COT 2000

Foundations of Computing

Summer 2024

Lecture 9 – part 1

Lab 5
Homework 3 - Due: 06/14/24
Exam 2 – 06/21/24

Lecture 9 – part 2

Review

Review

- Arguments
  - Test a valid argument
- Argument forms (MP, MT)
- Fallacies
  - Converse Error
  - Inverse Error
- Contradiction rule
- Predicates, Predicate Variables, Statements,
- Predicate Variables Domain
- Truth Set of a Predicate
- Universal Quantifier: $\forall$
- Existential Quantifier: $\exists$

Testing an Argument Form for Validity

1. Identify the premises and conclusion of the argument form.
2. Construct a truth table showing the truth values of all the premises and the conclusion.
3. A row of the truth table in which all the premises are true is called a **critical row**.
    - If there is a critical row in which the conclusion is false, then it is possible for an argument of the given form to have true premises and a false conclusion, and so the argument form is invalid.
    - If the conclusion in every critical row is true, then the argument form is valid.

Valid Argument Forms

Modus Ponens	$p \to q$ p $\therefore q$	**Elimination**	**a.** $p \vee q$ $\sim q$ $\therefore p$ **b.** $p \vee q$ $\sim p$ $\therefore q$
Modus Tollens	$p \to q$ $\sim q$ $\therefore \sim p$	**Transitivity**	$p \to q$ $q \to r$ $\therefore p \to r$
Generalization	**a.** p $\therefore p \vee q$ **b.** q $\therefore p \vee q$	**Proof by Division into Cases**	$p \vee q$ $p \to r$ $q \to r$ $\therefore r$
Specialization	**a.** $p \wedge q$ $\therefore p$ **b.** $p \wedge q$ $\therefore q$		
Conjunction	p q $\therefore p \wedge q$	**Contradiction Rule**	$\sim p \to c$ $\therefore p$

Converse error

Show that the following argument **is invalid**:

If Zeke is a cheater, **then** Zeke sits in the back row.

Zeke sits in the back row.

∴ Zeke is a cheater.

$$p \to q$$
$$q$$
$$\therefore p$$

Inverse error

Consider the following argument::

If interest rates are going up, stock market prices will go down.

Interest rates are not going up.

∴ Stock market prices will not go down.

$$p \rightarrow q$$
$$\sim p$$
$$\therefore \sim q$$

Contradiction Rule

Contradiction Rule

If you can show that the supposition that statement p is false leads logically to a contradiction, then you can conclude that p is true.

$\sim p \to$ **c**, where **c** is a contradiction
$\therefore p$

			premises	conclusion
p	$\sim p$	c	$\sim p \to$ c	p
T	F	F	T	T
F	T	F	F	

There is only one critical row in which the premise is true, and in this row the conclusion is also true. Hence this form of argument is valid.

If an assumption leads to a contradiction, then that assumption must be false.

Predicates

Predicate	**Predicate Variable**	**Statements**	
P(x): "$x^2 > x$"	x	*P(2): "$2^2 > 2$"*	*P(1/2): "$1/2^2 > 2$"*
		True	False
	Domain		
	$x \in \mathbb{R}$		

Truth set of a Predicate

Let $P(x)$: "n is a factor of 8. The domain of n is the set of all positive integers"

Truth Set $= \{x \in \mathbb{Z}^+ \mid P(x)\} = \{1, 2, 4, 8\}$

Review

Universal Quantifier Statement **Counterexample**

$\forall x \in \mathbb{R},\ x^2 > x.$ False $x = \frac{1}{2}$

Existential Quantifier Statement **Example**

$\exists x \in \mathbb{Z}^+,\ x^2 = x.$ True $x = 1$

Universal Conditional Statement

$\forall x \in \mathbb{R},\ (x > 2 \to x^2 > 4).$ True

Negation

$$\sim (\forall x \in D, Q(x)) \ \equiv\ \exists x \in D, \sim Q(x).$$

$$\sim (\exists x \in D, P(x)) \ \equiv\ \forall x \in D, \sim P(x).$$

$$\sim (\forall x \in D, P(x) \to Q(x)) \ \equiv\ \exists x \text{ such that } (P(x) \wedge \sim Q(x)).$$

Finding Truth Values of a Predicate

• Definition

If $P(x)$ is a predicate and x has domain D, the **truth set** of $P(x)$ is the set of all elements of D that make $P(x)$ true when they are substituted for x. The truth set of $P(x)$ is denoted

$$\{x \in D \mid P(x)\}.$$

The Universal Quantifier: ∀

The symbol ∀ denotes "for all" and is called the **universal quantifier**.

• **Definition**

Let $Q(x)$ be a predicate and D the domain of x. A **universal statement** is a statement of the form "$\forall x \in D, Q(x)$." It is defined to be true if, and only if, $Q(x)$ is true for every x in D. It is defined to be false if, and only if, $Q(x)$ is false for at least one x in D. A value for x for which $Q(x)$ is false is called a **counterexample** to the universal statement.

