COT 2000
Foundations of Computing

Spring 2024

Lecture 15 – part 1

Lab 8

Homework 5 – 06/05/24

Lecture 15 – part 2

Review

Review

- Mathematical Induction
- Closed Form Expressions
- Sum of Geometric Sequences
- Defining sequences recursively
- Fibonacci Numbers
- Compound Interest
- Hanoi Tower
- Functions revisited

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

$$\sum_{i=0}^{n} r^i = \frac{r^{n+1} - 1}{r - 1}$$

$$C_0 = 1$$

$$C_{n+1} = \sum_{i=0}^{n} C_i \cdot C_{n-i} \quad \text{for } n \geq 0$$

$F_0=1$, $F_1=1$, For $k \geq 2$,

$F_k = F_{k-1} + F_{k-2}$

$$A_k = A_{k-1} + i\,A_{k-1}$$

$$m_k = 2m_{k-1} + 1$$

$$m_k = 2^k - 1$$

Finding close form expressions

Method	Description	Nature
Characteristic Equation	Solve characteristic equation derived from the relation. Used for linear homogeneous relations with constant coefficients.	Science and Art
Iterative/Substitution	Iteratively substitute the recurrence into itself to reveal patterns, suitable for simpler recurrences.	Science
Generating Functions	Associate sequence with a power series and manipulate to find a closed-form expression for the sequence.	Science and Art
Matrix Methods	Represent the relation using matrices and apply linear algebra to find the solution. Useful for linear transformations.	Science
Master Theorem	Apply for recurrences from divide-and-conquer algorithms; provides a formula given the recurrence fits a specific form.	Science

Then apply rigorous proof methods

.........the process involves trial, error, and a lot of persistence !

Fibonacci Numbers

- The **Fibonacci sequence** is a series of numbers where each number is the sum of the two preceding ones.
- This sequence can be used to model a hypothetical rabbit population growth as described by Fibonacci.
- **Initial Conditions:**
  - $F_0=1$
  - $F_1=1$
- **Recurrence Relation:**
  - For $k \geq 2$, $F_k = F_{k-1} + F_{k-2}$
- **Calculating the Number of Rabbits:**
  - To find the number of rabbit pairs at the end of the 12th month, calculate F_{12} using the recurrence relation.

How many rabbits will there be at the end of the year?

A single pair of rabbits (male and female) is born at the beginning of a year.

Assume the following conditions:

1. Rabbit **pairs** are not fertile during their first month of life but thereafter give birth to one new male/female **pair** at the end of every month.
2. No rabbits die.

(1) $F_k = F_{k-1} + F_{k-2}$ recurrence relation

(2) $F_0 = 1, \quad F_1 = 1$ initial conditions. pair

To answer Fibonacci's question, compute F_2, F_3, and so forth through F_{12}:

(3)	$F_2 = F_1 + F_0 = 1 + 1$	$= 2$	by (1) and (2)
(4)	$F_3 = F_2 + F_1 = 2 + 1$	$= 3$	by (1), (2) and (3)
(5)	$F_4 = F_3 + F_2 = 3 + 2$	$= 5$	by (1), (3) and (4)
(6)	$F_5 = F_4 + F_3 = 5 + 3$	$= 8$	by (1), (4) and (5)
(7)	$F_6 = F_5 + F_4 = 8 + 5$	$= 13$	by (1), (5) and (6)
(8)	$F_7 = F_6 + F_5 = 13 + 8$	$= 21$	by (1), (6) and (7)
(9)	$F_8 = F_7 + F_6 = 21 + 13$	$= 34$	by (1), (7) and (8)
(10)	$F_9 = F_8 + F_7 = 34 + 21$	$= 55$	by (1), (8) and (9)
(11)	$F_{10} = F_9 + F_8 = 55 + 34$	$= 89$	by (1), (9) and (10)
(12)	$F_{11} = F_{10} + F_9 = 89 + 55$	$= 144$	by (1), (10) and (11)
(13)	$F_{12} = F_{11} + F_{10} = 144 + 89$	$= 233$	by (1), (11) and (12)

At the end of the twelfth month there are 233 rabbit pairs, or 466 rabbits in all.

