COT 2000
Foundations of Computing

Summer 2024

Lecture 13 – part 1

Homework 4 – 06/28/24

Lab 7

Lecture 13 – part 2

Review

Review

- Sequences
- Summation and Product Notation
- Summation and Product Properties
- Change of variables
- Factorial Notation
- (n choose r)
- Mathematical Induction

Properties

If $a_m, a_{m+1}, a_{m+2}, \ldots$ and $b_m, b_{m+1}, b_{m+2}, \ldots$ are sequences of real numbers and c is any real number, then the following equations hold for any integer $n \geq m$:

1. $\displaystyle\sum_{k=m}^{n} a_k + \sum_{k=m}^{n} b_k = \sum_{k=m}^{n} (a_k + b_k)$

2. $\displaystyle c \cdot \sum_{k=m}^{n} a_k = \sum_{k=m}^{n} c \cdot a_k$ generalized distributive law

3. $\displaystyle\left(\prod_{k=m}^{n} a_k\right) \cdot \left(\prod_{k=m}^{n} b_k\right) = \prod_{k=m}^{n} (a_k \cdot b_k).$

Factorial Notation

• Definition

For each positive integer n, the quantity ***n* factorial** denoted ***n*!**, is defined to be the product of all the integers from 1 to n:

$$n! = n \cdot (n-1) \cdots 3 \cdot 2 \cdot 1.$$

Zero factorial, denoted $0!$, is defined to be 1:

$$0! = 1.$$

Recursive definition

$$n! = \begin{cases} 1 & \text{if } n = 0 \\ n \cdot (n-1)! & \text{if } n \geq 1. \end{cases}$$

Example: Fist 10 factorials

$$\begin{aligned}
0! &= 1 & 1! &= 1 \\
2! &= 2 \cdot 1 = 2 & 3! &= 3 \cdot 2 \cdot 1 = 6 \\
4! &= 4 \cdot 3 \cdot 2 \cdot 1 = 24 & 5! &= 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120 \\
6! &= 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720 & 7! &= 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5{,}040 \\
8! &= 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 & 9! &= 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \\
&= 40{,}320 & &= 362{,}880
\end{aligned}$$

(n choose r) Notation

- Binomial coefficient $\binom{n}{r}$

• Definition

Let n and r be integers with $0 \leq r \leq n$. The symbol

$$\binom{n}{r}$$

is read "***n* choose *r***" and represents the number of subsets of size r that can be chosen from a set with n elements.

• Formula for Computing $\binom{n}{r}$

For all integers n and r with $0 \leq r \leq n$,

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}.$$

also called *combinations*

Method of Proof by Mathematical Induction

Consider a statement of the form, "For all integers $n \geq a$, a property $P(n)$ is true." To prove such a statement, perform the following two steps:

Step 1 (basis step): Show that $P(a)$ is true.

Step 2 (inductive step): Show that for all integers $k \geq a$, if $P(k)$ is true then $P(k+1)$ is true. To perform this step,

suppose that $P(k)$ is true, where k is any particular but arbitrarily chosen integer with $k \geq a$.

[This supposition is called the **inductive hypothesis.***]*

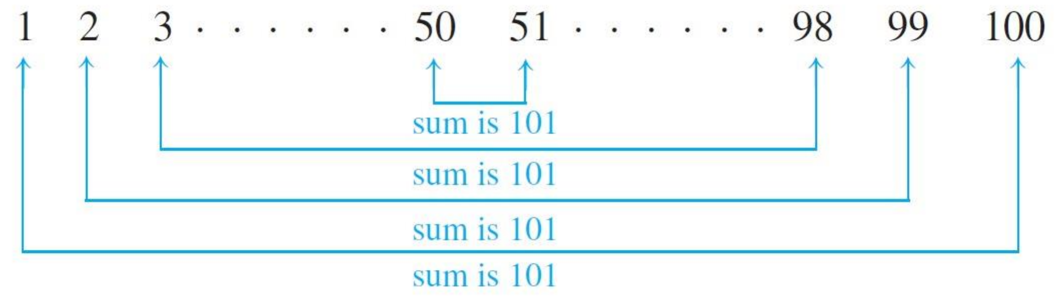
Then

show that $P(k+1)$ is true.

