COT 2000
Foundations of Computing

Summer 2024

Lecture 7 – part 1

Lab 4
Homework 2 - Due: 06/07/24
Homework 3 - Due: 06/14/24

Lecture 7 – part 2

Review

Review

- Conditional Identity
- Negation of a conditional statement
- Contrapositive
- Converse
- Inverse
- Biconditional
- Necessary and sufficient conditions

Logical equivalences Involving →

Exercise: Find the truth tables for each equivalence

$$p \to q \equiv \neg p \vee q \quad \text{(Conditional Identity)} \tag{1}$$

$$p \to q \equiv \neg q \to \neg p \quad \text{(Contrapositive)} \tag{2}$$

$$p \vee q \to r \equiv (p \to r) \wedge (q \to r) \quad \text{(Distributive Law of Implication)} \tag{3}$$

$$(p \wedge q) \to r \equiv p \to (q \to r) \quad \text{(Exportation)} \tag{4}$$

$$\neg(p \to q) \equiv p \wedge \neg q \quad \text{(Reduction)} \tag{5}$$

$$q \to p \quad \text{(Converse of } p \to q\text{)} \tag{6}$$

$$\neg p \to \neg q \quad \text{(Inverse of } p \to q\text{)} \tag{7}$$

Conditional & Converse

Conditional
$p \rightarrow q$

Not equivalent

Not equivalent

Inverse
$\sim p \rightarrow \sim q$

Equivalent
Why ?

Converse
$q \rightarrow p$

Truth Table Showing that $p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$

p	q	$p \to q$	$q \to p$	$p \leftrightarrow q$	$(p \to q) \wedge (q \to p)$
T	T	T	T	T	T
T	F	F	T	F	F
F	T	T	F	F	F
F	F	T	T	T	T

$p \leftrightarrow q$ and $(p \to q) \wedge (q \to p)$ always have the same truth values, so they are logically equivalent

Example: Write the following compound statement as two conditionals:

"This computer program is correct **if, and only if**, it produces correct answers for all possible sets of input data."

"**If** this program is correct, **then** it produces the correct answers for all possible sets of input data; **and if** this program produces the correct answers for all possible sets of input data, **then** it is correct."

Necessary and Sufficient Condition

• Definition

If r and s are statements:

r is a **sufficient condition** for s	means	"if r then s."
r is a **necessary condition** for s	means	"if not r then not s."

r is a necessary and sufficient condition for s means "r if, and only if, s."

$r \leftrightarrow s$

Biconditional and Exclusive OR

$p \oplus q \equiv (p \vee q) \wedge \sim(p \wedge q)$ By definition

$(p \vee q) \wedge (\sim p \vee \sim q)$ Morgan's Law

$[(p \vee q) \wedge \sim p] \vee [(p \vee q) \wedge \sim q]$ Distributive Law

$[(p \wedge \sim p) \vee (q \wedge \sim p)] \vee [(p \wedge \sim q) \vee (q \wedge \sim q)]$ Distributive Law

$[c \vee (q \wedge \sim p)] \vee [(p \wedge \sim q) \vee c]$ Negation Law

$(q \wedge \sim p) \vee (p \wedge \sim q)$ Identity Law

$\boxed{p \oplus q \equiv (p \wedge \sim q) \vee (q \wedge \sim p)}$ Commutative Law

$$p \leftrightarrow q \equiv p \rightarrow q \wedge q \rightarrow p$$ By definition

$$\equiv (\sim p \vee q) \wedge (\sim q \vee p) \quad (1)$$ Conditional identity

$$p \oplus q \equiv (p \wedge \sim q) \vee (q \wedge \sim p)$$

$$\sim(p \oplus q) \equiv \sim[(p \wedge \sim q) \vee (q \wedge \sim p)]$$

$$\equiv \sim(p \wedge \sim q) \wedge \sim(q \wedge \sim p)$$ De Morgan's Law

$$\equiv (\sim p \vee q) \wedge (\sim q \vee p) \quad (2)$$ De Morgan's Law

Since (1) $\equiv$ (2), therefore

$$\boxed{p \leftrightarrow q \equiv \sim(p \oplus q)}$$

Lecture 7 – part 3

Logic Arguments

Valid and Invalid Arguments

• Definition

An **argument** is a sequence of statements, and an **argument form** is a sequence of statement forms. All statements in an argument and all statement forms in an argument form, except for the final one, are called **premises** (or **assumptions** or **hypotheses**). The final statement or statement form is called the **conclusion.** The symbol ∴, which is read "therefore," is normally placed just before the conclusion.

