Homework 3 Solutions

1. **Hard-Coding a Network.** The idea is that each of the hidden units in the first layer will respond to a violation of one of the inequalities. The output unit will check that there are no violations, by checking that the hidden units are all off.

$\mathbf{W}^{(1)} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$, $\mathbf{b}^{(1)} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{W}^{(2)} = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}$ and $\mathbf{b}^{(2)} = \frac{1}{2}$.

2. **Backprop.**

- Computation graph:

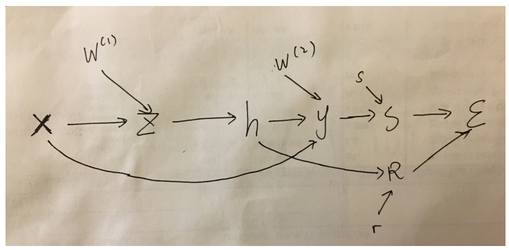

Figure 1: The computation graph for Problem 2. Showing parameters (e.g. **r**, **s**, weigths and biases) is optional.

- Backprop equations:

$$\begin{aligned}
\overline{\mathcal{E}} &= 1 \\
\overline{\mathcal{S}} &= \overline{\mathcal{E}} \\
\overline{\mathcal{R}} &= \overline{\mathcal{E}} \\
\overline{\mathbf{y}} &= \overline{\mathcal{S}} \frac{\partial \mathcal{S}}{\partial \mathbf{y}} \\
&= \overline{\mathcal{S}}(\mathbf{y} - \mathbf{s}) \\
\overline{\mathbf{h}} &= \overline{\mathbf{y}} \frac{\partial \mathbf{y}}{\partial \mathbf{h}} + \overline{\mathcal{R}} \frac{\partial \mathcal{R}}{\partial \mathbf{h}} \\
&= [\mathbf{W}^{(2)}]^\top \overline{\mathbf{y}} + \mathbf{r} \\
\overline{\mathbf{z}} &= \overline{\mathbf{h}} \frac{\partial \mathbf{h}}{\partial \mathbf{z}} \\
&= \overline{\mathbf{h}} \circ \sigma'(\mathbf{z}) \\
\overline{\mathbf{x}} &= \overline{\mathbf{z}} \frac{\partial \mathbf{z}}{\partial \mathbf{x}} + \overline{\mathbf{y}} \frac{\partial \mathbf{y}}{\partial \mathbf{x}} \\
&= [\mathbf{W}^{(1)}]^\top \overline{\mathbf{z}} + \overline{\mathbf{y}}
\end{aligned}$$

3. **Sparsifying Activation Function.** There are two ways to approach this problem. First, you could write out the backprop equations. Second, you could use the fact that $\partial\mathcal{E}/\partial w$ represents the effect on $\mathcal{E}$ of an infinitesimal change to w, and argue whether this effect is zero. Here, we'll denote a generic activation function with ϕ, ReLU with r, and the input to an activation function with z.

   - $\frac{\partial\mathcal{E}}{\partial w_1}$: YES.
     - Justification 1: $\frac{\partial\mathcal{E}}{\partial w_1} = \overline{y}\,\frac{\partial y}{\partial w_1} = \overline{y}\,\phi'(z)\,h_1 = 0$ (given h_1=0).
     - Justification 2: Since $y = \phi(w_1 h_3)$ and $h_3 = 0$, changing w_1 has no effect on the predictions.
   - $\frac{\partial\mathcal{E}}{\partial w_2}$: YES.
     - Justification 1: $\frac{\partial\mathcal{E}}{\partial w_2} = \overline{h_1}\,\frac{\partial h_1}{\partial w_2} = \overline{h_1}\,r'(z_1)\,h_3 = 0$, which is zero because $r'(-1) = 0$.
     - Justification 2: Changing w_2 by an infinitesimal amount has no effect, because it only affects the input to h_1, which is in the flat region of the ReLU.
   - $\frac{\partial\mathcal{E}}{\partial w_3}$: NO. Changing w_3 by a small amount can change h_3, which changes h_2, which changes y. Both h_3 and h_2 may be positive. (This argument can also be spelled out explicitly by writing out the backprop rules for each of these steps.)

Marking Rubrics

1. **Hard-Coding a Network.**
   - works only for $\mathbb{Z}$ but not for $\mathbb{R}$: -1 mark
   - doesn't work when some of the inputs are equal: -0.5 mark
   - doesn't work for some other issue: up to 0.5 mark from 2
2. **Backprop.**
   (a)
   - a missing edge: -0.25 mark
   - showing parameters (e.g. **r**, **s**, weigths and biases) is optional, no down-mark

   (b)
   - mistakes in the order of tensors in dot products or mismatched dimensions: -0.75 mark
   - missing some parts of gradients: -1.5 mark
3. **Sparsifying Activation Function.**
   - one wrong answer: -1 mark
   - mistake in the reasoning: -1 mark
   - all answers correct but no justification: -2 marks