COT 2000
Foundations of Computing

Spring 2024

Lecture 18 – part 1

Lab 9

Homework 6 – 07/19/24

Lecture 18 – part 2

Review

Review

- Counting and Probability
- The Rule of Products
- Permutations – P(n,k)
- Combinations – C(n,k)

$$P(n,k) = \underbrace{n \cdot (n-1) \cdot (n-2) \cdot \dots \cdot (n-k+1)}_{\text{The Rule of Products applied to unique elements}} = \frac{n!}{(n-k)!}.$$

Permutation

$$\binom{n}{k} = \frac{n!}{(n-k)! \cdot k!}$$

Extended Rule of Products

The extended rule of products

The Extended Rule of Products states that if **'n'** operations must be performed, and each operation has a respective number of options denoted by $\mathbf{p_1, p_2, \ldots, p_n}$, with each $\mathbf{p_i}$ being independent of the choices made in previous operations, then there are $\mathbf{p_1 \cdot p_2 \cdot \ldots \cdot p_n}$ different ways to perform these **'n'** operations

Permutation - P(n,k)

- **Permutation:** An ordered arrangement of k elements selected from a set of n elements, **$0 \leq k \leq n$**, where no two elements of the arrangement are the same, is called a permutation of **n** objects taken **k** at a time.
- The total number of such permutations is denoted by **P(n, k).**

(n choose r) Notation

- Binomial coefficient $\binom{n}{r}$

• Definition

Let n and r be integers with $0 \le r \le n$. The symbol

$$\binom{n}{r}$$

is read "***n* choose *r***" and represents the number of subsets of size r that can be chosen from a set with n elements.

• Formula for Computing $\binom{n}{r}$

For all integers n and r with $0 \le r \le n$,

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}.$$

also called *combinations*

Example:

How many ways can we order three letters from $A = \{a, b, c, d\}$?

Permutation

Order **is** important

$$\frac{n!}{(n-k)!}.$$

Solution:

By rule of products : 4 x 3 x 2 = 24

By permutation : P(4,3) = 4!/(4-3)! = 4! = 24

```
abc, acb, bca, bac, cab, cba
abd, adb, bda, bad, dab, dba
acd, adc, cda, cad, dac, dca
bcd, bdc, cdb, cbd, dbc, dcb
```

Example:

How many ways can we select a set of three letters from $A = \{a, b, c, d\}$?

Solution:

```
abc, acb, bca, bac, cab, cba --- > {a,b,c}
abd, adb, bda, bad, dab, dba --- > {a,b,d}
acd, adc, cda, cad, dac, dca --- > {a,c,d}
bcd, bdc, cdb, cbd, dbc, dcb --- > {b,c,d}
```

Order **is not** important in sets

Combinations !

$$\frac{P(4,3)}{6} = \frac{P(4,3)}{3!} = \frac{4!}{(4-3)!3!} = 4 = \binom{4}{3}$$

Lecture 18 – part 3

More Exercises

Example:

Consider the problem of choosing five members from a group of twelve to work as a team on a special project. How many distinct five-person teams can be chosen?

Solution The number of distinct five-person teams is the same as the number of subsets of size 5 (or 5-combinations) that can be chosen from the set of twelve. This number is $\binom{12}{5}$. By Theorem 9.5.1,

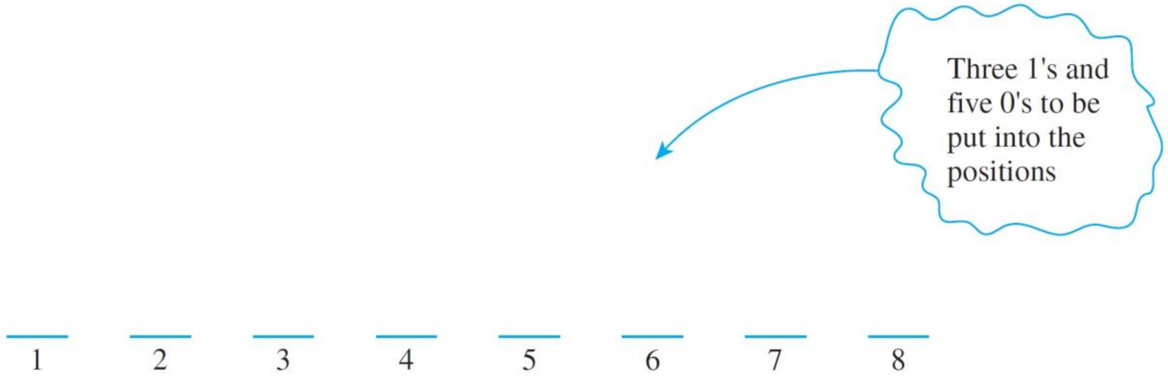
$$\binom{12}{5} = \frac{12!}{5!(12-5)!} = \frac{1\cancel{2} \cdot 11 \cdot 1\cancel{0} \cdot 9 \cdot 8 \cdot \cancel{7}!}{(\cancel{5} \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot 1) \cdot \cancel{7}!} = 11 \cdot 9 \cdot 8 = 792.$$