To say that an *argument form* is **valid** means that no matter what particular statements are substituted for the statement variables in its premises, if the resulting premises are all true, then the conclusion is also true. To say that an *argument* is **valid** means that its form is valid.

Example

If Socrates is a man, **then** Socrates is mortal.
Socrates is a man.
∴ Socrates is mortal.

If p **then** q
p
$\therefore q$

Valid and Invalid Arguments

The crucial fact about a <u>valid argument</u> is that the truth of its conclusion follows ***necessarily*** or ***inescapably*** or *by logical form alone* from the truth of its premises.

It is **impossible** to have a valid argument with true premises and a false conclusion.

When an argument is **valid** and **its premises are true**, the truth of the conclusion is said to be ***inferred* or *deduced*** from the truth of the premises.

Testing an Argument Form for Validity

1. Identify the premises and conclusion of the argument form.
2. Construct a truth table showing the truth values of all the premises and the conclusion.
3. A row of the truth table in which all the premises are true is called a **critical row**.
   - If there is a critical row in which the conclusion is false, then it is possible for an argument of the given form to have true premises and a false conclusion, and so the argument form is invalid.
   - If the conclusion in every critical row is true, then the argument form is valid.

Exercise: Determine whether the following argument form is valid or invalid

$$p \to q \lor \sim r$$
$$q \to p \land r$$
$$\therefore\ p \to r$$

						Premises		Conclusion
p	q	r	$\sim r$	$q \lor \sim r$	$p \land r$	$p \to q \lor \sim r$	$q \to p \land r$	$p \to r$
T	T	T						
T	T	F						
T	F	T						
T	F	F						
F	T	T						
F	T	F						
F	F	T						
F	F	F						

Exercise: Determine whether the following argument form is valid or invalid

$$p \to q \lor \sim r$$
$$q \to p \land r$$
$$\therefore p \to r$$

						Premises		Conclusion
p	q	r	$\sim r$	$q \lor \sim r$	$p \land r$	$p \to q \lor \sim r$	$q \to p \land r$	$p \to r$
T	T	T	F	T	T	T	T	T
T	T	F	T	T	F	T	F	F
T	F	T	F	F	T	F	T	T
T	F	F	T	T	F	T	T	F
F	T	T	F	T	F	T	F	T
F	T	F	T	T	F	T	F	T
F	F	T	F	F	F	T	T	T
F	F	F	T	T	F	T	T	T

INVALID

Modus Ponens

- An argument form consisting of two premises and a conclusion is called a **syllogism**.
- The first and second premises are called the **major premise** and **minor premise**, respectively.
- The most famous form of syllogism in logic is called **modus ponens.**

If p then q.
p
$\therefore q$

If the sum of the digits of 371,487 is divisible by 3,
then 371,487 is divisible by 3.
The sum of the digits of 371,487 is divisible by 3.
$\therefore$ 371,487 is divisible by 3.

Modus Ponens

		premises		conclusion		
p	q	$p \rightarrow q$	p	q		
T	T	T	T	T	⟵ critical row	VALID
T	F	F	T			
F	T	T	F			
F	F	T	F			

Modus Tollens

If p then q.
$\sim q$
$\therefore \sim p$

If Zeus is human, then Zeus is mortal.
Zeus is not mortal.
∴ Zeus is not human.

Exercise:

Use modus ponens or modus tollens to fill in the blanks of the following arguments so that they become valid inferences.

MP	MT
If p then q. p $\therefore q$	If p then q. $\sim q$ $\therefore \sim p$

a) **If** there are more pigeons than there are pigeonholes, **then** at least two pigeons roost in the same hole.
There are more pigeons than there are pigeonholes.
∴ ____.

b) **If** 870,232 is divisible by 6, **then** it is divisible by 3.
870,232 is not divisible by 3.
∴ ____.

