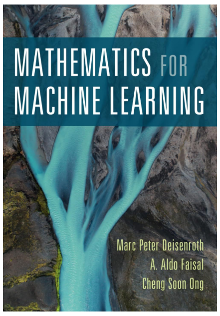
Mathematics for Machine Learning (Deisenroth, Faisal, Ong)

Chapter 2: Linear Algebra

Full solutions to exercises 2.1 - 2.20



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2.1) $(\mathbb{R}\setminus\{-1\}, *)$; $a * b = ab + a + b$.

a) – closure (i.e if $a, b \in \mathbb{R}\setminus\{-1\}$ then $a * b \in \mathbb{R}\setminus\{-1\}$):

Suppose $a, b \neq -1$ with $a * b = ab + a + b = -1$.

Then $a(b+1) = -1 - b \Leftrightarrow a = \frac{-(b+1)}{b+1} = -1$ ※

– associativity:

$$(a * b) * c = (ab + a + b) * c$$

$$= (ab + a + b) \cdot c + ab + a + b + c$$

$$= \underset{1}{abc} + \underset{2}{ac} + \underset{3}{bc} + \underset{4}{ab} + \underset{5}{a} + \underset{6}{b} + \underset{7}{c}.$$

OTOH,

$$a * (b * c) = a * (bc + b + c)$$

$$= a(bc + b + c) + a + bc + b + c$$

$$= \underset{1}{abc} + \underset{4}{ab} + \underset{2}{ac} + \underset{5}{a} + \underset{3}{bc} + \underset{6}{b} + \underset{7}{c}.$$

All 7 terms match.

– abelian:

$$a * b = ab + a + b$$

$$= ba + b + a$$

$$= b * a.$$

– identity:

need $e \in \mathbb{R}\setminus\{-1\}$ s.t $a * e = a \quad \forall a \in \mathbb{R}\setminus\{-1\}$.

$$a * e = ae + a + e = a$$

$\Leftrightarrow e(a+1) = 0 \Leftrightarrow e = 0$ (, $a \neq -1$ so can divide by $a+1$.)

identity is $e = 0$.

- inverse:

given $a \in \mathbb{R} \setminus \{-1\}$, need b s.t $a * b = 0$.

$$a * b = ab + a + b = 0$$

$$\Leftrightarrow b(a+1) = -a \Leftrightarrow b = \frac{-a}{a+1} = -1 + \frac{1}{a+1}.$$

b) $3 * x * x = 15$

(Inverse of 3 is $-1 + \frac{1}{4} = -\frac{3}{4}$.)

$$\Leftrightarrow x * x = -\frac{3}{4} * 15$$

$$\Leftrightarrow x * x = -\frac{45}{4} - \frac{3}{4} + \frac{60}{4} = \frac{12}{4} = 3.$$

$$\Leftrightarrow x^2 + x + x = 3$$

$$\Leftrightarrow x^2 + 2x - 3 = 0 \Leftrightarrow (x+3)(x-1) = 0$$

$$\Leftrightarrow x \in \{-3, 1\}.$$

2.2 a) Show $(\mathbb{Z}/n\mathbb{Z}, \oplus)$; $\overline{a} \oplus \overline{b} = \overline{a+b}$ is a group.

- closed (i.e if $\overline{a}, \overline{b} \in \mathbb{Z}/n\mathbb{Z}$ then $\overline{a+b} \in \mathbb{Z}/n\mathbb{Z}$).

  Obvious.

- associative:

$$(\overline{a} \oplus \overline{b}) \oplus \overline{c} = \overline{a+b} \oplus \overline{c} = \overline{(a+b)+c} = \overline{a+b+c}$$

$$\overline{a} \oplus (\overline{b} \oplus \overline{c}) = \overline{a} \oplus \overline{b+c} = \overline{a+(b+c)} = \overline{a+b+c}.$$

- identity:

$\overline{a} \oplus \overline{0} = \overline{a+0} = \overline{a}$ & $\overline{0} \oplus \overline{a} = \overline{0+a} = \overline{a}$. $\overline{0}$ is identity.

- inverse:

Let $\overline{a} \in \mathbb{Z}/n\mathbb{Z}$ (so $0 \leq a \leq n-1$)

Consider $\overline{n-a} \in \mathbb{Z}/n\mathbb{Z}$

Then $\overline{a} \oplus \overline{n-a} = \overline{a+(n-a)} = \overline{n} = \overline{0}$.

- abelian:

$\overline{a} \oplus \overline{b} = \overline{a+b} = \overline{b+a} = \overline{b} \oplus \overline{a}$.

b)

$\times$	$\overline{1}$	$\overline{2}$	$\overline{3}$	$\overline{4}$
$\overline{1}$	$\overline{1}$	$\overline{2}$	$\overline{3}$	$\overline{4}$
$\overline{2}$	$\overline{2}$	$\overline{4}$	$\overline{1}$	$\overline{3}$
$\overline{3}$	$\overline{3}$	$\overline{1}$	$\overline{4}$	$\overline{2}$
$\overline{4}$	$\overline{4}$	$\overline{3}$	$\overline{2}$	$\overline{1}$

- closed: ✓
- identity: $\overline{1}$
- inverses: $\overline{1}^{-1} = \overline{1}$
  $\overline{2}^{-1} = \overline{3}$
  $\overline{3}^{-1} = \overline{2}$
  $\overline{4}^{-1} = \overline{4}$
- associative: Yes, since "regular" $\times$ is associative
- commutative: multiplication table is symmetric.

c) $\overline{4} \cdot \overline{2} = \overline{8} = \overline{0}$, hence $\overline{2}$ (or $\overline{4}$) has no inverse.

[Proof: If $\overline{2}^{-1}$ existed then $\overline{2} \cdot \overline{4} = \overline{0} \Leftrightarrow \overline{2}^{-1} \cdot \overline{2} \cdot \overline{4} = \overline{0}$
$\Leftrightarrow \overline{4} = \overline{0}$ ※].

d) Want to show (W.T.S.) $(\mathbb{Z}/n\mathbb{Z} \setminus \{0\}, \times)$ group $\Leftrightarrow$ n prime.
Note: All conditions except inverses obviously hold.
$\Rightarrow$: use contrapositive.

