

Spherical Barotropic Primitive Equations Model

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$$\begin{cases} \frac{\partial U}{\partial t} + \frac{1}{2a \cos \theta} \left(u \frac{\partial U}{\partial \lambda} + \frac{\partial u U}{\partial \lambda} + v \cos \theta \frac{\partial U}{\partial \theta} + \frac{\partial v U \cos \theta}{\partial \theta} \right) + \frac{h}{a \cos \theta} \frac{\partial \phi}{\partial \lambda} - f^* V = 0 \\ \frac{\partial V}{\partial t} + \frac{1}{2a \cos \theta} \left(u \frac{\partial V}{\partial \lambda} + \frac{\partial u V}{\partial \lambda} + v \cos \theta \frac{\partial V}{\partial \theta} + \frac{\partial v V \cos \theta}{\partial \theta} \right) + \frac{h}{a} \frac{\partial \phi}{\partial \theta} + f^* U = 0 \\ \frac{\partial \phi}{\partial t} + \frac{1}{a \cos \theta} \left(\frac{\partial h U}{\partial \lambda} + \frac{\partial h V \cos \theta}{\partial \theta} \right) = 0 \end{cases}$$

Where $f^* = 2\Omega \sin \theta + \frac{u}{a} \tan \theta$

For discretization on C-grid, the equations have to be written as

$$\begin{cases} \frac{\partial U}{\partial t} + \frac{1}{2a \cos \theta} \left(u \frac{\partial U}{\partial \lambda} + \frac{\partial u U}{\partial \lambda} + 2 \frac{\partial v U \cos \theta}{\partial \theta} - U \frac{\partial v \cos \theta}{\partial \theta} \right) + \frac{h}{a \cos \theta} \frac{\partial \phi}{\partial \lambda} - f^* V = 0 \\ \frac{\partial V}{\partial t} + \frac{1}{2a \cos \theta} \left(2 \frac{\partial u V}{\partial \lambda} - V \frac{\partial u}{\partial \lambda} + v \cos \theta \frac{\partial V}{\partial \theta} + \frac{\partial v V \cos \theta}{\partial \theta} \right) + \frac{h}{a} \frac{\partial \phi}{\partial \theta} + f^* U = 0 \\ \frac{\partial \phi}{\partial t} + \frac{1}{a \cos \theta} \left(\frac{\partial h U}{\partial \lambda} + \frac{\partial h V \cos \theta}{\partial \theta} \right) = 0 \end{cases}$$

Discretization On C-grid

$$\begin{cases} \frac{\partial U}{\partial t} + \frac{1}{2a \cos \theta} \left[u \bar{U}_x^x + (\bar{u} U)_x^x + 2 \left(\bar{v} \cos \theta^x \bar{U}^y \right)_y - U \left(\bar{v} \cos \theta^x \right)_y \right] + \frac{\bar{h}^x}{a \cos \theta} \phi_x - 2\Omega \sin \theta \bar{V}_u^{xy} - \frac{u}{a} \tan \theta \bar{V}_u^{xy} = 0 \\ \frac{\partial V}{\partial t} + \frac{1}{2a \cos \theta} \left[2 \left(\bar{u}^y \bar{V}^x \right)_x - V \bar{u}_x^y + v \cos \theta \bar{V}_y^y + (\bar{v} \cos \theta V)_y^y \right] + \frac{\bar{h}^y}{a} \phi_y + 2\Omega \sin \theta \bar{U}^{xy} + \frac{1}{a} \bar{u} U \tan \theta^{xy} = 0 \\ \frac{\partial \phi}{\partial t} + \frac{1}{a \cos \theta} \left[(\bar{h}^x U)_x + (\bar{h}^y V \cos \theta)_y \right] = 0 \end{cases}$$

