COT 2000
Foundations of Computing

Summer 2024

Lecture 2 – part 1

Homework 1 – Due 05/24/24

Lecture 2 – part 2

Review

Review

- What are the foundational topics in mathematics?
- What is discrete mathematics?
- Why is discrete mathematics important for Computer Science?
- Which topics are we going to cover in this course?
- What are some core concepts of set theory?
- What is the set-roster and set-builder notations in set theory?
- What the symbols $\in$ and $\notin$ means?
- What does the symbols $\subseteq$, $\subset$ and $\nsubseteq$ means?

Lecture 2 – part 3

Language of Sets (cont)

Ordered pairs

- {a, b} and {b, a} represent the same set
- **How to deal with ordered pairs ?** In an ordered pair we want to be able to indicate which element comes first.
- **Solution:**
- An ordered pair is a set of the form {{a}, {a, b}}.
- This set has elements, {a} and {a, b}. This allows us to distinguish between a and b and say that a is the first element of the ordered pair and b is the second element of the pair.
- The usual notation is more simply as (a, b).

Two ordered pairs (a, b) and (c, d) are equal if, and only if,
$a = c$ and $b = d$.

Symbolically:
$(a, b) = (c, d)$ means that $a = c$ and $b = d$.

Ordered Pairs

a. Is $(1, 2) = (2, 1)$?

b. Is $\left(3, \frac{5}{10}\right) = \left(\sqrt{9}, \frac{1}{2}\right)$?

c. What is the first element of $(1, 1)$?

Solution

a. No. By definition of equality of ordered pairs,

$$(1, 2) = (2.1) \text{ if, and only if, } 1 = 2 \text{ and } 2 = 1.$$

But $1 \neq 2$, and so the ordered pairs are not equal.

b. Yes. By definition of equality of ordered pairs,

$$\left(3, \frac{5}{10}\right) = \left(\sqrt{9}, \frac{1}{2}\right) \text{ if, and only if, } 3 = \sqrt{9} \text{ and } \frac{5}{10} = \frac{1}{2}.$$

Because these equations are both true, the ordered pairs are equal.

c. In the ordered pair $(1, 1)$, the first and the second elements are both 1.

Cartesian product

• Definition

Given sets A and B, the **Cartesian product of A and B**, denoted $\mathbf{A \times B}$ and read "A cross B," is the set of all ordered pairs (a, b), where a is in A and b is in B. Symbolically:

$$\mathbf{A \times B} = \{(a, b) \mid a \in A \text{ and } b \in B\}.$$

Let $A = \{1, 2, 3\}$ and $B = \{u, v\}$.

a. Find $A \times B$

b. Find $B \times A$

c. Find $B \times B$

d. How many elements are in $A \times B$, $B \times A$, and $B \times B$?

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b. Find $B \times A$

c. Find $B \times B$

d. How many elements are in $A \times B$, $B \times A$, and $B \times B$?

SOLUTION

a. $A \times B = \{(1, u), (2, u), (3, u), (1, v), (2, v), (3, v)\}$

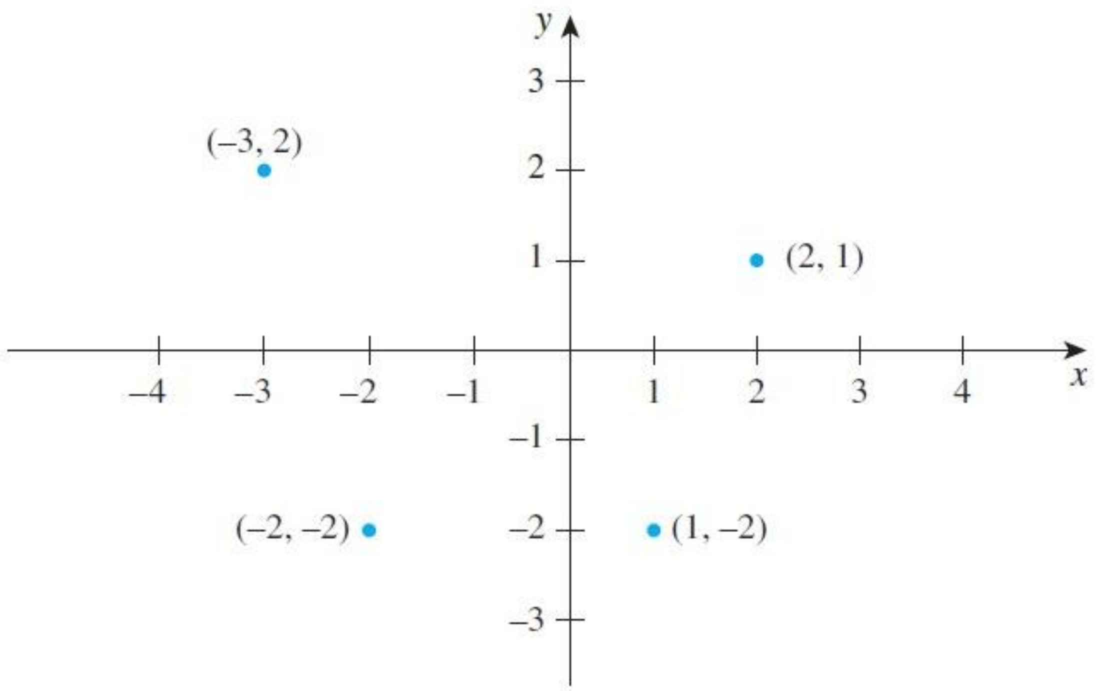
b. $B \times A = \{(u, 1), (u, 2), (u, 3), (v, 1), (v, 2), (v, 3)\}$