Suppose n not prime. Then $n = ab$ for some $a, b \in \mathbb{Z}$.
Then $\overline{a} \cdot \overline{b} = \overline{n} = \overline{0}$, and so $\overline{a}$ (or $\overline{b}$) doesn't have an inverse (see c) for why.).

$\Leftarrow$: Suppose $n = p$ prime. Then $\forall\, 1 \leq a < p\ \exists\, x, y \in \mathbb{Z}$ s.t $xa + yp = 1$.
Mod p this gives $\overline{x} \cdot \overline{a} = \overline{1}$, so $\overline{a}^{-1} = \overline{x}$. So inverses exist.

2.3) $\mathcal{G} = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \in \mathbb{R}^{3\times 3} \right\}$.

Multiplication example : $\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$

$$= \begin{pmatrix} 1 & a+x & c+ay+z \\ 0 & 1 & b+y \\ 0 & 0 & 1 \end{pmatrix}.$$

Firstly, the result is not invariant under mapping $\{a,b,c\} \leftrightarrow (x,y,z)$ (i.e. swapping roles of a,b,c & x,y,z) so it's not abelian.

- closed: Yes, see above calculation
- identity: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.
- inverse: $\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -x & xy-z \\ 0 & 1 & -y \\ 0 & 0 & 1 \end{pmatrix} \in \mathcal{G}$
- associativity: $\mathcal{G} \subset \mathbb{R}^{3\times 3}$ and $(\mathbb{R}^{3\times3}, \cdot)$ is associative $\therefore$ $(\mathcal{G}, \cdot)$ is too. (It inherits it).

2.4 a) Not possible, $2 \neq 3$.

b) $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} = \left(1\begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix} + 1\begin{pmatrix} 3 \\ 6 \\ 9 \end{pmatrix},\ 1\begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix} + 1\begin{pmatrix} 2 \\ 5 \\ 8 \end{pmatrix},\ 1\begin{pmatrix} 2 \\ 5 \\ 8 \end{pmatrix} + 1\begin{pmatrix} 3 \\ 6 \\ 9 \end{pmatrix} \right)$

$$= \begin{pmatrix} 4 & 3 & 5 \\ 10 & 9 & 11 \\ 16 & 15 & 17 \end{pmatrix}$$

Combos of columns of LHS matrix.

c) $$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = \begin{pmatrix} 1\,(1\ \ 2\ \ 3) + 1\,(4\ \ 5\ \ 6) \\ \hline 1\,(4\ \ 5\ \ 6) + 1\,(7\ \ 8\ \ 9) \\ \hline 1\,(1\ \ 2\ \ 3) + 1\,(7\ \ 8\ \ 9) \end{pmatrix} = \begin{pmatrix} 5 & 7 & 9 \\ 11 & 13 & 15 \\ 8 & 10 & 12 \end{pmatrix}$$

combos of rows of RHS matrix.

d) $$\begin{pmatrix} 1 & 2 & 1 & 2 \\ 4 & 1 & -1 & -4 \end{pmatrix}\begin{pmatrix} 0 & 3 \\ 1 & -1 \\ 2 & 1 \\ 5 & 2 \end{pmatrix} = \begin{pmatrix} 14 & 6 \\ -21 & 2 \end{pmatrix}$$

e) $$\begin{pmatrix} 0 & 3 \\ 1 & -1 \\ 2 & 1 \\ 5 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 & 1 & 2 \\ 4 & 1 & -1 & -4 \end{pmatrix} = \begin{pmatrix} 12 & 3 & -3 & -12 \\ -3 & 1 & 2 & 6 \\ 6 & 5 & 1 & 0 \\ 13 & 12 & 3 & 2 \end{pmatrix}.$$

2.5 a) $$A = \begin{pmatrix} 1 & 1 & -1 & -1 \\ 2 & 5 & -7 & -5 \\ 2 & -1 & 1 & 3 \\ 5 & 2 & -4 & 2 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ -2 \\ 4 \\ 6 \end{pmatrix}.$$

Form augmented matrix:

$$(A|b) = \left(\begin{array}{cccc|c} 1 & 1 & -1 & -1 & 1 \\ 2 & 5 & -7 & -5 & -2 \\ 2 & -1 & 1 & 3 & 4 \\ 5 & 2 & -4 & 2 & 6 \end{array}\right)$$

$$\left(\begin{array}{cccc|c} 1 & 1 & -1 & -1 & 1 \\ 2 & 5 & -7 & -5 & -2 \\ 2 & -1 & 1 & 3 & 4 \\ 5 & 2 & -4 & 2 & 6 \end{array}\right) \xrightarrow[R4 \to R4-5R1]{R2 \to R2-2R1,\ R3 \to R3-2R1} \left(\begin{array}{cccc|c} 1 & 1 & -1 & -1 & 1 \\ 0 & 3 & -5 & -3 & -4 \\ 0 & -3 & 3 & 5 & 2 \\ 0 & -3 & 1 & 7 & 1 \end{array}\right)$$

$$\xrightarrow{R2 \to \frac{1}{3}R2} \left(\begin{array}{cccc|c} 1 & 1 & -1 & -1 & 1 \\ 0 & 1 & -\frac{5}{3} & -1 & -\frac{4}{3} \\ 0 & -3 & 3 & 5 & 2 \\ 0 & -3 & 1 & 7 & 1 \end{array}\right) \xrightarrow[R4 \to R4+3R2]{R3 \to R3+3R2} \left(\begin{array}{cccc|c} 1 & 1 & -1 & -1 & 1 \\ 0 & 1 & -\frac{5}{3} & -1 & -\frac{4}{3} \\ 0 & 0 & -2 & 2 & -2 \\ 0 & 0 & -4 & 4 & -3 \end{array}\right)$$

$$\xrightarrow{R4 \to R4-2R3} \left(\begin{array}{cccc|c} 1 & 1 & -1 & -1 & 1 \\ 0 & 1 & -\frac{5}{3} & -1 & -\frac{4}{3} \\ 0 & 0 & -2 & 2 & -2 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right).$$

bottom row is inconsistent. i.e cannot have $0x_1 + \ldots + 0x_4 = 1$.

