Approximation of Fixed Arc Length Polynomial Coefficients

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***Abstract*—This paper is concerned with approximating polynomial functions of integer degree two or more which retain an arc length that is near to one over the interval (0, ϵ), as ϵ is shifted within the bounds of zero and plus or minus one. Some notation is introduced in order to discuss matters efficiently. The set $S_n(\epsilon)$ contains polynomials of degree n with n real roots: 0, those that divide the interval $(0, \epsilon)$ into n minus one equal partitions, and ϵ, and furthermore, whose arc length is measured over the interval $(0, \epsilon)$. The function $s_n\{f\}$ is that which measures the arc length of f over said interval, and $\Delta_n\{f\}$ measures its error. The coefficient function $c_n(\epsilon)$ gives $f \in S_n(\epsilon)$ an arc length of exactly one over the interval $(0, \epsilon)$, and is approximated by $c'_n(\epsilon)$.**

I. Introduction

Let $S_2(\epsilon)$ be the set of polynomials in the real value x of degree two which pass through both the origin, and another point on the x-axis, $(\epsilon, 0)$, and whose arc length over the interval $[0, \epsilon]$ is one. If $|\epsilon|$ is greater than one, then $S_2(\epsilon)$ can have no members as, obviously, no curve segment whose arc length is one could span a distance greater than one. If ϵ is equal to zero, then any function in $S_2(\epsilon)$ passes through the origin, proceeds one unit of distance onward in some manner, and then passes through the origin once more. Then if ϵ is equal to zero, $S_2(\epsilon)$ must again be empty; by definition, a function can not behave in this way. If $|\epsilon|$ is equal to one, $S_2(\epsilon)$ contains exactly one member, that being the zero function (supposing that polynomial sets admit the zero function). And finally, when $|\epsilon|$ is strictly less than one, but not equal to zero, $S_2(\epsilon)$ has precisely two members:

let $f \in S_2(\epsilon)$, where $|\epsilon| < 1$, and $\epsilon \neq 0$, then f has an arc length of one on the interval $[0, \epsilon]$, and there exists exactly one constant, c_2 such that:

$$f(x) = \pm c_2(\epsilon x - x^2)$$

If $s_2\{f\}$ is the arc length of f on the interval $[0, \epsilon]$, where $f \in S_2(\epsilon)$, then $s_2\{f\}$ is given by:

$$s_2\{f\} = \int_0^\epsilon \sqrt{1 + [f'(x)]^2}dx$$

Since $s_2\{f\}$ must equal one, the following equality is obtained:

$$s_2\{f\} = \frac{c_2\epsilon\sqrt{c_2^2\epsilon^2 + 1} + \sinh^{-1}(c_2\epsilon)}{2c_2} = 1$$

While $s_2\{f\}$ has a closed form representation, this equality can not be solved for the coefficient c_2 (at least not by any apparent means), as is desired.

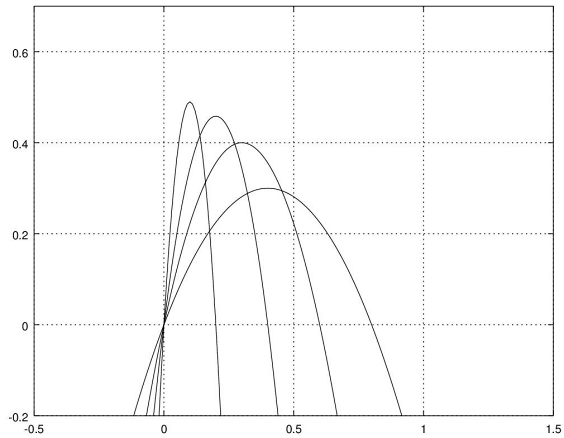

Fig. 1. Plots of f^+ from $S_2(0.2), S_2(0.4), S_2(0.6), andS(0.8)$

II. Construction of $c'_2(\epsilon)$

It is unlikely that the equality $s_2\{f\} = 1$ can be resolved by any algebraic expression for c_2 that has a closed form. However there can exist a closed form, algebraic function of epsilon that approximately resolves the equality. Let the exact solution to the above equality be called $c_2(\epsilon)$, and let $c'_2(\epsilon)$ approximate $c_2(\epsilon)$ on the intervals $(0, 1]$ or $[-1, 0)$. And so now, for some $f \in S_2(\epsilon)$:

$$f(x) = \pm c_2(\epsilon)(\epsilon x - x^2)$$
$$f(x) \approx \pm c'_2(\epsilon)(\epsilon x - x^2)$$