Thus there are 792 distinct five-person teams. ■

Example:

How many eight-bit strings have exactly three 1’s?

Solution To solve this problem, imagine eight empty positions into which the 0’s and 1’s of the bit string will be placed. In step 1, choose positions for the three 1’s, and in step 2, put the 0’s into place.



Once a subset of three positions has been chosen from the eight to contain 1’s, then the remaining five positions must all contain 0’s (since the string is to have exactly three 1’s). It follows that the number of ways to construct an eight-bit string with exactly three 1’s is the same as the number of subsets of three positions that can be chosen from the eight into which to place the 1’s. By Theorem 9.5.1, this equals

$$\binom{8}{3} = \frac{8!}{3! \cdot 5!} = \frac{8 \cdot 7 \cdot \cancel{6} \cdot \cancel{5!}}{\cancel{3 \cdot 2} \cdot \cancel{5!}} = 56. \quad \blacksquare$$

Example:

What is the probability that in a group of 25 people, at least two persons share the same birthday?

Solution:

Let P(shared birthday) denote the probability that at least two persons share the same birthday. This is calculated as the complement of the probability that everyone has unique birthdays.

$$P(\text{shared birthday}) = 1 - P(\text{no shared birthdays})$$

Consider the set $E \subseteq S$, where E is the set of all outcomes with unique birthdays (the event of interest), and S is the sample space containing all possible combinations of birthdays for the 25 people.

The probability of no shared birthdays is given by:

$$P(\text{no shared birthdays}) = \frac{N(E)}{N(S)}$$

Here, $N(E)$ is the number of ways to assign different birthdays to each of the 25 people, and $N(S)$ is the number of ways for possible assignments of birthdays.

$N(E)$ is determined by the permutation of 25 unique days out of 365:

$$N(E) = 365 \cdot 364 \cdot 363 \cdot ... \cdot 341 = P(365, 25) = \frac{365!}{(365-25)!}$$

$N(S)$ is simply $365_1 \cdot 365_2 \cdot 365_3 ... 365_{25} = 365^{25}$, as each person can be born on any of the 365 days, independently of others.

Thus, the probability that all 25 people have unique birthdays is:

$$P(\text{unique birthdays}) = \frac{N(E)}{N(S)} = \frac{P(365, 25)}{365^{25}} = 0.4313$$

And the probability that at least two persons share the same birthday is:
$P(\text{shared birthday}) = 1 - \dfrac{N(E)}{N(S)} = 1 - 0.4313 = 0.5687$ or 56.87 %

Exercise

Three people have been exposed to a certain illness. Once exposed, a person has a 50–50 chance of actually becoming ill.

a. What is the probability that exactly one of the people becomes ill?

b. What is the probability that at least two of the people become ill?

c. What is the probability that none of the three people becomes ill?

Solution

000
001
010
011
100
101
110
111

a. 3/8 = 37.5 %
b. 4/8 = 1/2 = 50%
c. 1/8 = 12.5%

Exercise

Two faces of a six-sided die are painted red, two are painted blue, and two are painted yellow. The die is rolled three times, and the colors that appear face up on the first, second, and third rolls are recorded.

a. Let BBR denote the outcome where the color appearing face up on the first and second rolls is blue and the color appearing face up on the third roll is red. Because there are as many faces of one color as of any other, the outcomes of this experiment are equally likely. List all possible outcomes.

b. Consider the event that all three rolls produce different colors. One outcome in this event is RBY and another RYB. List all outcomes in the event. What is the probability of the event?