No solutions.

b).

$$\left(\begin{array}{ccccc|c} 1 & -1 & 0 & 0 & 1 & 3 \\ 1 & 1 & 0 & -3 & 0 & 6 \\ 2 & -1 & 0 & 1 & -1 & 5 \\ -1 & 2 & 0 & -2 & -1 & -1 \end{array}\right) \xrightarrow[R3 \to R3-2R1,\ R4 \to R4+R1]{R2 \to R2-R1} \left(\begin{array}{ccccc|c} 1 & -1 & 0 & 0 & 1 & 3 \\ 0 & 2 & 0 & -3 & -1 & 3 \\ 0 & 1 & 0 & 1 & -3 & -1 \\ 0 & 1 & 0 & -2 & 0 & 2 \end{array}\right)$$

$$\xrightarrow{R2 \rightleftarrows R3} \left(\begin{array}{ccccc|c} 1 & -1 & 0 & 0 & 1 & 3 \\ 0 & 1 & 0 & 1 & -3 & -1 \\ 0 & 2 & 0 & -3 & -1 & 3 \\ 0 & 1 & 0 & -2 & 0 & 2 \end{array}\right) \xrightarrow[R4 \to R4-R2]{R1 \to R1+R2,\ R3 \to R3-2R2} \left(\begin{array}{ccccc|c} 1 & 0 & 0 & 1 & -2 & 2 \\ 0 & 1 & 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & -5 & 5 & 5 \\ 0 & 0 & 0 & -3 & 3 & 3 \end{array}\right)$$

$$\xrightarrow[\text{then } R4 \to R4+3R3,\ R1 \to R1-R3,\ R2 \to R2-R3]{R3 \to -\frac{1}{5}R3} \left(\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & -1 & 3 \\ 0 & 1 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right), \quad \text{Initial soln: } \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 0 \\ -1 \\ 0 \end{pmatrix}$$

"-1 trick": (solns $Ax=0$) $\lambda \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ -2 \\ 0 \\ -1 \\ -1 \end{pmatrix}$ $\Rightarrow$ soln. set

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 0 \\ -1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ -2 \\ 0 \\ -1 \\ -1 \end{pmatrix}$$

2.6) $(A|b) = \left(\begin{array}{cccccc|c} 0 & 1 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \end{array}\right)$

$$\xrightarrow[R3 \to -R3]{R3 \to R3 - R1 \text{ then}} \left(\begin{array}{cccccc|c} 0 & 1 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 1 \end{array}\right)$$

$$\xrightarrow[R2 \to R2 - R3]{R1 \to R1 - R3} \left(\begin{array}{cccccc|c} 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 1 & -1 & 1 \end{array}\right)$$

"-1 trick" : $\begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$

Initial soln: $x_2 = 1, \quad x_4 = -2, \quad x_5 = 1.$

Soln set: $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \xi \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ -1 \\ -1 \end{pmatrix}$

2.7) $\underline{x} \in \mathbb{R}^3$ s.t $Ax = 12x \iff \underline{x} \in \mathbb{R}^3$ s.t $(A - 12I)\underline{x} = 0.$

i.e $\begin{pmatrix} -6 & 4 & 3 \\ 6 & -12 & 9 \\ 0 & 8 & -12 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\sum x_i = 1$ corresponds to $\begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 1.$

Combining these:

Need solns $\underline{x}$ to $\left(\begin{array}{ccc|c} -6 & 4 & 3 & 0 \\ 6 & -12 & 9 & 0 \\ 0 & 8 & -12 & 0 \\ 1 & 1 & 1 & 1 \end{array}\right)$

$$\xrightarrow{R1 \leftrightarrows R4} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 6 & -12 & 9 & 0 \\ 0 & 8 & -12 & 0 \\ -6 & 4 & 3 & 0 \end{array}\right) \xrightarrow[R4 = R4+6R1]{R2 = R2-6R1} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -18 & 3 & -6 \\ 0 & 8 & -12 & 0 \\ 0 & 10 & 9 & 6 \end{array}\right)$$

$$\xrightarrow[\text{then } R2 \to -\frac{1}{18}R2]{R4 \to R4+R2+R3} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & -\frac{1}{6} & \frac{1}{3} \\ 0 & 8 & -12 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow[R3 = R3-8R2]{R1 = R1-R2} \left(\begin{array}{ccc|c} 1 & 0 & \frac{7}{6} & \frac{2}{3} \\ 0 & 1 & -\frac{1}{6} & \frac{1}{3} \\ 0 & 0 & -\frac{64}{6} & -\frac{8}{3} \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow[R1 \to R1 - \frac{7}{6}R3,\ R2 \to R2 + \frac{1}{6}R3]{R3 \to -\frac{6}{64}R3} \left(\begin{array}{ccc|c} 1 & 0 & 0 & \frac{3}{8} \\ 0 & 1 & 0 & \frac{3}{8} \\ 0 & 0 & 1 & \frac{1}{4} \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Soln: $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \frac{1}{8}\begin{pmatrix} 3 \\ 3 \\ 2 \end{pmatrix}$.

2.8 a) $\quad$ column 1 + column 3 $=$ 2 (colum 2)

$$\begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} + \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} = 2\begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}$$

(in each row, the middle entry is the average of the end entries.)

$\Rightarrow$ not invertible.

b) $(A \mid I_4) = \left(\begin{array}{cccc|cccc} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right)$

$$\xrightarrow[R4 = R4-R1]{R3 = R3-R1} \left(\begin{array}{cccc|cccc} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow[R4 = R4-R2]{R3 = R3-R2} \left(\begin{array}{cccc|cccc} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 & -1 & -1 & 0 & 1 \end{array}\right)$$

$$\xrightarrow[R3=-R3]{R3 \leftrightarrows R4 \text{ then}} \left(\begin{array}{cccc|cccc} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & -2 & 1 & -1 & -1 & 1 & 0 \end{array}\right)$$

$$\xrightarrow[R4=R4+2R3]{R1=R1-R3,\ R2=R2-R3} \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & -2 \end{array}\right)$$

so $A^{-1} = \begin{pmatrix} 0 & -1 & 0 & 1 \\ -1 & 0 & 0 & 1 \\ 1 & 1 & 0 & -1 \\ 1 & 1 & 1 & -2 \end{pmatrix}$

2.9 a) $A = \left\{ \lambda \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \mu^3 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} : \lambda, \mu \in \mathbb{R} \right\}.$

$f: \mathbb{R} \to \mathbb{R}$, $x \mapsto x^3$ bijective, set $\xi = \sqrt[3]{\mu}$, then

$A = \left\{ \lambda \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \xi \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} : \lambda, \xi \in \mathbb{R} \right\}.$

$\therefore$ A is a subspace.

