COT 2000
Foundations of Computing

Summer 2024

Lecture 8 – part 1

Lab 4

Homework 2 - Due: 06/07/24

Homework 3 - Due: 06/14/24

Lecture 8 – part 2

Review

Review

- Arguments
- Test a valid argument
- Modus Ponens
- Modus Tollen
- Generalization
- Specialization
- Elimination
- Transitivity
- Proof by Division into Cases
- others

Valid and Invalid Arguments

• Definition

An **argument** is a sequence of statements, and an **argument form** is a sequence of statement forms. All statements in an argument and all statement forms in an argument form, except for the final one, are called **premises** (or **assumptions** or **hypotheses**). The final statement or statement form is called the **conclusion.** The symbol $\therefore$, which is read "therefore," is normally placed just before the conclusion.

To say that an *argument form* is **valid** means that no matter what particular statements are substituted for the statement variables in its premises, if the resulting premises are all true, then the conclusion is also true. To say that an *argument* is **valid** means that its form is valid.

Example

If Socrates is a man, **then** Socrates is mortal.
Socrates is a man.
∴ Socrates is mortal.

If p **then** q
p
$\therefore q$

Testing an Argument Form for Validity

1. Identify the premises and conclusion of the argument form.
2. Construct a truth table showing the truth values of all the premises and the conclusion.
3. A row of the truth table in which all the premises are true is called a **critical row**.
   - If there is a critical row in which the conclusion is false, then it is possible for an argument of the given form to have <u>true premises and a false conclusion</u>, and so the argument form <u>is invalid</u>.
   - If the <u>conclusion</u> in every critical row is <u>true</u>, then the argument form <u>is valid</u>.

Exercise: Determine whether the following argument form is valid or invalid

$p \rightarrow q \vee \sim r$

$q \rightarrow p \wedge r$

$\therefore p \rightarrow r$

						Premises		Conclusion
p	q	r	$\sim r$	$q \vee \sim r$	$p \wedge r$	$p \rightarrow q \vee \sim r$	$q \rightarrow p \wedge r$	$p \rightarrow r$
T	T	T	F	T	T	T	T	T
T	T	F	T	T	F	T	F	F
T	F	T	F	F	T	F	T	T
T	F	F	T	T	F	T	T	F
F	T	T	F	T	F	T	F	T
F	T	F	T	T	F	T	F	T
F	F	T	F	F	F	T	T	T
F	F	F	T	T	F	T	T	T

INVALID

Modus Ponens

		premises		conclusion		
p	q	$p \to q$	p	q		
T	T	T	T	T	← critical row	VALID
T	F	F	T			
F	T	T	F			
F	F	T	F			

Valid Argument Forms

Modus Ponens	$p \to q$ p $\therefore q$	**Elimination**	**a.** $p \vee q$ $\sim q$ $\therefore p$ **b.** $p \vee q$ $\sim p$ $\therefore q$
Modus Tollens	$p \to q$ $\sim q$ $\therefore \sim p$	**Transitivity**	$p \to q$ $q \to r$ $\therefore p \to r$
Generalization	**a.** p $\therefore p \vee q$ **b.** q $\therefore p \vee q$	**Proof by Division into Cases**	$p \vee q$ $p \to r$ $q \to r$ $\therefore r$
Specialization	**a.** $p \wedge q$ $\therefore p$ **b.** $p \wedge q$ $\therefore q$		
Conjunction	p q $\therefore p \wedge q$	**Contradiction Rule**	$\sim p \to c$ $\therefore p$

Where are the glasses ?

Example:
You are about to leave for school in the morning and discover that you don't have your glasses.

You know the following statements are true:

a. **If** I was reading the newspaper in the kitchen, **then** my glasses are on the kitchen table.
b. **If** my glasses are on the kitchen table, **then** I saw them at breakfast.
c. I did not see my glasses at breakfast.
d. I was reading the newspaper in the living room **or** I was reading the newspaper in the kitchen.
e. **If** I was reading the newspaper in the living room **then** my glasses are on the coffee table.

Where are the glasses?

Where are the glasses ?

Solution:

Let

RK = I was reading the newspaper in the kitchen.

GK = My glasses are on the kitchen table.

SB = I saw my glasses at breakfast.

RL = I was reading the newspaper in the living room.

GC = My glasses are on the coffee table.

1. $RK \rightarrow GK$ by (a)

$GK \rightarrow SB$ by (d)

$\therefore RK \rightarrow SB$ by transitivity

2. $RK \rightarrow SB$ by the conclusion of (1)

$\sim SB$ by (c)

$\therefore \sim RK$ by modus tollens

3. $RL \vee RK$ by (d)

$\sim RK$ by the conclusion of (2)

$\therefore RL$ by elimination

4. $RL \rightarrow GC$ by (e)

RL by the conclusion of (3)

$\therefore GC$ by modus ponens

Thus, the glasses are on the coffee table.

