}^*})^T) + \boldsymbol{\nu}]) - \mathbf{u}\}$$

So we are having following as our controller as $\boldsymbol{u}(\mathbf{s}, \boldsymbol{\xi}, \boldsymbol{\nu}, \hat{\boldsymbol{\theta}}^*)$ and tuning function $\boldsymbol{\tau}$ by just putting the tuning function and control input so that we can make it negative definite. We can also use the direct formula for standard augmented systems:

$$\begin{aligned} \boldsymbol{u} &= -\frac{\partial V_a}{\partial x} g - z + \frac{\partial V_a}{\partial x}(f + F\theta + g\xi) + \frac{\partial \alpha}{\partial \theta}\Gamma((\frac{\partial V_1}{\partial x})F)^T + \frac{\partial V_a}{\partial \theta}\Gamma((\frac{\partial \alpha}{\partial x})F)^T \\ \tau &= ((\frac{\partial V_1}{\partial x})F)^T \end{aligned}$$

where f,g,F and alpha are standard system dynamic variables.