COT 2000
Foundations of Computing

Spring 2024

Lecture 17 – part 1

Lab 8

Exam 3– 07/12/24

Homework 6 – 07/19/24

Lecture 17 – part 2

Review

Review

- Counting and Probability
- Random process
- Sample space (set)
- Event space (set)
- Probability $P(E) = N(E)/N(S)$
- Possibility tree
- Multiplication rule or Rule of products

Random Process – Sample Space

- To say that a process is **random** means that when it takes place, one outcome from some set of outcomes is sure to occur, but it is **impossible to predict with certainty which outcome** that will be.
- The **set of outcomes** that can result from a random process or experiment is called a **sample space**.

• Definition

A **sample space** is the set of all possible outcomes of a random process or experiment. An **event** is a subset of a sample space.

Number of elements N(A):

• Notation

For any finite set A, $N(A)$ denotes the number of elements in A.

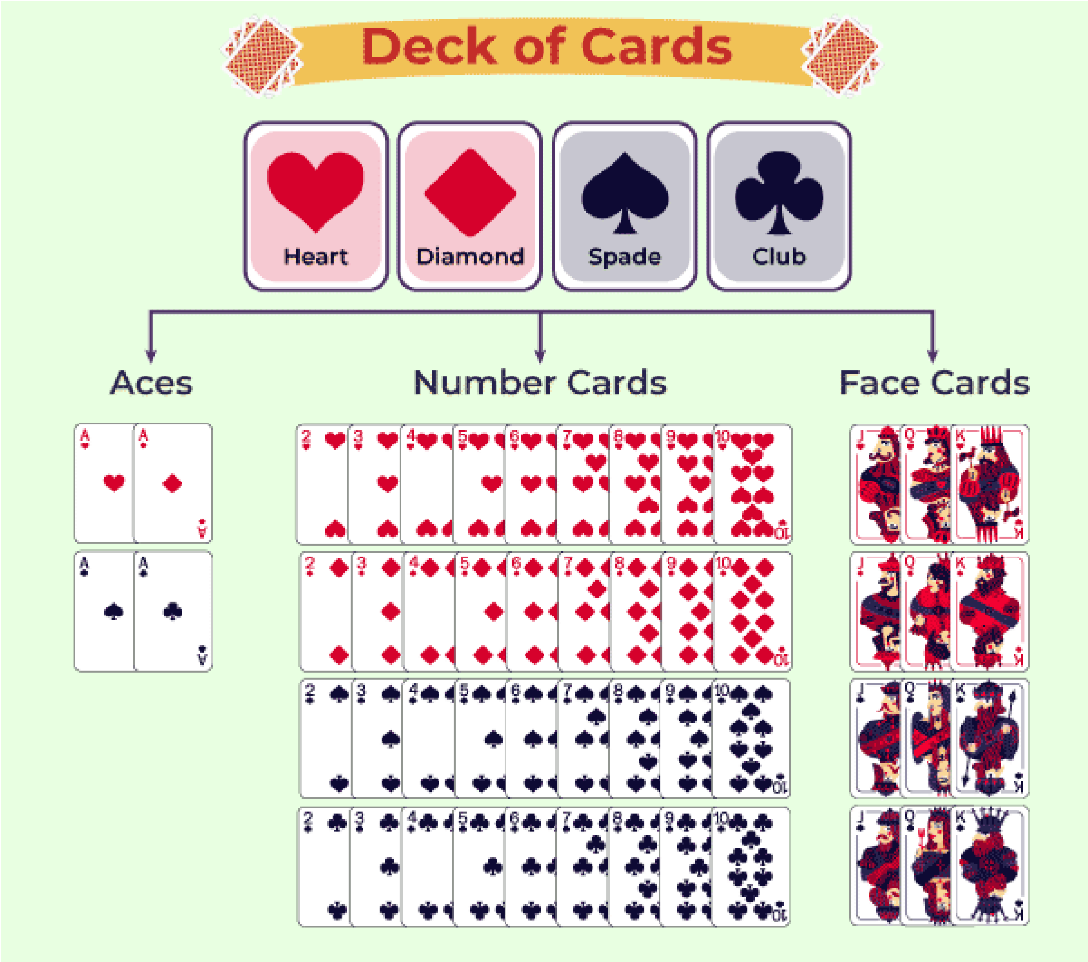
Probability:

$$P(E) = \frac{N(E)}{N(S)}.$$

Number of outcomes in E (set of outcomes)
E is a subset of S

Number of total outcomes in S (sample space)

