Spherical Barotropic Primitive Equations Model

$$
\begin{cases}
\dfrac{\partial U}{\partial t} + \dfrac{1}{2a\cos\theta}\left(u\dfrac{\partial U}{\partial \lambda} + \dfrac{\partial uU}{\partial \lambda} + v\cos\theta\dfrac{\partial U}{\partial \theta} + \dfrac{\partial vU\cos\theta}{\partial \theta}\right) + \dfrac{h}{a\cos\theta}\dfrac{\partial \phi^*}{\partial \lambda} - f^*V = 0 \\
\dfrac{\partial V}{\partial t} + \dfrac{1}{2a\cos\theta}\left(u\dfrac{\partial V}{\partial \lambda} + \dfrac{\partial uV}{\partial \lambda} + v\cos\theta\dfrac{\partial V}{\partial \theta} + \dfrac{\partial vV\cos\theta}{\partial \theta}\right) + \dfrac{h}{a}\dfrac{\partial \phi^*}{\partial \theta} + f^*U = 0 \\
\dfrac{\partial \phi}{\partial t} + \dfrac{1}{a\cos\theta}\left(\dfrac{\partial hU}{\partial \lambda} + \dfrac{\partial hV\cos\theta}{\partial \theta}\right) = 0
\end{cases}
$$

Where

$$f^* = 2\Omega\sin\theta + \frac{u}{a}\tan\theta$$

$$\phi^* = \phi + gh_s$$

h_s is the surface height.

For discretization on C-grid, the equations have to be written as

$$
\begin{cases}
\dfrac{\partial U}{\partial t} + \dfrac{1}{2a\cos\theta}\left(u\dfrac{\partial U}{\partial \lambda} + \dfrac{\partial uU}{\partial \lambda} + 2\dfrac{\partial vU\cos\theta}{\partial \theta} - U\dfrac{\partial v\cos\theta}{\partial \theta}\right) + \dfrac{h}{a\cos\theta}\dfrac{\partial \phi^*}{\partial \lambda} - f^*V = 0 \\
\dfrac{\partial V}{\partial t} + \dfrac{1}{2a\cos\theta}\left(2\dfrac{\partial uV}{\partial \lambda} - V\dfrac{\partial u}{\partial \lambda} + v\cos\theta\dfrac{\partial V}{\partial \theta} + \dfrac{\partial vV\cos\theta}{\partial \theta}\right) + \dfrac{h}{a}\dfrac{\partial \phi^*}{\partial \theta} + f^*U = 0 \\
\dfrac{\partial \phi}{\partial t} + \dfrac{1}{a\cos\theta}\left(\dfrac{\partial hU}{\partial \lambda} + \dfrac{\partial hV\cos\theta}{\partial \theta}\right) = 0
\end{cases}
$$

Discretization On C-grid

$$
\begin{cases}
\dfrac{\partial U}{\partial t} + \dfrac{1}{2a\cos\theta}\left[u\overline{U}_x^x + \overline{(uU)}_x^x + 2\left(\overline{v\cos\theta}^x\,\overline{U}^y\right)_y - U\left(\overline{v\cos\theta}^x\right)_y\right] + \dfrac{\overline{h}^x}{a\cos\theta}{\phi^*}_x - 2\Omega\sin\theta\,\overline{V_u}^{xy} - \dfrac{u}{a}\tan\theta\,\overline{V_u}^{xy} = 0 \\
\dfrac{\partial V}{\partial t} + \dfrac{1}{2a\cos\theta}\left[2\left(\overline{u}^y\overline{V}^x\right)_x - V\overline{u}_x^y + v\cos\theta\,\overline{V}_y^y + \overline{(v\cos\theta\,V)}_y^y\right] + \dfrac{\overline{h}^y}{a}{\phi^*}_y + 2\Omega\overline{\sin\theta\,\overline{U}^x}^y + \dfrac{1}{a}\overline{u\overline{U}^x\tan\theta}^y = 0 \\
\dfrac{\partial \phi}{\partial t} + \dfrac{1}{a\cos\theta}\left[\left(\overline{h}^x U\right)_x + \left(\overline{h}^y V\cos\theta\right)_y\right] = 0
\end{cases}
$$