Lecture 13 – part 3

Closed Form

The story is told that one of the greatest mathematicians of all time, Carl Friedrich Gauss (1777–1855), was given the problem of adding the numbers from 1 to 100 by his teacher when he was a young child. The teacher had asked his students to compute the sum, supposedly to gain himself some time to grade papers. But after just a few moments, Gauss produced the correct answer. Needless to say, the teacher was dumbfounded. How could young Gauss have calculated the quantity so rapidly? In his later years, Gauss explained that he had imagined the numbers paired according to the following schema.



The sum of the numbers in each pair is 101, and there are 50 pairs in all; hence the total sum is $50 \cdot 101 = 5{,}050$.

$$1 + 2 + 3 + \cdots + n = \frac{n}{2}(n+1) \text{ in closed form.}$$

Closed Form

• Definition Closed Form

If a sum with a variable number of terms is shown to be equal to a formula that does not contain either an ellipsis or a summation symbol, we say that it is written **in closed form**.

For example, writing $1 + 2 + 3 + \cdots + n = \dfrac{n(n+1)}{2}$

expresses the sum $1 + 2 + 3 + \cdots + n$ in closed form.

$$1 + 2 + 3 + \cdots + n = \frac{n}{2}(n+1)$$

Example:

a. Evaluate $2 + 4 + 6 + \cdots + 500$.

b. Evaluate $5 + 6 + 7 + 8 + \cdots + 50$.

c. For an integer $h \geq 2$, write $1 + 2 + 3 + \cdots + (h-1)$ in closed form.

Solution:

a.
$$\begin{aligned} 2 + 4 + 6 + \cdots + 500 &= 2 \cdot (1 + 2 + 3 + \cdots + 250) \\ &= 2 \cdot \left(\frac{250 \cdot 251}{2}\right) && \text{by applying the formula for the sum of the first } n \text{ integers with } n = 250 \\ &= 62{,}750. \end{aligned}$$

b.
$$\begin{aligned} 5 + 6 + 7 + 8 + \cdots + 50 &= (1 + 2 + 3 + \cdots + 50) - (1 + 2 + 3 + 4) \\ &= \frac{50 \cdot 51}{2} - 10 && \text{by applying the formula for the sum of the first } n \text{ integers with } n = 50 \\ &= 1{,}265 \end{aligned}$$

c.
$$\begin{aligned} 1 + 2 + 3 + \cdots + (h-1) &= \frac{(h-1) \cdot [(h-1)+1]}{2} && \text{by applying the formula for the sum of the first } n \text{ integers with } n = h - 1 \\ &= \frac{(h-1) \cdot h}{2} && \text{since } (h-1) + 1 = h. \end{aligned}$$

■

Sum of a geometric sequence

- In a geometric sequence, each term is obtained from the preceding one by multiplying by a constant factor.
- If the first term is 1 and the constant factor is *r* , then the sequence is 1, $r, r^2, r^3, \ldots, r^n, \ldots$.
- For all integers n ≥ 0 and real numbers r not equal to 1, the sum is given by:

$$\sum_{i=0}^{n} r^i = \frac{r^{n+1} - 1}{r - 1}$$

$$1 + r + r^2 + \cdots + r^n = \frac{r^{n+1}-1}{r-1}$$

Proof (by mathematical induction):

Suppose r is a particular but arbitrarily chosen real number that is not equal to 1, and let the property $P(n)$ be the equation

$$\sum_{i=0}^{n} r^i = \frac{r^{i+1} - 1}{r - 1} \quad \leftarrow P(n)$$

We must show that $P(n)$ is true for all integers $n \geq 0$. We do this by mathematical induction on n.

