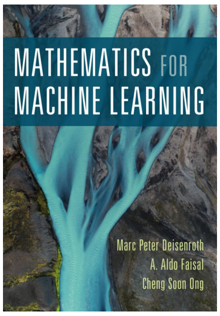
Mathematics for Machine Learning (Deisenroth, Faisal, Ong)

Chapter 3: Analytic Geometry

Full solutions to exercises 3.1 - 3.10



CONTENTS (clickable links)

3.1) $\langle \underline{x}, \underline{y} \rangle := x_1 y_1 - x_1 y_2 - x_2 y_1 + 2 x_2 y_2$

$$= \underline{x}^T \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \underline{y}$$

- symmetric: $\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$ is symmetric ✓

- bilinear: $\underline{x}^T \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \underline{y}$ is bilinear in $\underline{x}$ & $\underline{y}$ (since it's just matrix multiplication.) ✓.

- positive definite: $\langle \underline{x}, \underline{x} \rangle = x_1^2 - 2x_1 x_2 + 2x_2^2$

$$= (x_1 - x_2)^2 + x_2^2$$

This is always > 0 for $(x_1, x_2) \neq (0,0)$.

When $\underline{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ then $\langle \underline{x}, \underline{x} \rangle = 0^2 + 0^2 = 0$ ✓.

3.2) No, it's not symmetric

e.g $\langle e_1, e_2 \rangle = (1, 0) \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0$

$\langle e_2, e_1 \rangle = (0, 1) \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1$.

Thus $\langle e_1, e_2 \rangle \neq \langle e_2, e_1 \rangle$.

3.3) $d(\underline{x}, \underline{y}) = \sqrt{\langle \underline{x} - \underline{y}, \underline{x} - \underline{y} \rangle}$.

$\underline{x} - \underline{y} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} - \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 3 \end{pmatrix}$.

a) $d(\underline{x},\underline{y}) = \sqrt{(2\ 3\ 3)\begin{pmatrix}2\\3\\3\end{pmatrix}}$

$= \sqrt{4+9+9} = \sqrt{22}.$

b) $d(\underline{x},\underline{y}) = \sqrt{(2\ 3\ 3)\begin{pmatrix}2 & 1 & 0\\1 & 3 & -1\\0 & -1 & 2\end{pmatrix}\begin{pmatrix}2\\3\\3\end{pmatrix}}$

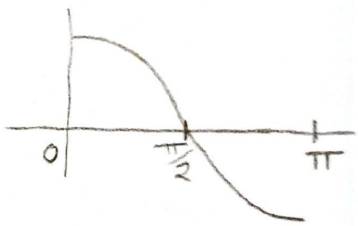
$= \sqrt{(7\ 8\ 3)\begin{pmatrix}2\\3\\3\end{pmatrix}} = \sqrt{14+24+9} = \sqrt{47}$

3.4 $\quad$ Angle: $\cos(\theta_{x,y}) = \dfrac{\langle x,y\rangle}{\sqrt{\langle x,x\rangle}\sqrt{\langle y,y\rangle}}$, $x = \begin{pmatrix}1\\2\end{pmatrix}$, $y = \begin{pmatrix}-1\\-1\end{pmatrix}$.

a) $\langle x,y\rangle = x^T y = (1\ 2)\begin{pmatrix}-1\\-1\end{pmatrix} = -3$

$\sqrt{\langle x,x\rangle} = \sqrt{1^2+2^2} = \sqrt{5}$

$\sqrt{\langle y,y\rangle} = \sqrt{(-1)^2+(-1)^2} = \sqrt{2}$



$\cos(\theta_{x,y}) = \dfrac{-3}{\sqrt{10}} \Rightarrow \theta_{x,y} = \arccos\left(\dfrac{-3}{\sqrt{10}}\right). \approx 2.82$ rads

$\approx 161.6°.$

b) $\langle x,y\rangle = x^T B y = (1\ 2)\begin{pmatrix}2 & 1\\1 & 3\end{pmatrix}\begin{pmatrix}-1\\-1\end{pmatrix} = (1\ 2)\begin{pmatrix}-3\\-4\end{pmatrix} = -11$

$\sqrt{\langle x,x\rangle} = \sqrt{(1\ 2)\begin{pmatrix}2 & 1\\1 & 3\end{pmatrix}\begin{pmatrix}1\\2\end{pmatrix}} = \sqrt{(1\ 2)\begin{pmatrix}4\\7\end{pmatrix}} = \sqrt{18}$