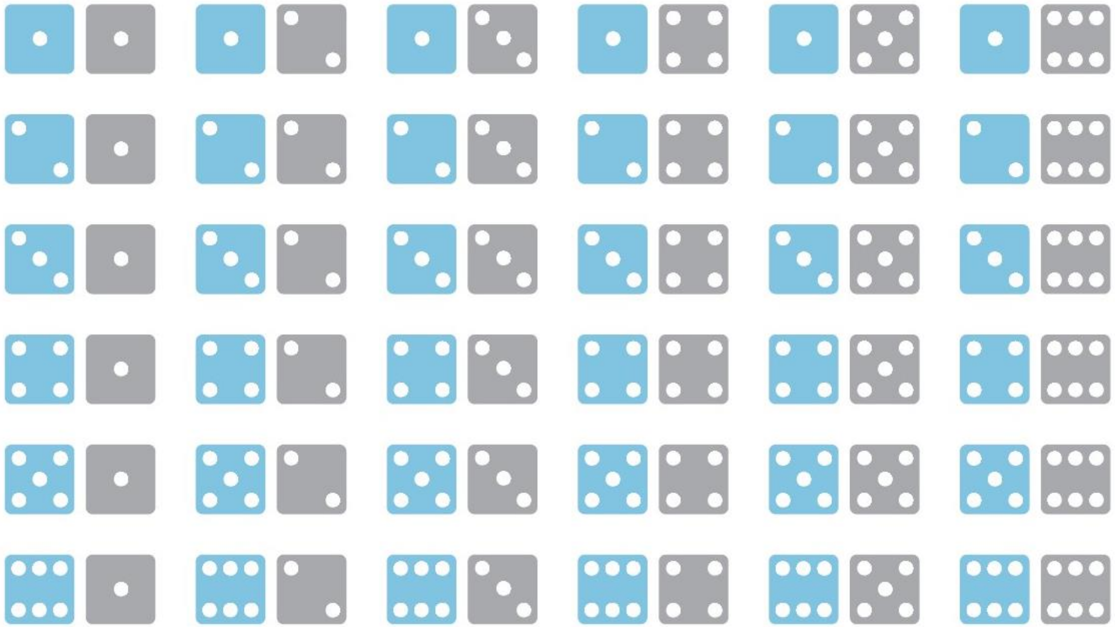
Example: An ordinary deck of cards contains 52 cards divided into four *suits*. The *red suits* are diamonds (♦) and hearts (♥) and the *black suits* are clubs (♣) and spades (♠). Each suit contains 13 cards of the following *denominations:* 2, 3, 4, 5, 6, 7, 8, 9, 10, J (jack), Q (queen), K (king), and *A* (ace). The cards J, Q, and K are called *face cards.*

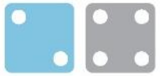


https://www.geeksforgeeks.org/card-probability/

a) What is the sample space of outcomes ? (S)
b) What is the event that the chosen card is a black face card ? (E)
c) What is the probability that the chosen card is a black face card ?