"All human beings are mortal"

$\forall$ human beings x, x is mortal.

$\forall x \in H, x \text{ is mortal}$

"For all x in the set of all human beings, x is mortal."

The Existential Quantifier: ∃

The symbol ∃ denotes “there exists” and is called the **existential quantifier**.

- **Definition**

Let $Q(x)$ be a predicate and D the domain of x. An **existential statement** is a statement of the form “$\exists x \in D$ such that $Q(x)$.” It is defined to be true if, and only if, $Q(x)$ is true for at least one x in D. It is false if, and only if, $Q(x)$ is false for all x in D.

“There is a student in Math 140”

$\exists p \in P$ such that p is a student in Math 140

Lecture 9 – part 3

More on Quantified Statements

Universal Conditional Statement

A reasonable argument can be made that the most important form of statement in mathematics is the **universal conditional statement:**

$\forall$*x,* if *P(x)* then *Q(x)*

$$\forall x \in \mathbb{R},\ (x > 2 \rightarrow x^2 > 4)$$

If a real number is greater than 2 then its square is greater than 4.

Negation of Quantified Statements

Negation of a Universal Statement

The negation of a statement of the form

$$\forall x \text{ in } D,\ Q(x)$$

is logically equivalent to a statement of the form

$$\exists x \text{ in } D \text{ such that } \sim Q(x).$$

Symbolically, $\sim(\forall x \in D,\ Q(x)) \equiv \exists x \in D \text{ such that } \sim Q(x).$

The negation of a universal statement (“all are”) is logically equivalent to an existential statement (“some are not” or “there is at least one that is not”).

Negation of Quantified Statements

Negation of an Existential Statement

The negation of a statement of the form

$$\exists x \text{ in } D \text{ such that } Q(x)$$

is logically equivalent to a statement of the form

$$\forall x \text{ in } D, \sim Q(x).$$

Symbolically, $\sim(\exists x \in D \text{ such that } Q(x)) \equiv \forall x \in D, \sim Q(x).$

The negation of an existential statement (“some are”) is logically equivalent to a universal statement (“none are” or “all are not”).

Write formal negations for the following statements:

a. $\forall$ primes p, p is odd.

b. $\exists$ a triangle T such that the sum of the angles of T equals 200°.

Solution:

a. By applying the rule for the negation of a $\forall$ statement, you can see that the answer is

$\exists$ a prime p such that p is not odd.

b. By applying the rule for the negation of a $\exists$ statement, you can see that the answer is

$\forall$ triangles T, the sum of the angles of T does not equal 200°.

Rewrite the following statement formally. Then write formal and informal negations.

No politicians are honest.

Solution

Formal version: $\forall$ politicians x, x is not honest.

Formal negation: $\exists$ a politician x such that x is honest.

Informal negation: Some politicians are honest.

Negations of Universal Conditional Statements

Negation of a Universal Conditional Statement

$$\sim(\forall x, \text{ if } P(x) \text{ then } Q(x)) \equiv \exists x \text{ such that } P(x) \text{ and } \sim Q(x).$$

By definition of the negation of a *for all* statement,

$$\sim (\forall x, P(x) \rightarrow Q(x)) \equiv \exists x \text{ such that } \sim (P(x) \rightarrow Q(x)). \qquad (1)$$

But the negation of an *if-then* statement is logically equivalent to an *and* statement. More precisely,

$$\sim (P(x) \rightarrow Q(x)) \equiv P(x) \wedge \sim Q(x). \qquad (2)$$

Substituting (2) into (1) gives

$$\sim (\forall x, P(x) \rightarrow Q(x)) \equiv \exists x \text{ such that } (P(x) \wedge \sim Q(x)).$$

Variants of Universal Conditional Statements

• Definition

Consider a statement of the form: $\forall x \in D$, if $P(x)$ then $Q(x)$.

1. Its **contrapositive** is the statement: $\forall x \in D$, if $\sim Q(x)$ then $\sim P(x)$.
2. Its **converse** is the statement: $\forall x \in D$, if $Q(x)$ then $P(x)$.
3. Its **inverse** is the statement: $\forall x \in D$, if $\sim P(x)$ then $\sim Q(x)$.

Lecture 9 – part 4

Arguments with Quantified Statements

The rule of universal instantiation

If some property is true of *everything* in a set, then it is true of *any particular* thing in the set.

The validity of this argument form follows immediately from the definition of truth values for a universal statement.

All men are mortal.
Socrates is a man.
∴ Socrates is mortal.

Universal **instantiation** is the fundamental tool of deductive reasoning.