Example: Compound Interest

On your twenty-first birthday you get a letter informing you that on the day you were born an eccentric rich aunt deposited $100,000 in a bank account earning 4% interest compounded annually and she now intends to turn the account over to you, provided you can figure out how much it is worth. What is the amount currently in the account?

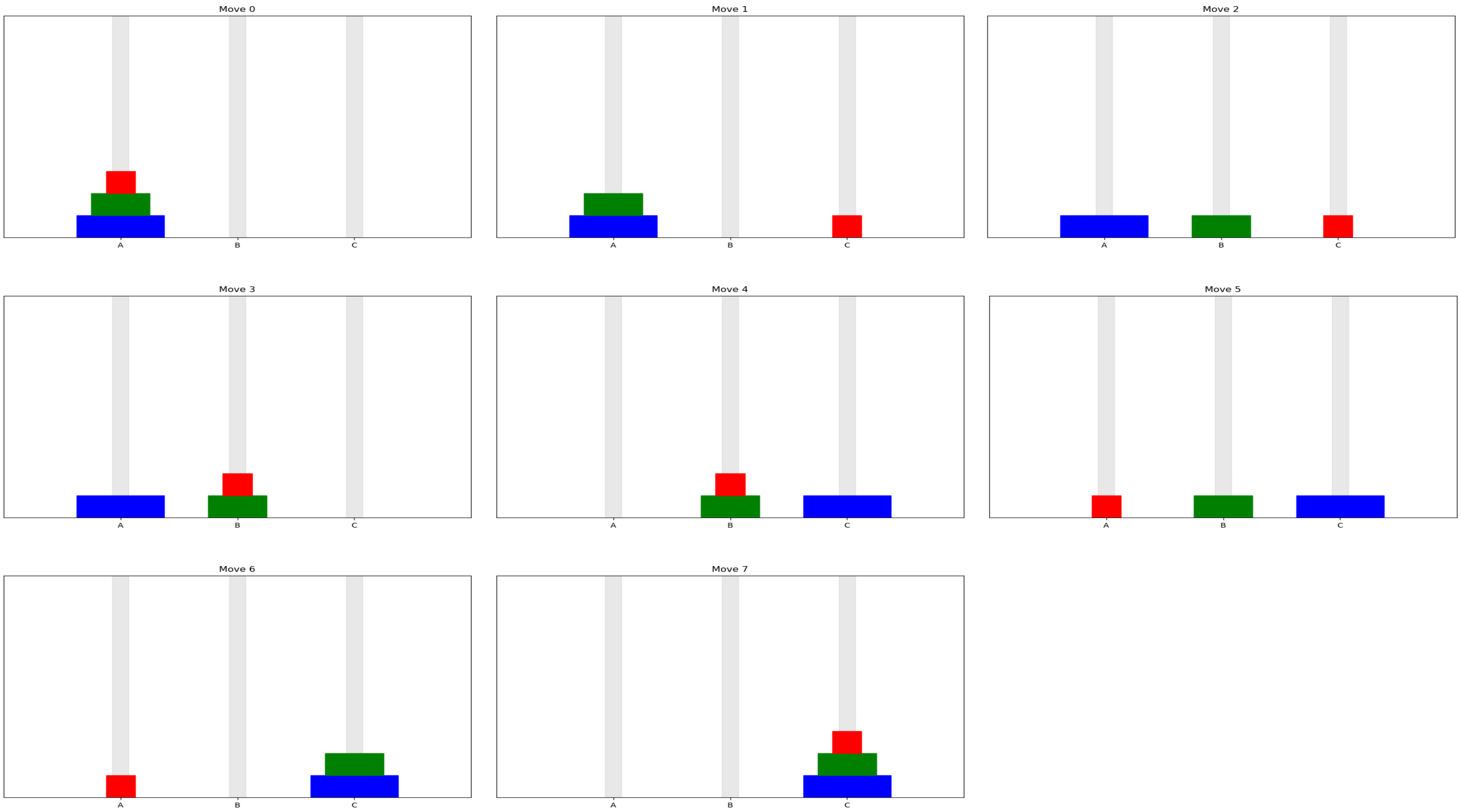
$$\begin{aligned} A_k &= A_{k-1} + (0.04) \cdot A_{k-1} \\ &= (1 + 0.04) \cdot A_{k-1} = (1.04) \cdot A_{k-1} \qquad \text{by factoring out } A_{k-1}. \end{aligned}$$