Step 1

Show that P(0) is true:

To establish $P(0)$, we must show that

$$\sum_{i=0}^{0} r^i = \frac{r^{0+1} - 1}{r - 1} \quad \leftarrow P(0)$$

The left-hand side of this equation is $r^0 = 1$ and the right-hand side is

$$\frac{r^{0+1} - 1}{r - 1} = \frac{r - 1}{r - 1} = 1$$

also because $r^1 = r$ and $r \neq 1$. Hence $P(0)$ is true.

Step 2

Show that for all integers $k \geq 0$, if $P(k)$ is true then $P(k+1)$ is also true:

[Suppose that P(k) is true for a particular but arbitrarily chosen integer $k \geq 0$. That is:]

Let k be any integer with $k \geq 0$, and suppose that

$$\sum_{i=0}^{k} r^i = \frac{r^{k+1} - 1}{r - 1} \qquad \leftarrow P(k) \text{ inductive hypothesis}$$

[We must show that $P(k+1)$ is true. That is:] We must show that

$$\sum_{i=0}^{k+1} r^i = \frac{r^{(k+1)+1} - 1}{r - 1},$$

or, equivalently, that

$$\sum_{i=0}^{k+1} r^i = \frac{r^{k+2} - 1}{r - 1}. \qquad \leftarrow P(k+1)$$

[We will show that the left-hand side of $P(k+1)$ equals the right-hand side.]

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The left-hand side of $P(k+1)$ is

$$\sum_{i=0}^{k+1} r^i = \sum_{i=0}^{k} r^i + r^{k+1}$$ by writing the $(k+1)$st term separately from the first k terms

$$= \frac{r^{k+1}-1}{r-1} + r^{k+1}$$ by substitution from the inductive hypothesis

$$= \frac{r^{k+1}-1}{r-1} + \frac{r^{k+1}(r-1)}{r-1}$$ by multiplying the numerator and denominator of the second term by $(r-1)$ to obtain a common denominator

$$= \frac{(r^{k+1}-1) + r^{k+1}(r-1)}{r-1}$$ by adding fractions

$$= \frac{r^{k+1} - 1 + r^{k+2} - r^{k+1}}{r-1}$$ by multiplying out and using the fact that $r^{k+1} \cdot r = r^{k+1} \cdot r^1 = r^{k+2}$

$$= \frac{r^{k+2}-1}{r-1}$$ by canceling the r^{k+1}'s.

which is the right-hand side of $P(k+1)$ *[as was to be shown.]*
[Since we have proved the basis step and the inductive step, we conclude that the theorem is true.]

Note: Two different ways to show that an equation is true:
(1) Transforming the left-hand side and the right-hand side independently until they are seen to be equal, and
(2) Transforming one side of the equation until it is seen to be the same as the other side of the equation.

Example: In each of (a) and (b) below, assume that m is an integer that is greater than or equal to 3. Write each of the sums in closed form.

$$1 + r + r^2 + \cdots + r^n = \frac{r^{n+1}-1}{r-1}$$

$$\sum_{i=0}^{n} r^i = \frac{r^{n+1}-1}{r-1}$$

a. $1 + 3 + 3^2 + \cdots + 3^{m-2}$

b. $3^2 + 3^3 + 3^4 + \cdots + 3^m$

Solution:

a. $$1 + 3 + 3^2 + \cdots + 3^{m-2} = \frac{3^{(m-2)+1} - 1}{3 - 1}$$ by applying the formula for the sum of a geometric sequence with $r = 3$ and $n = m - 2$

$$= \frac{3^{m-1} - 1}{2}.$$

b. $$3^2 + 3^3 + 3^4 + \cdots + 3^m = 3^2 \cdot (1 + 3 + 3^2 + \cdots + 3^{m-2})$$ by factoring out 3^2

$$= 9 \cdot \left(\frac{3^{m-1} - 1}{2}\right)$$ by part (a).