$$h_u = \bar{h}^x = \frac{h_{i+1,j} + h_{i,j}}{2}$$

$$h_v = \bar{h}^y = \frac{h_{i,j+1} + h_{i,j}}{2}$$

$$U = h_u u = \frac{h_{i+1,j} + h_{i,j}}{2} u$$

$$V = h_v v = \frac{h_{i,j+1} + h_{i,j}}{2} v$$

$$\bar{V}_u^{xy} = \frac{1}{4} [C_1 (V_{i+1,j} + V_{i,j}) + C_2 (V_{i+1,j-1} + V_{i,j-1})]$$

where

$$C_1 = \frac{\cos \theta_{vj}}{\cos \theta_{zj}}$$

$$C_2 = \frac{\cos \theta_{vj-1}}{\cos \theta_{zj}}$$

U equation

$$v \rightarrow u : i + \frac{1}{2}, j - \frac{1}{2}$$

$$h \rightarrow u : i + \frac{1}{2}$$

$$h \rightarrow v : j + \frac{1}{2}$$

$$u\overline{U}_x^x = u_{i,j} \frac{U_{i+1,j} - U_{i-1,j}}{2dx}$$

$$\overline{(uU)}_x^x = \frac{(uU)_{i+1,j} - (uU)_{i-1,j}}{2dx}$$

$$2 \left(\overline{v \cos \theta_v}^x \overline{U}^y \right)_y = \frac{[(v \cos \theta_v)_{i+1,j} + (v \cos \theta_v)_{i,j}](U_{i,j+1} + U_{i,j}) - [(v \cos \theta_v)_{i+1,j-1} + (v \cos \theta_v)_{i,j-1}](U_{i,j} + U_{i,j-1})}{2dy}$$

$$U \left(\overline{v \cos \theta_v}^x \right)_y = \frac{U_{i,j} [(v \cos \theta_v)_{i+1,j} + (v \cos \theta_v)_{i,j} - (v \cos \theta_v)_{i+1,j-1} - (v \cos \theta_v)_{i,j-1}]}{2dy}$$

$$2 \left(\overline{v \cos \theta_v}^x \overline{U}^y \right)_y - U \left(\overline{v \cos \theta_v}^x \right)_y = \frac{[(v \cos \theta_v)_{i+1,j} + (v \cos \theta_v)_{i,j}]U_{i,j+1} - [(v \cos \theta_v)_{i+1,j-1} + (v \cos \theta_v)_{i,j-1}]U_{i,j-1}}{2dy}$$

$$\phi_x = \frac{\phi_{i+1,j} - \phi_{i,j}}{dx}$$

$$2\Omega \sin \theta \overline{V}_u^{xy} = \frac{1}{2} \sin \theta [C_1(V_{i+1,j} + V_{i,j}) + C_2(V_{i+1,j-1} + V_{i,j-1})]$$

$$\frac{u}{a} \tan \theta \overline{V}_u^{xy} = \frac{u}{a} \tan \theta \frac{C_1(V_{i+1,j} + V_{i,j}) + C_2(V_{i+1,j-1} + V_{i,j-1})}{4}$$

V equation

$$h \rightarrow v : j + \frac{1}{2}$$

$$u \rightarrow v : i - \frac{1}{2}, j + \frac{1}{2}$$

$$2 \left(\overline{u}^y \overline{V}^x \right)_x = \frac{(u_{i,j+1} + u_{i,j})(V_{i+1,j} + V_{i,j}) - (u_{i-1,j+1} + u_{i-1,j})(V_{i,j} + V_{i-1,j})}{2dx}$$

$$V\overline{u}_x^y = \frac{V_{i,j}(u_{i,j+1} + u_{i,j} - u_{i-1,j+1} - u_{i-1,j})}{2dx}$$

$$2 \left(\overline{u}^y \overline{V}^x \right)_x - V\overline{u}_x^y = \frac{(u_{i,j+1} + u_{i,j})V_{i+1,j} - (u_{i-1,j+1} + u_{i-1,j})V_{i-1,j}}{2dx}$$

$$v \cos \theta \overline{V}_y^y = \frac{v \cos \theta (V_{i,j+1} - V_{i,j-1})}{2dy}$$

