COT 2000
Foundations of Computing

Summer 2024

Lecture 3 – part 1

Homework 1 - Due:05/24/24
Lab 2

Lecture 3 – part 2

Review

Review

- What are ordered pairs?
- What is the Cartesian product?
- What is the Cartesian plane?
- What is the concept of relations in set theory?
- What are arrow diagrams of relations?
- What is a function in set theory?
- What are the two conditions for a function?
- What are function machines?

Ordered pairs

An ordered pair is a set of the form **{{a}, {a, b}}**.

The usual notation is more simply as **(a, b).**

Two ordered pairs **(a, b)** and **(c, d)** are equal if, and only if, **a = c** and **b = d**.

Cartesian Product

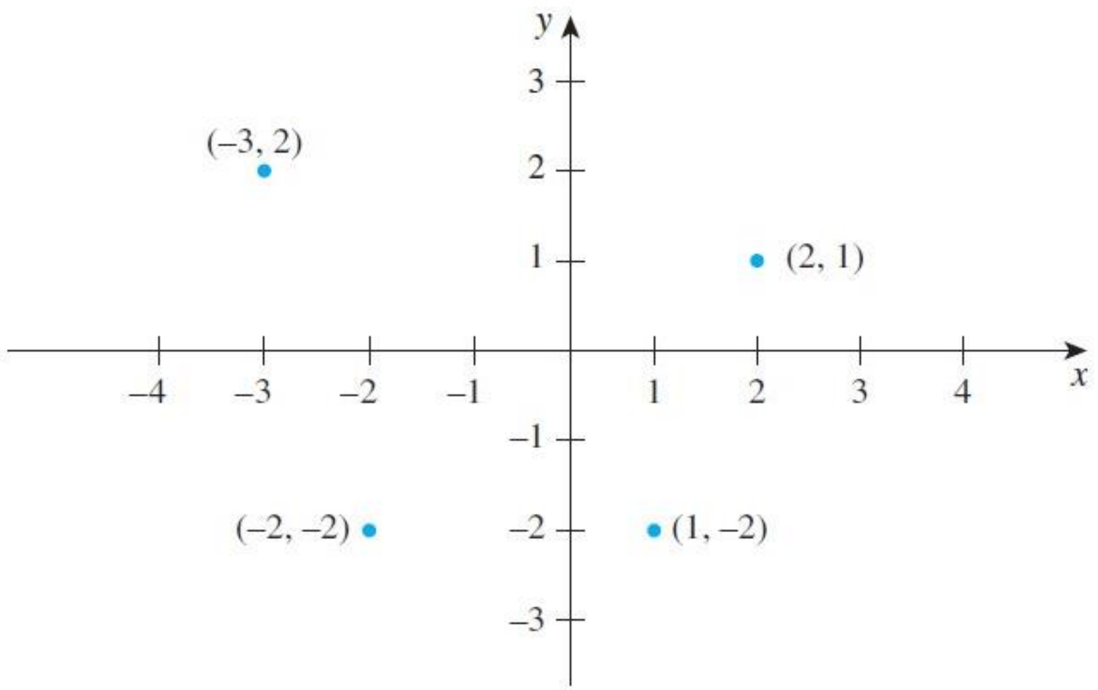
$A \times B = \{ (a,b) \mid a \in A \text{ and } b \in B \}$

A cross B equals the set of all ordered pairs (a, b) such that a is an element of A and b is an element of B.

For $A=\{1, 2, 3\}$ and $B = \{a, b\}$**,**

$A \times B = \{ (1,a), (1,b), (2,a), (2,b), (3,a), (3,b) \}$

Cartesian Plane



R × **R** is the set of all ordered pairs *(x, y)* where both *x* and *y* are real numbers.

