COT 2000
Foundations of Computing

Spring 2024

Lecture 16 – part 1

Lab 8
Exam 3– 07/12/24

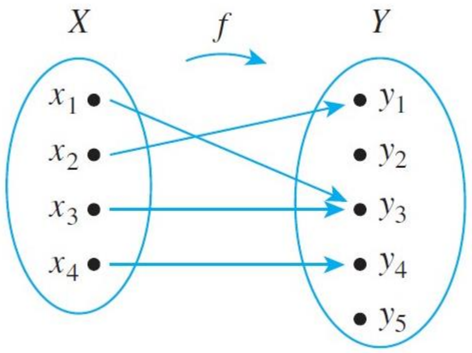
Lecture 16 – part 2

Review

Review

- Functions revisited, definition
- Equality of functions
- The identity function
- Functions to define sequences
- Boolean Functions
- One-to-one Functions
- Onto Functions
- One to One correspondence
- Inverse functions
- Composition of functions

Function definition



1. **Domain (X):** $X = \{x_1, x_2, x_3, x_4\}$.
2. **Co-domain (Y):** $Y = \{y_1, y_2, y_3, y_4, y_5\}$.
3. **Range:** The range of the function f is given by:

$$\text{Range of } f = \{y_1, y_3, y_4\}$$

4. **Image of X under f:**

$$f(X) = \{y_1, y_3, y_4\}$$

5. **Preimage of y:** For y_3 as an example,

$$f^{-1}(y_3) = \{x_1, x_3\}$$

6. **Inverse Image of y:** For y_3,

$$f^{-1}(y_3) = \{x_1, x_3\}$$

Functions to define sequences

- There are many functions that can be used to define a given sequence.

The sequence:

$$1, -\frac{1}{2}, -\frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, \ldots, (-1)^n \frac{1}{n+1}, \ldots$$

can be thought of as the function f from the nonnegative integers to the real numbers that associates:

$$\begin{aligned}
0 &\to 1, \\
1 &\to -\frac{1}{2}, \\
2 &\to -\frac{1}{3}, \\
3 &\to -\frac{1}{4}, \\
4 &\to \frac{1}{5}, \\
&\vdots \\
n &\to (-1)^n \frac{1}{n+1}.
\end{aligned}$$

In other words, $f : \mathbb{Z}_{\text{nonneg}} \to \mathbb{R}$ is the function defined as follows:

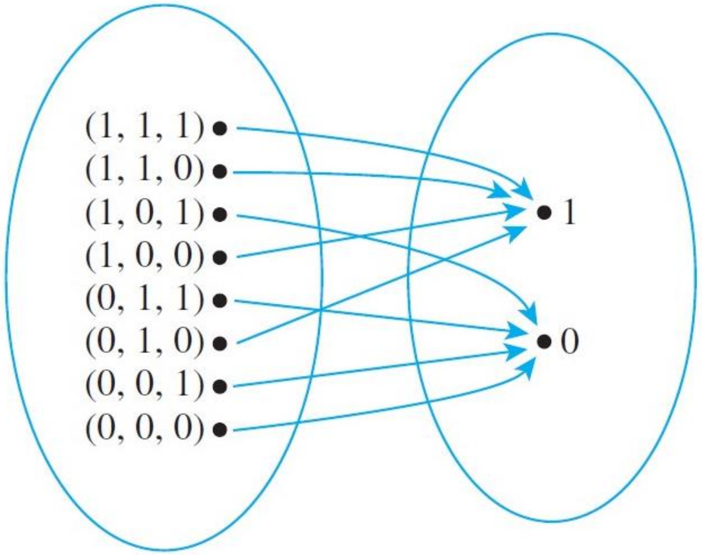
$$\forall n \geq 0 : f(n) = (-1)^n \frac{1}{n+1}$$

Boolean Function

• Definition

An **(n-place) Boolean function** f is a function whose domain is the set of all ordered n-tuples of 0's and 1's and whose co-domain is the set $\{0, 1\}$. More formally, the domain of a Boolean function can be described as the Cartesian product of n copies of the set $\{0, 1\}$, which is denoted $\{0, 1\}^n$. Thus $f: \{0, 1\}^n \to \{0, 1\}$.