(1) $A_k = (1.04) \cdot A_{k-1}$ recurrence relation

(2) $A_0 = \$100,000$ initial condition.

$$
\begin{array}{lllll}
(3) & A_1 = 1.04 \cdot A_0 & = (1.04) \cdot \$100,000 & = \$104,000 & \text{by (1) and (2)} \\
(4) & A_2 = 1.04 \cdot A_1 & = (1.04) \cdot \$104,000 & = \$108,160 & \text{by (1) and (3)} \\
(5) & A_3 = 1.04 \cdot A_2 & = (1.04) \cdot \$108,160 & = \$112,486.40 & \text{by (1) and (4)} \\
 & \vdots & & \vdots & \\
(22) & A_{20} = 1.04 \cdot A_{19} & \cong (1.04) \cdot \$210,684.92 & \cong \$219,112.31 & \text{by (1) and (21)} \\
(23) & A_{21} = 1.04 \cdot A_{20} & \cong (1.04) \cdot \$219,112.31 & \cong \$227,876.81 & \text{by (1) and (22)}
\end{array}
$$

Hanoi Tower Game



Recurrent function

```
def solve_hanoi(towers, n, start, end, aux, move):
    if n == 1:
        move_disk(towers, start, end, move[0])
        move[0] += 1
        return
    solve_hanoi(towers, n-1, start, aux, end, move)
    solve_hanoi(towers, 1, start, end, aux, move)
    solve_hanoi(towers, n-1, aux, end, start, move)
```

Lecture 15 – part 3

Functions - Revisited

Function definition

• Definition

A **function f from a set X to a set Y**, denoted $f: X \to Y$, is a relation from X, the **domain**, to Y, the **co-domain**, that satisfies two properties: (1) every element in X is related to some element in Y, and (2) no element in X is related to more than one element in Y. Thus, given any element x in X, there is a unique element in Y that is related to x by f. If we call this element y, then we say that "f sends x to y" or "f maps x to y" and write $x \xrightarrow{f} y$ or $f: x \to y$. The unique element to which f sends x is denoted

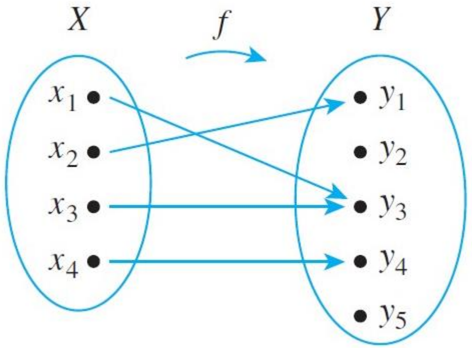
$f(x)$ and is called **f of x**, or
the output of f for the input x, or
the value of f at x, or
the image of x under f.

The set of all values of f taken together is called the *range of f* or the *image of X under f*. Symbolically,

$$\textbf{range of } f = \textbf{image of } X \textbf{ under } f = \{y \in Y \mid y = f(x), \text{ for some } x \text{ in } X\}.$$

Given an element y in Y, there may exist elements in X with y as their image. If $f(x) = y$, then x is called **a preimage of y** or **an inverse image of y**. The set of all inverse images of y is called *the inverse image of y*. Symbolically,

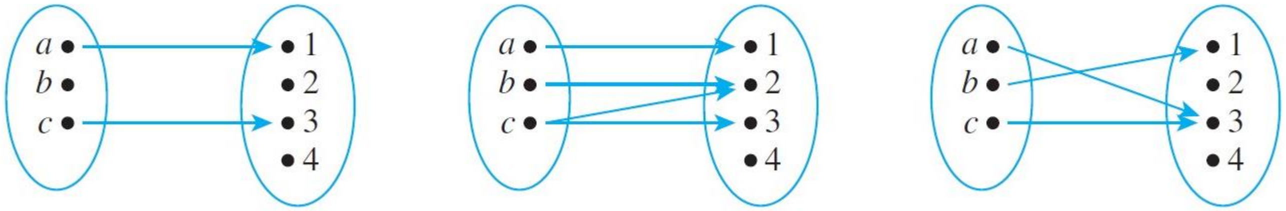
$$\textbf{the inverse image of } y = \{x \in X \mid f(x) = y\}.$$