As has been noted before, $S_2(\epsilon)$ is an empty set when $|\epsilon| > 0$. However, $f \in S_2(\epsilon)$, as was described previously is apparently defined for any $\epsilon \in \mathbb{R}$. It follows that $c_2(\epsilon)$ must give $f \in S_2(\epsilon)$ the property that it is not defined when $|\epsilon| > 0$.

then

$$c_2(\epsilon) \sim \sqrt{1 - \epsilon^2}$$

That is to say, $c_2(\epsilon)$ is similar to the the function that gives the unit circle, in that neither are defined for values larger than plus or minus one.

The set $S_2(\epsilon)$ must also be empty when $\epsilon = 0$. This quality, too, can be given to $f \in S_2(\epsilon)$ if and only if $c_2(\epsilon)$ is not defined when $\epsilon = 0$. Then $c_2(\epsilon)$ must have this property as well.

then

$$c_2(\epsilon) \sim \frac{\sqrt{1-\epsilon^2}}{\epsilon}$$

And so, $c_2(\epsilon)$ is similar to this expression, which by identity is equal to $\cot(\arcsin(\epsilon))$, in that $c_2(\epsilon)$ is not defined when $|\epsilon| > 0$ or when $\epsilon = 0$.

One final observation about $f \in S_2(\epsilon)$ is that $\forall x \in [-\epsilon, 0]$ or $[0, \epsilon]$, $|f(x)|$ can not exceed one half. If $|\max\{f(x)\}| > 1/2$ anywhere on the interval $[-\epsilon, 0]$ or $[0, \epsilon]$, it could not return to the x-axis and cross $(\epsilon, 0)$ without traversing a total distance that is greater than one. Obversely, as $|\epsilon|$ decreases toward zero, $|\max\{f(x)\}|$ must increase towards one half. For very small $|\epsilon|$, $f(x)$ is almost entirely vertical on $[-\epsilon, 0]$ or $[0, \epsilon]$

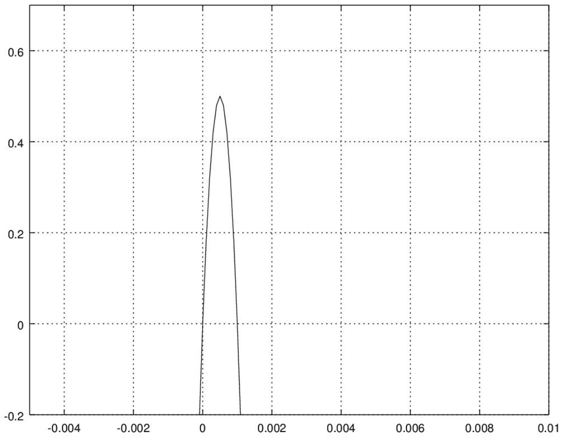

Fig. 2. A plot of $f^+ \in S_2(\epsilon)$ for small ϵ

Then for $f \in S_2(\epsilon)$, it follows that:

$$\lim_{\epsilon \to 0} |f(\frac{\epsilon}{2})| = \frac{1}{2}$$

Then

$$c_2(\epsilon) \sim 2\frac{\sqrt{1-\epsilon^2}}{\epsilon^2}$$

And as $c_2'(\epsilon)$ approximates $c_2(\epsilon)$, let

$$c_2'(\epsilon) = 2\frac{\sqrt{1-\epsilon^2}}{\epsilon^2}$$

then for $f \in S_2(\epsilon)$:

$$\begin{aligned} f(x) &\approx \pm c_2'(\epsilon)(\epsilon x - x^2) \\ &\approx \pm[2\frac{\sqrt{1-\epsilon^2}}{\epsilon^2}](\epsilon x - x^2) \\ &and \\ &\lim_{\epsilon \to 0} |\pm[2\frac{\sqrt{1-\epsilon^2}}{\epsilon^2}](\epsilon[\frac{\epsilon}{2}] - [\frac{\epsilon}{2}]^2)| = \frac{1}{2} \end{aligned}$$

Let $\Delta_2\{f\}$ be one less than the function of ϵ that is obtained when the constant, c_2 is substituted with the function $c_2'(\epsilon)$ in the function $s_2\{f\}$.