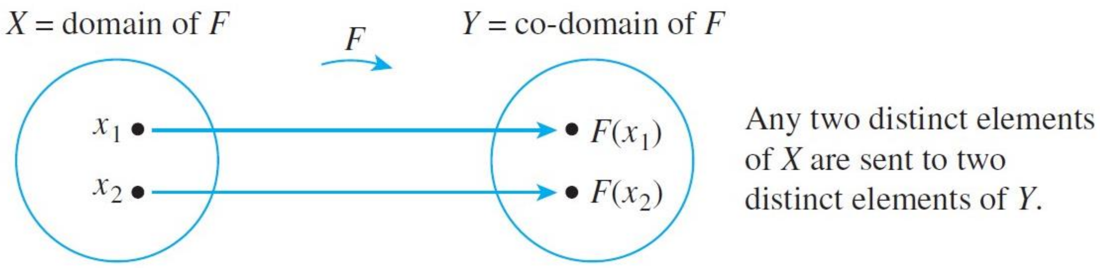
Input			Output
P	Q	R	S
1	1	1	1
1	1	0	1
1	0	1	0
1	0	0	1
0	1	1	0
0	1	0	1
0	0	1	0
0	0	0	0



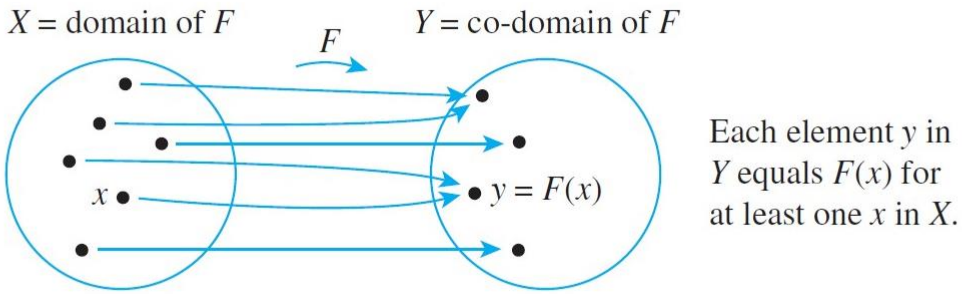
Tsetlin Machines

https://en.wikipedia.org/wiki/Tsetlin_machine

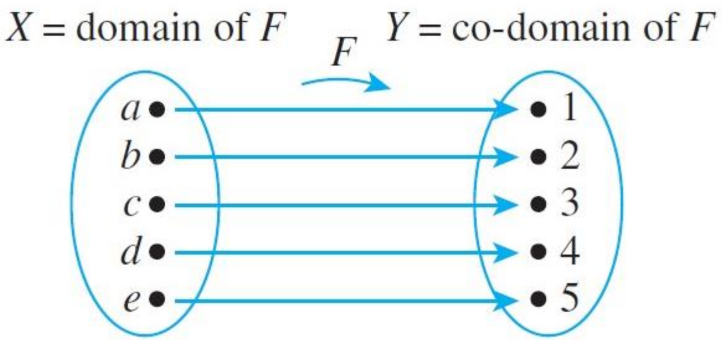
One-to-One Functions



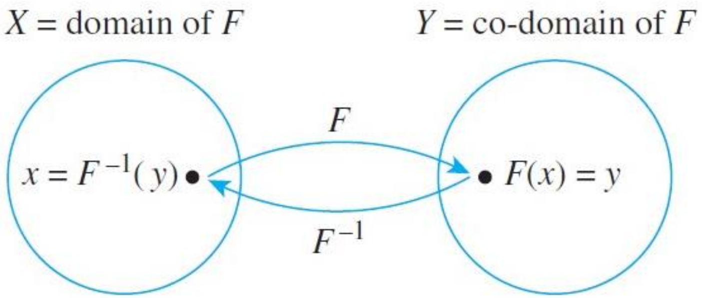
Onto Functions



One to One Correspondence



Inverse Function



Inverse Functions

Suppose $F: X \to Y$ is a one-to-one correspondence; that is, suppose F is one-to-one and onto. Then there is a function $\boldsymbol{F^{-1}: Y \to X}$ that is defined as follows:

Given any element y in Y,

$$\boldsymbol{F^{-1}(y)} = \text{that unique element } x \text{ in } X \text{ such that } F(x) \text{ equals } y.$$

In other words,

$$F^{-1}(y) = x \quad \Leftrightarrow \quad y = F(x).$$

If X and Y are sets and $F: X \to Y$ is one-to-one and onto, then $F^{-1}: Y \to X$ is also one-to-one and onto.