c. $B \times B = \{(u, u), (u, v), (v, u), (v, v)\}$

d. $A \times B$ has six elements. Note that this is the number of elements in A times the number of elements in B. $B \times A$ has six elements, the number of elements in B times the number of elements in A. $B \times B$ has four elements, the number of elements in B times the number of elements in B.

Cartesian Plane

Let **R** denote the set of all real numbers.
R × **R** is the set of all ordered pairs *(x, y)* where both *x* and *y* are real numbers.



Lecture 2 – part 4

Relations and Functions

Relations

Example:

Let $A = \{0, 1, 2\}$ and $B = \{1, 2, 3\}$ and let us say that an element x in A is related to an element y in B if, and only if, ***x* is less than *y***.

$0\ R\ 1$ since $0 < 1$,
$0\ R\ 2$ since $0 < 2$,
$0\ R\ 3$ since $0 < 3$,
$1\ R\ 2$ since $1 < 2$,
$1\ R\ 3$ since $1 < 3$, and
$2\ R\ 3$ since $2 < 3$.

$$A \times B = \{(0, 1), (0, 2), (0, 3), (1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3)\}.$$

$$\{(0, 1), (0, 2), (0, 3), (1, 2), (1, 3), (2, 3)\}.$$

Relations and the Cartesian Product

• Definition

Let A and B be sets. A **relation R from A to B** is a subset of $A \times B$. Given an ordered pair (x, y) in $A \times B$, **x is related to y by R**, written $x\ R\ y$, if, and only if, (x, y) is in R. The set A is called the domain of R and the set B is called its co-domain.

The notation for a relation R may be written symbolically as follows:

$$x\ R\ y \quad \text{means that} \quad (x, y) \in R.$$

Example

A Relation as a Subset

Let $A = \{1, 2\}$ and $B = \{1, 2, 3\}$ and define a relation R from A to B as follows: Given any $(x, y) \in A \times B$,

$$(x, y) \in R \quad \text{means that} \quad \frac{x - y}{2} \text{ is an integer.}$$

a. State explicitly which ordered pairs are in $A \times B$ and which are in R.

b. Is $1\ R\ 3$? Is $2\ R\ 3$? Is $2\ R\ 2$?

c. What are the domain and co-domain of R?

$A = \{1, 2\}$ and $B = \{1, 2, 3\}$

$(x, y) \in R$ means that $\dfrac{x - y}{2}$ is an integer.

a. $A \times B = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3)\}$. To determine explicitly the composition of R, examine each ordered pair in $A \times B$ to see whether its elements satisfy the defining condition for R.

$$(1, 1) \in R \text{ because } \tfrac{1-1}{2} = \tfrac{0}{2} = 0, \text{ which is an integer.}$$

$$(1, 2) \notin R \text{ because } \tfrac{1-2}{2} = \tfrac{-1}{2}, \text{ which is not an integer.}$$

$$(1, 3) \in R \text{ because } \tfrac{1-3}{2} = \tfrac{-2}{2} = -1, \text{ which is an integer.}$$

$$(2, 1) \notin R \text{ because } \tfrac{2-1}{2} = \tfrac{1}{2}, \text{ which is not an integer.}$$

$$(2, 2) \in R \text{ because } \tfrac{2-2}{2} = \tfrac{0}{2} = 0, \text{ which is an integer.}$$

$$(2, 3) \notin R \text{ because } \tfrac{2-3}{2} = \tfrac{-1}{2}, \text{ which is NOT an integer.}$$