This relies on $\mathbb{R}$ being closed under $\sqrt[3]{\ }$

If $\mathbb{R}$ was replaced by $\mathbb{Q}$, this would no longer be a subspace.

e.g. $(1)^3 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + (1)^3 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = 2 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$, but $\sqrt[3]{2} \notin \mathbb{Q}$.

b) $B = \left\{ \lambda^2 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} : \lambda \in \mathbb{R} \right\}.$

Issue: $f: \mathbb{R} \to \mathbb{R}$, $x \mapsto x^2$ isn't surjective (range is $\mathbb{R}_{\geq 0}$).

so, for example, $-2 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \notin B$

(since $\sqrt[2]{-2} \notin \mathbb{R}$).

c) $C = \left\{ \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} \in \mathbb{R}^3 : \begin{pmatrix} 1 & -2 & 3 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \gamma \right\}.$

$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ must be in C, so require $\gamma = 0$.

If $\gamma = 0$ this is a subspace since matrix multiplication (In particular, $\begin{pmatrix} 1 & -2 & 3 \end{pmatrix} : \mathbb{R}^3 \to \mathbb{R}$) is linear.

d) $D = \left\{ \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} \in \mathbb{R}^3 : \xi_2 \in \mathbb{Z} \right\}.$

Not a subspace, e.g $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \in D$, but $\frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \notin D$

since $\frac{1}{2} \notin \mathbb{Z}$.

Not closed under scalar multiplication

2.10 a) $\begin{pmatrix} | & | & | \\ x_1 & x_2 & x_3 \\ | & | & | \end{pmatrix} = \begin{pmatrix} 2 & 1 & 3 \\ -1 & 1 & -3 \\ 3 & -2 & 8 \end{pmatrix}$

rank is invariant under row operations.

$$\operatorname{rk}\begin{pmatrix} 2 & 1 & 3 \\ -1 & 1 & -3 \\ 3 & -2 & 8 \end{pmatrix} \xrightarrow[\substack{R2 \to R2 + R1 \\ R3 \to R3 - 3R1}]{R1 \to \frac{1}{2}R1} \operatorname{rk}\begin{pmatrix} 1 & \frac{1}{2} & \frac{3}{2} \\ 0 & \frac{3}{2} & -\frac{3}{2} \\ 0 & -\frac{7}{2} & \frac{7}{2} \end{pmatrix} = 2$$

(2nd & 3rd row are linearly dependent.)

b) Start with $x_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$.

$x_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}$ is non-zero in 2nd component (whereas x_3's 2nd component is 0).

$\therefore$ x_2, x_3 are linearly independent.

Both x_2 & x_3 have 0 in 3rd component, whereas x_1's 3rd component is 1 $\left(x_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}\right)$ so $x_1 \notin \operatorname{span}(x_2, x_3)$

$\therefore$ x_1, x_2, x_3 are linearly independent.

2.11) $a\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + b\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + c\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix} \Leftrightarrow \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & -1 \\ 1 & 3 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix}$

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 1 & 2 & -1 & -2 \\ 1 & 3 & 1 & 5 \end{array}\right) \xrightarrow[R3 \to R3 - R1]{R2 \to R2 - R1} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & 1 & -3 & -3 \\ 0 & 2 & -1 & 4 \end{array}\right)$$

$$\xrightarrow[R3 \to R3 - 2R2]{R1 \to R1 - R2} \left(\begin{array}{ccc|c} 1 & 0 & 5 & 4 \\ 0 & 1 & -3 & -3 \\ 0 & 0 & 5 & 10 \end{array}\right) \xrightarrow[\substack{R1 \to R1 - 5R3 \\ R2 \to R2 + 3R3}]{R3 = \frac{1}{5}R3} \left(\begin{array}{ccc|c} 1 & 0 & 0 & -6 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right), \quad \begin{array}{l} a = -6, \\ b = 3, \\ c = 2 \end{array}$$

so $\begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix} = -6\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + 3\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + 2\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$.

2.12) We load vectors into matrices as ~~rows~~ rows, as row operations do not change the row span.

$$A_1 = \begin{pmatrix} 1 & 1 & -3 & 1 \\ 2 & -1 & 0 & -1 \\ -1 & 1 & -1 & 1 \end{pmatrix} \qquad A_2 = \begin{pmatrix} -1 & -2 & 2 & 1 \\ 2 & -2 & 0 & 0 \\ -3 & 6 & -2 & -1 \end{pmatrix}$$

$$\xrightarrow[R3 \to R3+R1]{R2 \to R2-2R1} \begin{pmatrix} 1 & 1 & -3 & 1 \\ 0 & -3 & 6 & -3 \\ 0 & 2 & -4 & 2 \end{pmatrix} \qquad \xrightarrow[\substack{R2 \to R2-2R1 \\ R3 \to R3+3R1}]{R1 \to -R1} \begin{pmatrix} 1 & 2 & -2 & -1 \\ 0 & -6 & 4 & 2 \\ 0 & 12 & -8 & -4 \end{pmatrix}$$

$$\xrightarrow[\substack{R1 \to R1-R2 \\ R3 \to R3-2R2}]{R2 \to -\frac{1}{3}R2} \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \xrightarrow[\substack{R1 \to R1-2R2 \\ R3 \to R3-12R2}]{R2 \to -\frac{1}{6}R2} \begin{pmatrix} 1 & 0 & -\frac{2}{3} & -\frac{1}{3} \\ 0 & 1 & -\frac{2}{3} & -\frac{1}{3} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

so $U_1 = \text{span}\left\{\begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -2 \\ 1 \end{pmatrix}\right\}$, $U_2 = \text{span}\left\{\begin{pmatrix} 1 \\ 0 \\ -\frac{2}{3} \\ -\frac{1}{3} \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -\frac{2}{3} \\ -\frac{1}{3} \end{pmatrix}\right\}$.

If $v \in U_1 \cap U_2$ then $v = a\begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \\ -2 \\ 1 \end{pmatrix} = a\begin{pmatrix} 1 \\ 0 \\ -\frac{2}{3} \\ -\frac{1}{3} \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \\ -\frac{2}{3} \\ -\frac{1}{3} \end{pmatrix}$.