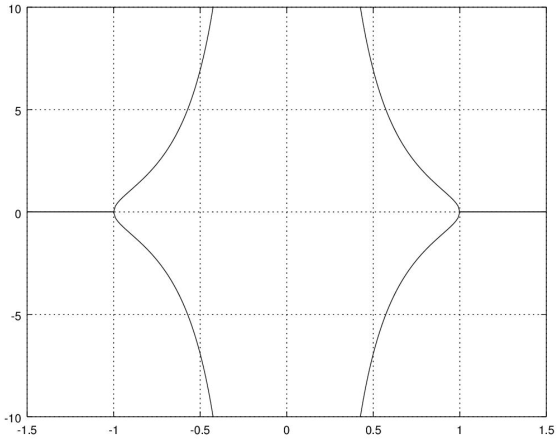

Fig. 3. A plot of $\pm c_2'(\epsilon)$, for the sake of elucidation

then where

$$\begin{aligned} s_2\{f\} &= \frac{c_2\epsilon\sqrt{c_2^2\epsilon^2+1} + \sinh^{-1}(c_2\epsilon)}{2c_2}, \\ \Delta_2\{f\} &= \frac{c_2'(\epsilon)\epsilon\sqrt{c_2'(\epsilon)^2\epsilon^2+1} + \sinh^{-1}(c_2'(\epsilon)\epsilon)}{2c_2'(\epsilon)} - 1 \\ &= \frac{[2\frac{\sqrt{1-\epsilon^2}}{\epsilon^2}]\epsilon\sqrt{[2\frac{\sqrt{1-\epsilon^2}}{\epsilon^2}]^2\epsilon^2+1} + \sinh^{-1}([2\frac{\sqrt{1-\epsilon^2}}{\epsilon^2}]\epsilon)}{2[2\frac{\sqrt{1-\epsilon^2}}{\epsilon^2}]} - 1 \end{aligned}$$

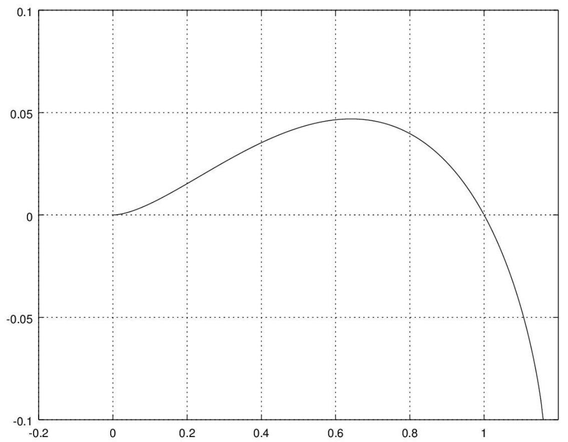

Fig. 4. $\Delta_2\{f\}$ with maximum $\approx 0.0468936737813141$ when $\epsilon \approx 0.643$ for $\epsilon \in (0, 1]$

if

$$\begin{aligned} &f(x) \in S_2(\epsilon) \\ &\quad and \\ &g(x) = [2\frac{\sqrt{1-\epsilon^2}}{\epsilon^2}](\epsilon x - x^2) \end{aligned}$$

then

$$g(x) \approx f(x)$$
$$and$$
$$s_2\{f\} = 1 = s_2\{g\} - \Delta_2\{g\}$$

Then the degree to which the arc length of the approximating quadratic function g deviates from that of its corresponding $f \in S_2(\epsilon)$ is given by $\Delta_2\{g\}$, and the maximum deviance is $\approx 0.0468936737813141$ when $\epsilon \approx 0.643$ for $\epsilon \in (0, 1]$.

III. A Generalized Approximation Scheme for $S_n(\epsilon)$

An additional constraint is now described for the set $S_n(\epsilon)$. For some function $f \in S_n(\epsilon)$, f has n real roots: zero, ϵ, and those remaining partition the interval $(0, \epsilon)$ into $n-1$ sub-intervals. Though the method of approximation may apply to any such polynomial, to narrow the scope of this study, $f \in S_n(\epsilon)$ are chosen such that f has roots zero, ϵ and those that partition the interval $(0, \epsilon)$ into $n-1$ equal length sub-intervals. That is:

if $f(x) \in S_n(\epsilon)$, then $f(x) = 0$ if and only if,

$$x \in \{0, \frac{\epsilon}{n-1}, \frac{2\epsilon}{n-1}, \frac{3\epsilon}{n-1}, ..., \frac{(n-2)\epsilon}{n-1}, \epsilon\}$$