$$\overline{(v \cos \theta V)}_y^y = \frac{(vV \cos \theta)_{i,j+1} - (vV \cos \theta)_{i,j-1}}{2dy}$$

$$\phi_y = \frac{\phi_{i,j+1} - \phi_{i,j}}{dy}$$

$$\overline{\sin \theta_u}^{xy} = \frac{(U_{i,j+1} + U_{i-1,j+1}) \sin \theta_{u,j+1} + (U_{i,j} + U_{i-1,j}) \sin \theta_{u,j}}{4}$$

$$\overline{uU \tan \theta_u}^{xy} = \frac{[(uU)_{i,j+1} + (uU)_{i-1,j+1}] \tan \theta_{u,j+1} + [(uU)_{i,j} + (uU)_{i-1,j}] \tan \theta_{u,j}}{4}$$

ϕ equation

$$u \rightarrow h : i - \frac{1}{2}$$

$$v \rightarrow h : j - \frac{1}{2}$$

$$\begin{aligned} (\bar{h}^x U)_x &= \frac{(h_{i+1,j} + h_{i,j})U_{i,j} - (h_{i,j} + h_{i-1,j})U_{i-1,j}}{2dx} \\ (\bar{h}^y V \cos \theta)_y &= \frac{(h_{i,j+1} + h_{i,j})(V \cos \theta)_{i,j} - (h_{i,j} + h_{i,j-1})(V \cos \theta)_{i,j-1}}{2dy} \end{aligned}$$

Final equations

$$\begin{aligned} LU &= \frac{1}{4a \cdot dx \cdot \cos \theta} [u(U_{i+1,j} - U_{i-1,j}) + (uU)_{i+1,j} - (uU)_{i-1,j}] \\ &\quad + \frac{1}{4a \cdot dy \cdot \cos \theta} \{[(v \cos \theta_v)_{i+1,j} + (v \cos \theta_v)_{i,j}]U_{i,j+1} - [(v \cos \theta_v)_{i+1,j-1} + (v \cos \theta_v)_{i,j-1}]U_{i,j-1}\} \\ &\quad + \frac{(h_{i+1,j} + h_{i,j})(\phi_{i+1,j} - \phi_{i,j})}{2a \cdot dx \cdot \cos \theta} - 2\Omega \sin \theta \bar{V}_u^{xy} - \frac{u}{a} \tan \theta \bar{V}_u^{xy} \end{aligned}$$

$$\begin{aligned} LV &= \frac{1}{4a \cdot dx \cdot \cos \theta} [(u_{i,j+1} + u_{i,j})V_{i+1,j} - (u_{i-1,j+1} + u_{i-1,j})V_{i-1,j}] \\ &\quad + \frac{1}{4a \cdot dy \cdot \cos \theta} [v \cos \theta (V_{i,j+1} - V_{i,j-1}) + (vV \cos \theta)_{i,j+1} - (vV \cos \theta)_{i,j-1}] \\ &\quad + \frac{(h_{i,j+1} + h_{i,j})(\phi_{i,j+1} - \phi_{i,j})}{2a \cdot dy} + \Omega \frac{(U_{i,j+1} + U_{i-1,j+1}) \sin \theta_{u_{j+1}} + (U_{i,j} + U_{i-1,j}) \sin \theta_{u_j}}{2} \\ &\quad + \frac{[(uU)_{i,j+1} + (uU)_{i-1,j+1}] \tan \theta_{u_{j+1}} + [(uU)_{i,j} + (uU)_{i-1,j}] \tan \theta_{u_j}}{4a} \end{aligned}$$

$$\begin{aligned} LZ &= \frac{1}{2a \cdot dx \cdot \cos \theta} [(h_{i+1,j} + h_{i,j})U_{i,j} - (h_{i,j} + h_{i-1,j})U_{i-1,j}] \\ &\quad + \frac{1}{2a \cdot dy \cdot \cos \theta} [(h_{i,j+1} + h_{i,j})(V \cos \theta)_{i,j} - (h_{i,j} + h_{i,j-1})(V \cos \theta)_{i,j-1}] \end{aligned}$$