$$h_u = \overline{h}^x = \frac{h_{i+1,j} + h_{i,j}}{2}$$

$$h_v = \overline{h}^y = \frac{h_{i,j+1} + h_{i,j}}{2}$$

$$U = h_u u = \frac{h_{i+1,j} + h_{i,j}}{2}u$$

$$V = h_v v = \frac{h_{i,j+1} + h_{i,j}}{2}v$$

$$\overline{V_u}^{xy} = \frac{1}{4}\left[C_1\left(V_{i+1,j} + V_{i,j}\right) + C_2\left(V_{i+1,j-1} + V_{i,j-1}\right)\right]$$

where

$$C_1 = \frac{\cos\theta_{vj}}{\cos\theta_{z_j}}$$

$$C_2 = \frac{\cos\theta_{vj-1}}{\cos\theta_{z_j}}$$

U equation

$$u \rightarrow v:\ i+\frac{1}{2}, j-\frac{1}{2}$$

$$u \rightarrow h:\ i+\frac{1}{2}$$

$$v \rightarrow h:\ j+\frac{1}{2}$$

$$u\overline{U_x}^x = u_{i,j}\frac{U_{i+1,j}-U_{i-1,j}}{2dx}$$

$$\overline{(uU)_x}^x = \frac{(uU)_{i+1,j}-(uU)_{i-1.j}}{2dx}$$

$$2\left(\overline{v\cos\theta_v}^x\,\overline{U}^y\right)_y = \frac{\left[(v\cos\theta_v)_{i+1,j}+(v\cos\theta_v)_{i,j}\right]\left(U_{i,j+1}+U_{i,j}\right)-\left[(v\cos\theta_v)_{i+1,j-1}+(v\cos\theta_v)_{i,j-1}\right]\left(U_{i,j}+U_{i,j-1}\right)}{2dy}$$

$$U\left(\overline{v\cos\theta_v}^x\right)_y = \frac{U_{i,j}\left[(v\cos\theta_v)_{i+1,j}+(v\cos\theta_v)_{i,j}-(v\cos\theta_v)_{i+1,j-1}-(v\cos\theta_v)_{i,j-1}\right]}{2dy}$$

$$2\left(\overline{v\cos\theta_v}^x\,\overline{U}^y\right)_y - U\left(\overline{v\cos\theta_v}^x\right)_y = \frac{\left[(v\cos\theta_v)_{i+1,j}+(v\cos\theta_v)_{i,j}\right]U_{i,j+1}-\left[(v\cos\theta_v)_{i+1,j-1}+(v\cos\theta_v)_{i,j-1}\right]U_{i,j-1}}{2dy}$$

$$\phi^*{}_x = \frac{\phi^*{}_{i+1,j}-\phi^*{}_{i,j}}{dx}$$

$$2\Omega\sin\theta\,\overline{V_u}^{xy} = \frac{1}{2}\sin\theta\left[C_1\left(V_{i+1,j}+V_{i,j}\right)+C_2\left(V_{i+1,j-1}+V_{i,j-1}\right)\right]$$

$$\frac{u}{a}\tan\theta\,\overline{V_u}^{xy} = \frac{u}{a}\tan\theta\,\frac{C_1\left(V_{i+1,j}+V_{i,j}\right)+C_2\left(V_{i+1,j-1}+V_{i,j-1}\right)}{4}$$

V equation

$$v \rightarrow h: j+\frac{1}{2}$$

$$v \rightarrow u: i-\frac{1}{2}, j+\frac{1}{2}$$

$$2\left(\overline{u}^y\overline{V}^x\right)_x = \frac{\left(u_{i,j+1}+u_{i,j}\right)\left(V_{i+1,j}+V_{i,j}\right)-\left(u_{i-1,j+1}+u_{i-1,j}\right)\left(V_{i,j}+V_{i-1,j}\right)}{2dx}$$

$$V\overline{u_x}^y = \frac{V_{i,j}\left(u_{i,j+1}+u_{i,j}-u_{i-1,j+1}-u_{i-1,j}\right)}{2dx}$$

$$2\left(\overline{u}^y\overline{V}^x\right)_x - V\overline{u_x}^y = \frac{\left(u_{i,j+1}+u_{i,j}\right)V_{i+1,j}-\left(u_{i-1,j+1}+u_{i-1,j}\right)V_{i-1,j}}{2dx}$$