Exercise:

Use modus ponens or modus tollens to fill in the blanks of the following arguments so that they become valid inferences.

MP	MT
If p then q. p $\therefore q$	If p then q. $\sim q$ $\therefore \sim p$

a) **If** there are more pigeons than there are pigeonholes, **then** at least two pigeons roost in the same hole.
There are more pigeons than there are pigeonholes.
∴ At least two pigeons roost in the same hole, by modus ponens.

b) **If** 870,232 is divisible by 6, **then** it is divisible by 3.
870,232 is not divisible by 3.
∴ 870,232 is not divisible by 6, by modus tollens.

Additional Valid Argument Forms: Rules of Inference

- A **rule of inference** is a form of argument that is valid.
- Thus modus ponens and modus tollens are both rules of inference.

Generalization

a. p
$\therefore p \vee q$

b. q
$\therefore p \vee q$

Specialization

a. $p \wedge q$
$\therefore p$

b. $p \wedge q$
$\therefore q$

Generalization

$$\begin{array}{l} \quad p \\ \therefore p \vee q \end{array}$$

As an **example**, suppose you are given the job of counting the upperclassmen at your school.

You ask what class Anton is in and are told he is a junior.

You reason as follows:

Anton is a junior.

∴ (more generally) Anton is a junior **or** Anton is a senior.

Knowing that upperclassman means junior or senior, you add Anton to your list.

Specialization

$$\begin{array}{l} \quad p \land q \\ \therefore p \end{array}$$

For instance, suppose you are looking for a person who knows graph algorithms to work with you on a project.

You discover that Ana knows both numerical analysis and graph algorithms.
You reason as follows:

Ana knows numerical analysis **and** Ana knows graph algorithms.
∴ (in particular) Ana knows graph algorithms.

Accordingly, you invite her to work with you on your project.

Elimination

These argument forms say that when you have only two possibilities and you can rule one out, the other must be the case.

a.	$p \vee q$
	$\sim q$
	$\therefore p$

b.	$p \vee q$
	$\sim p$
	$\therefore q$

For instance, suppose you know that for a particular number x,

$x - 3 = 0$ **or** $x + 2 = 0$.

If you also know that ***x* is not negative**, then $x \neq -2$, so

$x + 2 \neq 0$.

By elimination, you can then conclude that

$\therefore x - 3 = 0$.

Transitivity

- Many arguments in mathematics contain **chains of if-then statements**. From the fact that one statement implies a second and the second implies a third, you can conclude that the first statement implies the third.

$$
\begin{array}{l}
\quad p \rightarrow q \\
\quad q \rightarrow r \\
\therefore p \rightarrow r
\end{array}
$$

If 18,486 is divisible by 18, **then** 18,486 is divisible by 9.
If 18,486 is divisible by 9, **then** the sum of the digits of 18,486 is divisible by 9.
∴ **If** 18,486 is divisible by 18, **then** the sum of the digits of 18,486 is divisible by 9

Proof by Division into Cases

It often happens that you know one thing **or** another is true.
If you can show that in either case a certain conclusion follows,
then this conclusion must also be true.

$p \vee q$
$p \rightarrow r$
$q \rightarrow r$
$\therefore r$

x is positive or x is negative.
If x is positive, then $x^2 > 0$.
If x is negative, then $x^2 > 0$.
$\therefore x^2 > 0$.

Valid Argument Forms

Modus Ponens	$p \to q$ p $\therefore q$	**Elimination**	**a.** $p \vee q$ $\sim q$ $\therefore p$ **b.** $p \vee q$ $\sim p$ $\therefore q$
Modus Tollens	$p \to q$ $\sim q$ $\therefore \sim p$	**Transitivity**	$p \to q$ $q \to r$ $\therefore p \to r$
Generalization	**a.** p $\therefore p \vee q$ **b.** q $\therefore p \vee q$	**Proof by Division into Cases**	$p \vee q$ $p \to r$ $q \to r$ $\therefore r$
Specialization	**a.** $p \wedge q$ $\therefore p$ **b.** $p \wedge q$ $\therefore q$		
Conjunction	p q $\therefore p \wedge q$	**Contradiction Rule**	$\sim p \to c$ $\therefore p$

Where are the glasses ?