Solution

a. 3x3x3 = 27

RRR, RRB, RRY
RBR, RBB, RBY
RYR, RYB, RYY
BRR, BRB, BRY
BBR, BBB, BBY
BYR, BYB, BYY
YRR, YRB, YRY
YBR, YBB, YBY
YYR, YYB, YYY

b. 3x2x1 = 6

RBY
RYB
BRY
BYR
YRB
YBR

P = 6/27 = 2/9 = 22.22%

What is the probability of two distinct colors ?

3x2x3 / 27 = 18/27 = 2/3 = 66.6%

Exercise

An urn contains two blue balls (denoted B_1 and B_2) and one white ball (denoted W). One ball is drawn, its color is recorded, and it is replaced in the urn. Then another ball is drawn, and its color is recorded.

a. Let B_1W denote the outcome that the first ball drawn is B_1 and the second ball drawn is W. Because the first ball is replaced before the second ball is drawn, the outcomes of the experiment are equally likely. List all possible outcomes of the experiment.

b. Consider the event that the two balls that are drawn are both blue. List all outcomes in the event. What is the probability of the event?

c. Consider the event that the two balls that are drawn are of different colors. List all outcomes in the event. What is the probability of the event?

Solution

a. 3x3 = 9

B1B1, B1B2, B1W
B2B1, B2B2, B2W
WB1, WB2, WW

b.

B1B1, B1B2, B2B1, B2B2

P = 4/9 = 44.44%

c.

B1W, B2W, WB1, WB2

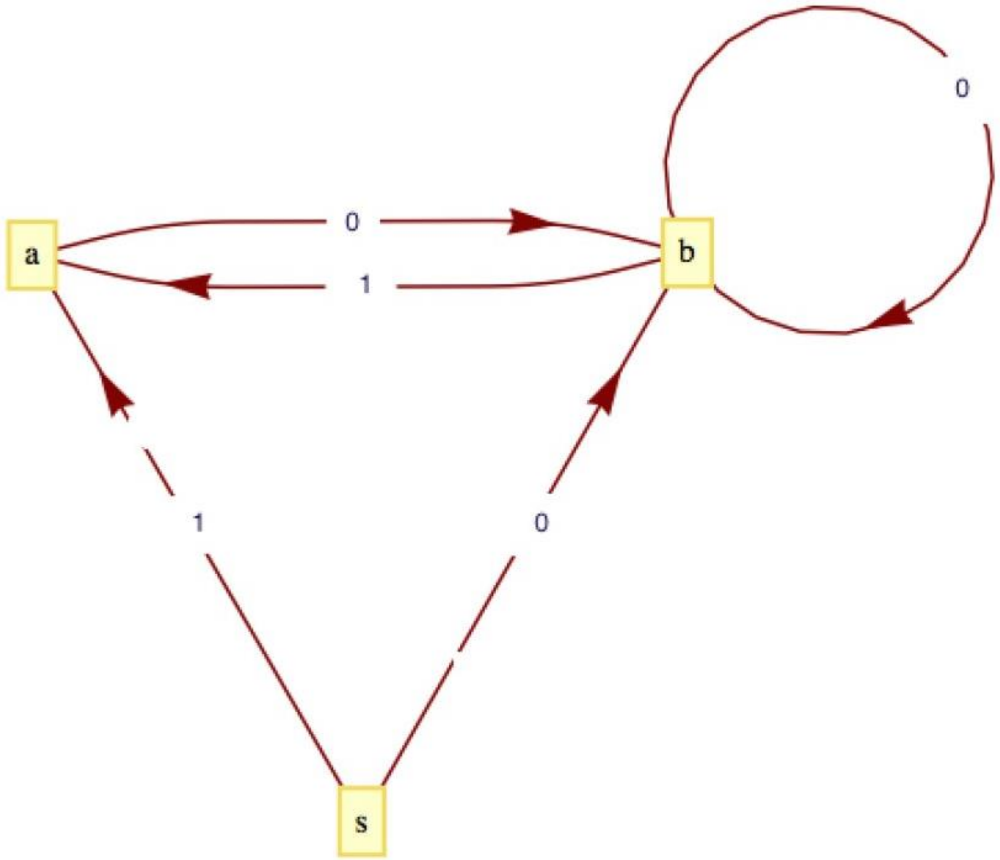
P = 4/9 = 44.44%

Lecture 18 – part 4

Graph Theory

Simple Directed Graph

- A simple directed graph consists of a nonempty **set of vertices**, V , and a **set of edges**, E, that is a subset of the set $V \times V$.
- Each edge is an ordered pair of elements from the vertex set.
- The first entry is the **initial vertex** of the edge and the second entry is the **terminal vertex**.