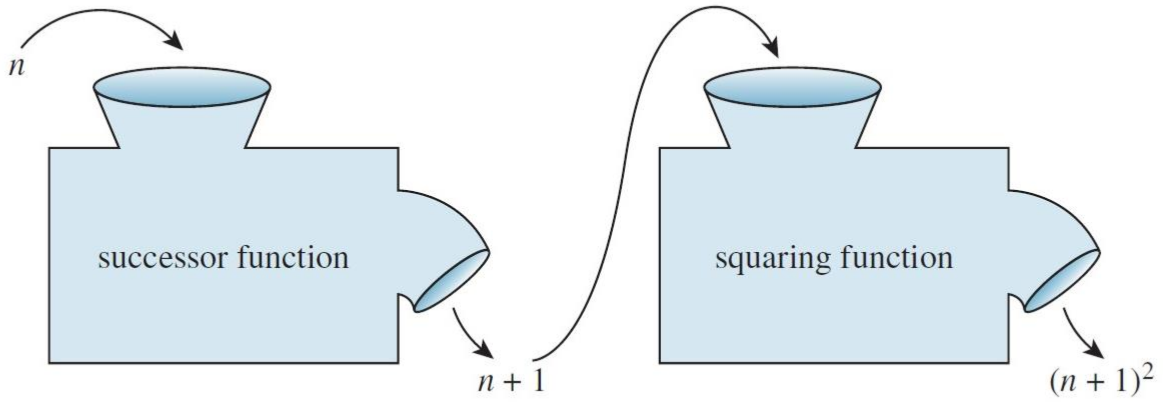
Composition of Functions

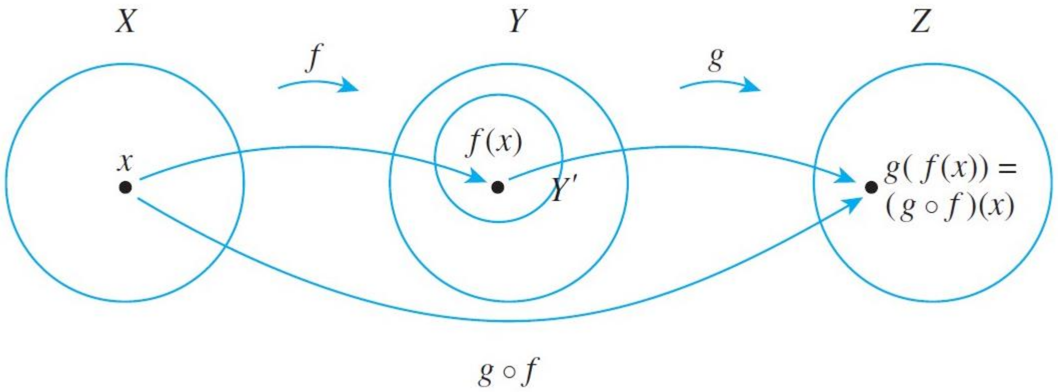
• Definition

Let $f: X \to Y'$ and $g: Y \to Z$ be functions with the property that the range of f is a subset of the domain of g. Define a new function $g \circ f: X \to Z$ as follows:

$$(g \circ f)(x) = g(f(x)) \quad \text{for all } x \in X,$$

where $g \circ f$ is read "g circle f" and $g(f(x))$ is read "g of f of x." The function $g \circ f$ is called the **composition of f and g.**





