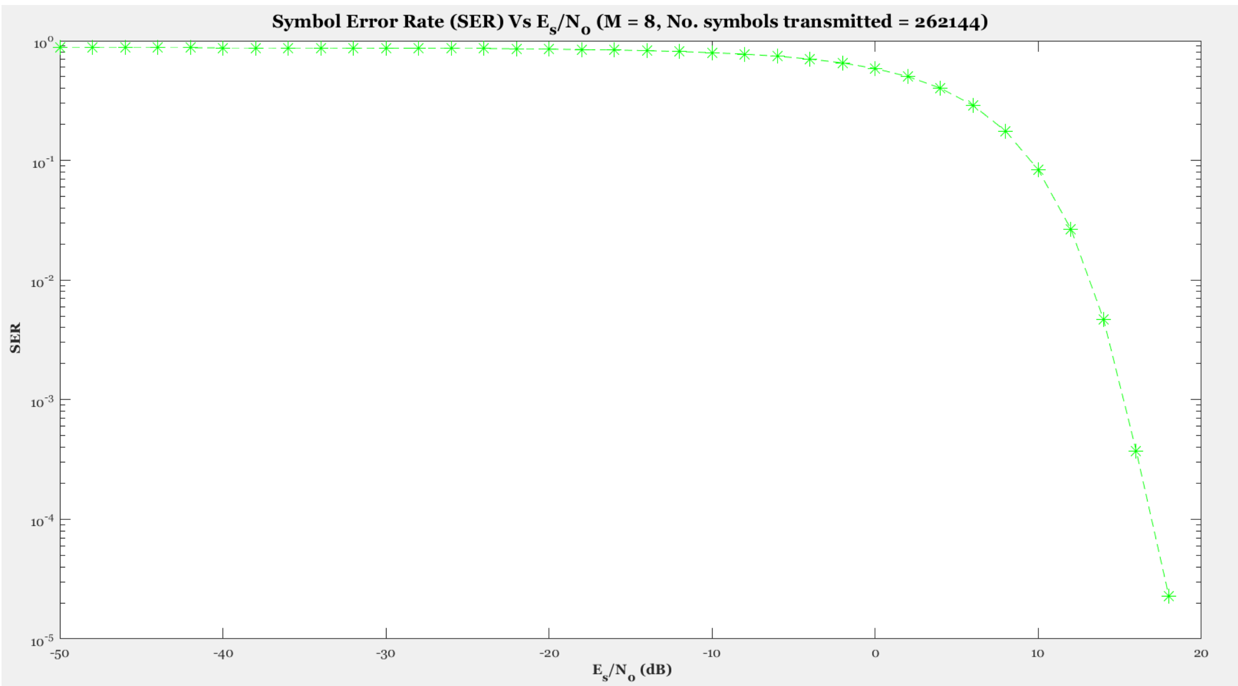
Digital Communications (EEEN40060): Assignment 1