1. **Domain (X):** $X = \{x_1, x_2, x_3, x_4\}$.
2. **Co-domain (Y):** $Y = \{y_1, y_2, y_3, y_4, y_5\}$.
3. **Range:** The range of the function f is given by:
$$\text{Range of } f = \{y_1, y_3, y_4\}$$
4. **Image of X under f:**
$$f(X) = \{y_1, y_3, y_4\}$$
5. **Preimage of y:** For y_3 as an example,
$$f^{-1}(y_3) = \{x_1, x_3\}$$
6. **Inverse Image of y:** For y_3,
$$f^{-1}(y_3) = \{x_1, x_3\}$$

Example:

Which of the arrow diagrams in Figure 7.1.2 define functions from $X = \{a, b, c\}$ to $Y = \{1, 2, 3, 4\}$?



(a) (b) (c)

Solution:

- Only (c) defines a function.
- In (a) there is an element of X, namely b, that is not sent to any element of Y; that is, there is no arrow coming out of b.
- And in (b) the element c is not sent to a *unique* element of Y; that is, there are two arrows coming out of c, one pointing to 2 and the other to 3.

Example:

Let $X = \{a, b, c\}$ and $Y = \{1, 2, 3, 4\}$. Define a function f from X to Y by the arrow diagram in Figure 7.1.3.

a. Write the domain and co-domain of f.

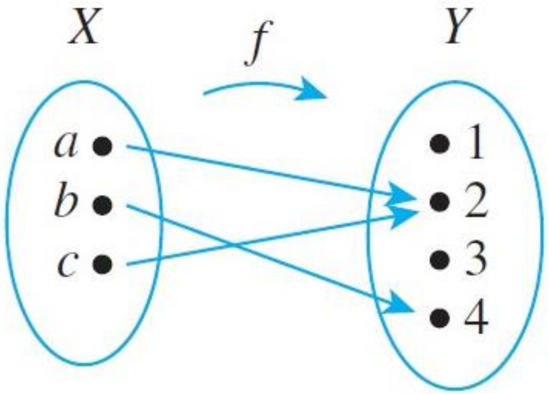
b. Find $f(a)$, $f(b)$, and $f(c)$.

c. What is the range of f?

d. Is c an inverse image of 2? Is b an inverse image of 3?

e. Find the inverse images of 2, 4, and 1.

f. Represent f as a set of ordered pairs.



Solution:

a. domain of $f = \{a, b, c\}$, co-domain of $f = \{1, 2, 3, 4\}$

b. $f(a) = 2$, $f(b) = 4$, $f(c) = 2$

c. range of $f = \{2, 4\}$

d. Yes, No

e. inverse image of $2 = \{a, c\}$
inverse image of $4 = \{b\}$
inverse image of $1 = \emptyset$ (*since no arrows point to* 1)

f. $\{(a, 2), (b, 4), (c, 2)\}$

Equality of functions

Theorem 7.1.1 A Test for Function Equality

If $F: X \to Y$ and $G: X \to Y$ are functions, then $F = G$ if, and only if, $F(x) = G(x)$ for all $x \in X$.

Proof:

Suppose $F: X \to Y$ and $G: X \to Y$ are functions, that is, F and G are binary relations from X to Y that satisfy the two additional function properties. Then F and G are subsets of $X \times Y$, and for (x, y) to be in F means that y is the unique element related to x by F, which we denote as $F(x)$. Similarly, for (x, y) to be in G means that y is the unique element related to x by G, which we denote as $G(x)$.

Now suppose that $F(x) = G(x)$ for all $x \in X$. Then if x is any element of X,

$$(x, y) \in F \Leftrightarrow y = F(x) \Leftrightarrow y = G(x) \Leftrightarrow (x, y) \in G \qquad \text{because } F(x) = G(x)$$

So F and G consist of exactly the same elements and hence $F = G$.