Problem:
Simplify the following expression

$$r^{k+1}r$$

Where r is a particular real number
and k is a particular integer

Universal true statements from Algebra:

1. For all real numbers x and all integers m and n, $x^m x^n = x^{m+n}$, or $\forall x \in \mathbb{R}, \forall m, n \in \mathbb{Z} : x^m x^n = x^{m+n}$.
2. For all real numbers x, $x^1 = x$, or $\forall x \in \mathbb{R} : x^1 = x$

$$\begin{aligned} r^{k+1}r &= r^{k+1}r^1 && \text{Step 1} \\ &= r^{(k+1)+1} && \text{Step 2} \\ &= r^{k+2} && \text{by basic algebra} \end{aligned}$$

The reasoning behind step 1 and step 2 is outlined as follows.

Step 1: For all real numbers x, $x^1 = x$. — universal truth
r is a particular real number. — particular instance
$\therefore r^1 = r$. — conclusion

Step 2: For all real numbers x and all integers m and n, $x^m \cdot x^n = x^{m+n}$. — universal truth
r is a particular real number and $k+1$ and 1 are particular integers. — particular instance
$\therefore r^{k+1} \cdot r^1 = r^{(k+1)+1}$. — conclusion

Both arguments are examples of universal instantiation.

Universal Modus Ponens

$p \rightarrow q$
p
$\therefore q$

The rule of universal instantiation can be combined with modus ponens to obtain the valid form of argument called **universal modus ponens**.

Universal Modus Ponens

Formal Version	*Informal Version*
$\forall x$, if $P(x)$ then $Q(x)$.	If x makes $P(x)$ true, then x makes $Q(x)$ true.
$P(a)$ for a particular a.	a makes $P(x)$ true.
$\therefore Q(a)$.	$\therefore$ a makes $Q(x)$ true.

Exercise:

Rewrite the following argument using quantifiers, variables, and predicate symbols.
Is this argument valid? Why?

If an integer is even, then its square is even.

k is a particular integer that is even.

$\therefore k^2$ is even.

$$\forall x \in \mathbb{Z}, (x \text{ is even} \implies x^2 \text{ is even})$$

Solution:

$$\forall x \in \mathbb{Z}, (\exists k \in \mathbb{Z} : x = 2k \implies \exists l \in \mathbb{Z} : x^2 = 2l)$$

Let E(x) be “x is an even integer,”
let S(x) be “x2 is even,” and
let k stand for a particular integer that is even.

$\forall x$, if $E(x)$ then $S(x)$.

$E(k)$, for a particular k.

$\therefore S(k)$.

Universal Modus Ponens
VALID

Universal Modus Tollens

$p \to q$
$\sim q$
$\therefore \sim p$

Universal modus tollens is the heart of **proof of contradiction**, which is one of the most important methods of mathematical argument.

Universal Modus Tollens

Formal Version	*Informal Version*
$\forall x$, if $P(x)$ then $Q(x)$.	If x makes $P(x)$ true, then x makes $Q(x)$ true.
$\sim Q(a)$, for a particular a.	a does not make $Q(x)$ true.
$\therefore \sim P(a)$.	$\therefore$ a does not make $P(x)$ true.

Exercise:

Rewrite the following argument using quantifiers, variables, and predicate symbols.
Is this argument valid? Why?

All human beings are mortal.
Zeus is not mortal.
∴ Zeus is not human.

Solution The major premise can be rewritten as

$$\forall x, \text{ if } x \text{ is human then } x \text{ is mortal.}$$

Let $H(x)$ be "x is human," let $M(x)$ be "x is mortal," and let Z stand for Zeus. The argument becomes

$$\begin{aligned} &\forall x, \text{ if } H(x) \text{ then } M(x) \\ &\sim M(Z) \\ \therefore\ &\sim H(Z). \end{aligned}$$

This argument has the form of universal modus tollens and is therefore valid. ■

Fallacies

$$p \rightarrow q$$
$$q$$
$$\therefore p$$

Converse Error (Quantified Form)

Formal Version	*Informal Version*
$\forall x$, if $P(x)$ then $Q(x)$.	If x makes $P(x)$ true, then x makes $Q(x)$ true.
$Q(a)$ for a particular a.	a makes $Q(x)$ true.
$\therefore P(a)$. ← invalid conclusion	$\therefore$ a makes $P(x)$ true. ← invalid conclusion

$$p \rightarrow q$$
$$\sim p$$
$$\therefore \sim q$$

Inverse Error (Quantified Form)

Formal Version	*Informal Version*
$\forall x$, if $P(x)$ then $Q(x)$.	If x makes $P(x)$ true, then x makes $Q(x)$ true.
$\sim P(a)$, for a particular a.	a does not make $P(x)$ true.
$\therefore \sim Q(a)$. ← invalid conclusion	$\therefore$ a does not make $Q(x)$ true. ← invalid conclusion

All the town criminals frequent the Den of Iniquity bar.
John frequents the Den of Iniquity bar.
∴ John is one of the town criminals.

Lecture 9 – part 5

Logic Exercises

74, 76, 77, 82, 87, 91, 106, 108, 118, 121...