$\sqrt{\langle y,y\rangle} = \sqrt{(-1\ -1)\begin{pmatrix}2 & 1\\1 & 3\end{pmatrix}\begin{pmatrix}-1\\-1\end{pmatrix}} = \sqrt{(-1\ -1)\begin{pmatrix}-3\\-4\end{pmatrix}} = \sqrt{7}$

$\cos(\theta_{x,y}) = \dfrac{-11}{\sqrt{18}\sqrt{7}} \Rightarrow \theta_{x,y} = \arccos\left(\dfrac{-11}{\sqrt{18}\sqrt{7}}\right) \approx 2.94$ rads

$\approx 168.5°.$

3.5 a) First, find a basis for U.

(vectors are the rows of the matrix since row operations preserve row space.).

$$\begin{pmatrix} 0 & -1 & 2 & 0 & 2 \\ 1 & -3 & 1 & -1 & 2 \\ -3 & 4 & 1 & 2 & 1 \\ -1 & -3 & 5 & 0 & 7 \end{pmatrix} \xrightarrow[\substack{R3 \to R3+3R1 \\ R4 \to R4+R1}]{R1 \rightleftarrows R2} \begin{pmatrix} 1 & -3 & 1 & -1 & 2 \\ 0 & -1 & 2 & 0 & 2 \\ 0 & -5 & 4 & -1 & 7 \\ 0 & -6 & 6 & -1 & 9 \end{pmatrix}$$

$$\xrightarrow[\text{then } R2 \to -R2]{R4 \to R4-R3-R2} \begin{pmatrix} 1 & -3 & 1 & -1 & 2 \\ 0 & 1 & -2 & 0 & -2 \\ 0 & -5 & 4 & -1 & 7 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{R3 \to R3+5R2} \begin{pmatrix} 1 & -3 & 1 & -1 & 2 \\ 0 & 1 & -2 & 0 & -2 \\ 0 & 0 & -6 & -1 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

So $\left\{ \begin{pmatrix} 1 \\ -3 \\ 1 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -2 \\ 0 \\ -2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1/6 \\ 1/2 \end{pmatrix} \right\}$ is a basis of U

Want to make this basis into an orthogonal one to make calculation of $B(B^TB)^{-1}B^T$ easier.

(basis orthogonal wrt dot product. $\Leftrightarrow$ B^TB diagonal.)

Even better; basis orthonormal $\Leftrightarrow$ B^TB is identity matrix.

Let b_1, b_2, b_3 be elements of basis above.

$$\|b_1\| = \sqrt{b_1^T b_1} = \sqrt{1^2 + 3^2 + 1^2 + 1^2 + 2^2} = \sqrt{16} = 4.$$

$$u_1 := \frac{1}{4}\begin{pmatrix} 1 \\ -3 \\ 1 \\ -1 \\ 2 \end{pmatrix}.$$

$v_2 := b_2 - \langle b_2, u_1 \rangle u_1$ (subtract projection to u_1 from b_2).

$$= \begin{pmatrix} 0 \\ 1 \\ -2 \\ 0 \\ -2 \end{pmatrix} - \frac{1}{4}(-9)\frac{1}{4}\begin{pmatrix} 1 \\ -3 \\ 1 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 9/16 \\ -11/16 \\ -23/16 \\ -9/16 \\ -7/8 \end{pmatrix} = \frac{1}{16}\begin{pmatrix} 9 \\ -11 \\ -23 \\ -9 \\ -14 \end{pmatrix}$$

$$\|v_2\| = \frac{1}{16}\sqrt{81+121+23^2+81+14^2} = \frac{1}{16}\sqrt{1008} = \frac{12}{16}\sqrt{7} = \frac{3}{4}\sqrt{7}$$

$$u_2 := \frac{4}{3\sqrt{7}} \cdot \frac{1}{16}\begin{pmatrix} 9 \\ -11 \\ -23 \\ -9 \\ -14 \end{pmatrix} = \frac{1}{12\sqrt{7}}\begin{pmatrix} 9 \\ -11 \\ -23 \\ -9 \\ -14 \end{pmatrix}.$$

$$v_3 := b_3 - \langle b_3, u_1\rangle u_1 - \langle b_3, u_2\rangle u_2 \qquad \left(\begin{array}{l}\text{subtract projections to} \\ u_1 \text{ \& } u_2 \text{ from } b_3.\end{array}\right).$$