Relations

$R \subseteq A \times B$

R is a subset of the Cartesian product $A \times B$, and it contains the specific ordered pairs that are related according to R.

x R y means that $(x,y) \in R$

$A = \{ 1,2,3 \},\ B = \{ 3,4 \}$

$R = \{ (x, y) \mid x \in A, y \in B, \text{ and } \boldsymbol{x < y} \}$

$R = \{ (1,3), (1,4), (2,3), (2,4), (3,4) \}$

Functions

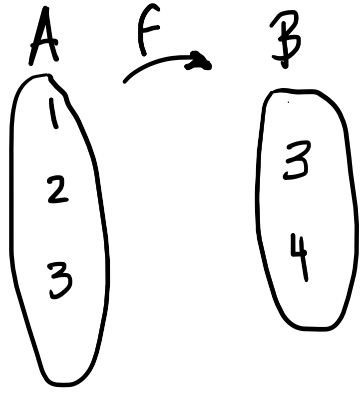
A **function** F from a set A to a set B is a relation with **domain** A and **co-domain** B that satisfies the following two properties:

1. For every element x in A, there is an element y in B such that $(x, y) \in F$.

2. For all elements x in A and y and z in B, if $(x, y) \in F$ and $(x, z) \in F$, then $y = z$.

Property 1 means that each element in the domain A **must be associated** with some element in the co-domain B.

Property 2 means that each element in the domain A can be paired with **only one** unique element in the co-domain B.



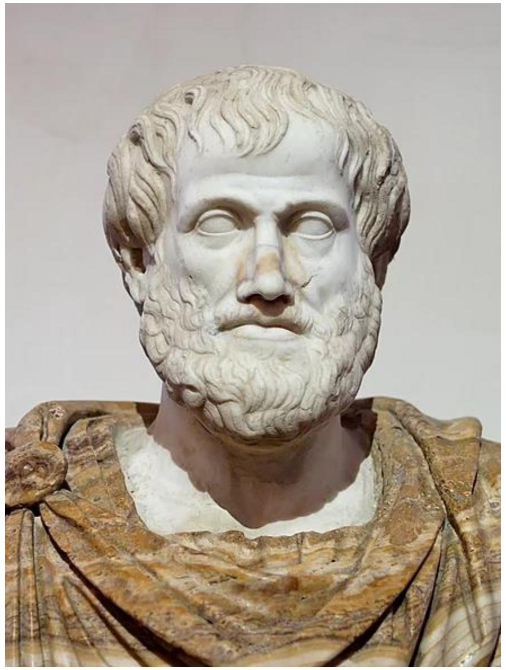
In essence, a function ensures that each input (from set A) is related to exactly one output (from set B), and no input is related to more than one output.

Lecture 3 – part 3

Logic of compound statements

Logic

- **Aristotle's Influence:** Pioneered rules for deductive reasoning across all knowledge branches.
- **Leibniz's Vision:** Proposed using symbols to mechanize deductive reasoning, akin to algebra.
- **19th Century Realization:** Boole & De Morgan established modern symbolic logic.
- **Symbolic Logic's Evolution:** Research ongoing and continuously expanding.
- **Modern Application:** Forms the theoretical basis for areas like digital logic circuit design in computer science.



https://en.wikipedia.org/wiki/Aristotle

Statements or propositions

• Definition

A **statement** (or **proposition**) is a sentence that is true or false but not both.

"Two plus two equals four" (True)

"Two plus two equals five" (False)

"He is a college student"

"$x + y > 0$"

Compound Statements

Europa supports life or Mars support life

Mars support life or Europa supports life

Compound Statements

The symbol ~denotes *not,*
∧ denotes *and,*
and ∨ denotes *or.*

Propositions
p, q, r
h,s

$$\sim p \wedge q = (\sim p) \wedge q$$

$$p \wedge q \vee r$$

$$(p \wedge q) \vee r \quad \text{or} \quad p \wedge (q \vee r)$$

Example

Write each of the following sentences symbolically, letting:
h = "It is hot" and s = "It is sunny."

a. It is not hot but it is sunny.

b. It is neither hot nor sunny.

a) "It is not hot <u>and</u> it is sunny" $\sim h \wedge s$

b) "it is not hot and it is not sunny" $\sim h \wedge \sim s$

Notation for Inequalities

$x \leq a$	means	$x < a$	or	$x = a$
$a \leq x \leq b$	means	$a \leq x$	and	$x \leq b$.

Suppose x is a particular real number.