Lecture 13 – part 4

Recurrence Relations

Recurrence Relation for a Sequence

• Definition

A **recurrence relation** for a sequence $a_0, a_1, a_2, \ldots$ is a formula that relates each term a_k to certain of its predecessors $a_{k-1}, a_{k-2}, \ldots, a_{k-i}$, where i is an integer with $k - i \geq 0$. The **initial conditions** for such a recurrence relation specify the values of $a_0, a_1, a_2, \ldots, a_{i-1}$, if i is a fixed integer, or $a_0, a_1, \ldots, a_m$, where m is an integer with $m \geq 0$, if i depends on k.

Example: Define a sequence $c_0, c_1, c_2, \ldots$ recursively as follows: For all integers $k \geq 2$,

$$(1) \quad c_k = c_{k-1} + kc_{k-2} + 1 \qquad \text{recurrence relation}$$

$$(2) \quad c_0 = 1 \quad \text{and} \quad c_1 = 2 \qquad \text{initial conditions.}$$

Find c_2, c_3, and c_4.

Solution:

$$\begin{aligned} c_2 &= c_1 + 2c_0 + 1 && \text{by substituting } k = 2 \text{ into (1)} \\ &= 2 + 2 \cdot 1 + 1 && \text{since } c_1 = 2 \text{ and } c_0 = 1 \text{ by (2)} \end{aligned}$$

$$\begin{aligned} (3) \quad \therefore c_2 &= 5 \\ c_3 &= c_2 + 3c_1 + 1 && \text{by substituting } k = 3 \text{ into (1)} \\ &= 5 + 3 \cdot 2 + 1 && \text{since } c_2 = 5 \text{ by (3) and } c_1 = 2 \text{ by (2)} \\ (4) \quad \therefore c_3 &= 12 \\ c_4 &= c_3 + 4c_2 + 1 && \text{by substituting } k = 4 \text{ into (1)} \\ &= 12 + 4 \cdot 5 + 1 && \text{since } c_3 = 12 \text{ by (4) and } c_2 = 5 \text{ by (3)} \\ (5) \quad \therefore c_4 &= 33 \end{aligned}$$

Example: Catalan numbers

Closed form

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{(n+1)!\cdot n!}$$

Recursive

$$C_0 = 1$$

$$C_{n+1} = \sum_{i=0}^{n} C_i \cdot C_{n-i} \quad \text{for } n \geq 0$$

Solution:

$$C_1 = \frac{1}{1+1}\binom{2\cdot 1}{1} = \frac{1}{2}\binom{2}{1} = \frac{1}{2}\cdot 2 = 1$$

$$C_2 = \frac{1}{2+1}\binom{2\cdot 2}{2} = \frac{1}{3}\binom{4}{2} = \frac{1}{3}\cdot 6 = 2$$

$$C_3 = \frac{1}{3+1}\binom{2\cdot 3}{3} = \frac{1}{4}\binom{6}{3} = \frac{1}{4}\cdot 20 = 5$$

$$C_1 = C_0 \cdot C_0 = 1\cdot 1 = 1$$

$$C_2 = C_0\cdot C_1 + C_1\cdot C_0 = 1\cdot 1 + 1\cdot 1 = 2$$

$$C_3 = C_0\cdot C_2 + C_1\cdot C_1 + C_2\cdot C_0 = 1\cdot 2 + 1\cdot 1 + 2\cdot 1 = 5$$

Example: **Find a close form expression for the sequence:**

$$m_k = 2m_{k-1} + 1$$

Initial condition : $m_0 = 0$

Recursive

$$m_1 = 1$$
$$m_2 = 2m_1 + 1 = 2 \cdot 1 + 1 = 3$$
$$m_3 = 2m_2 + 1 = 2 \cdot 3 + 1 = 7$$
$$m_4 = 2m_3 + 1 = 2 \cdot 7 + 1 = 15$$
$$m_5 = 2m_4 + 1 = 2 \cdot 15 + 1 = 31$$
$$m_6 = 2m_5 + 1 = 2 \cdot 31 + 1 = 63$$
$$m_7 = 2m_6 + 1 = 2 \cdot 63 + 1 = 127$$
$$m_8 = 2m_7 + 1 = 2 \cdot 127 + 1 = 255$$

Solution:

Recursive

$$
\begin{aligned}
m_k &= 2m_{k-1} + 1 && \text{(Recursive formula)} \\
m_k &= 2(2m_{k-2} + 1) + 1 && \text{(Expand } m_{k-1}\text{)} \\
m_k &= 2(2(2m_{k-3} + 1) + 1) + 1 && \text{(Expand } m_{k-2}\text{)} \\
m_k &= 2(2(2(2m_{k-4} + 1) + 1) + 1) + 1 && \text{(Expand } m_{k-3}\text{)} \\
m_k &= \ldots && \text{(Continue expanding)} \\
m_k &= 2^{k-1}m_1 + 2^{k-2} + 2^{k-3} + \ldots + 2^1 + 2^0 && \text{(General form)} \\
m_k &= 2^{k-1}(1) + 2^{k-2} + 2^{k-3} + \ldots + 2^1 + 2^0 && \text{(Base case)} \\
m_k + 2^k &= 1 + 2^1 + 2^2 + \ldots + 2^{k-1} + 2^k && \text{(Geometric series)} \\
m_k + 2^k &= \sum_{i=0}^{k-1} 2^i + 2^k && \text{(Geometric series sum)} \\
m_k + 2^k &= \sum_{i=0}^{k} 2^i && \text{(Simplifying)} \\
m_k + 2^k &= \frac{2^{k+1} - 1}{2 - 1} && \text{(Closed form)} \\
m_k &= 2^{k+1} - 2^k - 1 && \text{(Simplifying)} \\
m_k &= 2(2^k) - 2^k - 1 && \text{(Simplifying)} \\
m_k &= 2^k(2 - 1) - 1 && \text{(Simplifying)} \\
m_k &= 2^k - 1 && \text{(Simplifying)}
\end{aligned}
$$

$$1 + r + r^2 + \cdots + r^n$$

$$\sum_{i=0}^{n} r^i = \frac{r^{n+1} - 1}{r - 1}$$

Closed form

Example: **Can we find a close form expression for the sequence ?** $m_k = 2m_{k-1} + 1$

Recursive

$m_1 = 1$ Initial condition

$m_2 = 2m_1 + 1 = 2 \cdot 1 + 1 = 3$

$m_3 = 2m_2 + 1 = 2 \cdot 3 + 1 = 7$

$m_4 = 2m_3 + 1 = 2 \cdot 7 + 1 = 15$

$m_5 = 2m_4 + 1 = 2 \cdot 15 + 1 = 31$

$m_6 = 2m_5 + 1 = 2 \cdot 31 + 1 = 63$

$m_7 = 2m_6 + 1 = 2 \cdot 63 + 1 = 127$

$m_8 = 2m_7 + 1 = 2 \cdot 127 + 1 = 255$

Closed form

$m_1 = 2^1 - 1 = 1$

$m_2 = 2^2 - 1 = 3$

$m_3 = 2^3 - 1 = 7$

$m_4 = 2^4 - 1 = 15$

$m_5 = 2^5 - 1 = 31$

$m_6 = 2^6 - 1 = 63$

$m_7 = 2^7 - 1 = 127$

$m_8 = 2^8 - 1 = 255$

Fibonacci Numbers

- The **Fibonacci sequence** is a series of numbers where each number is the sum of the two preceding ones.
- This sequence can be used to model a hypothetical rabbit population growth as described by Fibonacci.
- **Initial Conditions:**
  - $F_0=1$
  - $F_1=1$
- **Recurrence Relation:**
  - For $k \geq 2$, $F_k = F_{k-1} + F_{k-2}$
- **Calculating the Number of Rabbits:**
  - To find the number of rabbit pairs at the end of the 12th month, calculate F_{12} using the recurrence relation.

How many rabbits will there be at the end of the year?

A single pair of rabbits (male and female) is born at the beginning of a year.

Assume the following conditions:

1. Rabbit **pairs** are not fertile during their first month of life but thereafter give birth to one new male/female **pair** at the end of every month.
2. No rabbits die.