Conversely, if $F = G$, then for all $x \in X$,

$$y = F(x) \Leftrightarrow (x, y) \in F \Leftrightarrow (x, y) \in G \Leftrightarrow y = G(x) \qquad \text{because } F \text{ and } G \text{ consist of exactly the same elements}$$

Thus, since both $F(x)$ and $G(x)$ equal y, we have that

$$F(x) = G(x).$$

Let's consider two sets X and Y defined as:

$$X = \{a, b, c\}$$

$$Y = \{1, 2, 3\}$$

The Cartesian product $X \times Y$ is given by:

$$X \times Y = \{(a,1),(a,2),(a,3),(b,1),(b,2),(b,3),(c,1),(c,2),(c,3)\}$$

Consider the function F from X to Y as:

$$F = \{(a,1),(b,2),(c,3)\}$$

Now, let $G : X \to Y$ be a function such that for every $x \in X$, $G(x)$ matches the second component of the corresponding ordered pair in F. That is, for all $x \in X$:

$$G(x) = F(x)$$

Without knowing the explicit pairs of G, we can infer from the above condition that:

$$(x, y) \in F \Rightarrow (x, y) \in G$$

And vice versa,

$$(x, y) \in G \Rightarrow (x, y) \in F$$

Given these implications, it's clear that the ordered pairs of F and G must be the same. Therefore, by the theorem, we conclude that:

$$F = G$$

Example:

a. Let $J_3 = \{0, 1, 2\}$, and define functions f and g from J_3 to J_3 as follows: For all x in J_3,

$$f(x) = (x^2 + x + 1) \mod 3 \quad \text{and} \quad g(x) = (x+2)^2 \mod 3.$$

Does $f = g$?

b. Let $F: \mathbf{R} \to \mathbf{R}$ and $G: \mathbf{R} \to \mathbf{R}$ be functions. Define new functions $F + G: \mathbf{R} \to \mathbf{R}$ and $G + F: \mathbf{R} \to \mathbf{R}$ as follows: For all $x \in \mathbf{R}$,

$$(F+G)(x) = F(x) + G(x) \quad \text{and} \quad (G+F)(x) = G(x) + F(x).$$

Does $F + G = G + F$?

Solution:

a. Yes, the table of values shows that $f(x) = g(x)$ for all x in J_3.

x	x^2+x+1	$f(x) = (x^2+x+1) \mod 3$	$(x+2)^2$	$g(x) = (x+2)^2 \mod 3$
0	1	$1 \mod 3 = 1$	4	$4 \mod 3 = 1$
1	3	$3 \mod 3 = 0$	9	$9 \mod 3 = 0$
2	7	$7 \mod 3 = 1$	16	$16 \mod 3 = 1$

b. Again the answer is yes. For all real numbers x,

$$\begin{aligned}(F+G)(x) &= F(x) + G(x) && \text{by definition of } F+G\\ &= G(x) + F(x) && \text{by the commutative law for addition of real numbers}\\ &= (G+F)(x) && \text{by definition of } G+F\end{aligned}$$

Hence $F + G = G + F$. ■

The Identity Function on a Set

Given a set X, define a function I_X from X to X by

$$I_X(x) = x \quad \text{for all } x \text{ in } X.$$

The function I_X is called the **identity function on X** because it sends each element of X to the element that is identical to it. Thus the identity function can be pictured as a machine that sends each piece of input directly to the output chute without changing it in any way.

Example: Let X be any set and suppose that a_{ij}^k and $\phi(z)$ are elements of X. Find $I_X\left(a_{ij}^k\right)$ and $I_X(\phi(z))$.

Solution Whatever is input to the identity function comes out unchanged, so $I_X\left(a_{ij}^k\right) = a_{ij}^k$ and $I_X(\phi(z)) = \phi(z)$. ■

Functions to define sequences

- There are many functions that can be used to define a given sequence.