Lecture 16 – part 3

Function Exercises

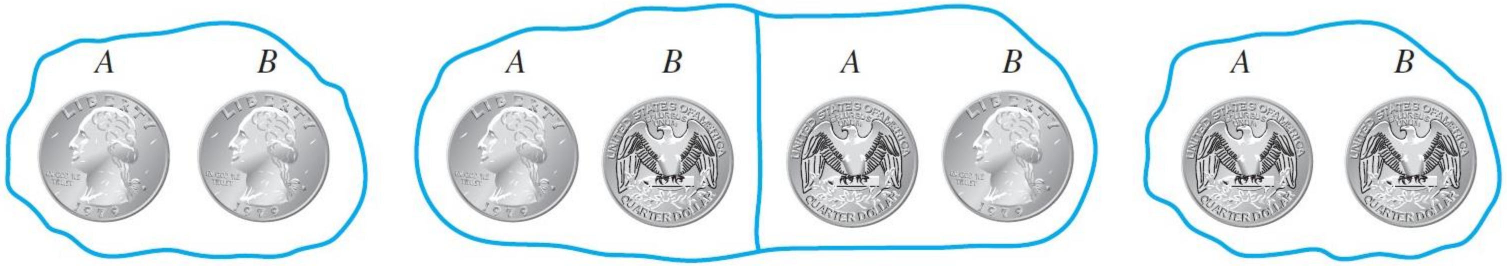
Lecture 16 – part 4

Counting and Probability

Introduction

Experiment: Tossing 2 quarters 50 times:

Event	Tally	Frequency (Number of times the event occurred)	Relative Frequency (Fraction of times the event occurred)
2 heads obtained	~~||||~~ ~~||||~~ |	11	22%
1 head obtained	~~||||~~ ~~||||~~ ~~||||~~ ~~||||~~ ~~||||~~ ||	27	54%
0 heads obtained	~~||||~~ ~~||||~~ ||	12	24%



2 heads obtained

1 head obtained

0 heads obtained

Random Process – Sample Space

- To say that a process is **random** means that when it takes place, one outcome from some set of outcomes is sure to occur, but it is **impossible to predict with certainty which outcome** that will be.
- The **set of outcomes** that can result from a random process or experiment is called a **sample space**.

• Definition

A **sample space** is the set of all possible outcomes of a random process or experiment. An **event** is a subset of a sample space.

Probability

- In case an experiment has finitely many outcomes and all outcomes are equally likely to occur, the **probability of an event** (set of outcomes) is just <u>the ratio of the number of outcomes in the event to the total number of outcomes</u>.

Equally Likely Probability Formula

If S is a finite sample space in which all outcomes are equally likely and E is an event in S, then the **probability of** $\boldsymbol{E}$, denoted $\boldsymbol{P(E)}$, is

$$P(E) = \frac{\text{the number of outcomes in } E}{\text{the total number of outcomes in } S}.$$

Number of elements N(A):

• Notation

For any finite set A, $N(A)$ denotes the number of elements in A.

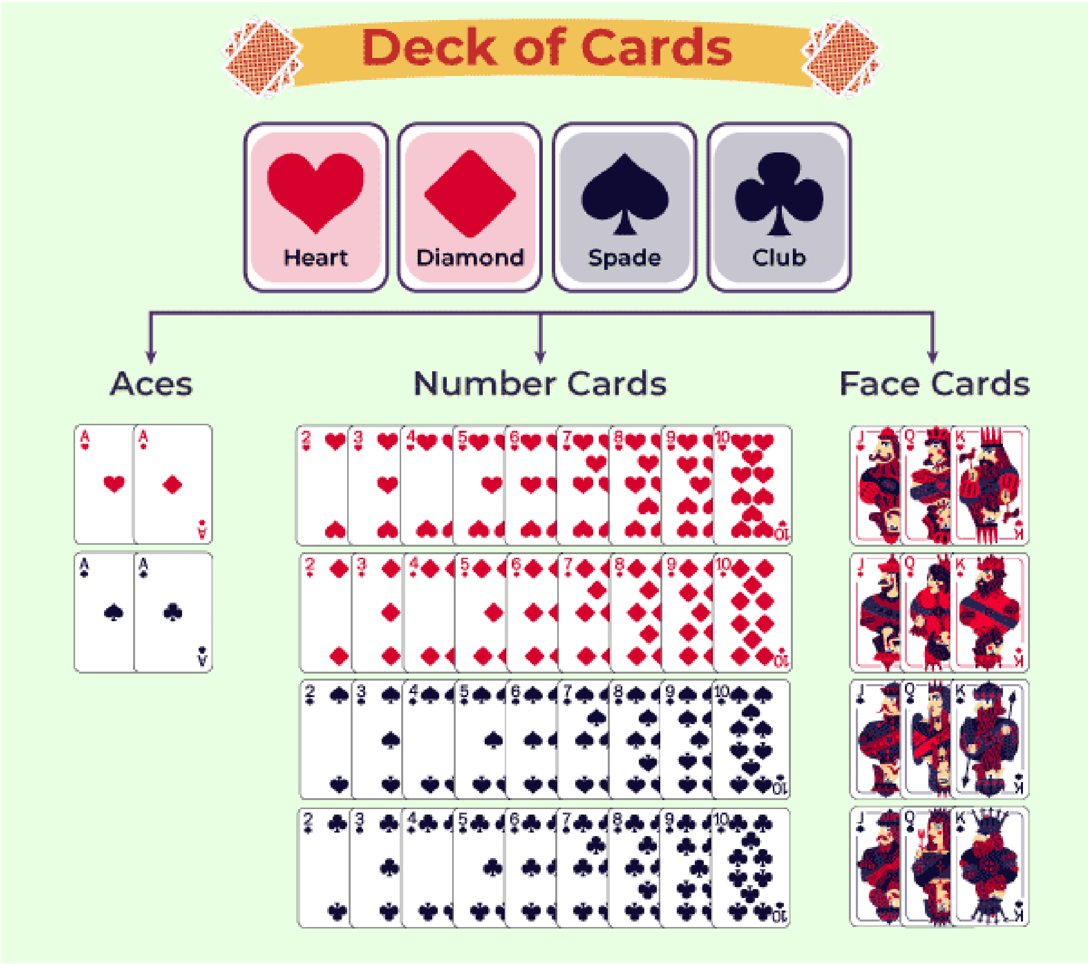
Probability:

$$P(E) = \frac{N(E)}{N(S)}.$$

Number of outcomes in E (set of outcomes)
E is a subset of S

Number of total outcomes in S (sample space)

Example: An ordinary deck of cards contains 52 cards divided into four *suits*. The *red suits* are diamonds (♦) and hearts (♥) and the *black suits* are clubs (♣) and spades (♠). Each suit contains 13 cards of the following *denominations:* 2, 3, 4, 5, 6, 7, 8, 9, 10, J (jack), Q (queen), K (king), and *A* (ace). The cards J, Q, and K are called *face cards*.



https://www.geeksforgeeks.org/card-probability/

a) What is the sample space of outcomes ? (S)
b) What is the event that the chosen card is a black face card ? (E)
c) What is the probability that the chosen card is a black face card ?

Solution:

(a) The outcomes in the sample space S are the 52 cards in the deck.

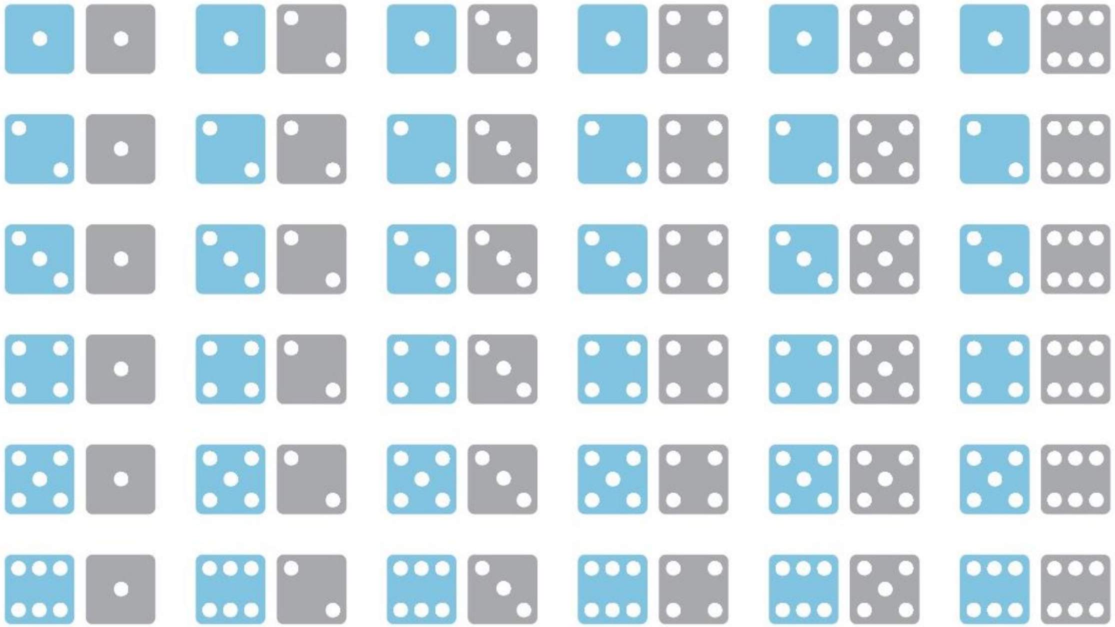
(b) Let E be the event that a black face card is chosen. The outcomes in E are the jack, queen, and king of clubs and the jack, queen, and king of spades. Symbolically,