Key! 1^{st} & 2^{nd} component force 'a' & 'b' to be equal on both sides.
aka it's the same 'a', 'b' on both sides.

$$\Rightarrow a\begin{pmatrix} 0 \\ 0 \\ -\frac{1}{3} \\ \frac{1}{3} \end{pmatrix} + b\begin{pmatrix} 0 \\ 0 \\ -\frac{4}{3} \\ \frac{4}{3} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \iff a = -4b.$$

Thus $\dim(U_1 \cap U_2) = 1$. To find basis we can just choose a specific value for b. **Set** $b = 1$; then $a = -4$.

and $U_1 \cap U_2 = \text{span}\left\{\begin{pmatrix} -4 \\ 1 \\ 2 \\ 1 \end{pmatrix}\right\}$.

2.13 a) $\dim(U_1) = \text{nullity}(A_1)$.

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & -2 & -1 \\ 2 & 1 & 3 \\ 1 & 0 & 1 \end{pmatrix} \xrightarrow[\substack{R3 \to R3-2R1 \\ R4 \to R4-R1}]{R2 \to R2-R1} \begin{pmatrix} 1 & 0 & 1 \\ 0 & -2 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow[R3 \to R3+2R2]{R2 \leftrightarrows R3} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$\text{nullity}(A_1) = 3 - 2 = 1$.

$\dim(U_2) = \text{nullity}(A_2)$.

$$\begin{pmatrix} 3 & -3 & 0 \\ 1 & 2 & 3 \\ 7 & -5 & 2 \\ 3 & -1 & 2 \end{pmatrix} \xrightarrow[\substack{R2 \to R2-3R1 \\ R3 \to R3-7R1 \\ R4 \to R4-3R1}]{R1 \rightleftarrows R2} \begin{pmatrix} 1 & 2 & 3 \\ 0 & -9 & -9 \\ 0 & -19 & -19 \\ 0 & -7 & -7 \end{pmatrix}$$

$$\xrightarrow[\substack{R3 \to R3+19R2 \\ R4 \to R4+7R2 \\ R1 \to R1-2R2}]{R2 \to -\frac{1}{9}R2} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$\text{nullity}(A_2) = 3 - 2 = 1$.

b) Ignore bottom 2 zero rows.
Use "−1 trick"

$U_1 = \text{span}\left\{\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}\right\}$

$U_2 = \text{span}\left\{\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}\right\}$.

c) $U_1 \cap U_2 = \text{span}\left\{\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}\right\}$.

2.14a) As we want to keep the column space intact (that is, the space spanned by the columns), we apply row operations to A_1^T & A_2^T.

$\dim(U_1) = rk(A_1) = rk(A_1^T)$.

$$A_1^T = \begin{pmatrix} 1 & 1 & 2 & 1 \\ 0 & -2 & 1 & 0 \\ 1 & -1 & 3 & 1 \end{pmatrix}$$

$$\xrightarrow{R3 \to R3-R1} \begin{pmatrix} 1 & 1 & 2 & 1 \\ 0 & -2 & 1 & 0 \\ 0 & -2 & 1 & 0 \end{pmatrix} \xrightarrow[\substack{R2 \to -\frac{1}{2}R2 \\ R1 \to R1-R2}]{R3 \to R3-R2} \begin{pmatrix} 1 & 0 & 5/2 & 1 \\ 0 & 1 & -1/2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$rk(A_1^T) = 2 \Leftrightarrow \dim(U_1) = 2$.

$\dim(U_2) = rk(A_2) = rk(A_2^T)$

$$A_2^T = \begin{pmatrix} 3 & 1 & 7 & 3 \\ -3 & 2 & -5 & -1 \\ 0 & 3 & 2 & 2 \end{pmatrix}$$

$$\xrightarrow[R1 \to \frac{1}{3}R1]{R2 \to R2+R1} \begin{pmatrix} 1 & 1/3 & 7/3 & 1 \\ 0 & 3 & 2 & 2 \\ 0 & 3 & 2 & 2 \end{pmatrix} \xrightarrow[R2 \to \frac{1}{3}R2]{R3 \to R3-R2} \begin{pmatrix} 1 & 1/3 & 7/3 & 1 \\ 0 & 1 & 2/3 & 2/3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{R1 \to R1-\frac{1}{3}R2} \begin{pmatrix} 1 & 0 & 19/9 & 7/9 \\ 0 & 1 & 2/3 & 2/3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$rk(A_2^T) = 2 \Leftrightarrow \dim(U_2) = 2$.

b) $U_1 = span\left\{\begin{pmatrix}1\\0\\5/2\\1\end{pmatrix}, \begin{pmatrix}0\\1\\-1/2\\0\end{pmatrix}\right\}$, $U_2 = span\left\{\begin{pmatrix}1\\0\\19/9\\7/9\end{pmatrix}, \begin{pmatrix}0\\1\\2/3\\2/3\end{pmatrix}\right\}$.

c) if $v \in U_1 \cap U_2$ then $v = a\begin{pmatrix}1\\0\\5/2\\1\end{pmatrix} + b\begin{pmatrix}0\\1\\-1/2\\0\end{pmatrix} = a\begin{pmatrix}1\\0\\19/9\\7/9\end{pmatrix} + b\begin{pmatrix}0\\1\\2/3\\2/3\end{pmatrix}$.

(same a & b on both sides!)

$$\Leftrightarrow a\begin{pmatrix}0\\0\\7/18\\2/9\end{pmatrix} + b\begin{pmatrix}0\\0\\-7/6\\-2/3\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix} \Leftrightarrow \begin{pmatrix}7/18 & -7/6\\ 2/9 & -2/3\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix} = \underline{0}$$

$$\Leftrightarrow \begin{pmatrix}a\\b\end{pmatrix} \in nullspace\begin{pmatrix}7 & -21\\ 4 & -12\end{pmatrix}$$

By inspection, $nullspace\begin{pmatrix}7 & -21\\ 4 & -12\end{pmatrix} = span\left\{\begin{pmatrix}3\\1\end{pmatrix}\right\}$.

so $U_1 \cap U_2 = span\left\{3\begin{pmatrix}1\\0\\5/2\\1\end{pmatrix} + 1\begin{pmatrix}0\\1\\-1/2\\0\end{pmatrix}\right\} = span\left\{\begin{pmatrix}3\\1\\7\\3\end{pmatrix}\right\}$.