(1) $F_k = F_{k-1} + F_{k-2}$ recurrence relation

(2) $F_0 = 1, \quad F_1 = 1$ initial conditions. pair

To answer Fibonacci's question, compute F_2, F_3, and so forth through F_{12}:

(3)	$F_2 = F_1 + F_0 = 1 + 1$	$= 2$	by (1) and (2)
(4)	$F_3 = F_2 + F_1 = 2 + 1$	$= 3$	by (1), (2) and (3)
(5)	$F_4 = F_3 + F_2 = 3 + 2$	$= 5$	by (1), (3) and (4)
(6)	$F_5 = F_4 + F_3 = 5 + 3$	$= 8$	by (1), (4) and (5)
(7)	$F_6 = F_5 + F_4 = 8 + 5$	$= 13$	by (1), (5) and (6)
(8)	$F_7 = F_6 + F_5 = 13 + 8$	$= 21$	by (1), (6) and (7)
(9)	$F_8 = F_7 + F_6 = 21 + 13$	$= 34$	by (1), (7) and (8)
(10)	$F_9 = F_8 + F_7 = 34 + 21$	$= 55$	by (1), (8) and (9)
(11)	$F_{10} = F_9 + F_8 = 55 + 34$	$= 89$	by (1), (9) and (10)
(12)	$F_{11} = F_{10} + F_9 = 89 + 55$	$= 144$	by (1), (10) and (11)
(13)	$F_{12} = F_{11} + F_{10} = 144 + 89$	$= 233$	by (1), (11) and (12)

At the end of the twelfth month there are 233 rabbit pairs, or 466 rabbits in all.

Example: Compound Interest

On your twenty-first birthday you get a letter informing you that on the day you were born an eccentric rich aunt deposited \$100,000 in a bank account earning 4% interest compounded annually and she now intends to turn the account over to you, provided you can figure out how much it is worth. What is the amount currently in the account?

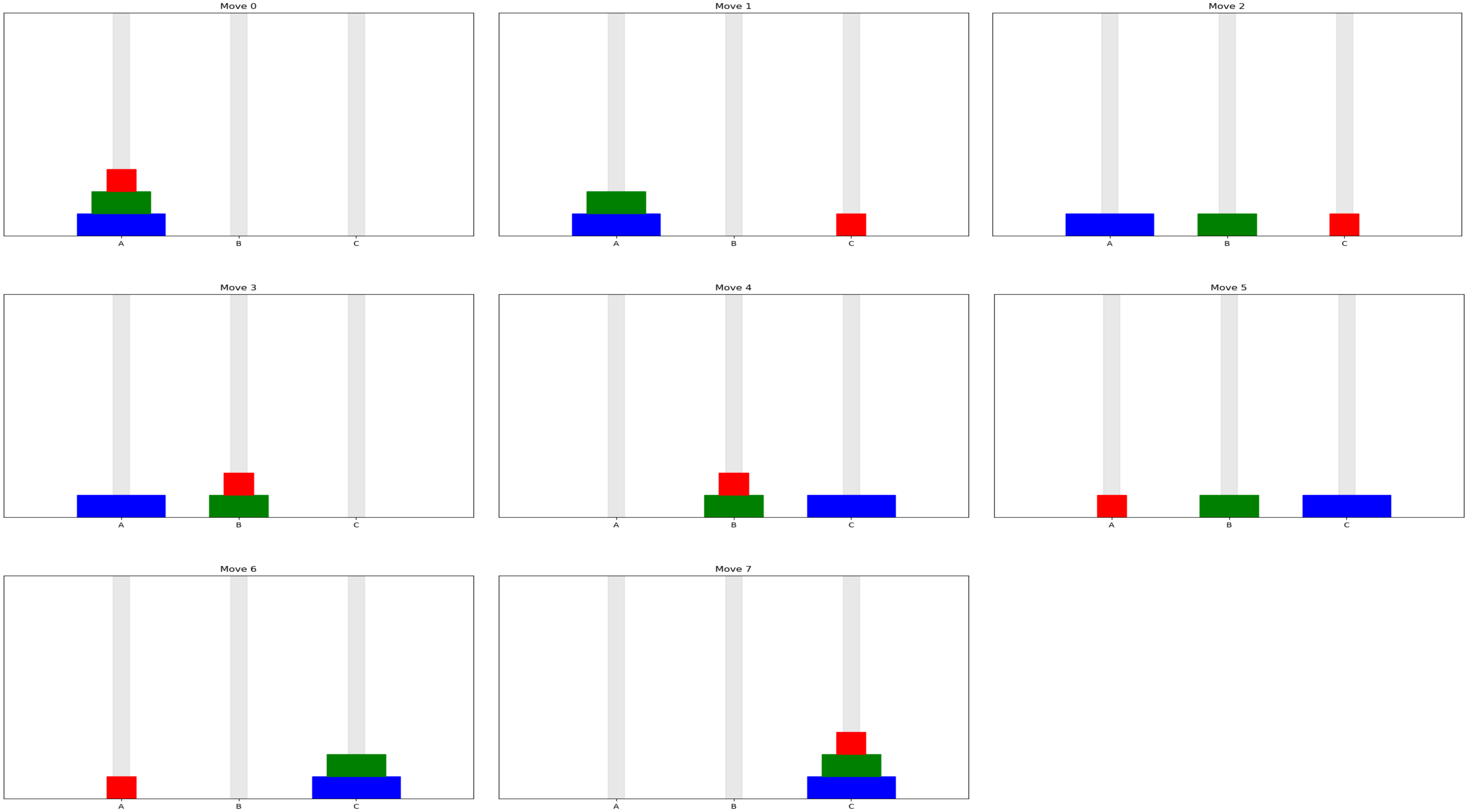
$$
\begin{aligned}
A_k &= A_{k-1} + (0.04) \cdot A_{k-1} \\
&= (1 + 0.04) \cdot A_{k-1} = (1.04) \cdot A_{k-1} \qquad \text{by factoring out } A_{k-1}.
\end{aligned}
$$

