Probabilistic Models

Triple : $(\Omega, \mathcal{F}, P)$

Ω — sample space

$\mathcal{F}$ — event class

P — probability measure

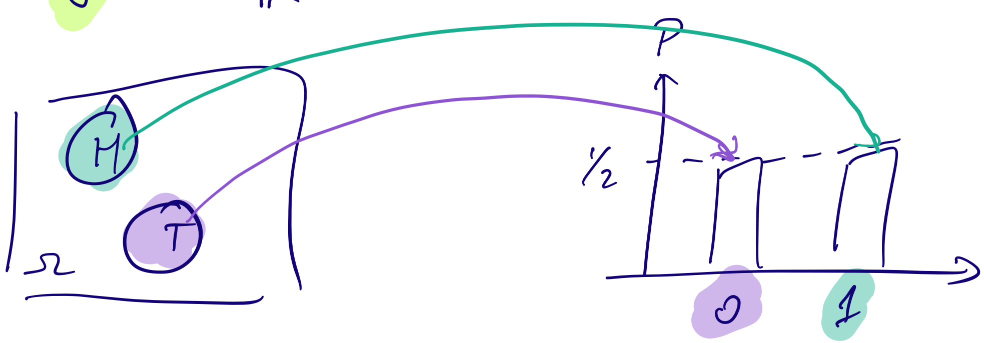
Probability measure, P

e.g. Experiment : flip a fair coin.

$\Omega = \{H, T\}$

$\mathcal{F} \equiv$ flip a fair coin

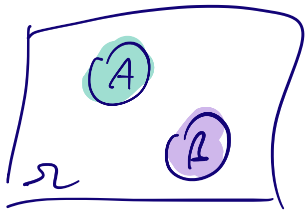
$P : \mathcal{F} \longrightarrow \mathbb{R}$



Mutually exclusive (M.E.)

A and B are M.E. if $A \cap B = \emptyset$

"intersection"



Fair experiment

An experiment with equally likely outcomes.

$$|\Omega| = N$$

If $E \subset \Omega$ is an event that consists of K equally likely outcomes (i.e., $E = \{O_1, O_2, \ldots, O_K\}$

$$\begin{aligned} P(E) &= \sum_{j=1}^{K} P(O_j) \\ &= \sum_{j=1}^{|E|} \frac{1}{|\Omega|} \\ &= \frac{|E|}{|\Omega|} \\ &= K/N \end{aligned}$$

$E \subset \Omega \equiv$ Event E is a subset of sample space.

"equivalence"

$$P(E) = \frac{|E|}{}$$

$\overline{|\Omega|}$

Example: prob. of getting a 1 or 2 when rolling a fair 6-sided die?

$\Omega = \{1, 2, 3, 4, 5, 6\}$

$E = \{1, 2\}$

$$P(E) = \frac{|E|}{|\Omega|} = \frac{2}{6} = \frac{1}{3}$$

Axioms of Probability

P must obey:

① $\forall E \in \mathcal{F}, \quad P(E) \geq 0$

"for all" "in"

② $P(\Omega) = 1$

③ Additivity axiom

$\forall E_1, E_2 \in \mathcal{F}$

$P(E_1 \cup E_2) = P(E_1) + P(E_2)$ if E_1 & E_2 are M.E.

$(E_1 \cap E_2 = \emptyset)$

"union"

If Ω is infinite, then the generalization of ③ is:

If $E_1, E_2, \dots, E_\infty$ is a sequence of events s.t.

$E_i \cap E_j = \emptyset, \ \forall i \neq j$, then:

$$P(E_1 \cup E_2 \cup \dots \cup E_\infty) = P\left(\bigcup_{i=1}^{\infty} E_i\right)$$

$$= \sum_{i=1}^{\infty} P(E_i)$$ countable index sets

Corollaries

Properties derived from axioms.

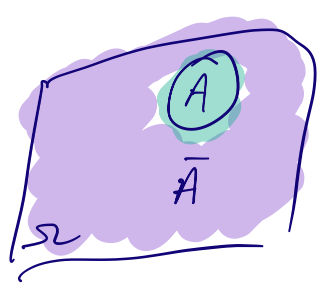
$A, B \in \mathcal{F}$

$P : \mathcal{F} \to \mathbb{R}$

$$\boxed{① \quad P(A^c) = P(\bar{A}) = 1 - P(A)}$$

"complement of A"