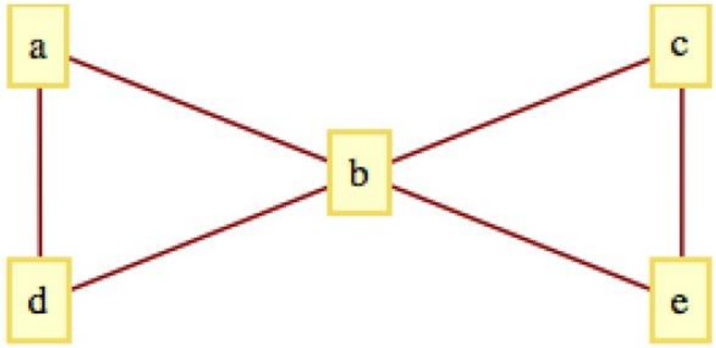


$V = \{s, a, b\}$

$E = \{(s, a), (s, b), (a, b), (b, a), (b, b)\}$.

Simple Undirected Graph

- The order of the vertices is not significant.
- A simple undirected graph consists of a nonempty set V , called a vertex set, and a set E of two-element subsets of V , called the edge set.
- When drawing an undirected graph, the two-element subsets are drawn as undirected lines or arcs connecting the vertices.
- It is customary to not allow "self loops" in undirected graphs since $\{v, v\}$ isn't a two element subset of vertices.
- Both directed and undirected graphs have nonempty vertex sets.