Lecture 8 – part 3

Fallacies

Fallacies

A **fallacy** is an error in reasoning that results in an invalid argument.

Three common fallacies are:

- **Using ambiguous premises,** and treating them as if they were unambiguous,
- **Circular reasoning** (assuming what is to be proved without having derived it from the premises), and
- **Jumping to a conclusion** (without adequate grounds).

Two other fallacies: ***converse error*** and ***inverse error***

- Give rise to arguments that superficially resemble those that are valid by modus ponens and modus tollens but are not, in fact, valid.

Truth table

or

For an argument to be valid, every argument of the same form whose premises are all true must have a true conclusion. It follows that for an argument to be invalid means that there is an argument of that form whose premises are all true and whose conclusion is false.

Converse error

Show that the following argument **is invalid**:

If Zeke is a cheater, **then** Zeke sits in the back row.

Zeke sits in the back row.

∴ Zeke is a cheater.

$$p \rightarrow q$$
$$q$$
$$\therefore p$$

$$p \to q$$

$$q$$

$$\therefore p$$

conclusion	premises	
p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Inverse error

Consider the following argument::

If interest rates are going up, stock market prices will go down.

Interest rates are not going up.

∴ Stock market prices will not go down.

$$p \rightarrow q$$
$$\sim p$$
$$\therefore \sim q$$

$p \to q$

$\sim p$

$\therefore \sim q$

		premise	conclusion	premise	
p	q	$\sim p$	$\sim q$	$p \to q$	
T	T	F	F	T	
T	F	F	T	F	
F	T	T	F	T	×
F	F	T	T	T	✓

critical rows

False premises and false conclusion

If John Lennon was a rock star, **then** John Lennon had red hair.
John Lennon was a rock star.
∴ John Lennon had red hair.

An Invalid Argument with True Premises and a True Conclusion

The argument below is invalid by the **converse error**, but it has a true conclusion.

If New York is a big city, **then** New York has tall buildings.
New York has tall buildings.
∴ New York is a big city.

• Definition

An argument is called **sound** if, and only if, it is valid *and* all its premises are true. An argument that is not sound is called **unsound**.

Contradiction Rule

Contradiction Rule

If you can show that the supposition that statement p is false leads logically to a contradiction, then you can conclude that p is true.

$\sim p \rightarrow$ **c**, where **c** is a contradiction

$\therefore p$

			premises	conclusion
p	$\sim p$	c	$\sim p \rightarrow$ c	p
T	F	F	T	T
F	T	F	F	

There is only one critical row in which the premise is true, and in this row the conclusion is also true. Hence this form of argument is valid.

If an assumption leads to a contradiction, then that assumption must be false.

Knights and Knaves

The logician Raymond Smullyan describes an island containing two types of people:

knights who always tell the truth and **knaves** who always lie.

You visit the island and are approached by two natives who speak to you as follows:

A says: *B* is a knight.
B says: *A* and I are of opposite type.

QUESTION: What are *A* and *B*?

Solution : A and B are both knaves.

A says: *B* is a knight.
B says: *A* and I are of opposite type.

To see this, reason as follows:

Suppose A is a knight.

∴ What A says is true.	by definition of knight
∴ B is also a knight.	That's what A said.
∴ What B says is true.	by definition of knight
∴ A and B are of opposite types.	That's what B said.
∴ **We have arrived at the following contradiction:** A and B are both knights and A and B are of opposite type.	
∴ The supposition is false.	by the contradiction rule
∴ A is not a knight.	negation of supposition
∴ A is a knave.	by elimination: It's given that all inhabitants are knights or knaves, so since A is not a knight, A is a knave.
∴ What A says is false.	
∴ B is not a knight.	
∴ B is also a knave.	by elimination

Sound Argument - Summary

Here's a step-by-step breakdown:

1. **Check the Validity of the Argument Form**:
    1. Determine the structure or form of the argument.
    2. Use truth tables, logical proofs, or other methods to check if the form is valid.
    3. An argument form is valid if, whenever its premises are true, its conclusion must also be true.
2. **Check the Truth of the Premises**:
    1. Investigate or ascertain the truth of each premise in the real world.
    2. If a premise is found to be false, the argument might not lead to a true conclusion, even if its form is valid. However, it's possible (in some cases) for an argument with false premises to accidentally have a true conclusion. This doesn't make the argument sound, though.
3. **Evaluate the Conclusion**:
    1. If the argument form is valid and all the premises are true, then the conclusion must also be true. In this case, the argument is not just valid but also sound.