2.15 a) $F = \left\{\begin{pmatrix}x\\y\\z\end{pmatrix} \in \mathbb{R}^3 : (1 \ 1 \ -1)\begin{pmatrix}x\\y\\z\end{pmatrix} = \underline{0}\right\}$.

So $F = nullspace\{(1 \ 1 \ -1)\}$, which is a subspace of $\mathbb{R}^3$.

$$G = \left\{\begin{pmatrix}a-b\\a+b\\a-3b\end{pmatrix} : a,b \in \mathbb{R}\right\} = \left\{a\begin{pmatrix}1\\1\\1\end{pmatrix} + b\begin{pmatrix}-1\\1\\-3\end{pmatrix} : a,b \in \mathbb{R}\right\} = span\left\{\begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}-1\\1\\-3\end{pmatrix}\right\}.$$

So G is the span of 2 vectors $\therefore$ it's a subspace.

b) Need $a,b \in \mathbb{R}$ s.t $\underbrace{(a-b)}_{'x'} + \underbrace{(a+b)}_{'y'} + \underbrace{(a-3b)}_{'z'} = 0$.

$$\Leftrightarrow 3a - 3b = 0 \Leftrightarrow a = b.$$

Thus $\dim(F \cap G) = 1$, $F \cap G = span\left\{\begin{pmatrix}0\\2\\-2\end{pmatrix}\right\}$. setting $a=b=1$.

c) basis for F (using "-1 trick").

$$F = \mathrm{span}\left\{\begin{pmatrix}1\\-1\\0\end{pmatrix},\begin{pmatrix}1\\0\\-1\end{pmatrix}\right\} = \mathrm{span}\left\{\begin{pmatrix}1\\0\\-1\end{pmatrix},\begin{pmatrix}0\\1\\-1\end{pmatrix}\right\}.$$

2nd basis vector

obtained by 2nd − 1st basis vector.

basis for G (by definition):

$$G = \mathrm{span}\left\{\begin{pmatrix}1\\1\\1\end{pmatrix},\begin{pmatrix}-1\\1\\-3\end{pmatrix}\right\}.$$

Get basis for G into "canonical form" i.e RREF.

$$\begin{pmatrix}1&1&1\\-1&1&-3\end{pmatrix}\xrightarrow{R2\to R2+R1}\begin{pmatrix}1&1&1\\0&2&-2\end{pmatrix}\xrightarrow[R1\to R1-R2]{R2\to\frac{1}{2}R2}\begin{pmatrix}1&0&2\\0&1&-1\end{pmatrix}$$

If $v\in F\cap G$ then $v = a\begin{pmatrix}1\\0\\-1\end{pmatrix}+b\begin{pmatrix}0\\1\\-1\end{pmatrix} = a\begin{pmatrix}1\\0\\2\end{pmatrix}+b\begin{pmatrix}0\\1\\-1\end{pmatrix}$

(same a,b on both sides).

$$\Leftrightarrow a\begin{pmatrix}0\\0\\-3\end{pmatrix}+b\begin{pmatrix}0\\0\\0\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix} \Leftrightarrow a=0.$$

Setting $b=1$, say, gives $v=\begin{pmatrix}0\\1\\-1\end{pmatrix}\in F\cap G$.

Thus $F\cap G = \mathrm{span}\left\{\begin{pmatrix}0\\1\\-1\end{pmatrix}\right\}$.

This is (of course) the same basis vector found in part b), up to scaling.

2.16) linear map $f: V \to W$ satisfies $f(v+u) = f(v) + f(u)$ $\lambda \in \mathbb{R}$, $v, u \in V$.
$f(\lambda v) = \lambda f(v)$

a) $\Phi(f+g) = \int_a^b (f+g)(x)\, dx$
$= \int_a^b (f(x) + g(x))\, dx = \int_a^b f(x)\, dx + \int_a^b g(x)\, dx = \Phi(f) + \Phi(g)$

$\Phi(\lambda f) = \int_a^b (\lambda f)(x)\, dx = \int_a^b \lambda f(x)\, dx = \lambda \int_a^b f(x)\, dx = \lambda \Phi(f)$.

b) $\Phi(f+g) = (f+g)' = f' + g' = \Phi(f) + \Phi(g)$
$\Phi(\lambda f) = (\lambda f)' = \lambda f' = \lambda \Phi(f)$.

c) No! e.g $\Phi(0) = \cos(0) = 1$.
(+ linear maps must send ~~take~~ $\underline{0} \mapsto \underline{0}$; $f(\underline{0}) = f(0 \cdot \underline{0}) = 0 f(\underline{0}) = \underline{0}$.)

d) Yes, matrix multiplication (on a vector) is linear.

e) Yes, matrix multiplication (on a vector) is linear.

2.17) $\Phi\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} 3x + 2y + z \\ x + y + z \\ x - 3y \\ 2x + 3y + z \end{pmatrix} = \begin{pmatrix} 3 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & -3 & 0 \\ 2 & 3 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

so $A_\Phi = \begin{pmatrix} 3 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & -3 & 0 \\ 2 & 3 & 1 \end{pmatrix}$.

To find $rk(A_\Phi)$, apply row operations.

$$\begin{pmatrix} 3 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & -3 & 0 \\ 2 & 3 & 1 \end{pmatrix} \xrightarrow[\substack{R2 \to R2 - 3R1 \\ R3 \to R3 - R1 \\ R4 \to R4 - 2R1}]{R2 \rightleftarrows R1} \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & -2 \\ 0 & -4 & -1 \\ 0 & 1 & -1 \end{pmatrix} \xrightarrow[\substack{R3 \to R3 + 4R2 \\ R4 \to R4 - R2}]{R2 \to -R2} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 7 \\ 0 & 0 & -3 \end{pmatrix} \xrightarrow[R4 \to R4 + 3R3]{R3 \to \frac{1}{7} R3} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

- $rk(A_\Phi) = \text{row rank}(A_\Phi) = \text{column rank}(A_\Phi) = 3$.
- $\dim(\mathrm{Im}(\Phi)) = rk(A_\Phi) = 3$ $\quad \mathrm{Im}(\Phi) = \mathrm{span}\left\{\begin{pmatrix} 3 \\ 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ -3 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}\right\}$.