Example:
You are about to leave for school in the morning and discover that you don’t have your glasses.

You know the following statements are true:

a. **If** I was reading the newspaper in the kitchen, **then** my glasses are on the kitchen table.
b. **If** my glasses are on the kitchen table, **then** I saw them at breakfast.
c. I did not see my glasses at breakfast.
d. I was reading the newspaper in the living room **or** I was reading the newspaper in the kitchen.
e. **If** I was reading the newspaper in the living room **then** my glasses are on the coffee table.

Where are the glasses?

Where are the glasses ?

Solution:
Let
RK = I was reading the newspaper in the kitchen.
GK = My glasses are on the kitchen table.
SB = I saw my glasses at breakfast.
RL = I was reading the newspaper in the living room.
GC = My glasses are on the coffee table.

1. $RK \rightarrow GK$ by (a)
$GK \rightarrow SB$ by (d)
$\therefore RK \rightarrow SB$ by transitivity

2. $RK \rightarrow SB$ by the conclusion of (1)
$\sim SB$ by (c)
$\therefore \sim RK$ by modus tollens

3. $RL \vee RK$ by (d)
$\sim RK$ by the conclusion of (2)
$\therefore RL$ by elimination

4. $RL \rightarrow GC$ by (e)
RL by the conclusion of (3)
$\therefore GC$ by modus ponens

Thus, the glasses are on the coffee table.

Lecture 7 – part 4

Fallacies

Fallacies

A **fallacy** is an error in reasoning that results in an invalid argument.

Three common fallacies are:

- **Using ambiguous premises,** and treating them as if they were unambiguous,
- **Circular reasoning** (assuming what is to be proved without having derived it from the premises), and
- **Jumping to a conclusion** (without adequate grounds).

Two other fallacies: ***converse error*** and ***inverse error***

- Give rise to arguments that superficially resemble those that are valid by modus ponens and modus tollens but are not, in fact, valid.

Truth table

or

For an argument to be valid, every argument of the same form whose premises are all true must have a true conclusion. It follows that for an argument to be invalid means that there is an argument of that form whose premises are all true and whose conclusion is false.

Converse error

Show that the following argument **is invalid**:

If Zeke is a cheater, **then** Zeke sits in the back row.

Zeke sits in the back row.

∴ Zeke is a cheater.

$$p \rightarrow q$$
$$q$$
$$\therefore p$$

$p \to q$

q

$\therefore p$

conclusion	premises	
p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Inverse error

Consider the following argument::

If interest rates are going up, stock market prices will go down.

Interest rates are not going up.

∴ Stock market prices will not go down.

$$p \to q$$
$$\sim p$$
$$\therefore \sim q$$

$p \to q$

$\sim p$

$\therefore \sim q$

p	q	$\overbrace{\sim p}^{\text{Premise}}$	$\overbrace{\sim q}^{\text{conclusion}}$	$\overbrace{p \to q}^{\text{premise}}$	
T	T	F	F	T	
T	F	F	T	F	
F	T	T	F	T	✗
F	F	T	T	T	✓

critical rows

False premises and false conclusion

If John Lennon was a rock star, **then** John Lennon had red hair.

John Lennon was a rock star.

∴ John Lennon had red hair.

An Invalid Argument with True Premises and a True Conclusion

The argument below is invalid by the **converse error**, but it has a true conclusion.

If New York is a big city, **then** New York has tall buildings.
New York has tall buildings.
∴ New York is a big city.

• Definition

An argument is called **sound** if, and only if, it is valid *and* all its premises are true. An argument that is not sound is called **unsound**.

Contradiction Rule

Contradiction Rule

If you can show that the supposition that statement p is false leads logically to a contradiction, then you can conclude that p is true.

$\sim p \rightarrow$ **c**, where **c** is a contradiction
$\therefore p$

			premises	conclusion
p	$\sim p$	c	$\sim p \rightarrow$ c	p
T	F	F	T	T
F	T	F	F	

There is only one critical row in which the premise is true, and in this row the conclusion is also true. Hence this form of argument is valid.

If an assumption leads to a contradiction, then that assumption must be false.