So, in short, for the conclusion to be confidently deemed true, both the validity of the argument form and the truth of the premises need to be established

Lecture 8 – part 4

Quantified Statements

Predicates

• Definition

A **predicate** is a sentence that contains a finite number of variables and becomes a statement when specific values are substituted for the variables. The **domain** of a predicate variable is the set of all values that may be substituted in place of the variable.

In logic, predicates can be obtained by removing some or all of the nouns from a statement.

For instance:
Sentence "James is a student at Bedford College,"

let P stand for "is a student at Bedford College", → $P(x)$
let Q stand for "is a student at." → $Q(x,y)$

Then both P and Q are *predicate symbols*.

x and y are *predicate variables* that take values in appropriate sets.

Let $P(x)$ be the predicate "$x^2 > x$" with domain the set **R** of all real numbers.

Write $P(2)$, $P(1/2)$, $P(-1/2)$ and

indicate which of these statements are true and which are false.

Solution:

$$P(2):\quad 2^2 > 2, \quad \text{or} \quad 4 > 2. \quad \text{True.}$$

$$P\left(\tfrac{1}{2}\right):\quad \left(\tfrac{1}{2}\right)^2 > \tfrac{1}{2}, \quad \text{or} \quad \tfrac{1}{4} > \tfrac{1}{2}. \quad \text{False.}$$

$$P\left(-\tfrac{1}{2}\right):\quad \left(-\tfrac{1}{2}\right)^2 > -\tfrac{1}{2}, \quad \text{or} \quad \tfrac{1}{4} > -\tfrac{1}{2}. \quad \text{True.}$$

Finding Truth Values of a Predicate

• Definition

If $P(x)$ is a predicate and x has domain D, the **truth set** of $P(x)$ is the set of all elements of D that make $P(x)$ true when they are substituted for x. The truth set of $P(x)$ is denoted

$$\{x \in D \mid P(x)\}.$$

Let $Q(n)$ be the predicate "n is a factor of 8." Find the truth set of $Q(n)$ if

a. the domain of n is the set $\mathbf{Z}^+$ of all positive integers

b. the domain of n is the set $\mathbf{Z}$ of all integers.

Solution:

a. The truth set is $\{1, 2, 4, 8\}$ because these are exactly the positive integers that divide 8 evenly.

b. The truth set is $\{1, 2, 4, 8, -1, -2, -4, -8\}$ because the negative integers -1, -2, -4, and -8 also divide into 8 without leaving a remainder.

The Universal Quantifier: ∀

- One sure way to change predicates into statements is to assign specific values to all their variables.
- Another way to obtain statements from predicates is to add **quantifiers**.
- **Quantifiers** are words that refer to quantities such as "some" or "all" and tell for how many elements a given predicate is true.

The symbol ∀ denotes "for all" and is called the **universal quantifier**.

"All human beings are mortal"

∀ human beings *x*, *x* is mortal.

$$\forall x \in H, x \text{ is mortal}$$

"For all *x* in the set of all human beings, *x* is mortal."

Truth and Falsity of Universal Statements

• Definition

Let $Q(x)$ be a predicate and D the domain of x. A **universal statement** is a statement of the form "$\forall x \in D, Q(x)$." It is defined to be true if, and only if, $Q(x)$ is true for every x in D. It is defined to be false if, and only if, $Q(x)$ is false for at least one x in D. A value for x for which $Q(x)$ is false is called a **counterexample** to the universal statement.

a. Let $D = \{1, 2, 3, 4, 5\}$, and consider the statement

$$\forall x \in D, x^2 \geq x.$$

Show that this statement is true.

b. Consider the statement

$$\forall x \in \mathbf{R}, x^2 \geq x.$$

Find a counterexample to show that this statement is false.

Solution:

a. Check that "$x^2 \geq x$" is true for each individual x in D.

$$1^2 \geq 1, \qquad 2^2 \geq 2, \qquad 3^2 \geq 3, \qquad 4^2 \geq 4, \qquad 5^2 \geq 5.$$

Hence "$\forall x \in D, x^2 \geq x$" is true.

b. *Counterexample:* Take $x = \frac{1}{2}$. Then x is in $\mathbf{R}$ (since $\frac{1}{2}$ is a real number) and

$$\left(\frac{1}{2}\right)^2 = \frac{1}{4} \ngeq \frac{1}{2}.$$

Hence "$\forall x \in \mathbf{R}, x^2 \geq x$" is false.

The Existential Quantifier: ∃

The symbol ∃ denotes "there exists" and is called the **existential quantifier**.