Let p, q, and r symbolize "$0 < x$," "$x < 3$," and "$x = 3$," respectively.

Write the following inequalities symbolically:

a. $x \leq 3$, b. $0 < x < 3$, c. $0 < x \leq 3$

Solution

a. $q \vee r$ b. $p \wedge q$ c. $p \wedge (q \vee r)$

Truth Values

Negation

• **Definition**

If p is a statement variable, the **negation** of p is "not p" or "It is not the case that p" and is denoted $\sim p$. It has opposite truth value from p: if p is true, $\sim p$ is false; if p is false, $\sim p$ is true.

Truth Table for $\sim p$

p	$\sim p$
T	F
F	T

Truth Values

Conjunction

- **Definition**

If p and q are statement variables, the **conjunction** of p and q is "p and q," denoted $p \wedge q$. It is true when, and only when, both p and q are true. If either p or q is false, or if both are false, $p \wedge q$ is false.

Truth Table for $p \wedge q$

p	q	$p \wedge q$
T	T	
T	F	
F	T	
F	F	

Truth Values

Disjunction

• Definition

If p and q are statement variables, the **disjunction** of p and q is "p or q," denoted $p \vee q$. It is true when either p is true, or q is true, or both p and q are true; it is false only when both p and q are false.

Truth Table for $p \vee q$

p	q	$p \vee q$
T	T	
T	F	
F	T	
F	F	

Proposition forms

$(p \vee q) \wedge \sim(p \wedge q)$

• Definition

A **statement form** (or **propositional form**) is an expression made up of statement variables (such as p, q, and r) and logical connectives (such as $\sim$, $\wedge$, and $\vee$) that becomes a statement when actual statements are substituted for the component statement variables. The **truth table** for a given statement form displays the truth values that correspond to all possible combinations of truth values for its component statement variables.

True Table for Exclusive OR

"*p* or *q* but not both"
"*p* or *q* and not both *p* and *q*,"

$$(p \vee q) \wedge \sim(p \wedge q)$$

p	q	p v q	p ∧ q	~(p ∧ q)	(p ∨ q) ∧ ~(p ∧ q)
T	T				
T	F				
F	T				
F	F				

Example

Truth Table for $(p \wedge q) \vee \sim r$

p	q	r	$p \wedge q$	$\sim r$	$(p \wedge q) \vee \sim r$

Logical Equivalence

6 is greater than 2 and 2 is less than 6

(1) Dogs bark and cats meow and (2) Cats meow and dogs bark

p	q	$p \wedge q$	$q \wedge p$
T	T	T	T
T	F	F	F
F	T	F	F
F	F	F	F

Example

- Are $\sim(p \wedge q)$ and $\sim p \wedge \sim q$ logically equivalent ?

p	q	$\sim p$	$\sim q$	$p \wedge q$	$\sim(p \wedge q)$		$\sim p \wedge \sim q$
T	T	F	F	T	F		F
T	F	F	T	F	T	$\neq$	F
F	T	T	F	F	T	$\neq$	F
F	F	T	T	F	T		T

$\sim(p \wedge q)$ and $\sim p \wedge \sim q$ have different truth values in rows 2 and 3, so they are not logically equivalent

De Morgan's Laws

"John is tall and Jim is redheaded"

"John is not tall or Jim is not redheaded."

$\sim(p \wedge q)$ and $\sim p \vee \sim q$ are logically equivalent

p	q	$\sim p$	$\sim q$	$p \wedge q$	$\sim(p \wedge q)$	$\sim p \vee \sim q$
T	T	F	F	T	F	F
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T

$\sim(p \wedge q)$ and $\sim p \vee \sim q$ always have the same truth values, so they are logically equivalent

De Morgan’s Laws

Symbolically,

$$\sim(p \wedge q) \equiv \sim p \vee \sim q.$$

$$\sim(p \vee q) \equiv \sim p \wedge \sim q.$$

https://en.wikipedia.org/wiki/Augustus_De_Morgan

De Morgan’s Laws

The negation of an *and* statement is logically equivalent to the *or* statement in which each component is negated.

The negation of an *or* statement is logically equivalent to the *and* statement in which each component is negated.