$$= \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1/6 \\ 1/2 \end{pmatrix} - \frac{1}{4}\left(\frac{11}{6}\right)\frac{1}{4}\begin{pmatrix} 1 \\ -3 \\ 1 \\ -1 \\ 2 \end{pmatrix} - \frac{1}{12\sqrt{7}}(-31.5)\frac{1}{12\sqrt{7}}\begin{pmatrix} 9 \\ -11 \\ -23 \\ -9 \\ -14 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1/6 \\ 1/2 \end{pmatrix} + \begin{pmatrix} -11/96 \\ 33/96 \\ -11/96 \\ +11/96 \\ -22/96 \end{pmatrix} + \begin{pmatrix} 9/32 \\ -11/32 \\ -23/32 \\ -9/32 \\ -14/32 \end{pmatrix} = \begin{pmatrix} 1/6 \\ 0 \\ 1/6 \\ 0 \\ -1/6 \end{pmatrix}.$$

$$\|v_3\| = \frac{1}{6}\sqrt{1^2+1^2+1^2} = \frac{\sqrt{3}}{6}.$$

$$u_3 := \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}$$

So $\left\{ \frac{1}{4}\begin{pmatrix} 1 \\ -3 \\ 1 \\ -1 \\ 2 \end{pmatrix}, \frac{1}{12\sqrt{7}}\begin{pmatrix} 9 \\ -11 \\ -23 \\ -9 \\ -14 \end{pmatrix}, \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} \right\}$ is O.N.B of U.

Let $B = \begin{pmatrix} | & | & | \\ u_1 & u_2 & u_3 \\ | & | & | \end{pmatrix} = \begin{pmatrix} 1/4 & 9/12\sqrt{7} & 1/\sqrt{3} \\ -3/4 & -11/12\sqrt{7} & 0 \\ 1/4 & -23/12\sqrt{7} & 1/\sqrt{3} \\ -1/4 & -9/12\sqrt{7} & 0 \\ 2/4 & -14/12\sqrt{7} & -1/\sqrt{3} \end{pmatrix}.$

Then $B(B^TB)^{-1}B^T = BB^T$ (since $B^TB = \mathrm{Id}$, by construction of u_1, u_2, u_3 (ie. they are ONB)).

$$BB^T = \begin{pmatrix} 1/4 & 9/12\sqrt{7} & 1/\sqrt{3} \\ -3/4 & -11/12\sqrt{7} & 0 \\ 1/4 & -23/12\sqrt{7} & 1/\sqrt{3} \\ -1/4 & -9/12\sqrt{7} & 0 \\ 2/4 & -14/12\sqrt{7} & -1/\sqrt{3} \end{pmatrix}\begin{pmatrix} 1/4 & -3/4 & 1/4 & -1/4 & 2/4 \\ 9/12\sqrt{7} & -11/12\sqrt{7} & \frac{-23}{12\sqrt{7}} & \frac{-9}{12\sqrt{7}} & \frac{-14}{12\sqrt{7}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{3}} & 0 & -\frac{1}{\sqrt{3}} \end{pmatrix}$$

$$BB^T = \begin{pmatrix} \frac{10}{21} & -\frac{2}{7} & \frac{4}{21} & -\frac{1}{7} & -\frac{1}{3} \\ -\frac{2}{7} & \frac{43}{63} & \frac{4}{63} & \frac{2}{7} & -\frac{2}{9} \\ \frac{4}{21} & \frac{4}{63} & \frac{58}{63} & \frac{1}{7} & \frac{1}{9} \\ -\frac{1}{7} & \frac{2}{7} & \frac{1}{7} & \frac{1}{7} & 0 \\ -\frac{1}{3} & -\frac{2}{9} & \frac{1}{9} & 0 & \frac{7}{9} \end{pmatrix}$$

$$\pi_u(\underline{x}) = BB^T\underline{x} = \begin{pmatrix} \frac{10}{21} & -\frac{2}{7} & \frac{4}{21} & -\frac{1}{7} & -\frac{1}{3} \\ -\frac{2}{7} & \frac{43}{63} & \frac{4}{63} & \frac{2}{7} & -\frac{2}{9} \\ \frac{4}{21} & \frac{4}{63} & \frac{58}{63} & \frac{1}{7} & \frac{1}{9} \\ -\frac{1}{7} & \frac{2}{7} & \frac{1}{7} & \frac{1}{7} & 0 \\ -\frac{1}{3} & -\frac{2}{9} & \frac{1}{9} & 0 & \frac{7}{9} \end{pmatrix} \begin{pmatrix} -1 \\ -9 \\ -1 \\ 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -5 \\ -1 \\ -2 \\ 3 \end{pmatrix}$$