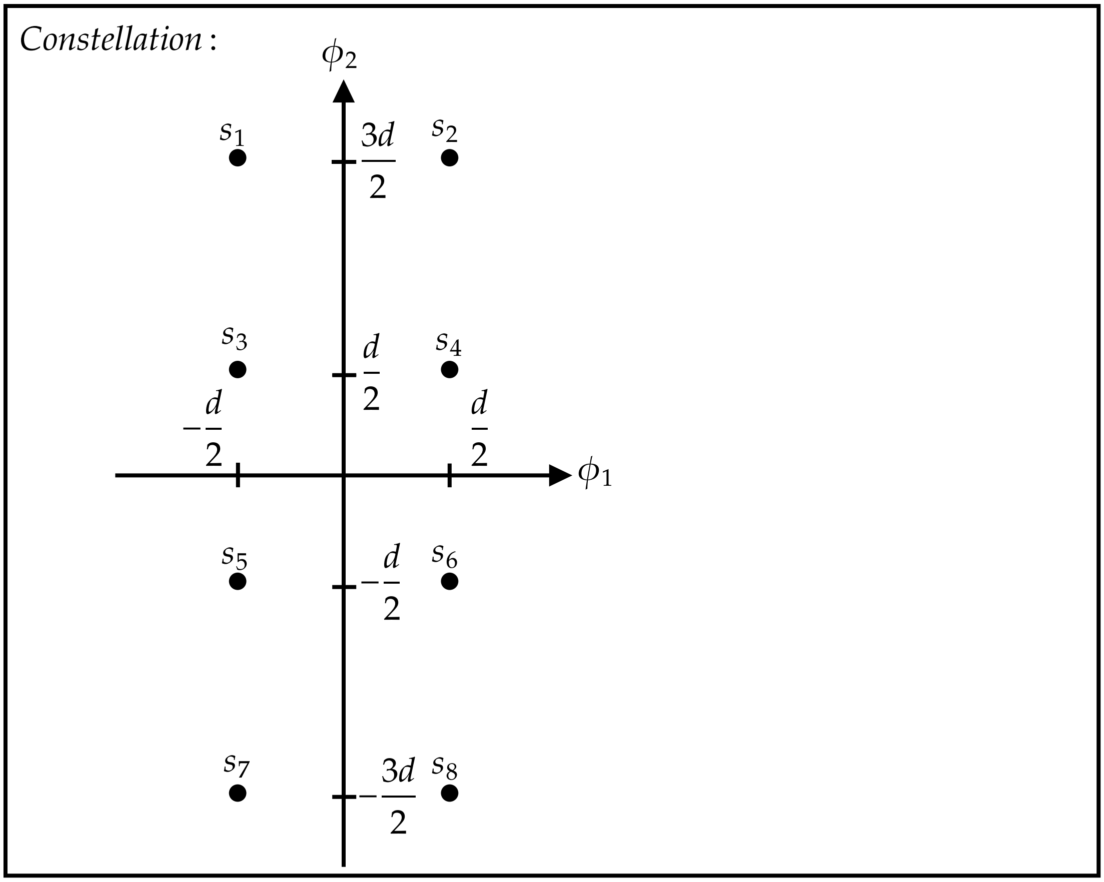
Student Name: Dylan Boland
Student Number: 17734349

(Q.1) Shown below is the simulated symbol error rate curve generated by the MATLAB script:



As stated in the title of the plot above, the MATLAB script simulated the transmission of 262144 (2^{18}) symbols (a higher number could have been used). This allows one to see the necessary value for (E_s / N_0) in order to achieve an SER as low as 1 in 10000.

(Q.2) **Theoretical Symbol Error Rate (SER):**
Shown below is the constellation of the 8-ary communication system. Due to the symmetry in the constellation, the probability of error associated with s_1, s_2, s_7, and s_8 is the same. Likewise, the probability of error is the same for s_3, s_4, s_5, and s_6. Therefore, in order to work out the overall probability of error, one can work out the probability of error for one of the outer symbols, say, s_1, and for one of the inner symbols, say, s_3. An average can then be taken to get the overall probability of error:

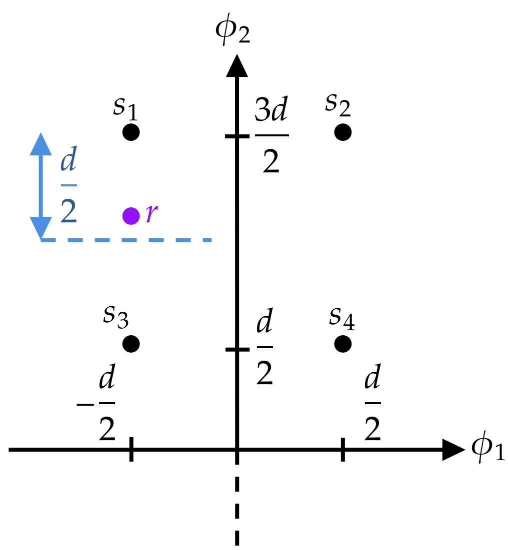


Let us begin with the probability of error of an outer symbol, say, s_1:

If s_1 is sent, and r is received, then it must not land too close to s_3 along the vertical... otherwise the receiver will make the wrong decision...