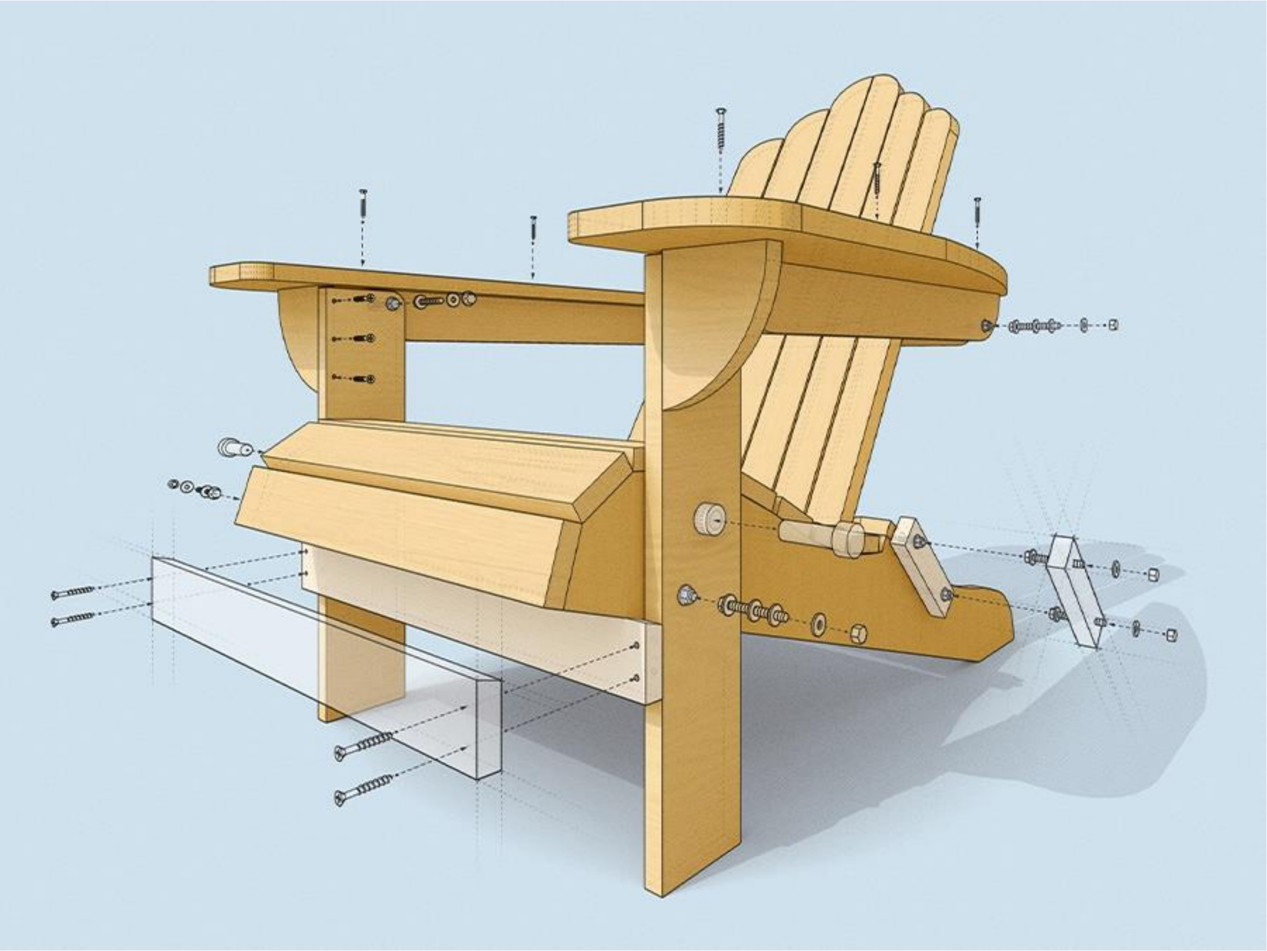


$V = \{a, b, c, d, e\}$

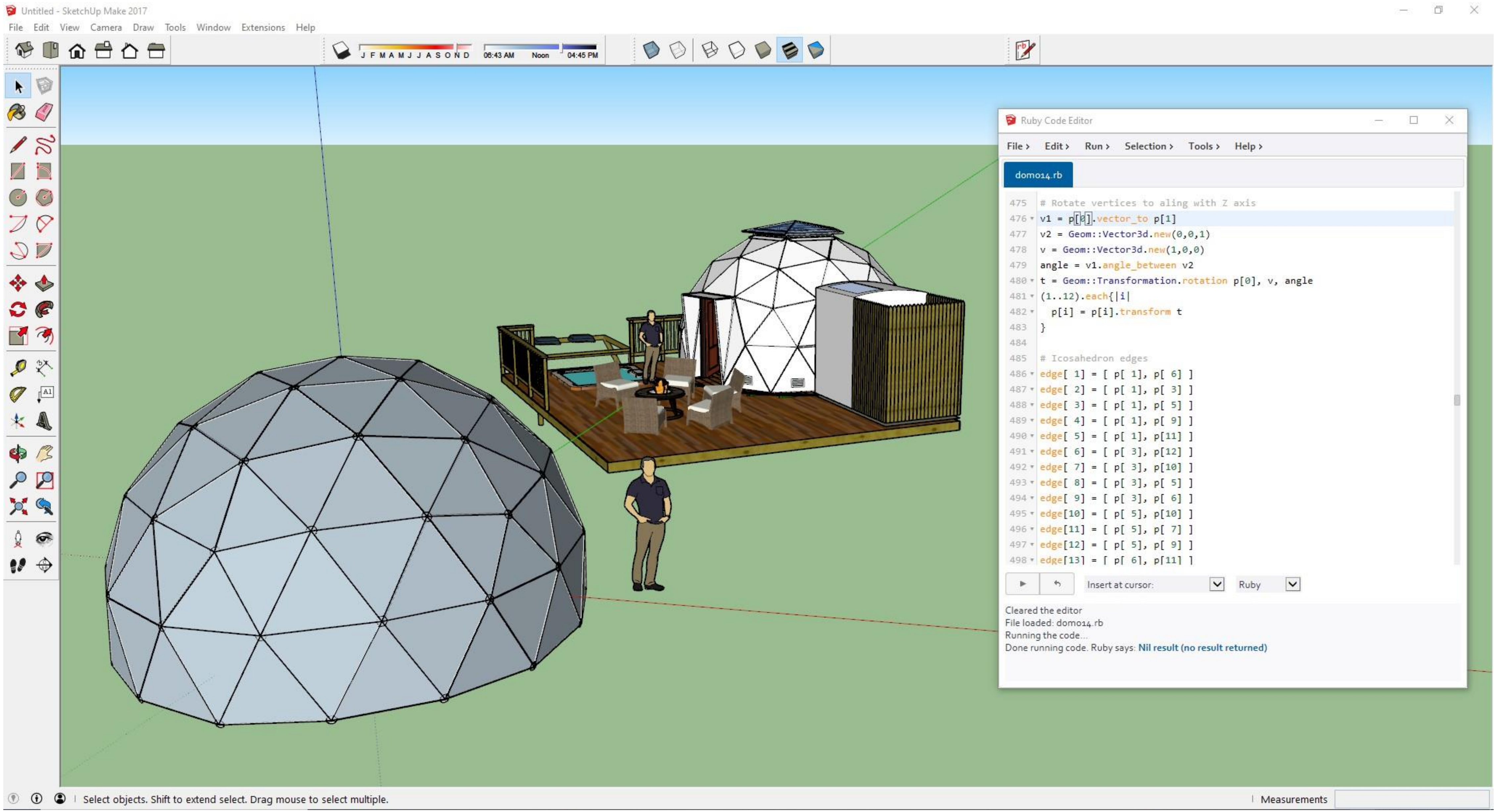
$E = \{\{a, b\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, e\}, \{b, e\}\}$