Example: Rolling a pair of dice

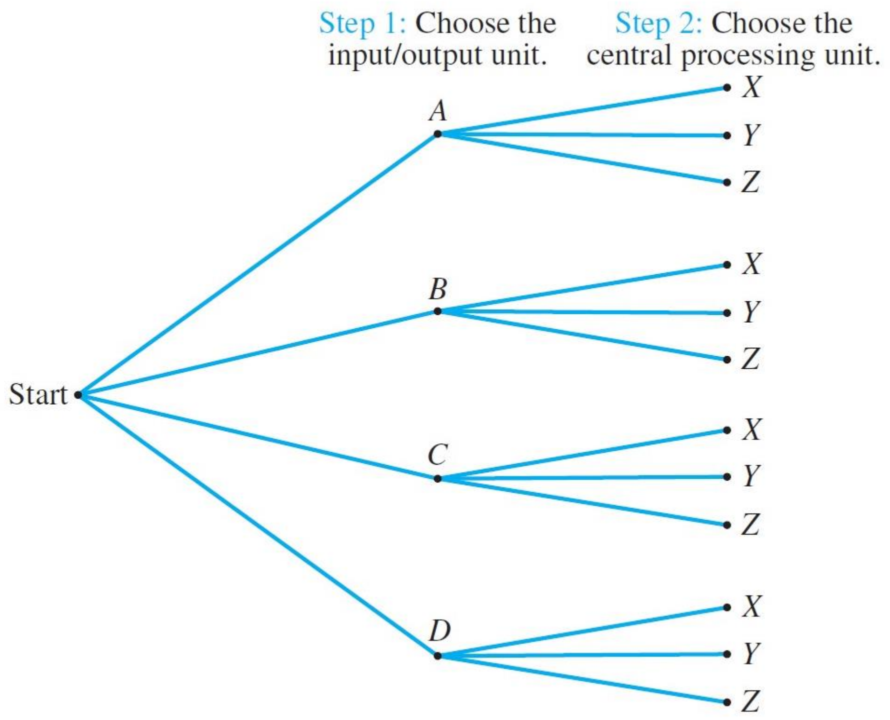


A more compact notation identifies, say,  with the notation 24,  with 53, and so forth.

a. Use the compact notation to write the sample space S of possible outcomes.

b. Use set notation to write the event E that the numbers showing face up have a sum of 6 and find the probability of this event.

Solution:



Thus the total number of ways to pair the two types of units is the same as the number of branches of the tree, which is:

$3 + 3 + 3 + 3 = 4 \cdot 3 = 12.$

The Multiplication Rule

If an operation consists of k steps and

the first step can be performed in n_1 ways,

the second step can be performed in n_2 ways *[regardless of how the first step was performed]*,

$\vdots$

the kth step can be performed in n_k ways *[regardless of how the preceding steps were performed]*,

then the entire operation can be performed in $n_1 n_2 \cdots n_k$ ways.

In simple terms, the rule helps you find out how many different ways you can complete a series of steps or make choices when each step or choice is independent of the others.

Example: Suppose a computer installation has:

- Four input/output units (*A, B, C,* and *D*) and
- Three central processing units (*X, Y,* and *Z*).

Any input/output unit can be paired with any central processing unit.

How many ways are there to pair an input/output unit with a central processing unit?

Lecture 17 – part 3

More on Counting

The rule of products

If two operations must be performed, and if the first operation can always be performed $\mathbf{p_1}$ different ways and the second operation can always be performed $\mathbf{p_2}$ different ways, then there are $\mathbf{p_1 p_2}$ different ways that the two operations can be performed.

Note: **Assume Independent operations**

It is important that p_2 does not depend on the option that is chosen in the first operation. Another way of saying this is that p_2 is independent of the first operation. If p_2 is dependent on the first operation, then the rule of products does not apply.

Example

Imagine you're making a sandwich. You have to choose the type of bread, the kind of filling, and the type of sauce. The options available for each part of the sandwich are:

- Choose the type of bread: 3 options
- Choose the kind of filling: 5 options (and this choice doesn't depend on which bread you chose)
- Choose the type of sauce: 2 options (and this doesn't depend on the bread or filling)

Question: find out how many unique sandwiches you can make.

Solution

To find out how many unique sandwiches you can make, you just multiply the number of options at each step:

$$\text{Number of types of bread} \times \text{Number of kinds of fillings} \times \text{Number of types of sauces}$$
$$3 \times 5 \times 2 = 30 \text{ different sandwiches}$$

Extended Rule of Products

The extended rule of products

The Extended Rule of Products states that if **'n'** operations must be performed, and each operation has a respective number of options denoted by $\mathbf{p_1, p_2, \ldots, p_n}$, with each $\mathbf{p_i}$ being independent of the choices made in previous operations, then there are $\mathbf{p_1 \cdot p_2 \cdot \ldots \cdot p_n}$ different ways to perform these **'n'** operations

Example:

A questionnaire contains four questions, each with two possible answers, and three questions, each with five possible answers. Considering that the answer to each question is independent of the answers to the other questions, how many different ways are there to answer the questionnaire?

Solution:

The extended rule of products applies in this scenario because each question's possible answers are independent of the others. For the four questions with two possible answers each, and the three questions with five possible answers each, the total number of different ways to answer the questionnaire is calculated as follows:

$$\mathbf{2 \cdot 2 \cdot 2 \cdot 2 \cdot 5 \cdot 5 \cdot 5 = 2^4 \cdot 5^3 = 16 \cdot 125 = 2000.}$$

Therefore, there are 2000 different ways to answer the questionnaire.

Question:
Ten people are invited to a dinner party. How many different ways can they be seated at a linear table?

Solution:
We are arranging ten people in a line. The first position can be filled by any of the ten people, the second position by any of the remaining nine people, and so on, until the last position, which can only be filled by the remaining person.