Proof: $\Omega = \bar{A} \cup A$



$$P(\Omega) = 1 = P(\bar{A} \cup A)$$

$$= P(\bar{A}) + P(A)$$

$$\Rightarrow P(\bar{A}) = 1 - P(A)$$

x —— x

$\boxed{(2)\ P(A) \leq 1}$

Proof: $\Omega = A \cup \bar{A}$

$$P(A) = P(\Omega) - P(\bar{A})$$
$$= 1 - \underbrace{P(\bar{A})}_{\geq 0}$$
$$\leq 1$$

x —— x

$\boxed{(3)\ P(\phi) = 0}$

ϕ = null space

Proof: $\phi = \bar{\Omega} \iff \bar{\phi} = \Omega$

$$P(\phi) = 1 - P(\bar{\phi}) = 1 - \underbrace{P(\Omega)}_{=1}$$
$$= 1 - 1$$
$$= 0$$

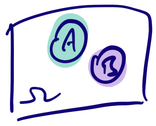
x —— x

(4) If $A_1, \dots, A_n$ are pairwise M.E., then
$$P\left(\bigcup_{k=1}^{n} A_k\right) = \sum_{k=1}^{n} P(A_k)$$

Proof by induction.

⑤ $P(A \cup B) = P(A) + P(B) + P(A \cap B)$

<u>Case 1</u> : $A \cap B = \emptyset$: A and B are M.E.

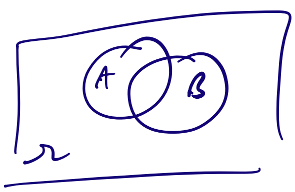


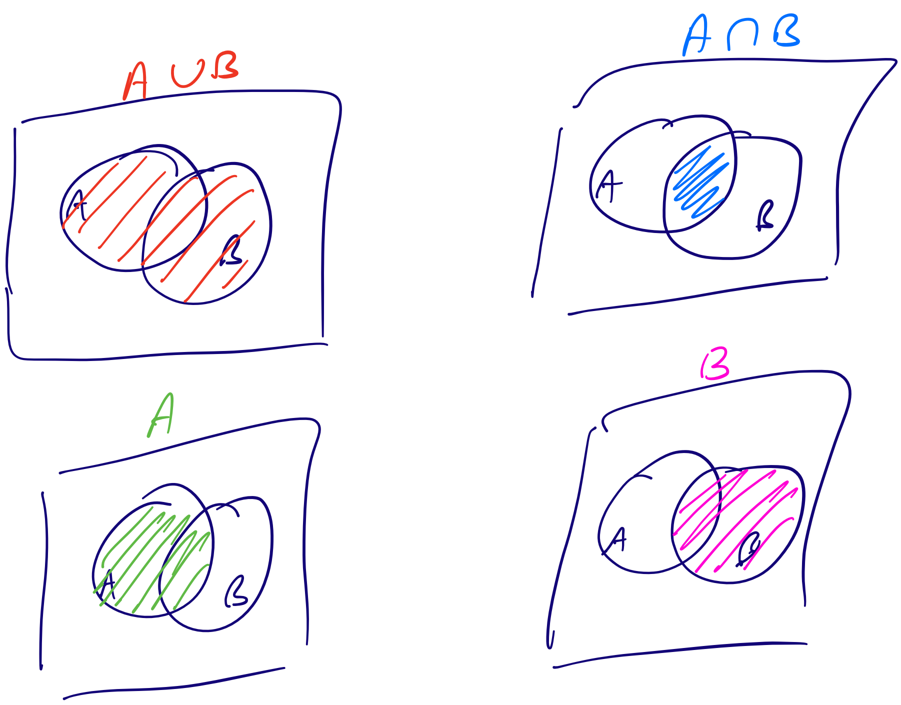
$A \cap B = \emptyset$

$P(A \cap B) = 0$

$\therefore P(A \cup B) = P(A) + P(B)$

<u>Case 2</u> : $A \cap B \neq \emptyset$





$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Example: Roll a fair-sided die twice.

$E_i \equiv$ observe a 1 or a 2 on roll i

E_1 : observe a 1 or a 2 on 1^{st} roll

Probability of 1 or 2 on either roll?

$$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$$

$$= \frac{1}{3} + \frac{1}{3} - \frac{4}{36}$$

$E_1 \cap E_2$

$\boxed{\text{Roll 2}}$

$\boxed{\text{Roll 1}}$	1	2	3	4	5	6
1	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
2	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
3	(3,1)	(3,2)	.			
4	(4,1)	(4,2)		.		
5	(5,1)	(5,2)			.	
6	(6,1)	(6,2)				(6,6)

E_1

E_2

$$P(E_1 \cap E_2) = \frac{|E_1 \cap E_2|}{|\Omega|} = \frac{4}{36}$$

$$P(E_1) = \frac{|E_1|}{|\Omega|} = \frac{12}{36}$$

$$P(E_2) = \frac{|E_2|}{|\Omega|} = \frac{12}{36}$$