Complete Undirected Graph. A complete undirected graph on n vertices is an undirected graph with the property that each pair of distinct vertices are connected to one another. Such a graph is usually denoted by K_n.

https://ruby.sketchup.com/Sketchup/Vertex.html



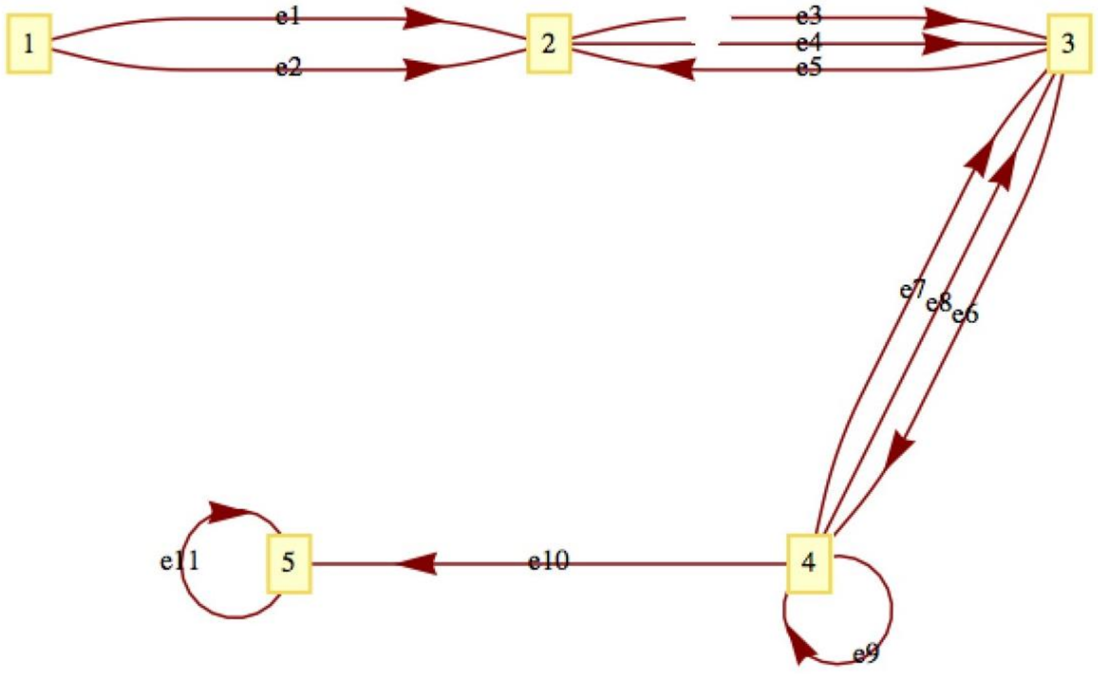
```ruby
edge = entities[0]
# returns array of vertices that make up the line
verticies = edge.vertices
vertex1 = verticies[0]
vertex2 = verticies[1]
edge = vertex1.common_edge vertex2
if (edge)
  UI.messagebox edge
else
  UI.messagebox "Failure"
end
```