b) $$d(\underline{x}, u) = d(\underline{x}, \pi_u(\underline{x}))$$

$$= d\left(\begin{pmatrix} -1 \\ -9 \\ -1 \\ 4 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -5 \\ -1 \\ -2 \\ 3 \end{pmatrix}\right).$$

$$= \sqrt{(2,-4,0,6,-2)\begin{pmatrix} 2 \\ -4 \\ 0 \\ 6 \\ -2 \end{pmatrix}} = \sqrt{4+16+36+4} = \sqrt{60} = 2\sqrt{15}$$

3.6) Formula for $\pi_u(\underline{x})$ for general inner product (given by A).

Let $U = \{u_1, .., u_k\}$ be a basis. Let $B = \begin{pmatrix} | & & | \\ u_1 & \cdots & u_k \\ | & & | \end{pmatrix}$

$$\langle u_i, x - \pi_u(x) \rangle = 0 \quad \forall i$$

$$\Leftrightarrow u_i^T A(x - \pi_u(x)) = 0 \quad \forall i$$

$$\Leftrightarrow B^T A(x - \pi_u(x)) = 0.$$

Let $\Pi_u(x) = B\lambda_u(x)$ (i.e. $\lambda_u(x)$ are co-ords of $\Pi_u(x)$ wrt $\{u_1, \dots, u_k\}$.)

Then $B^T A(\underline{x} - B\lambda_u(x)) = 0$

$\Leftrightarrow \quad B^T A\underline{x} = B^T A B \lambda_u(x)$

$\Leftrightarrow \quad \lambda_u(x) = (B^T A B)^{-1} B^T A \underline{x}$.

(Note: $B^T A B$ is invertible; follows from pos. definiteness of the inner product)

$\Rightarrow \quad \Pi_u(x) = B(B^T A B)^{-1} B^T A \underline{x}$.

a) $B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$, $A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$.

$$B^T A B = \begin{pmatrix} \langle e_1, e_1 \rangle & \langle e_1, e_3 \rangle \\ \langle e_3, e_1 \rangle & \langle e_3, e_3 \rangle \end{pmatrix} = \begin{pmatrix} A_{11} & A_{13} \\ A_{31} & A_{33} \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.$$

so $(B^T A B)^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

$$B^T A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 0 \\ 0 & -1 & 2 \end{pmatrix}.$$

$$\Pi_u(e_2) = B(B^T A B)^{-1} B^T A \underline{e}_2$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 & 0 \\ 0 & -1 & 2 \end{pmatrix} \underline{e}_2 = \begin{pmatrix} 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 1 \end{pmatrix} \underline{e}_2 = \begin{pmatrix} \frac{1}{2} \\ 0 \\ -\frac{1}{2} \end{pmatrix}.$$

<u>Check orthogonality</u>: (sanity check.)

$\Pi_u(\underline{e}_2) - \underline{e}_2 = \begin{pmatrix} \frac{1}{2} \\ -1 \\ -\frac{1}{2} \end{pmatrix}$.

$$\langle \Pi_u(e_2) - e_2, e_1 \rangle = \left(\tfrac{1}{2}, -1, -\tfrac{1}{2}\right)\begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \left(\tfrac{1}{2}, -1, -\tfrac{1}{2}\right)\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = 0. \checkmark$$

$$\langle \Pi_u(e_2) - e_2, e_3 \rangle = \left(\tfrac{1}{2}, -1, -\tfrac{1}{2}\right)\begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} = 0 \checkmark$$