Then $\forall n \in \mathbb{N}$, functions $f(x) \in S_n(\epsilon)$ are determined up to one free coefficient which scales $f(x)$. That coefficient is made a function of ϵ so as to scale $f(x)$ such that $s_n\{f\} = 1$. And so $f(x) \in S_n(\epsilon)$ may be given as:

$$f(x) = c_n(\epsilon) \prod_{i=0}^{n-1} (x - \frac{i\epsilon}{n-1})$$
$$and$$
$$f(x) \approx c'_n(\epsilon) \prod_{i=0}^{n-1} (x - \frac{i\epsilon}{n-1})$$

where

$$c'_n(\epsilon) = \pm k_n \frac{\sqrt{1-\epsilon^2}}{\epsilon^n}$$

Let α_i be a root of $\frac{df(x)}{dx}$, where $f(x) \in S_n(\epsilon)$. In the limit as ϵ approaches zero,

$$s_n\{f\} = 2 \sum_{i=1}^{n-1} |\lim_{\epsilon \to 0} f(\alpha_i)|$$

For instance, if $f \in S_4(\epsilon)$,

$$f(x) \approx \pm[k_4 \frac{\sqrt{1-\epsilon^2}}{\epsilon^4}] \prod_{i=0}^{3} (x - \frac{i\epsilon}{3})$$
$$\approx \pm[k_4 \frac{\sqrt{1-\epsilon^2}}{\epsilon^4}](\frac{2\epsilon^3}{9}x - \frac{11\epsilon^2}{9}x^2 + 2\epsilon x^3 - x^4)$$

and

$$\frac{df}{dx} \approx \pm[k_4 \frac{\sqrt{1-\epsilon^2}}{\epsilon^4}](\frac{2}{9}\epsilon^3 - \frac{22}{9}\epsilon^2 x + 6\epsilon x - 4x^3)$$

To find the roots of the above equation, we may first let ϵ equal one, and solve for the roots of the resulting equation. this gives:

$$\alpha'_1 = \frac{3-\sqrt{5}}{6}, \alpha'_2 = \frac{1}{2}, \alpha'_3 = \frac{3+\sqrt{5}}{6}$$

Then the general solution for the roots of $\frac{df}{dx}$ are obtained by multiplying by ϵ

$$\alpha_1 = \epsilon\frac{3-\sqrt{5}}{6}, \alpha_2 = \frac{\epsilon}{2}, \alpha_3 = \epsilon\frac{3+\sqrt{5}}{6}$$

And so the arc length of $f \in S_4(\epsilon)$ for very small ϵ is approximately the vertical distance from the x-axis up to $f(\alpha_1)$, then back down to the x-axis again, then the vertical distance down to $f(\alpha_2)$ and back, and finally the distance up to and back down from $f(\alpha_3)$.

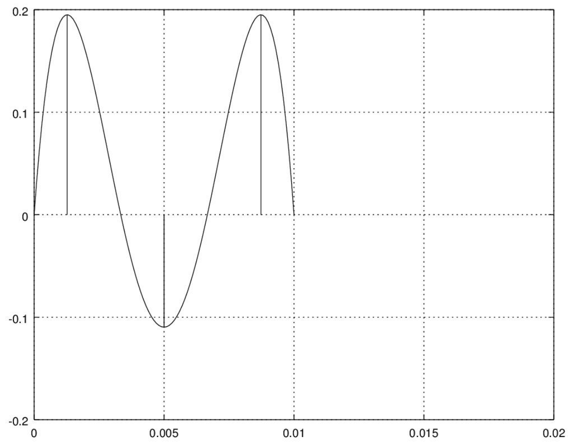

Fig. 5. approximation of $f \in S_4(\epsilon)$ for small ϵ

Since the condition that the arc length of $f \in S_n(\epsilon)$ equals one must be met, this gives:

$$2\sum_{1}^{3} |\lim_{\epsilon \to 0}[k_4 \frac{\sqrt{1-\epsilon^2}}{\epsilon^4}](\frac{2\epsilon^3}{9}\alpha_i - \frac{11\epsilon^2}{9}\alpha_i^2 + 2\epsilon\alpha_i^3 - \alpha_i^4)| = 1$$

$$2(k_4\frac{1}{81} + k_4\frac{1}{144} + k_4\frac{1}{81}) = 1$$
$$k_4 = \frac{1}{2}\frac{1}{\frac{1}{81} + \frac{1}{144} + \frac{1}{81}}$$
$$k_4 = \frac{648}{41}$$

then if $f \in S_4(\epsilon)$,

$$f(x) \approx \pm\frac{648}{41}\frac{\sqrt{1-\epsilon^2}}{\epsilon^4}\prod_{i=0}^{3}(x - \frac{i\epsilon}{3})$$

