Spherical Barotropic Primitive Equations Model

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$$\begin{cases} \dfrac{\partial U}{\partial t} + \dfrac{1}{2a\cos\theta}\left(u\dfrac{\partial U}{\partial \lambda} + \dfrac{\partial uU}{\partial \lambda} + v\cos\theta\dfrac{\partial U}{\partial \theta} + \dfrac{\partial vU\cos\theta}{\partial \theta}\right) + \dfrac{h}{a\cos\theta}\dfrac{\partial \phi}{\partial \lambda} - f^*V = 0 \\[2mm] \dfrac{\partial V}{\partial t} + \dfrac{1}{2a\cos\theta}\left(u\dfrac{\partial V}{\partial \lambda} + \dfrac{\partial uV}{\partial \lambda} + v\cos\theta\dfrac{\partial V}{\partial \theta} + \dfrac{\partial vV\cos\theta}{\partial \theta}\right) + \dfrac{h}{a}\dfrac{\partial \phi}{\partial \theta} + f^*U = 0 \\[2mm] \dfrac{\partial \phi}{\partial t} + \dfrac{1}{a\cos\theta}\left(\dfrac{\partial hU}{\partial \lambda} + \dfrac{\partial hV\cos\theta}{\partial \theta}\right) = 0 \end{cases}$$

Where $f^* = 2\Omega\sin\theta + \dfrac{u}{a}\tan\theta$

For discretization on C-grid, the equations have to be written as

$$\begin{cases} \dfrac{\partial U}{\partial t} + \dfrac{1}{2a\cos\theta}\left(u\dfrac{\partial U}{\partial \lambda} + \dfrac{\partial uU}{\partial \lambda} + 2\dfrac{\partial vU\cos\theta}{\partial \theta} - U\dfrac{\partial v\cos\theta}{\partial \theta}\right) + \dfrac{h}{a\cos\theta}\dfrac{\partial \phi}{\partial \lambda} - f^*V = 0 \\[2mm] \dfrac{\partial V}{\partial t} + \dfrac{1}{2a\cos\theta}\left(2\dfrac{\partial uV}{\partial \lambda} - V\dfrac{\partial u}{\partial \lambda} + v\cos\theta\dfrac{\partial V}{\partial \theta} + \dfrac{\partial vV\cos\theta}{\partial \theta}\right) + \dfrac{h}{a}\dfrac{\partial \phi}{\partial \theta} + f^*U = 0 \\[2mm] \dfrac{\partial \phi}{\partial t} + \dfrac{1}{a\cos\theta}\left(\dfrac{\partial hU}{\partial \lambda} + \dfrac{\partial hV\cos\theta}{\partial \theta}\right) = 0 \end{cases}$$

Discretization On C-grid

$$\begin{cases} \dfrac{\partial U}{\partial t} + \dfrac{1}{2a\cos\theta}\left[u\overline{U}_x^{\,x} + \overline{(uU)}_x^{\,x} + 2\left(\overline{v\cos\theta}^{\,x}\,\overline{U}^{\,y}\right)_y - U\left(\overline{v\cos\theta}^{\,x}\right)_y\right] + \dfrac{\overline{h}^{\,x}}{a\cos\theta}\phi_x - 2\Omega\sin\theta\,\overline{V_u}^{\,xy} - \dfrac{u}{a}\tan\theta\,\overline{V_u}^{\,xy} = 0 \\[2mm] \dfrac{\partial V}{\partial t} + \dfrac{1}{2a\cos\theta}\left[2\left(\overline{u}^{\,y}\overline{V}^{\,x}\right)_x - V\overline{u}_x^{\,y} + v\cos\theta\,\overline{V}_y^{\,y} + \overline{(v\cos\theta V)}_y^{\,y}\right] + \dfrac{\overline{h}^{\,y}}{a}\phi_y + 2\Omega\,\overline{\sin\theta\,U}^{\,xy} + \dfrac{1}{a}\overline{uU\tan\theta}^{\,xy} = 0 \\[2mm] \dfrac{\partial \phi}{\partial t} + \dfrac{1}{a\cos\theta}\left[\left(\overline{h}^{\,x}U\right)_x + \left(\overline{h}^{\,y}V\cos\theta\right)_y\right] = 0 \end{cases}$$