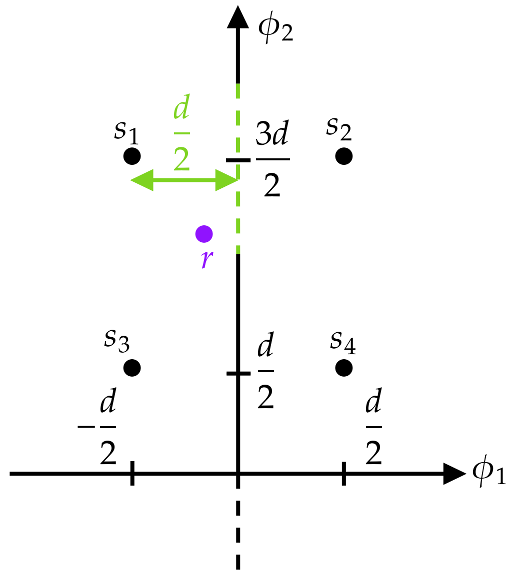
$$\rightarrow P_{error\ vertical} = Q\left(\frac{\frac{d}{2}}{\sigma}\right)$$

The Q-function...



If s_1 is sent, and r is received, then it must not land too close to s_2 along the horizontal either... otherwise the receiver will again make the wrong decision...

$$\rightarrow P_{error\ horiztonal} = Q\left(\frac{\frac{d}{2}}{\sigma}\right)$$



In order for the receiver to make the correct decision when s_1 is transmitted, the received symbol, *r*, must lie within the decision boundary in both the vertical *and* horizontal dimensions. Therefore, the probability of the receiver making a correct decision is equal to:

$$P_{correct} = (P_{correct\ vertical})(P_{correct\ horizontal})$$

The probability that the received symbol, *r*, does not lie too close to s_3 along the vertical can be found by subtracting the probability that it *does* lie too close to s_3 along the vertical from one. The same reasoning can also be applied to the horizontal dimension:

$$\rightarrow P_{correct\ vertical} = 1 - P_{error\ vertical}$$

$$\rightarrow P_{correct\ horizontal} = 1 - P_{error\ horizontal}$$

$$\therefore P_{correct} = (1 - P_{error\ vertical})(1 - P_{error\ horizontal})$$

As is also shown in the two diagrams above, for the outer symbols the probability of the received symbol drifting over the vertical and horizontal decision boundaries is the same. Therefore, we can reference this probability simply as *P*:

$$\rightarrow P_{error\ vertical} = P_{error\ horizontal} = P = Q\left(\frac{\frac{d}{2}}{\sigma}\right)$$

$$\therefore P_{correct} = (1 - P)(1 - P) = 1 + P^2 - 2P$$

Lastly, now that we have an expression for the probability of the receiver making a correct decision given that an outer symbol was transmitted, we can get the probability of an error as being:

$$\rightarrow P_{error} = 1 - P_{correct} = 1 - \left(1 + P^2 - 2P\right) = 2P - P^2$$

If we assume that there is a reasonable signal-to-noise ratio (SNR), then the following linear approximation can be made:

$$P_{error} = 2P - P^2 \approx 2P$$

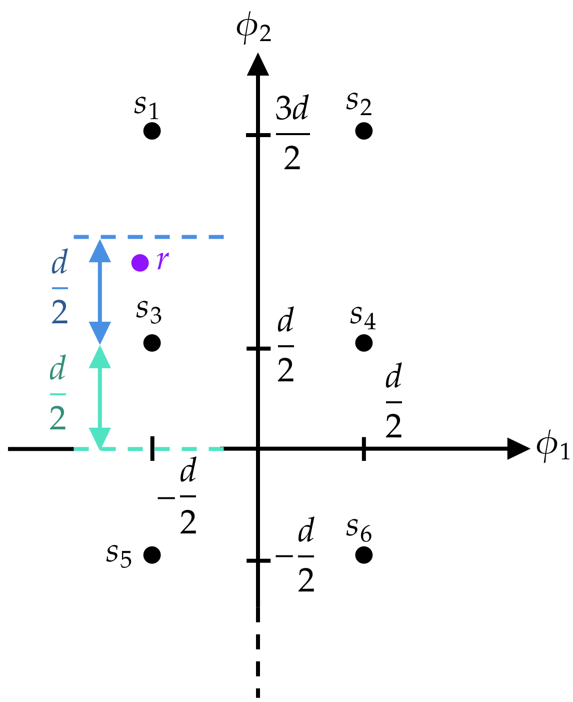
This is because at a high SNR, the value of *P* is quite low, meaning the square of *P* is very low.