The sequence:

$$1, -\frac{1}{2}, -\frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, \ldots, (-1)^n \frac{1}{n+1}, \ldots$$

can be thought of as the function f from the nonnegative integers to the real numbers that associates:

$$\begin{aligned}
0 &\to 1, \\
1 &\to -\frac{1}{2}, \\
2 &\to -\frac{1}{3}, \\
3 &\to -\frac{1}{4}, \\
4 &\to \frac{1}{5}, \\
&\vdots \\
n &\to (-1)^n \frac{1}{n+1}.
\end{aligned}$$

In other words, $f : \mathbb{Z}_{\text{nonneg}} \to \mathbb{R}$ is the function defined as follows:

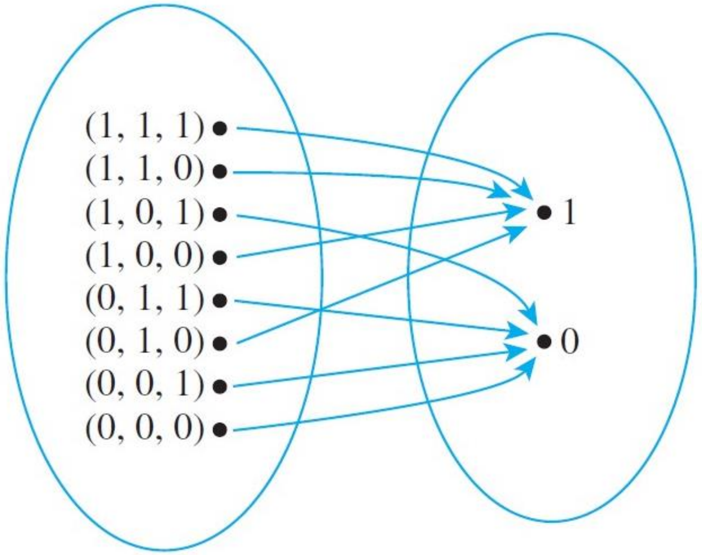
$$\forall n \geq 0 : f(n) = (-1)^n \frac{1}{n+1}$$

Boolean Function

- **Definition**

An **(n-place) Boolean function** f is a function whose domain is the set of all ordered n-tuples of 0's and 1's and whose co-domain is the set $\{0, 1\}$. More formally, the domain of a Boolean function can be described as the Cartesian product of n copies of the set $\{0, 1\}$, which is denoted $\{0, 1\}^n$. Thus $f\colon \{0, 1\}^n \to \{0, 1\}$.

Input			Output
P	***Q***	***R***	***S***
1	1	1	1
1	1	0	1
1	0	1	0
1	0	0	1
0	1	1	0
0	1	0	1
0	0	1	0
0	0	0	0



Example: Consider the three-place Boolean function defined from the set of all 3-tuples of 0's and 1's to {0, 1} as follows: For each triple (x_1, x_2, x_3) of 0's and 1's,

$$f(x_1, x_2, x_3) = (x_1 + x_2 + x_3)\ mod\ 2.$$

Describe f using an input/output table.

Solution

$$f(1, 1, 1) = (1 + 1 + 1)\ mod\ 2 = 3\ mod\ 2 = 1$$
$$f(1, 1, 0) = (1 + 1 + 0)\ mod\ 2 = 2\ mod\ 2 = 0$$

The rest of the values of f can be calculated similarly to obtain the following table.

Input			**Output**
x_1	x_2	x_3	$(x_1 + x_2 + x_3)\ mod\ 2$
1	1	1	1
1	1	0	0
1	0	1	0
1	0	0	1
0	1	1	0
0	1	0	1
0	0	1	1
0	0	0	0

■

One-to-One Functions

• Definition

Let F be a function from a set X to a set Y. F is **one-to-one** (or **injective**) if, and only if, for all elements x_1 and x_2 in X,