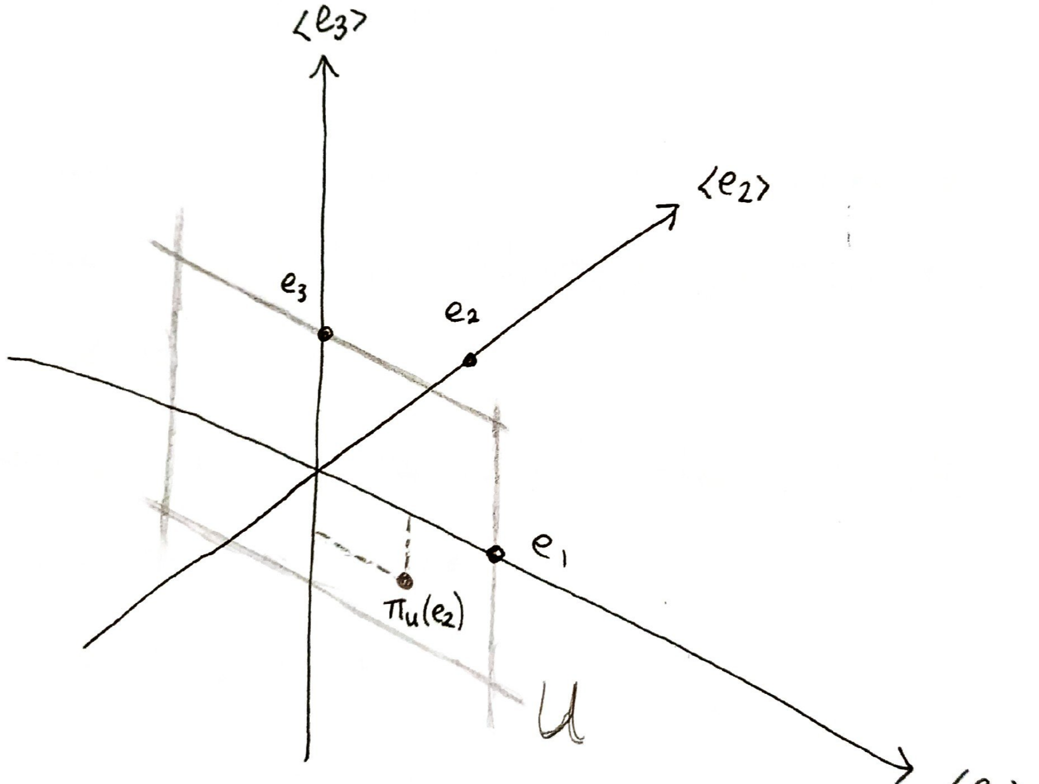
b) $d(e_2, U) = d(e_2, \pi_U(e_2))$

$$= \sqrt{\langle e_2 - \pi_U(e_2), e_2 - \pi_U(e_2) \rangle}$$

$$= \sqrt{\left(-\tfrac{1}{2}, 1, \tfrac{1}{2}\right) \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} -\frac{1}{2} \\ 1 \\ \frac{1}{2} \end{pmatrix}}$$

$$= \sqrt{\left(-\tfrac{1}{2}, 1, \tfrac{1}{2}\right) \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}} = 1.$$

c).



3.7 a) '$\Rightarrow$' Assume π is a projection i.e $\pi^2 = \pi$.

$$\begin{aligned} \text{Then } (id_V - \pi)^2 &= (id_V - \pi)(id_V - \pi) \\ &= id_V^2 - id_V \circ \pi - \pi \circ id_V + \pi^2 \\ &= id_V - \pi - \pi + \pi \quad \text{since } \pi^2 = \pi \\ &= id_V - \pi. \end{aligned}$$

'$\Leftarrow$' Suppose $id_V - \pi$ is a projection.

This means $(id_V - \pi)^2 = id_V - \pi$

$\Leftrightarrow$ $id_V - 2\pi + \pi^2 = id_V - \pi$

$\Leftrightarrow$ $\pi^2 = \pi$, so π is a projection.

b) Let $w \in Im(\pi)$, then $w = \pi(v)$ for some $v \in V$.

We have $(id_V - \pi)(w) = (id_V - \pi)(\pi(v)) = \pi(v) - \pi^2(v) = 0$.

so $w \in ker(id_V - \pi)$.

Conversely, if $w \in ker(id_V - \pi)$ then $(id_V - \pi)(w) = 0$

$\Leftrightarrow$ $w = \pi(w)$ so $w \in Im(\pi)$.

Thus $ker(id_V - \pi) = Im(\pi)$.

OTOH, let $v \in ker(\pi)$, then $(Id_V - \pi)(v) = v$ and so $v \in Im(Id_V - \pi)$.

Conversely, let $w \in Im(Id_V - \pi)$ i.e $w = (Id_V - \pi)(v)$ for some $v \in V$

Then $\pi(w) = \pi((Id_V - \pi)(v)) = \pi(v - \pi(v)) = \pi(v) - \pi^2(v) = 0$, so $w \in ker(\pi)$

Thus $Im(Id_V - \pi) = ker(\pi)$.

Next page: Alternative proof using well chosen basis of V

3.7 b (alternative proof)

$\pi : V \to V$. Let $U = \text{Im}(\pi)$.