Now we can work out the probability of error associated with the inner symbols. Let us assume that s_3 was transmitted:

With the inner symbols the situation is different. There are two ways in which an error can be made along the vertical: in this example, if s_3 were sent, and r was received, then it could land too close to s_1 or s_5...

$$\rightarrow P_{error\ vertical} = Q\left(\frac{\frac{d}{2}}{\sigma}\right) + Q\left(\frac{\frac{d}{2}}{\sigma}\right)$$

$$\therefore P_{error\ vertical} = 2Q\left(\frac{\frac{d}{2}}{\sigma}\right) = 2P$$



$$\rightarrow P_{correct} = (P_{correct\ vertical})(P_{correct\ horizontal})$$

$$\therefore P_{correct} = (1 - P_{error\ vertical})(1 - P_{error\ horizontal})$$

And the probability of error in the horizontal dimension is *P*, like before:

$$\therefore P_{correct} = (1-2P)(1-P) = 1 + 2P^2 - 3P$$

$$\therefore P_{error} = 1 - P_{correct} = 1 - \left(1 + 2P^2 - 3P\right) = 3P - 2P^2$$

Again, assuming that there is a reasonably good SNR then the following is a good approximation of this probability of error:

$$\rightarrow P_{error} \approx 3P$$

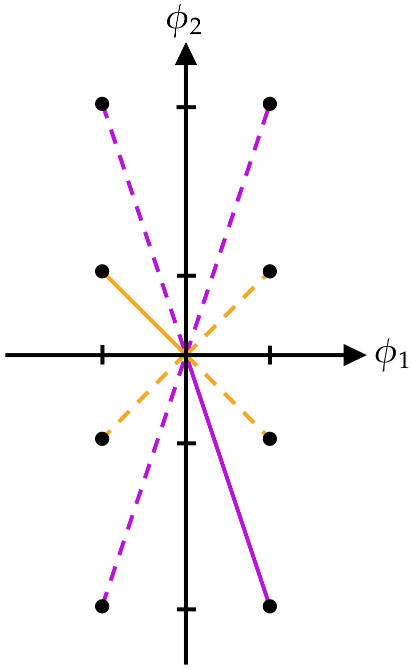
Since there are four inner symbols, and four outer symbols, the overall probability of error can be found in the following way:

$$\rightarrow P_{error} = \frac{1}{8}((4)(2P) + (4)(3P)) = \frac{5}{2}P = \frac{5}{2}Q\left(\frac{\frac{d}{2}}{\sigma}\right)$$

One can equivalently express this as:

$$\rightarrow P_{error} = \frac{5}{2}Q\left(\frac{\sqrt{\frac{d^2}{4}}}{\sqrt{\frac{N_0}{2}}}\right) = \frac{5}{2}Q\left(\sqrt{\frac{d^2}{2N_0}}\right)$$

In order to relate the probability of error to the average symbol energy, one can express the average symbol energy in terms of *d*:



$$E_{inner\ symbols} = \left(\sqrt{\left(\frac{d}{2}\right)^2 + \left(\frac{d}{2}\right)^2}\right)^2 = \frac{2d^2}{4}$$

$$E_{outer\ symbols} = \left(\sqrt{\left(\frac{d}{2}\right)^2 + \left(\frac{3d}{2}\right)^2}\right)^2 = \frac{10d^2}{4}$$

