CSC321: Assignment #4

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February 12, 2018

Problem 1

1. $$\frac{\partial C}{\partial \theta_i} = a_i(\theta_i - r_i)$$

   then

   $$\begin{aligned}\theta_i^{(t+1)} &= \theta_i^{(t)} - \alpha\frac{\partial C}{\partial \theta_i} \\ &= \theta_i^{(t)} - \alpha a_i(\theta_i^{(t)} - r_i)\end{aligned}$$

2. $$\begin{aligned}e_i^{(t+1)} &= \theta_i^{(t+1)} - r_i \\ &= \theta_i^{(t)} - \alpha a_i(\theta_i^{(t)} - r_i) - r_i \\ &= e_i^{(t)} - \alpha a_i(\theta_i^{(t)} - r_i) \\ &= e_i^{(t)} - \alpha a_i e_i^{(t)}\end{aligned}$$

3. solving the equation,

   $$e_i^{(t)} = e_i^{(0)}(1 - \alpha a_i)^t$$

   For $0 < \alpha < \frac{2}{a_i}$, $e_i^{(t)}$ will converge, so $e_i^{(t)}$ will be stable.
   For $\alpha < 0$ or $\alpha > \frac{2}{a_i}$, $e_i^{(t)}$ will diverge and become unstable.

4. $$\begin{aligned}\mathcal{C}(\theta^{(t)}) &= \sum_{i=0}^{N} \frac{a_i}{2}(e_i^{(t)})^2 \\ &= \sum_{i=0}^{N} \frac{a_i}{2}(e_i^{(0)}(1 - \alpha a_i)^t)^2\end{aligned}$$

   As $t \to \infty$, a_i will dominate

Problem 2

1.

$$\begin{aligned}
\mathbb{E}[y] &= \mathbb{E}[\sum_j m_j w_j x_j] \\
&= \sum_j \mathbb{E}[m_j w_j x_j] \\
&= \sum_j w_j x_j \mathbb{E}[m_j] \\
&= \sum_j \frac{1}{2} w_j x_j \\
&= \frac{1}{2} \mathbf{w}^\top \mathbf{x}
\end{aligned}$$

$$\begin{aligned}
Var[y] &= Var[\sum_j m_j w_j x_j] \\
&= \sum_j Var[m_j w_j x_j] + \sum_{i \neq j} Cov[w_i, w_j] \\
&= \sum_j (w_j x_j)^2 Var[m_j] \\
&= \sum_j \frac{1}{4} (w_j x_j)^2 \\
&= \sum_j (\frac{1}{2} w_j x_j)^2 \\
&= \frac{1}{4} (\mathbf{w}^\top \mathbf{x})^2
\end{aligned}$$

2.

$$\begin{aligned}
\mathbb{E}[y] &= \sum_j \frac{1}{2} w_j x_j \\
&= \sum_j \tilde{w_j} x_j
\end{aligned}$$

then

$$\tilde{w_j} = \frac{1}{2} w_j$$

3.

$$\begin{aligned}
\mathcal{E} &= \frac{1}{2N}\sum_{i=1}^{N}\mathbb{E}[(y^{(i)} - t_{(i)})^2] \\
&= \frac{1}{2N}\sum_{i=1}^{N}\mathbb{E}[{y^{(i)}}^2 - 2y^{(i)}t^{(i)} + {t^{(i)}}^2] \\
&= \frac{1}{2N}\sum_{i=1}^{N}[\mathbb{E}[{y^{(i)}}^2] - \mathbb{E}[2y^{(i)}t^{(i)}] + \mathbb{E}[{t^{(i)}}^2]] \\
&= \frac{1}{2N}\sum_{i=1}^{N}[\mathbb{E}[y^{(i)}]^2 + Var[y^{(i)}] - 2\mathbb{E}[y^{(i)}]\mathbb{E}[t^{(i)}] + \mathbb{E}[t^{(i)}]^2 + Var[t^{(i)}]] \\
&= \frac{1}{2N}\sum_{i=1}^{N}[(\sum_j \frac{1}{2}w_j x_j^{(i)})^2 + (\sum_j(\frac{1}{2}w_j x_j^{(i)})^2) - t^{(i)}(\sum_j w_j x_j^{(i)}) + {t^{(i)}}^2]
\end{aligned}$$

substituting $\tilde{w}_j = \frac{1}{2}w_j$,

$$\mathcal{E} = \frac{1}{2N}\sum_{i=1}^{N}[(\sum_j \tilde{w}_j x_j^{(i)})^2 + (\sum_j(\tilde{w}_j x_j^{(i)})^2) - 2t^{(i)}(\sum_j \tilde{w}_j x_j^{(i)}) + {t^{(i)}}^2]$$

substituting $\tilde{y} = \sum_j \tilde{w}_j x_j$,

$$\begin{aligned}
\mathcal{E} &= \frac{1}{2N}\sum_{i=1}^{N}[\tilde{y}^{(i)2} + (\sum_j(\tilde{w}_j x_j^{(i)})^2) - 2t^{(i)}\tilde{y}^{(i)} + {t^{(i)}}^2] \\
&= \frac{1}{2N}\sum_{i=1}^{N}(\tilde{y}^{(i)} - t^{(i)})^2 + \frac{1}{2N}\sum_{i=1}^{N}(\sum_j(\tilde{w}_j x_j^{(i)})^2)
\end{aligned}$$

Then $\mathcal{R} = \sum_{i=1}^{N}(\sum_j(\tilde{w}_j x_j^{(i)})^2)$