And so, for $f \in S_n(\epsilon)$, the general approximating formula for f is:

$$f(x) \approx \pm[\frac{1}{2}\frac{1}{\sum_{i=1}^{n-1} g(\alpha_i)}\frac{\sqrt{1-\epsilon^2}}{\epsilon^n}]\prod_{i=0}^{n-1}(x-\frac{i\epsilon}{n-1})$$

where $g(x)=\prod_{i=0}^{n-1}(x-\frac{i}{n-1})$ and α_i is a root of $\frac{dg}{dx}$.

IV. Analysis

The positive members of $S_2(\epsilon)$, $S_3(\epsilon)$, $S_4(\epsilon)$, $S_5(\epsilon)$, and $S_6(\epsilon)$, for $\epsilon \in (0,1]$ are selected for analysis here and are approximated by f_2, f_3, f_4, f_5, and f_6. Via the algorithm described above, the coefficient function, $c'_n(\epsilon)$ respective to each f_n has been computed, and gives:

$$\begin{cases} c'_2(\epsilon) = 2\frac{\sqrt{1-\epsilon^2}}{\epsilon^2} \\ c'_3(\epsilon) = 3\sqrt{3}\frac{\sqrt{1-\epsilon^2}}{\epsilon^3} \\ c'_4(\epsilon) = \frac{648}{41}\frac{\sqrt{1-\epsilon^2}}{\epsilon^4} \\ c'_5(\epsilon) = (50.96177607\dots)\frac{\sqrt{1-\epsilon^2}}{\epsilon^5} \\ c'_6(\epsilon) = (164.7668005\dots)\frac{\sqrt{1-\epsilon^2}}{\epsilon^6} \end{cases}$$

and so

$$\begin{cases} f_2 = 2\frac{\sqrt{1-\epsilon^2}}{\epsilon^2}(\epsilon x - x^2) \\ f_3 = 3\sqrt{3}\frac{\sqrt{1-\epsilon^2}}{\epsilon^3}(-\frac{1}{2}\epsilon^2 x + \frac{3}{2}\epsilon x^2 - x^3) \\ f_4 = \frac{648}{41}\frac{\sqrt{1-\epsilon^2}}{\epsilon^4}(\frac{2}{9}\epsilon^3 x - \frac{11}{9}\epsilon^2 x^2 + 2\epsilon x^3 - x^4) \\ f_5 = (50.96177607\dots)\frac{\sqrt{1-\epsilon^2}}{\epsilon^5}(-\frac{3}{32}\epsilon^4 x + \frac{25}{32}\epsilon^3 x^2 \\ \qquad - \frac{35}{16}\epsilon^2 x^3 + \frac{5}{2}\epsilon x^4 - x^5) \\ f_6 = (164.7668005\dots)\frac{\sqrt{1-\epsilon^2}}{\epsilon^6}(\frac{24}{625}\epsilon^5 x - \frac{274}{625}\epsilon^4 x^2 \\ \qquad + \frac{9}{5}\epsilon^3 x^3 - \frac{17}{5}\epsilon^2 x^4 + 3\epsilon x^5 - x^6) \end{cases}$$