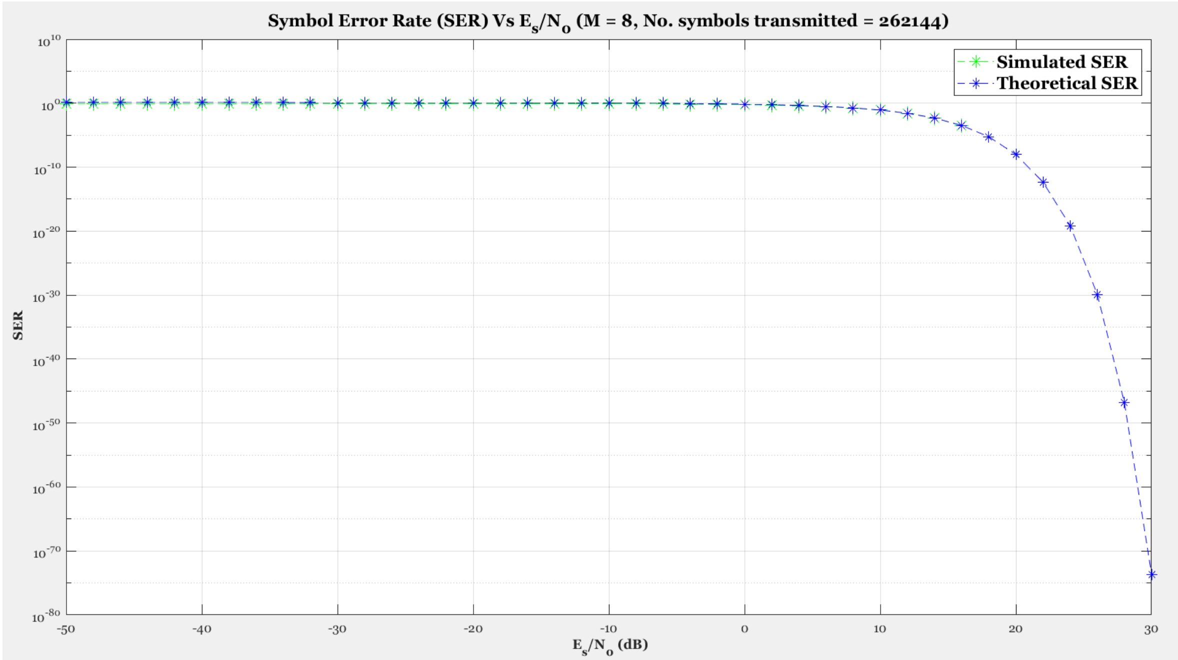
$$\therefore E_s = \frac{1}{8}\left(4\left(\frac{2d^2}{4}\right) + 4\left(\frac{10d^2}{4}\right)\right) = \frac{12d^2}{8}$$

$$\therefore d^2 = \frac{2}{3}E_s$$

Using the expression at the bottom of the diagram above, we can re-write the average probability of error as:

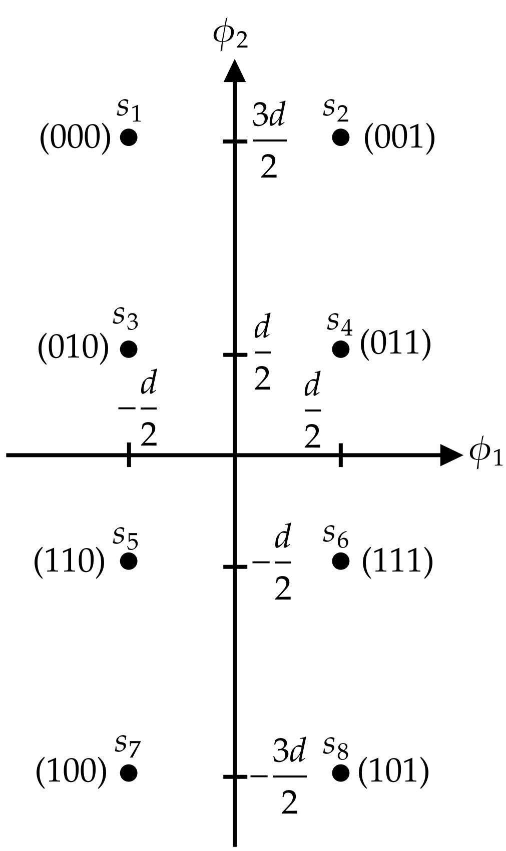
$$\rightarrow P_{error} = \frac{5}{2}Q\left(\sqrt{\frac{\frac{2E_s}{3}}{2N_0}}\right) = \frac{5}{2}Q\left(\sqrt{\frac{1}{3}\left(\frac{E_s}{N_0}\right)}\right)$$