$\dim(\ker(\Phi)) = \dim(\mathbb{R}^3) - rk(A_\Phi) = 3 - 3 = 0$. $\quad \ker(\Phi) = \{\underline{0}\}$.

2.18) Need to show i) $\ker(f) = \ker(g \circ f)$

ii) $\mathrm{Im}(g) = \mathrm{Im}(g \circ f)$

iii) $\ker(f) \cap \mathrm{im}(g) = \{\underline{0}\}$.

i) If $\underline{v} \in \ker(f)$ then $(g \circ f)(\underline{v}) = g(f(\underline{v})) = g(\underline{0}) = \underline{0}$ so $\underline{v} \in \ker(g \circ f)$.

If $\underline{v} \in \ker(g \circ f)$, i.e $(g \circ f)(\underline{v}) = \underline{0}$ then $(f \circ g \circ f)(\underline{v}) = f(\underline{0}) = \underline{0}$.

$\Leftrightarrow ((f \circ g) \circ f)(\underline{v}) = \underline{0} \Leftrightarrow (id_E \circ f)(\underline{v}) = 0 \Leftrightarrow f(\underline{v}) = 0 \Rightarrow \underline{v} \in \ker(f)$.

ii) Suppose $w \in \mathrm{Im}(g)$ i.e $\exists\, v \in E$ s.t $w = g(v)$. Then $w = (g \circ id_E)(v)$

$\Leftrightarrow w = (g \circ f \circ g)(v) \Leftrightarrow w = (g \circ f)(g(v)) \Rightarrow w \in \mathrm{Im}(g \circ f)$.

If $w \in \mathrm{Im}(g \circ f)$ then $\exists\, v \in E$ s.t $w = (g \circ f)(v) \Rightarrow w = g(f(v)) \Rightarrow w \in \mathrm{Im}(g)$.

iii) Let $\underline{v} \in \ker(f) \cap \mathrm{im}(g)$. Then $v = g(w)$ for some $w \in E$

and $f(\underline{v}) = \underline{0}$. Combining these gives $f(g(w)) = (f \circ g)(w) = \underline{0}$.

Since $f \circ g = id_E$, this implies $w = \underline{0}$ & so $v = g(\underline{0}) = \underline{0}$.

2.19) $A_\Phi : \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 1 & 1 & 1 \end{pmatrix} : \mathbb{R}^3_{\mathcal{E}} \to \mathbb{R}^3_{\mathcal{E}}$, $\mathcal{E} = \{e_1, e_2, e_3\}$ is the std basis.

a) To find basis of image we apply row operations to A_Φ^T.

$$A_\Phi^T = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R2 \to R2 - R1} \begin{pmatrix} 1 & 1 & 1 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow[\substack{R1 \to R1 - R2 \\ R1 \to R1 - R3}]{R2 \to -\frac{1}{2}R2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

so $\mathrm{Im}(\Phi) = \mathbb{R}^3$, $\ker(\Phi) = \{\underline{0}\}$.

b) Change of basis matrix from $\mathbb{R}^3_\beta$ to $\mathbb{R}^3_\mathcal{E}$ is

$$P = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 1 & 0 \end{pmatrix}.$$

Diagram :

$$\begin{tikzcd} \mathbb{R}^3_\mathcal{E} \arrow[r, "A_\Phi"] & \mathbb{R}^3_\mathcal{E} \arrow[d, bend left=60, dashed, "P^{-1}"] \\ \mathbb{R}^3_\beta \arrow[u, "P"] \arrow[r, "\widehat{A_\Phi}"'] & \mathbb{R}^3_\beta \arrow[u, "P"'] \end{tikzcd}$$

so $\widehat{A_\Phi} = P^{-1} A_\Phi P$.

Calculate P^{-1}:

$$(P \mid I_3) = \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \xrightarrow[R3 \to R3 - R1]{R2 \to R2 - R1} \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & -1 & -1 & 0 & 1 \end{array}\right)$$

$$\xrightarrow[R3 \to -R3]{R1 \to R1 - R2} \left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 2 & -1 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array}\right) \xrightarrow[R2 \to R2 + R3]{R1 \to R1 - 2R3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -1 & 2 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array}\right)$$

so $P^{-1} = \begin{pmatrix} 0 & -1 & 2 \\ 0 & 1 & -1 \\ 1 & 0 & -1 \end{pmatrix}$.

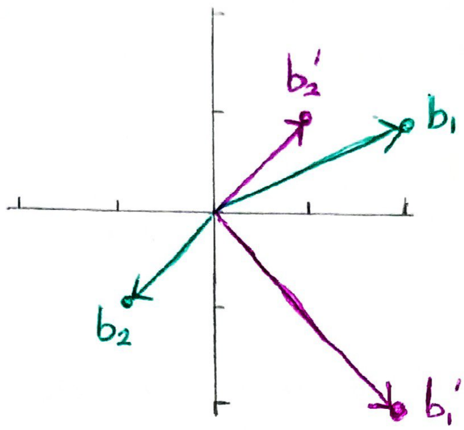
$$\widehat{A_\Phi} = P^{-1} A_\Phi P = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -1 & 2 \\ 0 & 1 & -1 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 2 & 3 & 1 \\ 0 & -1 & 1 \\ 3 & 4 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 6 & 9 & 1 \\ -3 & -5 & 0 \\ -1 & -1 & 0 \end{pmatrix}$$

2.20) $\mathcal{B} = (b_1, b_2) = \left(\begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \end{pmatrix} \right), \quad \mathcal{B}' = (b_1', b_2') = \left(\begin{pmatrix} 2 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right).$

a) both $\mathcal{B}$ & $\mathcal{B}'$ contain 2 non-zero vectors which are not colinear (i.e multiplies of eachother). Thus, they are linearly independent and therefore a basis.



b) Let $P_B = \begin{pmatrix} 2 & -1 \\ 1 & -1 \end{pmatrix}$ be change of basis matrix from B to $\mathcal{E} = \{e_1, e_2\}$ std basis.

