Normal Distributions

How a Continuous Probability Distribution Differs from a Discrete Probability Distribution?

1. A continuous probability distribution cannot be expressed in tabular form. Instead, an equation or formula is used to describe it.
2. The probability that a continuous random variable will assume a particular value is zero.

Normal distribution (aka Gaussian distribution) is the most popular and useful continuous probability distribution for a random variable, x. It describes data by two parameters:

- ***The mean of the distribution (μ)*** – The mean of the distribution determines the location of the center of the graph, thus, changing its values will shift the average or center of the normal distribution. The overall shape of the distribution remains the same.
- ***The standard deviation of the distribution (δ)*** – The standard deviation determines the height and width of the graph, hence, differing its values will either flattens out the normal curve or the normal curve becomes steeper. As the standard deviation, becomes smaller, the normal distribution becomes steeper. When the standard deviation becomes larger, the normal distribution tends to flatten out or become broader.

The graph of a normal distribution is called the ***normal curve***.

Properties of the Normal Curve

1. The normal distribution curve is symmetric about the mean (the shape are same on both sides).
   For a distribution that is symmetrical and bell-shaped (in particular, for a normal distribution):
   - Approximately 68% of the data values will lie within 1 standard deviation on each side of the mean.
   - Approximately 95% of the data values will lie within 2 standard deviations on each side of the mean.
   - Approximately 99.7% (or almost all) of the data values will lie within 3 standard deviations on each side of the mean.
2. The mean, median, and mode are equal.
3. The normal curve is asymptotic (it never touches the x-axis).
4. The total area under the part of a normal distribution curve is 1.00 or 100%.
5. The normal curve area may be sub-divided into at least three standard scores each to the left and to the right of the vertical axis. Along the horizontal line, the distance from one integral standard score to the next integral standard score is measured by the standard deviation.

The Relationship Between Z and X

This shifts the mean of X to zero

Formula:

$$Z = \frac{X - \mu}{\delta}$$

This changes the shape of the curve

where Z = standard score

- X = the value of any particular observation or measurement
- μ = the mean of the distribution
- δ = standard deviation of the distribution

Using the Z table

1. Go to the row that represents the leading digit of the Z-value and the first digit after the decimal point.
2. Go to the column that represents the second digit after the decimal point of the Z-value.
3. Intersect the row and column. The "intersection" number represents P(0 < z < Z).

Uniform Probability Distribution

Uniform distribution (aka Rectangular probability distribution) is a continuous distribution in which the same height, of the function, is obtained over a range of values.

$$f\,(x) \;=\; \frac{1}{(b-\;a)} \quad \textit{for } a \leq x \leq b$$

Where:

- a = the smallest value the variable can accept
- b = the largest value the variable can accept
- μ = the mean of the distribution
- δ = standard deviation of the distribution

The length of the rectangular base is (bb−aa) and the total area of the under the curve of probability distribution function should be 1. Thus,

$$(height\ of\ the\ rectangle)\ \times\ (b-a)\ =\ 1$$

Therefore,

$$Height\ of\ the\ Rectangle\ =\ \frac{1}{b-a}$$

Sample Problem 1: You arrive into a building and are about to take an elevator to your floor. Once you call the elevator, it will take between 0 and 40 seconds to arrive to you. We will assume that the elevator arrives uniformly between 0 and 40 seconds after you press the button. What is the probability that elevator takes less than 15 seconds to arrive?

Solution:

- In this case the interval of probability distribution = [a = 0 and b = 40].

  $$f\,(x)\;=\;\frac{1}{(b-\;a)}$$

  $$f\,(x)\;=\;\frac{1}{(40-\;0)}$$

  $$f\,(x)\;=\;\frac{1}{40}=\;0.025$$

- Interval of probability distribution of successful event = [0 seconds, 15 seconds]. The probability $P(x < 15)$

  $$The\ probability\ ratio\ =\frac{15}{40}=0.375$$

- Hence the probability that elevator takes less than 15 seconds to arrive = 0.375

Exponential Probability Distribution

Exponential distribution **(aka negative exponential distribution)** is a continuous distribution often used to measure the time that elapses between two occurrences of an event (the time between arrivals). Some examples are:

- Time required to complete a questionnaire
- Time between vehicle arrivals at a toll booth
- Distance between major defects in a highway

$$\boldsymbol{f(X) = \ \mu e^{-\mu x}, X \geq 0}$$

$$\boldsymbol{Expected\ value = \frac{1}{\mu}\ ;\ \ Variance = \frac{1}{\mu^2}}$$

Where:

- X = random variable (service times).
- x = time
  If X is an exponential random variable, then we can calculate probabilities by:
  - $P\ (X > x) = e^{-\mu x}$
  - $P\ (X < x) = 1 - e^{-\mu x}$
  - $P\ (x_1 < X < x_2) = P\ (X < x_2) - P\ (X < x_1) = e^{-\mu x_1} - e^{-\mu x_2}$
- µ = average number of units the service facility can handle in a specific period of time
- e = 2.718 (the base of natural logarithms)

Sample Problem 2: The mechanic of Joemuff Mufflers and Headers can install three (3) new mufflers per hour, and this service time is exponentially distributed. What is the probability that the time to install a new muffler would be ½ hour or less?

Solution:

Given:

X = exponentially distributed service time
µ = 3 per hour
x = ½ hour = 0.5 hour

$$P\ (X < x) = 1 - e^{-\mu x}$$

$$P(X \leq 0.5) = 1 - e^{-3(0.5)}$$

$$P(X \leq 0.5) = 1 - e^{-1.5}$$

$$P(X \leq 0.5) = 1 - 0.22313$$

$$P(X \leq 0.5) = 0.77687 = 77.687\%$$

Thus, the probability that the mechanic will install a muffler in 0.5 hour is about a 77.687%.

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