Simulated Vs Theoretical (SER):
Shown below is the graph comparing the simulated SER with the theoretical one.
One point to keep in mind is that the theoretical expression does not rely on the actual generation and simulation of transmission of symbols. It uses the equation above to compute the bit error rate. As a result, the corresponding blue curve goes much further down than the green one, which represents the simulated symbol error rate. Like before, a total of 262144 (2^{18}) symbols were generated, and their transmission was simulated. This number of symbols allows one to see the necessary (E_s / N_0) value for the simulated symbol error rate to drop below 1 in 10000:



(Q.3) **Bit-to-Symbol Mapping:**
A bit-to-symbol mapping that I would consider using is shown below:

Bit-to-Symbol Mapping:



This bit-to-symbol mapping scheme uses gray code. A benefit of this bit-to-symbol mapping is that only one bit will be in error for nearest-neighbour errors, which are the most likely and most common errors. For example, say s_1 were sent and then due to the noise in the channel it is corrupted, and the receiver decides that it was most likely s_3 that was sent by the transmitter. In this case, only the second bit is in error:

What the Tx side sent (s_1): (**000**)

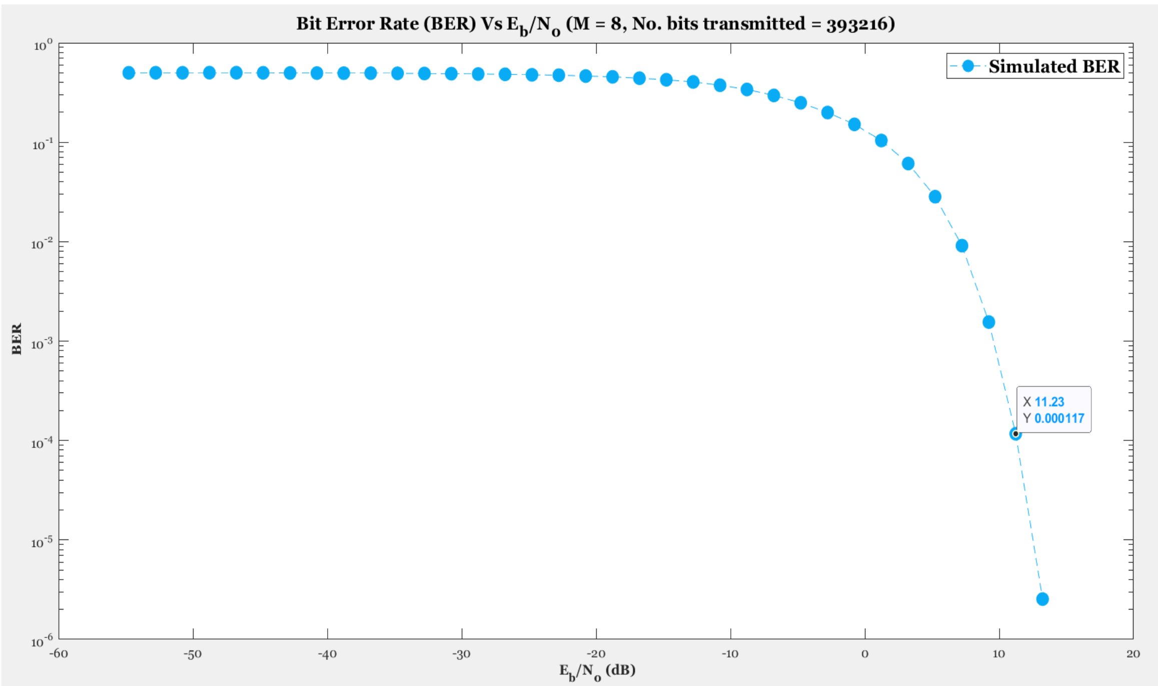
What the Rx side decided was sent (s_3): (**010**)

The same is true for other neighbouring pairs in the constellation...