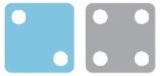
$$E = \{\text{J}\clubsuit, \text{Q}\clubsuit, \text{K}\clubsuit, \text{J}\spadesuit, \text{Q}\spadesuit, \text{K}\spadesuit\}.$$

(c) By part (b), $N(E) = 6$, and according to the description of the situation, all 52 outcomes in the sample space are equally likely. Therefore, by the equally likely probability formula, the probability that the chosen card is a black face card is

$$P(E) = \frac{N(E)}{N(S)} = \frac{6}{52} \approx 11.5\%.$$

Example: Rolling a pair of dice



A more compact notation identifies, say,  with the notation 24,  with 53, and so forth.

a. Use the compact notation to write the sample space S of possible outcomes.

b. Use set notation to write the event E that the numbers showing face up have a sum of 6 and find the probability of this event.

Solution:

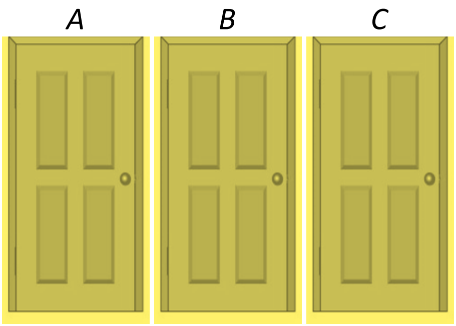
(a) $S = \{11, 12, 13, 14, 15, 16, 21, 22, 23, 24, 25, 26, 31, 32, 33, 34, 35, 36, 41, 42, 43, 44, 45, 46, 51, 52, 53, 54, 55, 56, 61, 62, 63, 64, 65, 66\}$.

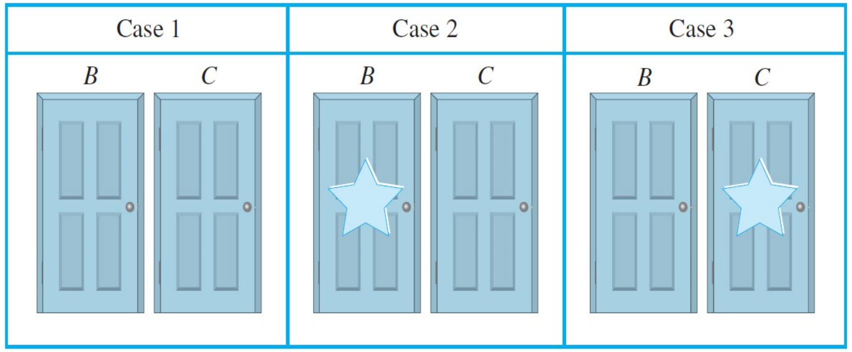
(b) $E = \{15, 24, 33, 42, 51\}$.

The probability that the sum of the numbers is 6:

$$P(E) = \frac{N(E)}{N(S)} = \frac{5}{36} = 13.9\%.$$

The Monty Hall Problem





Situation:

There are three doors on the set for a game show. Let's call them A, B, and C.

If you pick the right door you win the prize.

You pick door A. The host of the show, Monty Hall, then opens one of the other doors and reveals that there is no prize behind it.

Keeping the remaining two doors closed, he asks you whether you want to switch your choice to the other closed door or stay with your original choice of door A.

Questions:

- What should you do if you want to maximize your chance of winning the prize: stay with door A or switch
- Would the likelihood of winning be the same either way?

Solution:

The Monty Hall Problem: To Switch or Not to Switch?

- **Initial Situation:** Prize behind one of three doors (A, B, C). You pick door A.
- **Host's Action:** Host, who knows what's behind the doors, opens one (say B or C) with no prize.
- **Cases:**
  * **Case 1:** Prize is behind A. Host opens B or C. *Staying wins.*
  * **Case 2:** Prize is behind B. Host opens C. *Switching wins.*
  * **Case 3:** Prize is behind C. Host opens B. *Switching wins.*
- **Analysis:** In 2 out of 3 cases, switching leads to a win.
- **Strategy:** Always switch for a **2/3 win chance**.