Thus, by the rule of products, the total number of arrangements is:

For the 1st position: 10 choices
For the 2nd position: 9 choices (after one person is seated)
...
For the 10th position: 1 choice (the last person)

The total number of arrangements is
$10 \times 9 \times 8 \times ... \times 1 = 10!$ which equals 3,628,800 ways.

Question:

Assuming a group of ten people consists of five men and five women, in how many different ways can they be seated at a linear table so that no consecutive seats are occupied by persons of the same gender?

Solution:

The sequence can start with either a man or a woman, offering two choices for the first person's gender. Once the gender of the first person is chosen, the seating arrangement must alternate between men and women to ensure that no two consecutive seats are occupied by people of the same gender. This means if we start with a man, the sequence must be man-woman-man-woman, and so on, until all seats are filled.

For each group:

5 x 4 x 3 x 2 x 1 = 5!

Since there are **two ways** to start, the total number of arrangements is: **2 x 5! x 5! = 28,800**.

Question:

A club consists of twenty-five members. An election is to be held to select a president, secretary, and treasurer, in that specific order. It is assumed that a person can hold only one position. Using the rule of products, how many different ways can these three officers be chosen?

Solution:

Given that a person can only hold one position, the election process for choosing a president, secretary, and treasurer among twenty-five members adheres to the rule of products.

- Initially, any of the 25 members can be chosen as president.
- Once the president is chosen, only 24 members remain eligible for the position of secretary.
- After selecting a secretary, 23 members are left as potential candidates for the position of treasurer.

Therefore, the total number of ways to choose these three officers is calculated as:

$25 \cdot 24 \cdot 23 = 13{,}800$ different ways

Permutation - P(n,k)

- **Permutation:** An ordered arrangement of k elements selected from a set of n elements, $\mathbf{0 \le k \le n}$, where no two elements of the arrangement are the same, is called a permutation of **n** objects taken **k** at a time.
- The total number of such permutations is denoted by **P(n, k).**

Theorem 2.2.8 Permutation Counting Formula. *The number of possible permutations of k elements taken from a set of n elements is*

$$P(n,k) = n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot (n-k+1) = \prod_{j=0}^{k-1} (n-j) = \frac{n!}{(n-k)!}.$$

Proof. Case I: If $k = n$ we have $P(n,n) = n! = \frac{n!}{(n-n)!}$.

Case II: If $0 \leq k < n$,then we have k positions to fill using n elements and

(a) Position 1 can be filled by any one of $n - 0 = n$ elements

(b) Position 2 can be filled by any one of $n - 1$ elements

(c) $\cdots$

(d) Position k can be filled by any one of $n - (k-1) = n - k + 1$ elements

Hence, by the rule of products,

$$P(n,k) = n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot (n-k+1) = \frac{n!}{(n-k)!}.$$

■

It is important to note that the derivation of the permutation formula given above was done solely through the rule of products. This serves to reiterate our introductory remarks in this section that permutation problems are really rule-of-products problems. We close this section with several examples.

Question:
Another example of choosing officers. A club has eight members eligible to serve as president, vice-president, and treasurer. How many ways are there of choosing these officers?

Solution 1: Using the rule of products.
There are eight possible choices for the presidency, seven for the vice-presidency, and six for the office of treasurer.

By the rule of products there are $\mathbf{8 \cdot 7 \cdot 6 = 336}$ ways of choosing these officers.

Solution 2: Using the permutation formula.
We want the total number of permutations of eight objects taken three at a time:

$$\mathbf{P(8, 3) = 8!/(8 - 3)! = 8 \cdot 7 \cdot 6 = 336}$$

Lecture 17 – part 4

Counting and Probability Exercises

Lecture 17 – part 5

Combinations

Example:
How many ways can we order three letters from $A = \{a, b, c, d\}$?

Permutation

Order **is** important

$$\frac{n!}{(n-k)!}.$$

Solution:

By rule of products : 4 x 3 x 2 = 24

By permutation : P(4,3) = 4!/(4-3)! = 4! = 24

```
abc, acb, bca, bac, cab, cba
abd, adb, bda, bad, dab, dba
acd, adc, cda, cad, dac, dca
bcd, bdc, cdb, cbd, dbc, dcb
```

Example:
How many ways can we select a set of three letters from $A = \{a, b, c, d\}$?