Untitled - SketchUp Make 2017
File Edit View Camera Draw Tools Window Extensions Help
J F M A M J J A S O N D
06:43 AM Noon 04:45 PM
Ruby Code Editor
File › Edit › Run › Selection › Tools › Help ›
domo14.rb
# Rotate vertices to aling with Z axis
v1 = p[0].vector_to p[1]
v2 = Geom::Vector3d.new(0,0,1)
v = Geom::Vector3d.new(1,0,0)
angle = v1.angle_between v2
t = Geom::Transformation.rotation p[0], v, angle
(1..12).each{|i|
p[i] = p[i].transform t
}
# Icosahedron edges
edge[1] = [p[1], p[6]]
edge[2] = [p[1], p[3]]
edge[3] = [p[1], p[5]]
edge[4] = [p[1], p[9]]
edge[5] = [p[1], p[11]]
edge[6] = [p[3], p[12]]
edge[7] = [p[3], p[10]]
edge[8] = [p[3], p[5]]
edge[9] = [p[3], p[6]]
edge[10] = [p[5], p[10]]
edge[11] = [p[5], p[7]]
edge[12] = [p[5], p[9]]
edge[13] = [p[6], p[11]]
Insert at cursor:
Ruby
Cleared the editor
File loaded: domo14.rb
Running the code...
Done running code. Ruby says: Nil result (no result returned)
Select objects. Shift to extend select. Drag mouse to select multiple.
Measurements

