15.C57/15.C571/6.C57/6.C571/IDS.C57: Optimization **Fall 2024**

Assignment 2: Duality in linear optimization

Assigned: September 23; Due: October 09.

True or False? [10 pts]

Consider a linear optimization problem in standard form and assume that the rows of $\boldsymbol{A}$ are linearly independent. For each of the following statements, state whether it is true (with a short proof) or whether it is false (with a counterexample).

(a) Let $\boldsymbol{x}^*$ be a basic feasible solution. Suppose that for every basis corresponding to $\boldsymbol{x}^*$, the associated basic solution to the dual is infeasible. Then, the optimal cost must be strictly less than $\boldsymbol{c}^\top \boldsymbol{x}^*$.

(b) Let p_i be the dual variable associated with the i^{th} equality constraint in the problem. Eliminating the i^{th} equality constraint is equivalent to introducing the additional constraint $p_i = 0$ in the dual problem.

(c) If the unboundedness criterion in the primal simplex algorithm is satisfied, then the dual problem is infeasible.

A direct construction of the dual [20 pts]

Consider the following problem:

$$\min \left\{ \max_{i=1,\cdots,m} (\boldsymbol{a}_i^\top \boldsymbol{x} - b_i) : \boldsymbol{x} \in \mathbb{R}^n \right\}$$

Let v be the optimal objective value, which we assume is finite. Let $\boldsymbol{A}$ be the matrix with rows $\boldsymbol{a}_1, \cdots, \boldsymbol{a}_m$ and let $\boldsymbol{b}$ be the vector with components $b_1, \cdots, b_m$.

(a) Consider a vector $\boldsymbol{p} \in \mathbb{R}^m$ such that $\boldsymbol{p}^\top \boldsymbol{A} = \boldsymbol{0}^\top$, $\boldsymbol{p} \geq \boldsymbol{0}$, and $\sum_{i=1}^{m} p_i = 1$. Show that $-\boldsymbol{p}^\top \boldsymbol{b} \leq v$.

(b) In order to obtain the best possible lower bound from that construction, we form the following linear optimization problem:

$$\begin{aligned} \max \quad & -\boldsymbol{p}^\top \boldsymbol{b} \\ \text{s.t.} \quad & \boldsymbol{p}^\top \boldsymbol{A} = \boldsymbol{0}^\top \\ & \sum_{i=1}^{m} p_i = 1 \\ & \boldsymbol{p} \geq \boldsymbol{0} \end{aligned}$$

Show that the optimal cost in this problem is equal to v.

Back-propagation of dual variables in a multi-period problem [30 pts]

A company makes a product that can be either sold or stored to meet future demand. Let $t = 1, \cdots, T$ denote the periods of the planning horizon. Let b_t be the production volume during period t, which is assumed to be known in advance. The company can sell the product at a unit price of d_t. Furthermore, it can send the product to long-term storage at a unit transportation cost of $c \geq 0$, and retrieve it from storage at zero cost. We assume that when the product is prepared for long-term storage, it is partly damaged, and only a fraction $0 < f < 1$ of the total survives. Demand is assumed to be unlimited. The main question is whether it is profitable to store some of the production, in anticipation of higher prices in the future. We define the following variables:

$$\begin{aligned}
x_t &: \text{quantity of the product sold in period } t \\
y_t &: \text{quantity of the product sent to long-term storage in period } t \\
w_t &: \text{quantity of the product retrieved from storage in period } t \\
z_t &: \text{quantity of the product kept in long-term storage at the end of period } t
\end{aligned}$$

We formulate the following linear optimization problem, where d_{T+1} is the salvage price for whatever inventory is left at the end of the horizon, and $0 < \alpha < 1$ is a discount factor reflecting that future revenues are valued less than current ones.

$$\begin{aligned}
\max \quad & \sum_{t=1}^{T} \alpha^{t-1} \left(d_t x_t - c y_t \right) + \alpha^T d_{T+1} z_T \\
\text{s.t.} \quad & x_t + y_t - w_t = b_t, && \forall t = 1, \cdots, T \\
& z_t + w_t - z_{t-1} - f y_t = 0, && \forall t = 1, \cdots, T \\
& z_0 = 0 \\
& x_t, y_t, w_t, z_t \geq 0, && \forall t = 1, \cdots, T
\end{aligned}$$

(a) Interpret the formulation by justifying the objective function and each constraint.