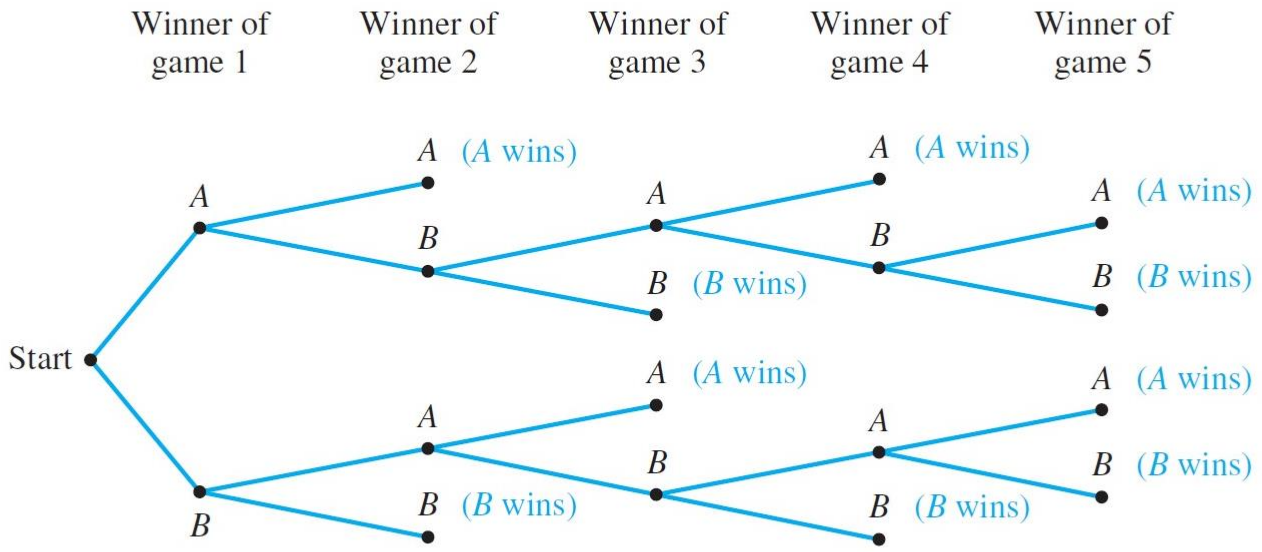
Possibility Trees

This is a **tree structure tool** that is useful for keeping systematic track of all possibilities in situations in which events happen in order.

Example:

Teams A and B are to play each other repeatedly until one wins two games in a row **or** a total of three games.

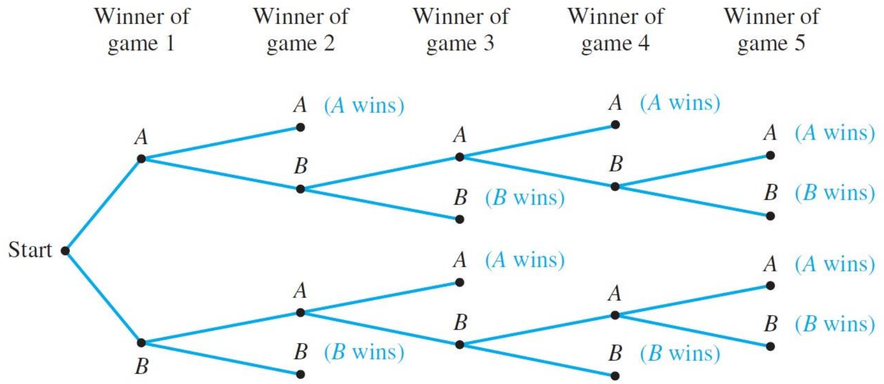
Sample case:
A win the first game, B win the second, and A win the third and fourth games.
Denoted by A–B–A–A.



Example: Tournament Play

a. How many ways can the tournament be played?

b. Assuming that all the ways of playing the tournament are equally likely, what is the probability that five games are needed to determine the tournament winner?