Thus

$$R = \{(1, 1), (1, 3), (2, 2)\}$$

The Circle Relation

Define a relation C from $\mathbf{R}$ to $\mathbf{R}$ as follows: For any $(x, y) \in \mathbf{R} \times \mathbf{R}$,

$$(x, y) \in C \quad \text{means that} \quad x^2 + y^2 = 1.$$

a. Is $(1, 0) \in C$? Is $(0, 0) \in C$? Is $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right) \in C$? Is $-2\ C\ 0$? Is $0\ C\ (-1)$? Is $1\ C\ 1$?

b. What are the domain and co-domain of C?

c. Draw a graph for C by plotting the points of C in the Cartesian plane.

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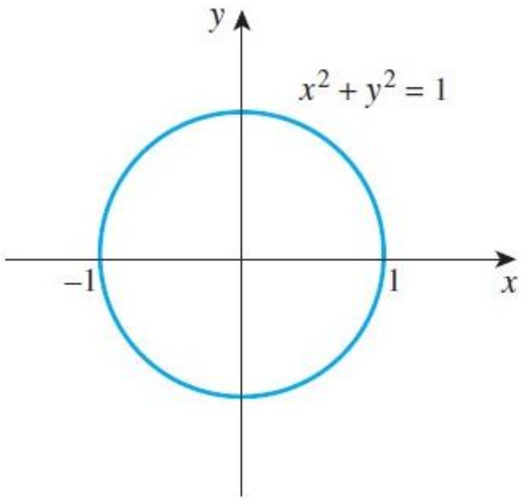
Solution



a. Yes, $(1, 0) \in C$ because $1^2 + 0^2 = 1$.
No, $(0, 0) \notin C$ because $0^2 + 0^2 = 0 \neq 1$.
Yes, $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right) \in C$ because $\left(-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{1}{4} + \frac{3}{4} = 1$.
No, $-2 \not\!C\ 0$ because $(-2)^2 + 0^2 = 4 \neq 1$.
Yes, $0\ C\ (-1)$ because $0^2 + (-1)^2 = 1$.
No, $1 \not\!C\ 1$ because $1^2 + 1^2 = 2 \neq 1$.

b. The domain and co-domain of C are both $\mathbf{R}$, the set of all real numbers.

Arrow Diagrams of Relations

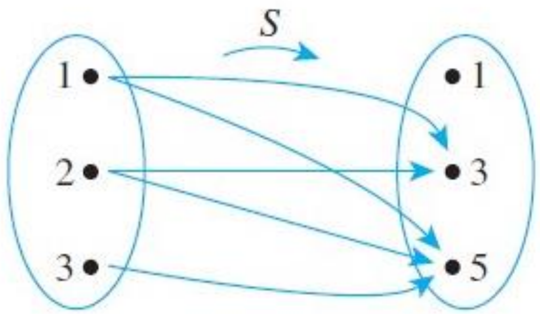
Let $A = \{1, 2, 3\}$ and $B = \{1, 3, 5\}$ and define relations S and T from A to B as follows: For all $(x, y) \in A \times B$,

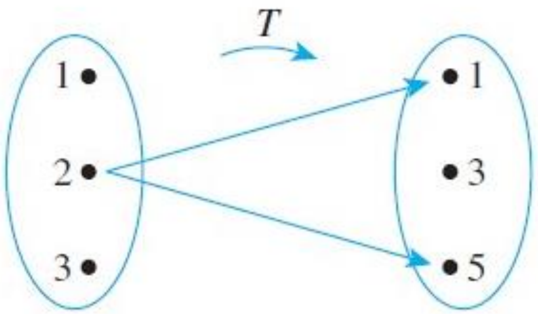
$$(x, y) \in S \quad \text{means that} \quad x < y \qquad \text{S is a "less than" relation.}$$

$$T = \{(2, 1), (2, 5)\}.$$

Draw arrow diagrams for S and T.

