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UNIVERSITY OF GHANA

BSc/BA, FIRST SEMESTER EXAMINATIONS: 2016/2017

DEPARTMENT OF MATHEMATICS

MATH 223: CALCULUS II (4 credits)

INSTRUCTION:

ANSWER ANY FIVE(5) OUT OF THE FOLLOWING SEVEN(7) QUESTIONS

TIME ALLOWED:

TWO HOURS AND THIRTY MINUTES $\left(2\frac{1}{2}\text{ hours}\right)$

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1. State the Fundamental Theorem of Calculus (Part I and Part II). Hence or otherwise:

   (a) If u is a differentiable function of x over I, obtain a formula for
   $$\frac{d}{dx}\int_a^u f(t).$$
   Hence find $\dfrac{d^2}{dx^2}\left[\int_0^x \left(\int_1^{\sin t} \sqrt{1+u^4}du\right) dt\right]$.

   (b) Evaluate
   $$\lim_{x\to 3}\left(\frac{x}{x-3}\int_3^x \frac{\sin t}{t}dt\right).$$
   (Hint: You may use $\int_3^x \frac{\sin t}{t}dt = F(x) - F(3)$ for some antidervative F of $\frac{\sin t}{t}$)

   (c) By Using L'hopital's rule,

      i. Evaluate the limit $\lim\limits_{z\to 0}\dfrac{\sin(2z)+7z^2-2z}{z^2(z+1)^2}$.

2. Let a, b, $c \in \mathbb{R}^+$ and distinct.

   (a) Let $x > 0$, By using the definition of the natural logarithmic function,

      i. Prove that $x(1+x) > (1+x)\ln(1+x) > x$.

      ii. Hence show that $\ln(1+x) > x - x^2 + x^3 - x^4 + \ldots$

   (b) Let a, b, $c \in \mathbb{R}^+$ and distinct. Show that there are two different real values of x which satisfy the equation
   $$a\cosh x + b\sinh x = 0$$
   a cosh x + b sinh x = c

   only if $b^2 < a^2 < b^2 + c^2$.

   (c) Let $x \in \mathbb{R}\backslash\{0\}$. Find the values of x that satisfy the equation
   $$\sinh 2x + \sinh x\left(\cosh^2 x - 1\right) = 0.$$

EXAMINERS: Prof. A. Aidoo, Mr. J.S.G. Jackson, Ms L.F. Kyei, and Mr. F.L. Amoah-Darko

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3. (a) Using $n = 6$ subintervals, approximate the value of

$$\int_0^4 \cos(1+\sqrt{x})dx,$$

using the Upper, Lower, and Midpoint Riemann sums. Compare the results and describe.

(b) By interpreting as an integral, evaluate

$$\lim_{n\to\infty} \frac{1}{n^{3/2}}\left(1+\sqrt{2}+\ldots+\sqrt{n-1}\right).$$

4. (a) Show that

$$\int t^3 \sin\left(2t^4\right) dt = -\frac{1}{8}\cos\left(2t^4\right) + C$$

hence show that

$$\int_0^{\pi^{1/4}} t^7 \sin\left(2t^4\right) dt = -\frac{\pi}{8}.$$

$$\int_0^{\pi^{(1/4)}} t^7 \sin(2t^4)dt =$$

(b) Show that if $I_n = \int \sec^n x dx$

$$(n-1)I_n = \sec^{n-2} x \tan x + (n-2)I_{n-2}.$$

Hence or otherwise, evaluate

$$\int_0^{\frac{\pi}{4}} \sec^4 x dx.$$

5. (a) Determine the exact value of the definite integral

$$\int_{\infty}^{-\infty} \frac{6w^3}{(w^4+1)^2} dw.$$

(b) Find the surface area produced when the curve

$$x = 3\cos(\pi t) \qquad y = 5t+2, \qquad 0 \le t \le \frac{1}{2}$$

is rotated about the y-axis.
(You may use without proof $\int \sec^3 x dx = \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta + \tan\theta| + C$.)

(c) A particle travels along a path defined by the curve

$$x = 3t+1 \qquad y = 4-t^2, \qquad -2 \le t \le 0.$$

Determine the total length of the curve.

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6. (a) Given that

$$y = x + \ln\left(\frac{(x-3)^5}{(y-1)^2}\right),$$

show that

$$(xy - 3y + x - 3)\frac{dy}{dx} = xy + 2y - x - 2.$$

(b) Suppose an object of mass m kg falling near the surface of the earth is retarded by air resistance proportional to its velocity $v = v(t)$ at any given time t. Then Newton's second law of motion states that

$$m\frac{dv}{dt} = mg - kv,$$

where g is the acceleration due to gravity near the surface of the earth and k is a constant. If the object starts from rest,

i. find the velocity for any $t > 0$;

ii. what is the limiting value of $v(t)$?

(c) Let $g(x)$ be a differentiable function over an interval I and $g(x) > 0$ on I. Show that

$$\frac{d}{dx}2\sqrt{g(x)} = \frac{g'(x)}{\sqrt{g(x)}}.$$

Hence or otherwise, show that the integral of the function

$$\frac{2x+1}{\sqrt{x^2+2x-1}}dx$$

is

$$2\sqrt{x^2+2x-1} - \ln|\cot\theta + \csc\theta| + K,$$

for some angle θ.

7. (a) Consider the substitution $t = \tanh\frac{x}{2}$.

i. Find expressions for dx, $\sinh x$ and $\cosh x$ in terms of t.

ii. Setup the integral

$$\int \frac{\sinh x + \cosh x + 1}{\sinh x + \cosh x + 2}dx.$$

as the integral of a rational function of t using the substitution above. *(There is no need to evaluate the integral.)*

(b) Solve the initial value problem

$$2u\frac{du}{dx} - \frac{6x}{x^2+1}u^2 = 2x^3u,$$

for $u(0) = 2$.

(c) Find the volume of the solid formed when the plane figure bounded by $r = 2a\cos\theta$ and the radius vectors at $\theta = 0$ and $\theta = \frac{\pi}{2}$ rotates about the initial line. Leave your answer in terms of π and a.

$$V = \int_{\alpha}^{\beta} \frac{2}{3}\pi r^3 \sin\theta d\theta$$

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