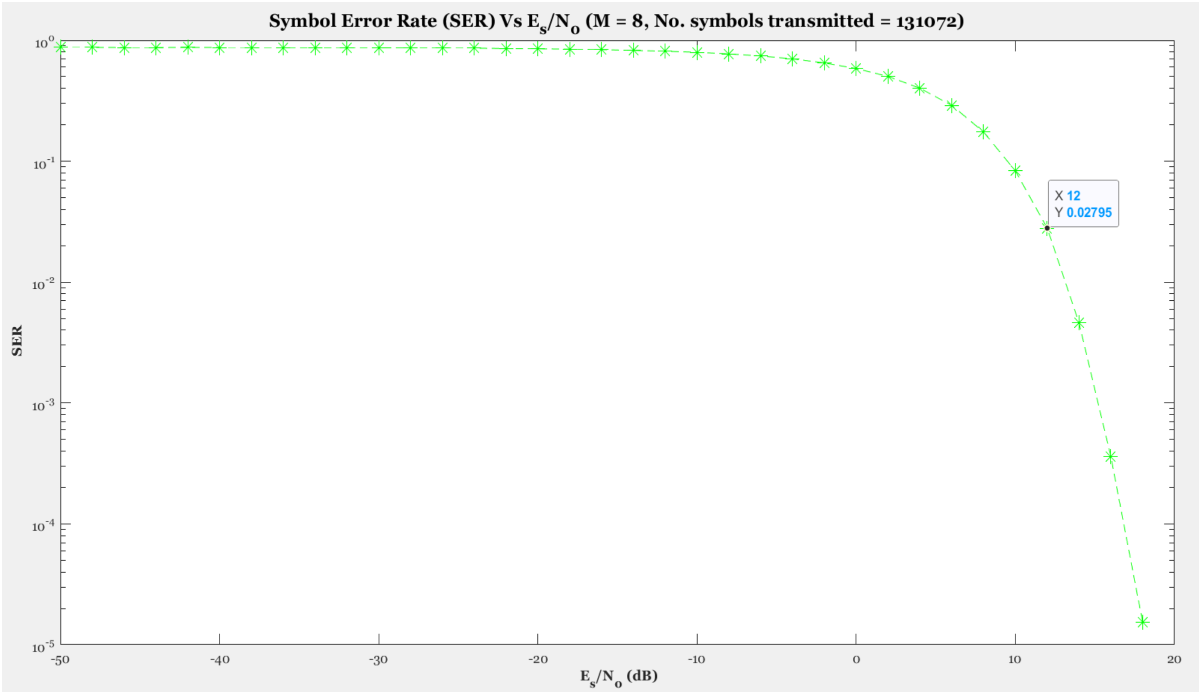
As discussed in the diagram above, I would use a gray code in order to ensure that nearest-neighbour errors in the symbol transmissions only resulted in single bit errors.

(Q.4) **Bit Error Rate (BER) Curve:**

Below is shown a graph generated from the same MATLAB script. It depicts the bit error rate against (E_b / N_0) in dB:



In the majority of cases of a symbol error, there will only be one bit error. This is because the symbol-to-bit mapping scheme uses a gray code. However, since there are three times as many bits as symbols (as each symbol represents three bits) we expect the bit error rate to be lower than the symbol error rate. The above graph was generated by simulating the transmission of 131072 (2^{17}) symbols, which corresponds to 393216 bits. Shown below is the corresponding SER curve:



The above graph, when compared with the BER curve, shows that the symbol error rate is higher. For example, the SER is around 0.027 at 12 dB on the horizontal. At the 11.2 dB tick mark on the horizontal of the BER plot the bit error rate is around 0.0001.

(Q.5) As can hopefully be seen in the BER plot above, the value of (E_b / N_0) above which the bit error rate (BER) falls below 0.0001 (equivalent to 1 bit in every 10000 being in error!) is around 11.2 dB.