(b) Let p_t and q_t be the dual variables associated with the first and second equality constraints, respectively. Write down the dual problem.

(c) Show that the following solution is optimal in the dual problem:

$$\begin{aligned}
q_T &= \max\left\{ \alpha^T d_{T+1}, \alpha^{T-1} d_T \right\} \\
p_T &= \max\left\{ f q_T - \alpha^{T-1} c, \alpha^{T-1} d_T \right\} \\
q_t &= \max\left\{ q_{t+1}, \alpha^{t-1} d_t \right\}, \ \forall t = 1, \cdots, T-1 \\
p_t &= \max\left\{ f q_t - \alpha^{t-1} c, \alpha^{t-1} d_t \right\}, \ \forall t = 1, \cdots, T-1
\end{aligned}$$

(d) Using this solution, propose a back-propagation algorithm to compute an optimal solution to the original (primal) problem. Primal and dual non-degeneracy can be assumed.

Cloud computing [40 pts]

A cloud computing provider needs to process 1000 jobs, indexed by $i = 1, \ldots, 1000$ with 20 machines, indexed by $j = 1, \ldots, 20$. We denote by c_{ij} the energy consumption resulting from performing job i on machine j. The objective of the problem is to allocate jobs to machines, while minimizing the total energy consumption and ensuring that each job gets fully processed.

a. Assume that each machine j has finite capacity, and can process up to C_j jobs. Formulate the problem as a linear optimization model. Is it a network flow? If so, propose a network representation, including the supply and/or demand at each node, the cost of each arc, and the capacity on each arc. If not, explain why not. Is the solution guaranteed to be integral?

b. You have access to the following data files:
   - energy.csv: A matrix of size 1000×20 that indicates the energy consumption c_{ij} for each job i (rows) and each machine j (columns).
   - capacity.csv: A vector of size 20 that indicates the capacity C_j of machine j.

   Implement the model, and report (i) the optimal energy consumption; (ii) the energy consumption that would be achieved if each job could be assigned to any machine without any consideration for machine capacities; and (iii) the number of jobs that are not (fully) assigned to the machine with the lowest energy consumption.

c. Write the dual of your linear optimization formulation. Propose a metric β_{ij} such that each job $i = 1, \ldots, 1000$ is served by a machine $j = 1, \ldots, 20$ that minimizes β_{ij}. Explain the underlying intuition in simple terms.

d. Actually, all jobs are not created equal. Accordingly, we replace the machine capacity constraint by a machine utilization constraint. Let r_{ij} be the utilization of machine j when performing job i, and let U_j be the maximal utilization of machine j. Formulate the problem as a linear optimization model. Is it a network flow? If so, propose a network representation, including the supply and/or demand at each node, the cost of each arc, and the capacity on each arc. If not, explain why not. Is the solution guaranteed to be integral?

e. You have access to the following data files:
   - utilization.csv: A matrix of size 1000×20 that indicates the utilization r_{ij} for each job i (rows) and each machine j (columns).
   - maxutil.csv: A vector of size 20 that indicates the maximal utilization U_j of machine j.

   Implement the model computationally, and report (i) the optimal energy consumption; (ii) the energy consumption that would be achieved if each job could be assigned to any machine without any consideration for machine utilization; and (iii) the number of jobs that are not (fully) assigned to the machine with the lowest energy consumption.

f. Provide a scatter plot showing the maximal utilization of each machine on the horizontal axis and the dual variable of the corresponding utilization constraint on the vertical axis. Comment briefly on the values and the trends that you observe.