Solution:

```
abc, acb, bca, bac, cab, cba --- > {a,b,c}
abd, adb, bda, bad, dab, dba --- > {a,b,d}
acd, adc, cda, cad, dac, dca --- > {a,c,d}
bcd, bdc, cdb, cbd, dbc, dcb --- > {b,c,d}
```

Order **is not** important in sets

Combinations !

$$\frac{P(4,3)}{6} = \frac{P(4,3)}{3!} = \frac{4!}{(4-3)!3!} = 4 = \binom{4}{3}$$

(n choose r) Notation

- Binomial coefficient $\binom{n}{r}$

• Definition

Let n and r be integers with $0 \leq r \leq n$. The symbol

$$\binom{n}{r}$$

is read "***n* choose *r***" and represents the number of subsets of size r that can be chosen from a set with n elements.

• Formula for Computing $\binom{n}{r}$

For all integers n and r with $0 \leq r \leq n$,

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}.$$

also called *combinations*

Combinations

- A combination can be seen as a subset of a larger set. Specifically, it's about selecting <u>a subset </u>of size k from a larger set of size n, without regard to the order of the elements.
- <u>The binomial coefficient :</u>
- (n choose k) represents the number of distinct k-element <u>subsets</u> that can be formed from an n-element set.

Binomial Coefficient Formula

If n and k are nonnegative integers with $0 \leq k \leq n$, then the number k-element subsets of an n element set is equal to

$$\binom{n}{k} = \frac{n!}{(n-k)! \cdot k!}$$

Proof

Proof 1: There are $k!$ ways of ordering the elements of any k element set. Therefore,

$$\binom{n}{k} = \frac{P(n,k)}{k!} = \frac{n!}{(n-k)! \cdot k!}.$$

Proof 2: To "construct" a permutation of k objects from a set of n elements, we can first choose one of the subsets of objects and second, choose one of the $k!$ permutations of those objects. By the rule of products,

$$P(n,k) = \binom{n}{k} \cdot k!$$

and solving for $\binom{n}{k}$ we get the desired formula.

Example:

Assume an evenly balanced coin is tossed five times. In how many ways can three heads be obtained?

Solution:

This is a combination problem, because the order in which the heads appear does not matter.
We can think of this as a situation involving sets by considering the set of flips of the coin, 1 through 5, in which heads comes up.

The number of ways to get three heads is:

$$\binom{5}{3} = \frac{5!}{3!(5-3)!} = \frac{5 \cdot 4 \cdot 3!}{3!2!} = 5 \cdot 2 = 10$$

Rule of products,

2x2x2x2x2 = 32,

but not a permutation.

HHHHH	HTHHH	THHHH	TTHHH
HHHHT	HTHHT	THHHT	TTHHT
HHHTH	HTHTH	THHTH	TTHTH
HHHTT	HTHTT	THHTT	TTHTT
HHTHH	HTTHH	THTHH	TTTHH
HHTHT	HTTHT	THTHT	TTTHT
HHTTH	HTTTH	THTTH	TTTTH
HHTTT	HTTTT	THTTT	TTTTT

Example:

a) How many possible social committees of 5 persons can be formed from a club of 25 members?

Solution: A committee is a combination of members where order does not matter. The number of ways to choose 5 members from 25 is given by the binomial coefficient:

$$\binom{25}{5} = \frac{25!}{5!(25-5)!} = \frac{25 \cdot 24 \cdot 23 \cdot 22 \cdot 21}{5!} = 53,130.$$

There are 53,130 different possible social committees.

b) If the club rules state that the treasurer must be on the social committee, how does this affect the number of possible committees?

Solution: With the treasurer on the committee, we now choose the remaining 4 members from the remaining 24 members:

$$\binom{24}{4} = \frac{24!}{4!(24-4)!} = \frac{24 \cdot 23 \cdot 22 \cdot 21}{4!} = 10,626.$$

The presence of the treasurer reduces the number of possible committees to 10,626.

c) If a chairperson, who is not the treasurer, must also be selected, what are the number of possible committees?

Solution: After choosing the 4 non-treasurer members, any one of them can be the chairperson. Thus, the number of possible committees is:

$$\binom{24}{4} \times 4 = 10,626 \times 4 = 42,504.$$

Therefore, if a chairperson must be selected in addition to the treasurer, there are 42,504 different possible social committees.

Example:

Consider the problem of choosing five members from a group of twelve to work as a team on a special project. How many distinct five-person teams can be chosen?