$$h_u = \overline{h}^{\,x} = \frac{h_{i+1,j} + h_{i,j}}{2}$$

$$h_v = \overline{h}^{\,y} = \frac{h_{i,j+1} + h_{i,j}}{2}$$

$$U = h_u u = \frac{h_{i+1,j} + h_{i,j}}{2}u$$

$$V = h_v v = \frac{h_{i,j+1} + h_{i,j}}{2}v$$

$$\overline{V_u}^{\,xy} = \frac{1}{4}\left[C_1\left(V_{i+1,j} + V_{i,j}\right) + C_2\left(V_{i+1,j-1} + V_{i,j-1}\right)\right]$$

where

$$C_1 = \frac{\cos\theta_{vj}}{\cos\theta_{zj}}$$

$$C_2 = \frac{\cos\theta_{vj-1}}{\cos\theta_{zj}}$$

U equation

$$v \to u : i + \frac{1}{2}, j - \frac{1}{2}$$

$$h \to u : i + \frac{1}{2}$$

$$h \to v : j + \frac{1}{2}$$

$$u \overline{U}_x^x = u_{i,j} \frac{U_{i+1,j} - U_{i-1,j}}{2dx}$$

$$\overline{(uU)}_x^x = \frac{(uU)_{i+1,j} - (uU)_{i-1,j}}{2dx}$$

$$2 \left(\overline{v \cos \theta_v}^x \, \overline{U}^y \right)_y = \frac{[(v \cos \theta_v)_{i+1,j} + (v \cos \theta_v)_{i,j}](U_{i,j+1} + U_{i,j}) - [(v \cos \theta_v)_{i+1,j-1} + (v \cos \theta_v)_{i,j-1}](U_{i,j} + U_{i,j-1})}{2dy}$$

$$U \left(\overline{v \cos \theta_v}^x \right)_y = \frac{U_{i,j}[(v \cos \theta_v)_{i+1,j} + (v \cos \theta_v)_{i,j} - (v \cos \theta_v)_{i+1,j-1} - (v \cos \theta_v)_{i,j-1}]}{2dy}$$

$$2 \left(\overline{v \cos \theta_v}^x \, \overline{U}^y \right)_y - U \left(\overline{v \cos \theta_v}^x \right)_y = \frac{[(v \cos \theta_v)_{i+1,j} + (v \cos \theta_v)_{i,j}]U_{i,j+1} - [(v \cos \theta_v)_{i+1,j-1} + (v \cos \theta_v)_{i,j-1}]U_{i,j-1}}{2dy}$$

$$\phi_x = \frac{\phi_{i+1,j} - \phi_{i,j}}{dx}$$

$$2\Omega \sin \theta \, \overline{V_u}^{xy} = \frac{1}{2} \sin \theta_u \left[C_1(V_{i+1,j} + V_{i,j}) + C_2(V_{i+1,j-1} + V_{i,j-1}) \right]$$

$$\frac{u}{a} \tan \theta \, \overline{V_u}^{xy} = \frac{u}{a} \tan \theta_u \frac{C_1(V_{i+1,j} + V_{i,j}) + C_2(V_{i+1,j-1} + V_{i,j-1})}{4}$$

V equation

$$h \to v : j + \frac{1}{2}$$

$$u \to v : i - \frac{1}{2}, j + \frac{1}{2}$$

$$2 \left(\overline{u}^y \overline{V}^x \right)_x = \frac{(u_{i,j+1} + u_{i,j})(V_{i+1,j} + V_{i,j}) - (u_{i-1,j+1} + u_{i-1,j})(V_{i,j} + V_{i-1,j})}{2dx}$$

$$V \overline{u}_x^y = \frac{V_{i,j}(u_{i,j+1} + u_{i,j} - u_{i-1,j+1} - u_{i-1,j})}{2dx}$$

$$2 \left(\overline{u}^y \overline{V}^x \right)_x - V \overline{u}_x^y = \frac{(u_{i,j+1} + u_{i,j})V_{i+1,j} - (u_{i-1,j+1} + u_{i-1,j})V_{i-1,j}}{2dx}$$

$$v \cos \theta \, \overline{V}_y^y = \frac{v \cos \theta_v (V_{i,j+1} - V_{i,j-1})}{2dy}$$