Solution:

- **Possible Outcomes:** 10 ways the tournament can end.
- **Outcomes Detailed:**
  - Two quick wins: A–A, B–B.
  - Four games needed: A–B–A–A, A–B–B, B–A–A, B–A–B–B.
  - Full five games: A–B–A–B–A, A–B–A–B–B, B–A–B–A–A, B–A–B–A–B.
- **Probability Analysis:**
  - **Minimum Games:** 2 games to declare a winner.
  - **Maximum Games:** 5 games for a conclusive result.
  - **Probability for 5 Games:** $\frac{4}{10} = 40\%$ since 4 out of 10 paths require all 5 games.

The Multiplication Rule

If an operation consists of k steps and

the first step can be performed in n_1 ways,

the second step can be performed in n_2 ways *[regardless of how the first step was performed]*,

$\vdots$

the kth step can be performed in n_k ways *[regardless of how the preceding steps were performed]*,

then the entire operation can be performed in $n_1 n_2 \cdots n_k$ ways.

In simple terms, the rule helps you find out how many different ways you can complete a series of steps or make choices when each step or choice is independent of the others.

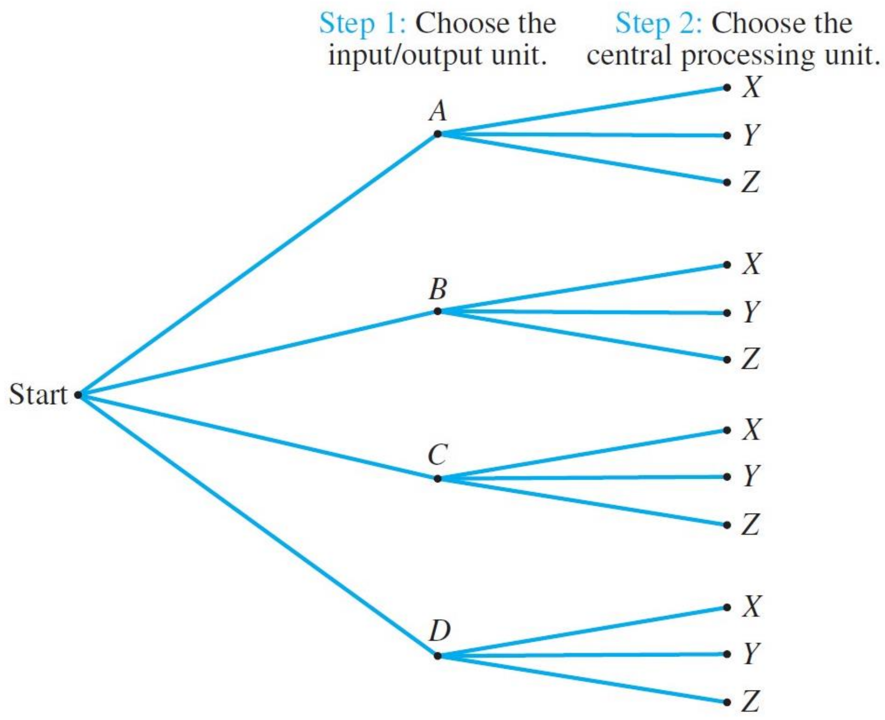
Example: Suppose a computer installation has:

- Four input/output units (*A, B, C,* and *D*) and
- Three central processing units (*X, Y,* and *Z*).

Any input/output unit can be paired with any central processing unit.

How many ways are there to pair an input/output unit with a central processing unit?

Solution:



Thus the total number of ways to pair the two types of units is the same as the number of branches of the tree, which is:

$3 + 3 + 3 + 3 = 4 \cdot 3 = 12.$

Example

Imagine you're making a sandwich. You have to choose the type of bread, the kind of filling, and the type of sauce. The options available for each part of the sandwich are:

- Choose the type of bread: 3 options
- Choose the kind of filling: 5 options (and this choice doesn't depend on which bread you chose)
- Choose the type of sauce: 2 options (and this doesn't depend on the bread or filling)

Question: find out how many unique sandwiches you can make.

Solution

To find out how many unique sandwiches you can make, you just multiply the number of options at each step:

$$\text{Number of types of bread} \times \text{Number of kinds of fillings} \times \text{Number of types of sauces}$$
$$3 \times 5 \times 2 = 30 \text{ different sandwiches}$$