A. Error Analysis of $c_n(\epsilon)'$

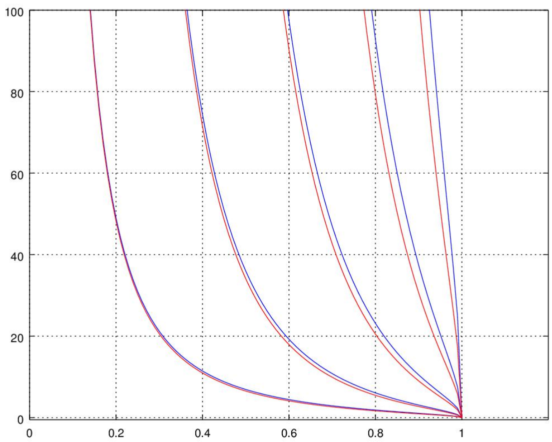

Fig. 6. $c'_n(e)$: $blue$, $c_n(e)$ (actual): red

The above plot shows pairs of numerically computed $c_n(\epsilon)$s in red, and their approximating $c'_n(\epsilon)$s in blue. Bending nearest the origin are the pair $c_2(\epsilon)$, and its approximant, $c'_2(\epsilon)$. Those bending second nearest the the origin are the $c'_3(\epsilon)$ and $c_3(\epsilon)$ pair, and so on, with the $c_6(\epsilon)$ and $c'_6(\epsilon)$ pair furthest from the origin, descending most steeply as ϵ increases toward one.

There is an apparent similarity among each $c_n(\epsilon)$ and $c'_n(\epsilon)$ pair, though with a clear trend of diminishing returns in accuracy as n increases. Analysis of the error in each $c_n(\epsilon)$ approximant has been conducted so as to determine the average discrepancy among each $c_n(\epsilon)$ and $c'_n(\epsilon)$ pair. This is done by computing both $c_n(\epsilon)$ and $c'_n(\epsilon)$ on a discretized interval for $epsilon$, for which the difference of their absolute values is summed and lastly divided by the size of the discretized interval. The largest discrepancy among each $c_n(\epsilon)$ and $c'_n(\epsilon)$ pair, and the ϵ for which the maximum discrepancy occurs is also reported.

TABLE I
$c'_n(\epsilon)$ Error Analysis

$E\{c'_n(\epsilon)\}$	Average Error	Max Error	ϵ_{max}
$c'_n(\epsilon)$	5.17587788727138e-001	2.24615455982348e+000	0.020
$c'_n(\epsilon)$	1.55273515937968e+001	5.24065869844519e+002	0.020
$c'_n(\epsilon)$	2.41041153218448e+003	1.65123159109116e+005	0.020
$c'_n(\epsilon)$	7.11525580370445e+005	6.07321035718994e+007	0.020
$c'_n(\epsilon)$	2.62777831224667e+008	2.43612122666250e+010	0.020

The data procured from the analysis of $c'_n(\epsilon)$ is not as expected. For each $c'_n(\epsilon)$, the maximum error occurs when ϵ is equal to 0.020. This value for ϵ is the lower bound of the interval on which the $c_n(\epsilon)$ and $c'_n(\epsilon)$ pairs are evaluated (choosing a lower bound closer to zero yields a maximum error at that bound). The average error of $c'_n(\epsilon)$ also becomes poorer and poorer as the lower bound of the interval is extended nearer to zero. These results suggest that the $c'_n(\epsilon)$ approximant diverges as ϵ decreases toward zero. This is unexpected since the constant coefficient of the $c'_n(\epsilon)$ approximant is calculated such that $c'_n(\epsilon)$ is asymptotic with $c_n(\epsilon)$ as ϵ decreases toward zero.

B. Error Analysis of $s\{f_n\}$

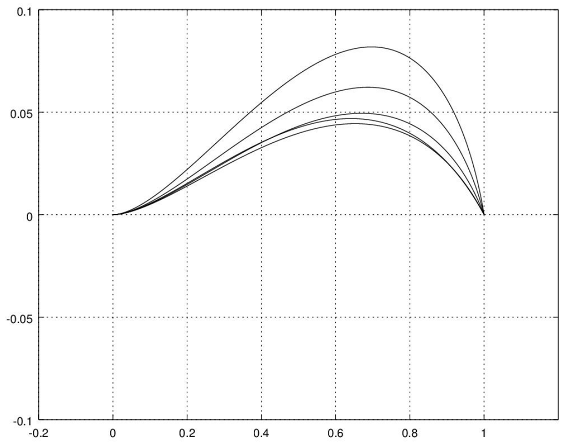

Fig. 7. $\Delta\{f_n\}$

Figure 7 shows the degree to which the arc length of each f_n deviates from one as a function of ϵ on the interval $(0, 1]$. The curve reaching highest is the error in arc length as ϵ is varied for the approximant f_6. Immediately below that is the curve representing the error in arc length of f_5, then f_4, then oddly enough, f_2 and finally f_3. And so, for instance, the arc length for the approximant f_2 for $\epsilon \in (0, 1]$ is bounded between one and 1.0468936737813141.