$$\overline{(v \cos \theta \, V)}_y^y = \frac{(vV \cos \theta_v)_{i,j+1} - (vV \cos \theta_v)_{i,j-1}}{2dy}$$

$$\phi_y = \frac{\phi_{i,j+1} - \phi_{i,j}}{dy}$$

$$\overline{\sin \theta_u \, U}^{xy} = \frac{(U_{i,j+1} + U_{i-1,j+1}) \sin \theta_{u\,j+1} + (U_{i,j} + U_{i-1,j}) \sin \theta_{u\,j}}{4}$$

$$\overline{uU \tan \theta_u}^{xy} = \frac{[(uU)_{i,j+1} + (uU)_{i-1,j+1}] \tan \theta_{u\,j+1} + [(uU)_{i,j} + (uU)_{i-1,j}] \tan \theta_{u\,j}}{4}$$

ϕ equation

$$u \to h : i - \frac{1}{2}$$

$$v \to h : j - \frac{1}{2}$$

$$\left(\bar{h}^x U\right)_x = \frac{(h_{i+1,j} + h_{i,j})U_{i,j} - (h_{i,j} + h_{i-1,j})U_{i-1,j}}{2dx}$$

$$\left(\bar{h}^y V \cos\theta\right)_y = \frac{(h_{i,j+1} + h_{i,j})(V\cos\theta_v)_{i,j} - (h_{i,j} + h_{i,j-1})(V\cos\theta_v)_{i,j-1}}{2dy}$$

Final equations

$$LU = \frac{1}{4a \cdot dx \cdot \cos\theta_u}\left[u\left(U_{i+1,j} - U_{i-1,j}\right) + (uU)_{i+1,j} - (uU)_{i-1,j}\right]$$

$$+ \frac{1}{4a \cdot dy \cdot \cos\theta_u}\left\{\left[(v\cos\theta_v)_{i+1,j} + (v\cos\theta_v)_{i,j}\right]U_{i,j+1} - \left[(v\cos\theta_v)_{i+1,j-1} + (v\cos\theta_v)_{i,j-1}\right]U_{i,j-1}\right\}$$

$$+ \frac{(h_{i+1,j} + h_{i,j})(\phi_{i+1,j} - \phi_{i,j})}{2a \cdot dx \cdot \cos\theta_u} - 2\Omega\sin\theta\, \overline{V_u}^{xy} - \frac{u}{a}\tan\theta\, \overline{V_u}^{xy}$$

$$LV = \frac{1}{4a \cdot dx \cdot \cos\theta_v}\left[\left(u_{i,j+1} + u_{i,j}\right)V_{i+1,j} - \left(u_{i-1,j+1} + u_{i-1,j}\right)V_{i-1,j}\right]$$

$$+ \frac{1}{4a \cdot dy \cdot \cos\theta_v}\left[v\cos\theta\left(V_{i,j+1} - V_{i,j-1}\right) + (vV\cos\theta_v)_{i,j+1} - (vV\cos\theta_v)_{i,j-1}\right]$$

$$+ \frac{(h_{i,j+1} + h_{i,j})(\phi_{i,j+1} - \phi_{i,j})}{2a \cdot dy} + \Omega\frac{\left(U_{i,j+1} + U_{i-1,j+1}\right)\sin\theta_{u\,j+1} + \left(U_{i,j} + U_{i-1,j}\right)\sin\theta_{u\,j}}{2}$$

$$+ \frac{\left[(uU)_{i,j+1} + (uU)_{i-1,j+1}\right]\tan\theta_{u\,j+1} + \left[(uU)_{i,j} + (uU)_{i-1,j}\right]\tan\theta_{u\,j}}{4a}$$

$$LZ = \frac{1}{2a \cdot dx \cdot \cos\theta}\left[\left(h_{i+1,j} + h_{i,j}\right)U_{i,j} - \left(h_{i,j} + h_{i-1,j}\right)U_{i-1,j}\right]$$

$$+ \frac{1}{2a \cdot dy \cdot \cos\theta}\left[\left(h_{i,j+1} + h_{i,j}\right)(V\cos\theta_v)_{i,j} - \left(h_{i,j} + h_{i,j-1}\right)(V\cos\theta_v)_{i,j-1}\right]$$