- **Definition**

Let $Q(x)$ be a predicate and D the domain of x. An **existential statement** is a statement of the form "$\exists x \in D$ such that $Q(x)$." It is defined to be true if, and only if, $Q(x)$ is true for at least one x in D. It is false if, and only if, $Q(x)$ is false for all x in D.

"There is a student in Math 140"

$\exists p \in P$ such that p is a student in Math 140

a. Consider the statement

$$\exists m \in \mathbf{Z}^+ \text{ such that } m^2 = m.$$

Show that this statement is true.

b. Let $E = \{5, 6, 7, 8\}$ and consider the statement

$$\exists m \in E \text{ such that } m^2 = m.$$

Show that this statement is false.

Solution:

a. Observe that $1^2 = 1$. Thus "$m^2 = m$" is true for at least one integer m. Hence "$\exists m \in \mathbf{Z}$ such that $m^2 = m$" is true.

b. Note that $m^2 = m$ is not true for any integers m from 5 through 8:

$$5^2 = 25 \neq 5, \qquad 6^2 = 36 \neq 6, \qquad 7^2 = 49 \neq 7, \qquad 8^2 = 64 \neq 8.$$

Thus "$\exists m \in E$ such that $m^2 = m$" is false.

Universal Conditional Statement

A reasonable argument can be made that the most important form of statement in mathematics is the **universal conditional statement:**

$\forall$*x,* if *P(x)* then *Q(x)*

$$\forall x \in \mathbb{R},\ (x > 2 \rightarrow x^2 > 4)$$

If a real number is greater than 2 then its square is greater than 4.

Negation of Quantified Statements

Negation of a Universal Statement

The negation of a statement of the form

$$\forall x \text{ in } D,\ Q(x)$$

is logically equivalent to a statement of the form

$$\exists x \text{ in } D \text{ such that } \sim Q(x).$$

Symbolically, $\sim(\forall x \in D,\ Q(x)) \equiv \exists x \in D \text{ such that } \sim Q(x).$

The negation of a universal statement (“all are”) is logically equivalent to an existential statement (“some are not” or “there is at least one that is not”).

Negation of Quantified Statements

Negation of an Existential Statement

The negation of a statement of the form

$$\exists x \text{ in } D \text{ such that } Q(x)$$

is logically equivalent to a statement of the form

$$\forall x \text{ in } D, \sim Q(x).$$

Symbolically, $\sim(\exists x \in D \text{ such that } Q(x)) \equiv \forall x \in D, \sim Q(x)$.

The negation of an existential statement (“some are”) is logically equivalent to a universal statement (“none are” or “all are not”).

Write formal negations for the following statements:

a. $\forall$ primes p, p is odd.

b. $\exists$ a triangle T such that the sum of the angles of T equals $200°$.

Solution:

a. By applying the rule for the negation of a $\forall$ statement, you can see that the answer is

$$\exists \text{ a prime } p \text{ such that } p \text{ is not odd.}$$

b. By applying the rule for the negation of a $\exists$ statement, you can see that the answer is

$$\forall \text{ triangles } T, \text{ the sum of the angles of } T \text{ does not equal } 200°.$$

Rewrite the following statement formally. Then write formal and informal negations.

No politicians are honest.

Solution

Formal version: $\forall$ politicians x, x is not honest.

Formal negation: $\exists$ a politician x such that x is honest.

Informal negation: Some politicians are honest.

Negations of Universal Conditional Statements

Negation of a Universal Conditional Statement

$$\sim(\forall x, \text{ if } P(x) \text{ then } Q(x)) \equiv \exists x \text{ such that } P(x) \text{ and } \sim Q(x).$$

By definition of the negation of a *for all* statement,

$$\sim (\forall x, P(x) \to Q(x)) \equiv \exists x \text{ such that } \sim (P(x) \to Q(x)). \tag{1}$$

But the negation of an *if-then* statement is logically equivalent to an *and* statement. More precisely,

$$\sim (P(x) \to Q(x)) \equiv P(x) \wedge \sim Q(x). \tag{2}$$

Substituting (2) into (1) gives

$$\sim (\forall x, P(x) \to Q(x)) \equiv \exists x \text{ such that } (P(x) \wedge \sim Q(x)).$$

Variants of Universal Conditional Statements

• Definition

Consider a statement of the form: $\forall x \in D$, if $P(x)$ then $Q(x)$.

1. Its **contrapositive** is the statement: $\forall x \in D$, if $\sim Q(x)$ then $\sim P(x)$.
2. Its **converse** is the statement: $\forall x \in D$, if $Q(x)$ then $P(x)$.
3. Its **inverse** is the statement: $\forall x \in D$, if $\sim P(x)$ then $\sim Q(x)$.