TABLE II
$\Delta\{f_n\}$ ERROR ANALYSIS

$\Delta\{f_n\}$	Average Error	Max Error	ϵ_{max}
f_2	0.0285001312269215	0.0468936737813141	0.643
f_3	0.0269615573192636	0.0444099684307782	0.653
f_4	0.0299880571299484	0.0494667561026323	0.671
f_5	0.0375048539247200	0.0621744553494021	0.688
f_6	0.0492458946732432	0.0818600318750098	0.698

And so where $s\{f\}$ gives the arc length of f for $\epsilon \in (0, 1]$,

$$\begin{cases} 1 \le s\{f_2\} \le 1.0468936737813141 \\ 1 \le s\{f_3\} \le 1.0444099684307782 \\ 1 \le s\{f_4\} \le 1.0494667561026323 \\ 1 \le s\{f_5\} \le 1.0621744553494021 \\ 1 \le s\{f_6\} \le 1.0818600318750098 \end{cases}$$

Despite the tendency for $c'_n(\epsilon)$ to diverge from $c_n(\epsilon)$ as ϵ decreases toward zero, as observed from the results in Table I, the arc length of f_n still converges to one as ϵ decreases toward zero. One conjecture that might explain this very peculiar behavior is that if for very small $|\epsilon|$, $\frac{ds\{f_n\}}{dc'_n(\epsilon)}$ is nearly zero, then the increasing error of the $c'_n(\epsilon)$ approximation of $c_n(\epsilon)$ as ϵ decreases toward zero has a diminishing, and eventually negligible effect on $s\{f_n\}$.

V. CONCLUSION

Based on the results obtained in analysis, this method of approximation yields fair results for polynomials of lesser degree. There is an apparent trend of increasing error as the degree of the approximated polynomial increases.

Though the function $c'_n(\epsilon)$ contains one division and one square root, which for computers are relatively cycle-expensive operations, this computation outpaces, by far, the numerical methods that were used to obtain the results with which it was compared. The method of approximation established here may easily be generalized for curves in $\mathbb{R}^n$ and for any arc length. This is then a plausible candidate for real-time computer graphics, or approximate calculation by hand when small error is acceptable.

A measure of the curvature [1] of some $f \in S_n(\epsilon)$ and its product with a measure of a material's propensity to right itself may serve to model, and give quick approximation of forces involved in spring mass systems, for instance, Hooke's Law for the restoring force of an ideal spring [2],

$$\begin{aligned} F &= -kx \\ F &\approx -k[\int_0^{\epsilon} \frac{|f''(x)|}{(1 + [f'(x)]^2)^{3/2}} dx] \\ x'' &\approx -\frac{k}{m}[\int_0^{\epsilon} \frac{|f''(x)|}{(1 + [f'(x)]^2)^{3/2}} dx] \end{aligned}$$

Functions $f \in S_n(\epsilon)$ may also serve to model, and approximate the physical deformation of objects undergoing elastic collisions, the simulation of pendulums suspended by strings, and perhaps longitudinal waves.

It may be so that the function $c_n(\epsilon)$ has some Laurent Series [3], of which $c'_n(\epsilon)$ is a principal term. There could be room for improvement in the approximation at the expense of one division per additional term if we suppose that

$$\begin{aligned} c_n(\epsilon) = \sqrt{1 - \epsilon^2}(\frac{k_{-n}}{\epsilon^n} + \frac{k_{-(n-1)}}{\epsilon^{n-1}} + \cdots + k_0 \\ + k_1\epsilon + k_2\epsilon^2 + \ldots) \end{aligned}$$

and determine the coefficients, k_n, of some truncation of the above, perhaps by a least squares method.

There is also the possibility that $c'_n(\epsilon)$ has superior results for some p and q such that:

$$c_n(\epsilon) = k_n \frac{\sqrt[1/2q]{1 - \epsilon^{2p}}}{\epsilon^n}$$

Approximation of $f \in S_n(\epsilon)$ may profit from further research along these avenues.

REFERENCES

[1] Stewart, James. *Essential Calculus*. 2nd ed. Belmont, CA: Brooks/Cole; 2013

[2] Cutnell, John D., and Kenneth W. Johnson. *Physics. 9th ed*. Hoboken, NJ: Wiley, 2012

[3] Laurent series: https://en.wikipedia.org/wiki/Laurent_series.