Solution The number of distinct five-person teams is the same as the number of subsets of size 5 (or 5-combinations) that can be chosen from the set of twelve. This number is $\binom{12}{5}$. By Theorem 9.5.1,

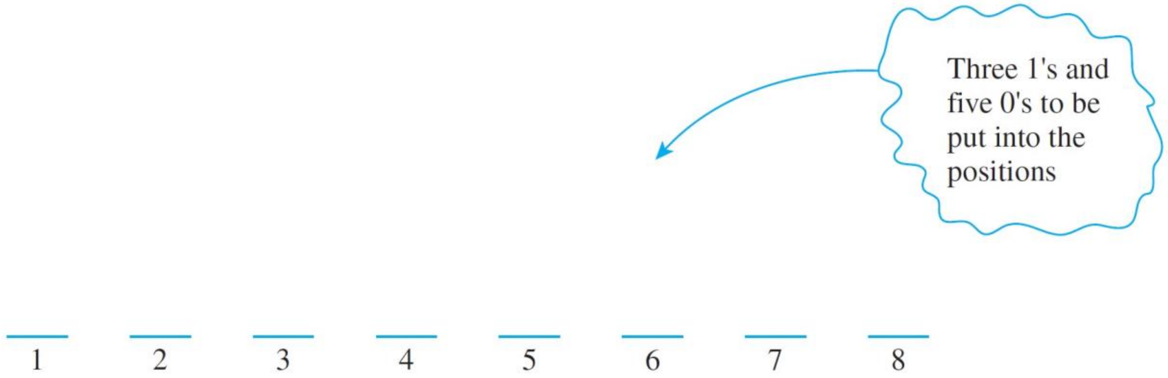
$$\binom{12}{5} = \frac{12!}{5!(12-5)!} = \frac{\cancel{12}\cdot 11\cdot \cancel{10}\cdot 9\cdot 8\cdot \cancel{7!}}{(\cancel{5}\cdot\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot 1)\cdot\cancel{7!}} = 11\cdot 9\cdot 8 = 792.$$

Thus there are 792 distinct five-person teams. ■

Example:

How many eight-bit strings have exactly three 1's?

Solution To solve this problem, imagine eight empty positions into which the 0's and 1's of the bit string will be placed. In step 1, choose positions for the three 1's, and in step 2, put the 0's into place.



Once a subset of three positions has been chosen from the eight to contain 1's, then the remaining five positions must all contain 0's (since the string is to have exactly three 1's). It follows that the number of ways to construct an eight-bit string with exactly three 1's is the same as the number of subsets of three positions that can be chosen from the eight into which to place the 1's. By Theorem 9.5.1, this equals

$$\binom{8}{3} = \frac{8!}{3! \cdot 5!} = \frac{8 \cdot 7 \cdot \cancel{6} \cdot \cancel{5!}}{\cancel{3 \cdot 2} \cdot \cancel{5!}} = 56. \qquad \blacksquare$$

Example:

What is the probability that in a group of 25 people, at least two persons share the same birthday?

Solution:

Let $P(\text{shared birthday})$ denote the probability that at least two persons share the same birthday. This is calculated as the complement of the probability that everyone has unique birthdays.

$$P(\text{shared birthday}) = 1 - P(\text{no shared birthdays})$$

Consider the set $E \subseteq S$, where E is the set of all outcomes with unique birthdays (the event of interest), and S is the sample space containing all possible combinations of birthdays for the 25 people.

The probability of no shared birthdays is given by:

$$P(\text{no shared birthdays}) = \frac{N(E)}{N(S)}$$

Here, $N(E)$ is the number of ways to assign different birthdays to each of the 25 people, and $N(S)$ is the number of ways for possible assignments of birthdays.

$N(E)$ is determined by the permutation of 25 unique days out of 365:

$$N(E) = 365 \cdot 364 \cdot 363 \cdot ... \cdot 341 = P(365, 25) = \frac{365!}{(365-25)!}$$

$N(S)$ is simply $365_1 \cdot 365_2 \cdot 365_3 ... 365_{25} = 365^{25}$, as each person can be born on any of the 365 days, independently of others.

Thus, the probability that all 25 people have unique birthdays is:

$$P(\text{unique birthdays}) = \frac{N(E)}{N(S)} = \frac{P(365, 25)}{365^{25}} = 0.4313$$

And the probability that at least two persons share the same birthday is:
$P(\text{shared birthday}) = 1 - \dfrac{N(E)}{N(S)} = 1 - 0.4313 = 0.5687$ or $56.87\ \%$