Solution





These example relations illustrate that it is possible to have several arrows coming out of the same element of A pointing in different directions. Also, it is quite possible to have an element of A that does not have an arrow coming out of it. ■

Functions

• Definition

A **function** F **from a set** A **to a set** B is a relation with domain A and co-domain B that satisfies the following two properties:

1. For every element x in A, there is an element y in B such that $(x, y) \in F$.
2. For all elements x in A and y and z in B,

$$\text{if} \quad (x, y) \in F \text{ and } (x, z) \in F, \quad \text{then} \quad y = z.$$

Properties (1) and (2) can be stated less formally as follows:

A relation F from A to B is a function if, and only if:

1. Every element of A is the first element of an ordered pair of F.
2. No two distinct ordered pairs in F have the same first element.

Each element in the domain corresponds to one and only one element of the co-domain.

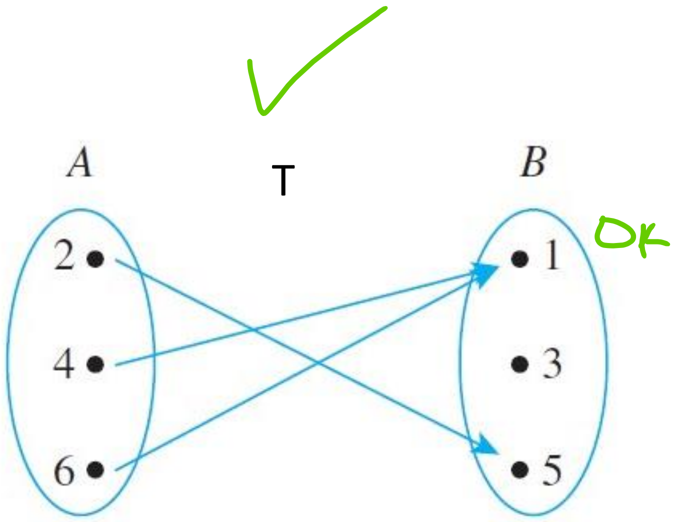
More precisely, if F is a function from a set A to a set B, then given any element x in A,

property (1) from the function definition guarantees that there is at least one element of B that is related to x by F, and

property (2) guarantees that there is at most one such element.

• Notation

If A and B are sets and F is a function from A to B, then given any element x in A, the unique element in B that is related to x by F is denoted $\boldsymbol{F(x)}$, which is read "***F* of *x***."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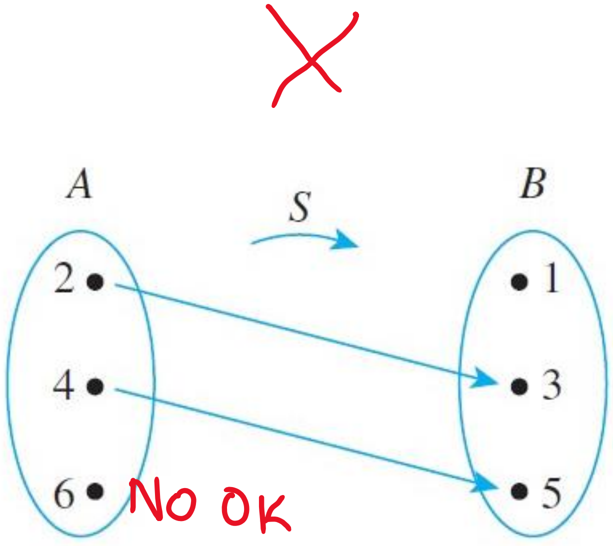
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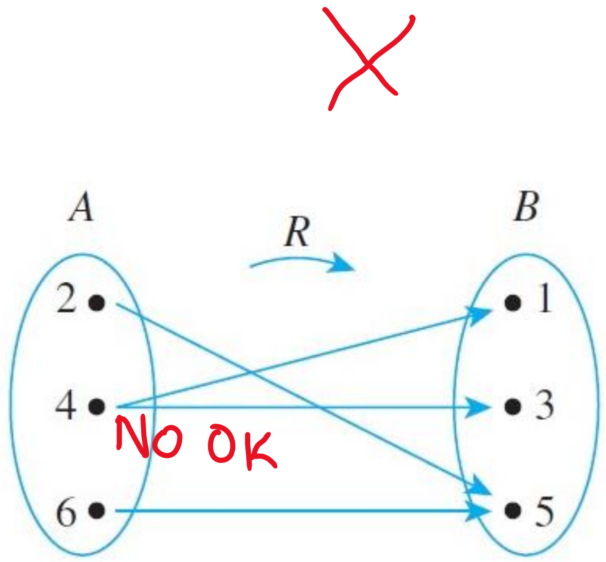
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Function Machines