$$v\cos\theta\,\overline{V_y}^y = \frac{v\cos\theta\left(V_{i,j+1}-V_{i,j-1}\right)}{2dy}$$

$$\overline{(v\cos\theta\,V)_y}^y = \frac{(vV\cos\theta)_{i,j+1}-(vV\cos\theta)_{i,j-1}}{2dy}$$

$$\phi^*{}_y = \frac{\phi^*{}_{i,j+1}-\phi^*{}_{i,j}}{dy}$$

$$\overline{\sin\theta_u\,\overline{U}^x}^y = \frac{\left(U_{i,j+1}+U_{i-1,j+1}\right)\sin\theta_{u_{j+1}}+\left(U_{i,j}+U_{i-1,j}\right)\sin\theta_{u_j}}{4}$$

$$\overline{\overline{uU}^x\tan\theta_u}^y = \frac{\left[(uU)_{i,j+1}+(uU)_{i-1,j+1}\right]\tan\theta_{u_{j+1}}+\left[(uU)_{i,j}+(uU)_{i-1,j}\right]\tan\theta_{u_j}}{4}$$

$\boldsymbol{\phi}$ equation

$$h \to u : i - \frac{1}{2}$$

$$h \to v : j - \frac{1}{2}$$

$$\left(\overline{h}^{x} U\right)_x = \frac{\left(h_{i+1,j} + h_{i,j}\right)U_{i,j} - \left(h_{i,j} + h_{i-1,j}\right)U_{i-1,j}}{2dx}$$

$$\left(\overline{h}^{y} V\cos\theta\right)_y = \frac{\left(h_{i,j+1} + h_{i,j}\right)(V\cos\theta)_{i,j} - \left(h_{i,j} + h_{i,j-1}\right)(V\cos\theta)_{i,j-1}}{2dy}$$

Final equations

$$LU = \frac{1}{4a \cdot dx \cdot \cos\theta}\left[u\left(U_{i+1,j} - U_{i-1,j}\right) + (uU)_{i+1,j} + (uU)_{i-1.j}\right] + \frac{1}{4a \cdot dy \cdot \cos\theta}\left\{\left[(v\cos\theta_v)_{i+1,j} + (v\cos\theta_v)_{i,j}\right]U_{i,j+1} - \left[(v\cos\theta_v)_{i+1,j-1} + (v\cos\theta_v)_{i,j-1}\right]U_{i,j-1}\right\} + \frac{\left(h_{i+1,j} + h_{i,j}\right)\left({\phi^*}_{i+1,j} - {\phi^*}_{i,j}\right)}{2a \cdot dx \cdot \cos\theta} - 2\Omega\sin\theta\,\overline{V_u}^{xy} - \frac{u}{a}\tan\theta\,\overline{V_u}^{xy}$$

$$LV = \frac{1}{4a \cdot dx \cdot \cos\theta}\left[\left(u_{i,j+1} + u_{i,j}\right)V_{i+1,j} - \left(u_{i-1,j+1} + u_{i-1,j}\right)V_{i-1,j}\right] + \frac{1}{4a \cdot dy \cdot \cos\theta}\left[v\cos\theta\left(V_{i,j+1} - V_{i,j-1}\right) + (vV\cos\theta)_{i,j+1} - (vV\cos\theta)_{i,j-1}\right] + \frac{\left(h_{i,j+1} + h_{i,j}\right)\left({\phi^*}_{i,j+1} - {\phi^*}_{i,j}\right)}{2a \cdot dy} + 2\Omega\,\overline{\sin\theta\,\overline{U}^{x}}^{y} + \frac{1}{a}\overline{\overline{uU}^{x}\tan\theta}^{y}$$

$$LZ = \frac{1}{2a \cdot dx \cdot \cos\theta}\left[\left(h_{i+1,j} + h_{i,j}\right)U_{i,j} - \left(h_{i,j} + h_{i-1,j}\right)U_{i-1,j}\right] + \frac{1}{2a \cdot dy \cdot \cos\theta}\left[\left(h_{i,j+1} + h_{i,j}\right)(V\cos\theta)_{i,j} - \left(h_{i,j} + h_{i,j-1}\right)(V\cos\theta)_{i,j-1}\right]$$