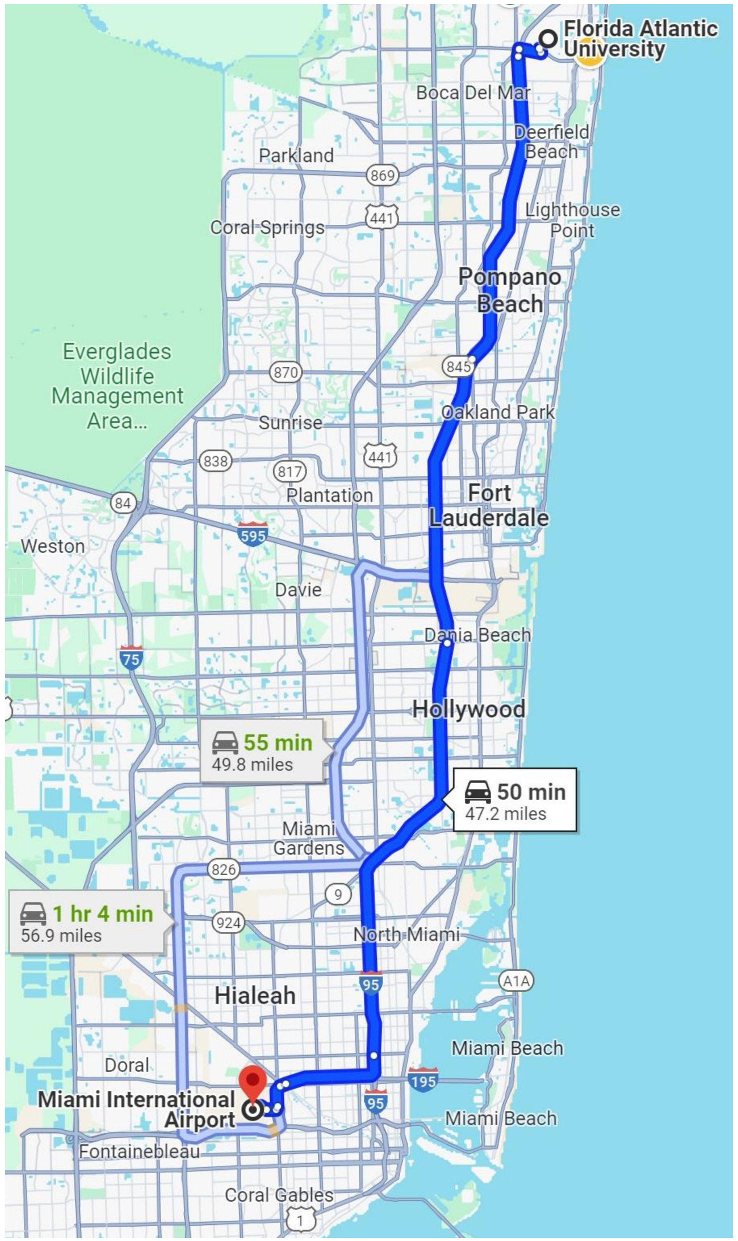
Multigraph

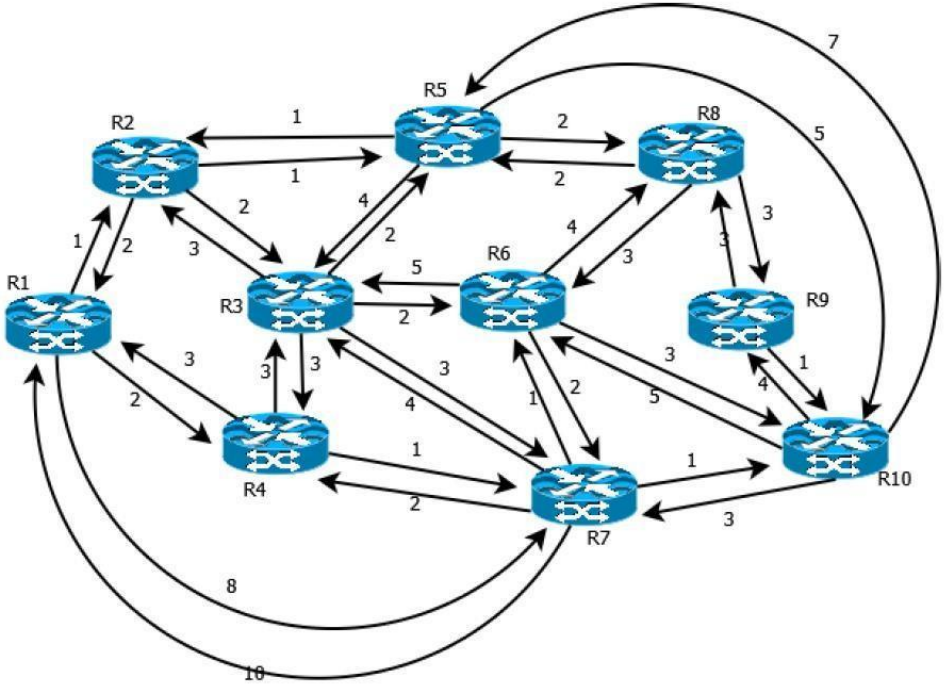
- A multigraph is a set of vertices V with a set of edges that can contain more than one edge between the vertices.

- Example: A road map: The cities and towns on the map can be thought of as vertices, while the roads are the edges. It is not uncommon to have more than one road connecting two cities.



Labels like e3 are used to avoid ambiguity, edge (2, 3) would be ambiguous.





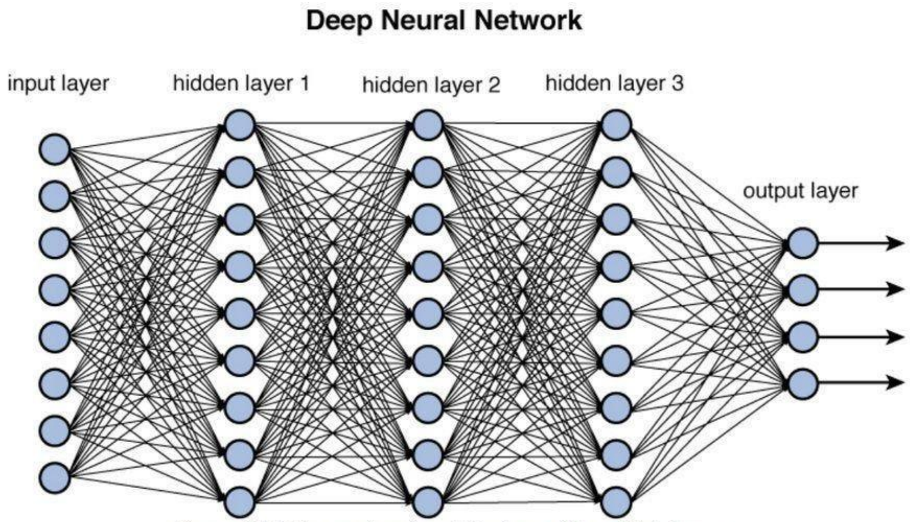


Figure 12.2 Deep network architecture with multiple layers.