(1) $A_k = (1.04) \cdot A_{k-1}$ recurrence relation

(2) $A_0 = \$100,000$ initial condition.

$$
\begin{array}{lllll}
(3)\ A_1 & = 1.04 \cdot A_0 & = (1.04) \cdot \$100,000 & = \$104,000 & \text{by (1) and (2)} \\
(4)\ A_2 & = 1.04 \cdot A_1 & = (1.04) \cdot \$104,000 & = \$108,160 & \text{by (1) and (3)} \\
(5)\ A_3 & = 1.04 \cdot A_2 & = (1.04) \cdot \$108,160 & = \$112,486.40 & \text{by (1) and (4)} \\
 & \vdots & & \vdots & \\
(22)\ A_{20} & = 1.04 \cdot A_{19} & \cong (1.04) \cdot \$210,684.92 & \cong \$219,112.31 & \text{by (1) and (21)} \\
(23)\ A_{21} & = 1.04 \cdot A_{20} & \cong (1.04) \cdot \$219,112.31 & \cong \$227,876.81 & \text{by (1) and (22)}
\end{array}
$$

Hanoi Tower Game



How many moves for k disks ?

The formula for the Tower of Hanoi puzzle can also be derived recursively. (How many moves m_k for k disks).

1. **Base Case:** For a single disk ($k=1$), it only takes one move to transfer it to another pole. So, $m_1=1$.
2. **Recursive Step:** Suppose you want to move k disks from one pole to another. The process can be broken down into three key steps:
   1. **Step 1:** First, you need to move the top $k-1$ disks from the starting pole to the auxiliary pole. This is a smaller instance of the same problem, so it takes m_{k-1} moves.
   2. **Step 2:** Next, you move the largest disk (the k^{th} disk) directly to the target pole. This takes 1 move.
   3. **Step 3:** Finally, you need to move the $k-1$ disks from the auxiliary pole to the target pole, on top of the largest disk. Again, this is a smaller instance of the same problem, requiring m_{k-1} moves.