Let $P_{B'} = \begin{pmatrix} 2 & 1 \\ -2 & 1 \end{pmatrix}$ be change of basis matrix from B' to $\mathcal{E}$.

$$\begin{tikzcd} \mathbb{R}^2_B \arrow[r, "P_B"] & \mathbb{R}^2_{\mathcal{E}} & \mathbb{R}^2_{B'} \arrow[l, "P_{B'}"'] \arrow[ll, dashed, bend left, "P_1"] \end{tikzcd}$$

P_1 (change of basis from B' to B).

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

$$P_1 = P_B^{-1} P_{B'} = \begin{pmatrix} 2 & -1 \\ 1 & -1 \end{pmatrix}^{-1} \begin{pmatrix} 2 & 1 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 0 \\ 6 & -1 \end{pmatrix}$$

Check: Is $b_1' = 4b_1 + 6b_2$ ✓

Is $b_2' = -b_2$ ✓.

c i) Define $A = \begin{pmatrix} | & | & | \\ c_1 & c_2 & c_3 \\ | & | & | \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 2 & -1 & 0 \\ -1 & 2 & -1 \end{pmatrix}$.

Then (c_1, c_2, c_3) basis $\Leftrightarrow \det(A) \neq 0 \Leftrightarrow rk(A) = 3$.

Adding 1 row to another doesn't change determinant

$$\det \begin{pmatrix} 1 & 0 & 1 \\ 2 & -1 & 0 \\ -1 & 2 & -1 \end{pmatrix} \underset{}{\overset{R3 \to R3 + R1}{=\!=}} \det \begin{pmatrix} 1 & 0 & 1 \\ 2 & -1 & 0 \\ 0 & 2 & 0 \end{pmatrix}$$

$$\underset{\text{third column}}{\overset{\text{expand along}}{=\!=}} 1 \det \begin{pmatrix} 2 & -1 \\ 0 & 2 \end{pmatrix} = 1(4) = 4.$$

ii) Changing base to the standard basis is easy. Columns of change of basis matrix are the basis vectors.

$$P_2 = \begin{pmatrix} 1 & 0 & 1 \\ 2 & -1 & 0 \\ -1 & 2 & -1 \end{pmatrix}.$$

d) $$\Phi(b_1) = \tfrac{1}{2}\left(\Phi(b_1 + b_2) + \Phi(b_1 - b_2)\right)$$
$$= \tfrac{1}{2}\left(c_2 + c_3 + 2c_1 - c_2 + 3c_3\right) = c_1 + 2c_3$$

$$\Phi(b_2) = \tfrac{1}{2}\left(\Phi(b_1 + b_2) - \Phi(b_1 - b_2)\right)$$
$$= \tfrac{1}{2}\left(c_2 + c_3 - 2c_1 + c_2 - 3c_3\right) = \frac{-2c_1 + 2c_2 - 2c_3}{2} = -c_1 + c_2 - c_3$$

so $A_\Phi : \mathbb{R}^2_B \longrightarrow \mathbb{R}^3_C$ is $A_\Phi = \begin{pmatrix} 1 & -1 \\ 0 & 1 \\ 2 & -1 \end{pmatrix}$.

e)

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\mathbb{R}^2_{\mathcal{B}'} \arrow[r, "A'"] \arrow[d, "\begin{pmatrix}4&0\\6&-1\end{pmatrix} = P_1"'] \arrow[r, dashed, bend right=60] & \mathbb{R}^3_{\mathcal{C}'} \\
\mathbb{R}^2_{\mathcal{B}} \arrow[r, "A_{\Phi} = \begin{pmatrix}1&-1\\0&1\\2&-1\end{pmatrix}"'] & \mathbb{R}^3_{\mathcal{C}} \arrow[u, "P_2 = \begin{pmatrix}1&0&1\\2&-1&0\\-1&2&-1\end{pmatrix}"']
\end{tikzcd}$$

$$A' = P_2 A_{\Phi} P_1$$

$$= \begin{pmatrix}1&0&1\\2&-1&0\\-1&2&-1\end{pmatrix}\begin{pmatrix}1&-1\\0&1\\2&-1\end{pmatrix}\begin{pmatrix}4&0\\6&-1\end{pmatrix} = \begin{pmatrix}1&0&1\\2&-1&0\\-1&2&-1\end{pmatrix}\begin{pmatrix}-2&1\\6&-1\\2&1\end{pmatrix}$$

$$= \begin{pmatrix}0&2\\-10&3\\12&-4\end{pmatrix}$$

f i) $x = 2b_1' + 3b_2' = \begin{pmatrix}2\\3\end{pmatrix}_{\mathcal{B}'} = \left[\begin{pmatrix}4&0\\6&-1\end{pmatrix}\begin{pmatrix}2\\3\end{pmatrix}\right]_{\mathcal{B}} = \begin{pmatrix}8\\9\end{pmatrix}_{\mathcal{B}} = 8b_1 + 9b_2.$

ii) $\Phi(x)_{\mathcal{C}} = \left[\begin{pmatrix}1&-1\\0&1\\2&-1\end{pmatrix}\begin{pmatrix}8\\9\end{pmatrix}_{\mathcal{B}}\right]_{\mathcal{C}} = \begin{pmatrix}-1\\9\\7\end{pmatrix}_{\mathcal{C}} = -c_1 + 9c_2 + 7c_3.$

iii) $\Phi(x)_{\mathcal{C}'} = \left[\begin{pmatrix}1&0&1\\2&-1&0\\-1&2&-1\end{pmatrix}\Phi(x)_{\mathcal{C}}\right]_{\mathcal{C}'} = \left[\begin{pmatrix}1&0&1\\2&-1&0\\-1&2&-1\end{pmatrix}\begin{pmatrix}-1\\9\\7\end{pmatrix}_{\mathcal{C}}\right]_{\mathcal{C}'} = \begin{pmatrix}6\\-11\\12\end{pmatrix}_{\mathcal{C}'}$

iv) $\Phi(x)_{\mathcal{C}'} = \left(A' x_{\mathcal{B}'}\right)_{\mathcal{C}'} = \left[\begin{pmatrix}0&2\\-10&3\\12&-4\end{pmatrix}\begin{pmatrix}2\\3\end{pmatrix}_{\mathcal{B}'}\right]_{\mathcal{C}'} = \begin{pmatrix}6\\-11\\12\end{pmatrix}_{\mathcal{C}'}.$