Let $\{u_1, \dots u_k\}$ be a basis of U.

Extend $\{u_1, \dots, u_k\}$ to a basis of V such that the added vectors are orthogonal to each u_i.

i.e $\{u_1, \dots, u_k, v_{k+1}, \dots, v_n\}$ is a basis of V with

$\langle u_i, v_{k+j} \rangle = 0 \quad \forall\ i \in \{1, \dots, k\},\ j \in \{1, n-k\}$.

Then, w.r.t. this basis $id_V = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$ &

$$\pi = \left(\begin{array}{ccc|ccc} 1 & & & & & \\ & \ddots & & & 0 & \\ & & 1 & & & \\ \hline & & & 0 & & \\ & 0 & & & \ddots & \\ & & & & & 0 \end{array}\right) = \left(\begin{array}{c|c} I_k & \underline{0} \\ \hline \underline{0} & \underline{0}_{n-k} \end{array}\right)$$

clearly $im(\pi) = span\{u_1, \dots, u_k\}$, $ker(\pi) = span\{v_{k+1}, \dots, v_n\}$.

Now, $id_V - \pi = \left(\begin{array}{ccc|ccc} 0 & & & & & \\ & \ddots & & & & \\ & & 0 & & & \\ \hline & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{array}\right) = \left(\begin{array}{c|c} \underline{0}_k & \underline{0} \\ \hline \underline{0} & I_{n-k} \end{array}\right)$

& so $im(id_V - \pi) = span\{v_{k+1}, \dots, v_n\}$, $ker(id_V - \pi) = span\{u_1, \dots, u_k\}$.

In other words: $\quad ker(id_V - \pi) = Im(\pi)$

$\quad Im(id_V - \pi) = ker(\pi)$.

3.8) $b_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $b_2 = \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}$

$$u_1 := \frac{b_1}{\|b_1\|} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

$$\begin{aligned} v_2 := b_2 - \pi_{b_1}(b_2) &= b_2 - u_1^T b_2 \, u_1 \\ &= \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} - \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} - \frac{1}{3} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} -4 \\ 5 \\ -1 \end{pmatrix} \end{aligned}$$

$$u_2 := \frac{v_2}{\|v_2\|} = \frac{1}{\cancel{\frac{1}{3}} \sqrt{4^2+5^2+1^2}} \cancel{\frac{1}{3}} \begin{pmatrix} -4 \\ 5 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{42}} \begin{pmatrix} -4 \\ 5 \\ -1 \end{pmatrix}.$$

O.N.B. $(c_1, c_2) = \left(\frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{42}} \begin{pmatrix} -4 \\ 5 \\ -1 \end{pmatrix} \right).$

3.9) C.Z. inequality : $\langle u, v \rangle^2 \leq \langle u, u \rangle \langle v, v \rangle$

a) Let $u = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, $v = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$, $\langle \bullet, \bullet \rangle$ = dot product.

Then $\langle u, v \rangle = x_1 + \dots + x_n = 1$

$\langle u, u \rangle = x_1^2 + \dots + x_n^2$

$\langle v, v \rangle = 1 + \dots + 1 = n.$

Thus, $1 \leq n(x_1^2 + \dots + x_n^2) \iff \sum_{i=1}^{n} x_i^2 \geq \frac{1}{n}.$

b) $u = \begin{pmatrix} 1/\sqrt{x_1} \\ \vdots \\ 1/\sqrt{x_n} \end{pmatrix}$, $v = \begin{pmatrix} \sqrt{x_1} \\ \vdots \\ \sqrt{x_n} \end{pmatrix}$.

$\langle u, v \rangle^2 = (1 + \dots + 1)^2 = n^2$

$\langle u, u \rangle = \sum \frac{1}{x_i}$, $\langle v, v \rangle = \sum x_i = 1$

Thus $n^2 \leq \sum_{i=1}^{n} \frac{1}{x_i}.$

3.10) Rotation by 30°:

$$R_{30°} = \begin{pmatrix} \cos 30 & -\sin 30 \\ \sin 30 & \cos 30 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{pmatrix}$$

$$R_{30°}\,\underline{x}_1 = \frac{1}{2}\begin{pmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} \sqrt{3} - \frac{3}{2} \\ 1 + \frac{3}{2}\sqrt{3} \end{pmatrix}$$

$$R_{30°}\,\underline{x}_2 = \frac{1}{2}\begin{pmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{pmatrix}\begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ -\frac{\sqrt{3}}{2} \end{pmatrix}$$