Example

Use De Morgan's laws to write the negation of $-1 < x \leq 4$.

$$-1 < x \quad \text{and} \quad x \leq 4.$$

The negation is:

$$-1 \not< x \quad \text{or} \quad x \not\leq 4,$$

Equivalent to:

$$-1 \geq x \quad \text{or} \quad x > 4.$$

Tautologies and Contradictions

Example:
The statement form $p \vee \sim p$ is a tautology and the statement form $p \wedge \sim p$ is a contradiction.

p	$\sim p$	$p \vee \sim p$	$p \wedge \sim p$
T	F	T	F
F	T	T	F
		↑ all T's so $p \vee \sim p$ is a tautology	↑ all F's so $p \wedge \sim p$ is a contradiction

• Definition

A **tautology** is a statement form that is always true regardless of the truth values of the individual statements substituted for its statement variables. A statement whose form is a tautology is a **tautological statement.**

A **contradication** is a statement form that is always false regardless of the truth values of the individual statements substituted for its statement variables. A statement whose form is a contradication is a **contradictory statement.**

Example

If **t** is a tautology and **c** is a contradiction, show that $p \wedge \mathbf{t} \equiv p$ and $p \wedge \mathbf{c} \equiv \mathbf{c}$.

Solution

p	$\mathbf{t}$	$p \wedge \mathbf{t}$	p	$\mathbf{c}$	$p \wedge \mathbf{c}$
T	T	T	T	F	F
F	T	F	F	F	F

same truth values, so $p \wedge \mathbf{t} \equiv p$

same truth values, so $p \wedge \mathbf{c} \equiv \mathbf{c}$

Logical Equivalences

Theorem 2.1.1 Logical Equivalences

Given any statement variables p, q, and r, a tautology $\mathbf{t}$ and a contradiction $\mathbf{c}$, the following logical equivalences hold.

1. *Commutative laws:*	$p \wedge q \equiv q \wedge p$	$p \vee q \equiv q \vee p$
2. *Associative laws:*	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	$(p \vee q) \vee r \equiv p \vee (q \vee r)$
3. *Distributive laws:*	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
4. *Identity laws:*	$p \wedge \mathbf{t} \equiv p$	$p \vee \mathbf{c} \equiv p$
5. *Negation laws:*	$p \vee {\sim}p \equiv \mathbf{t}$	$p \wedge {\sim}p \equiv \mathbf{c}$
6. *Double negative law:*	${\sim}({\sim}p) \equiv p$	
7. *Idempotent laws:*	$p \wedge p \equiv p$	$p \vee p \equiv p$
8. *Universal bound laws:*	$p \vee \mathbf{t} \equiv \mathbf{t}$	$p \wedge \mathbf{c} \equiv \mathbf{c}$
9. *De Morgan's laws:*	${\sim}(p \wedge q) \equiv {\sim}p \vee {\sim}q$	${\sim}(p \vee q) \equiv {\sim}p \wedge {\sim}q$
10. *Absorption laws:*	$p \vee (p \wedge q) \equiv p$	$p \wedge (p \vee q) \equiv p$
11. *Negations of* $\mathbf{t}$ *and* $\mathbf{c}$*:*	${\sim}\mathbf{t} \equiv \mathbf{c}$	${\sim}\mathbf{c} \equiv \mathbf{t}$

Simplifying Statements Forms

Verify the logical equivalence of:

$$\sim(\sim p \wedge q) \wedge (p \vee q) \equiv p.$$

Solution:

$$\begin{aligned}
\sim(\sim p \wedge q) \wedge (p \vee q) &\equiv (\sim(\sim p) \vee \sim q) \wedge (p \vee q) && \text{by De Morgan's laws} \\
&\equiv (p \vee \sim q) \wedge (p \vee q) && \text{by the double negative law} \\
&\equiv p \vee (\sim q \wedge q) && \text{by the distributive law} \\
&\equiv p \vee (q \wedge \sim q) && \text{by the commutative law for } \wedge \\
&\equiv p \vee \mathbf{c} && \text{by the negation law} \\
&\equiv p && \text{by the identity law.}
\end{aligned}$$