$$\text{if } F(x_1) = F(x_2), \text{ then } x_1 = x_2,$$

or, equivalently, $$\text{if } x_1 \neq x_2, \text{ then } F(x_1) \neq F(x_2).$$

Symbolically,

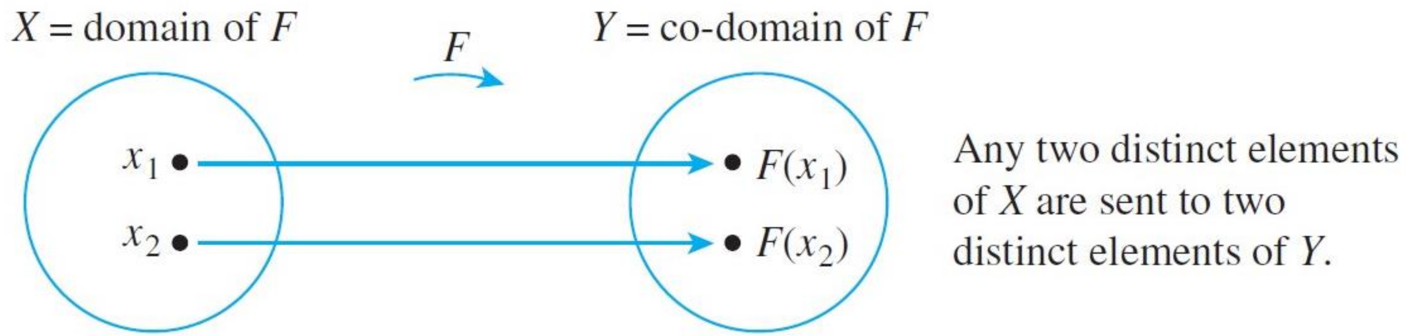
$$F\colon X \to Y \text{ is one-to-one} \quad \Leftrightarrow \quad \forall x_1, x_2 \in X, \text{if } F(x_1) = F(x_2) \text{ then } x_1 = x_2.$$

If no two arrows that start in the domain point to the same element of the co-domain then the function is called *one-to-one* or *injective*

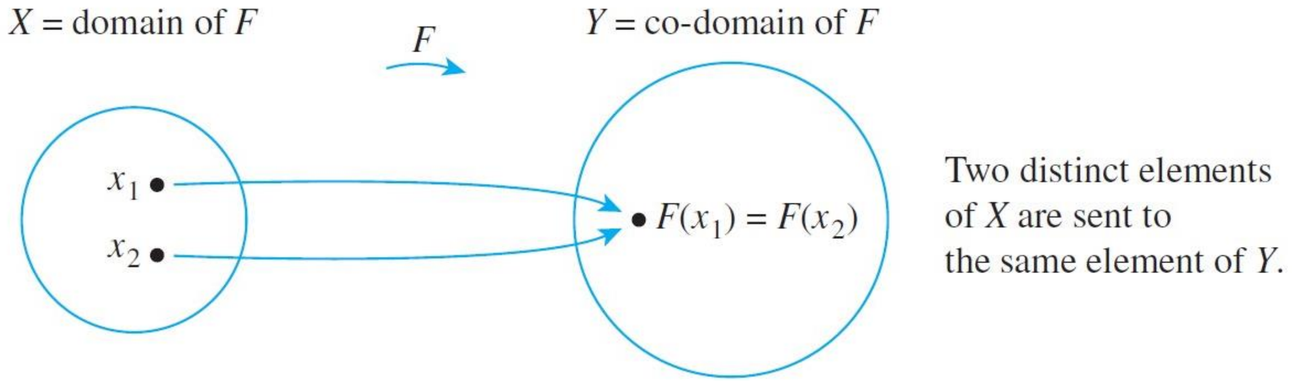
Negation:

A function $F\colon X \to Y$ is *not* one-to-one $\Leftrightarrow$ $\exists$ elements x_1 and x_2 in X with $F(x_1) = F(x_2)$ and $x_1 \neq x_2$.

A function that is one-to-one



A function that is not one-to-one



Onto Functions

• Definition

Let F be a function from a set X to a set Y. F is **onto** (or **surjective**) if, and only if, given any element y in Y, it is possible to find an element x in X with the property that $y = F(x)$.

Symbolically:

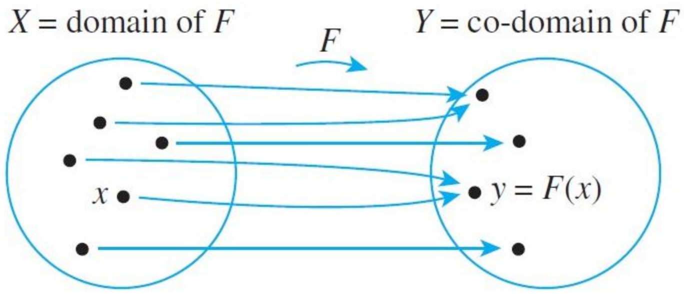
$$F\colon X \to Y \text{ is onto} \quad \Leftrightarrow \quad \forall y \in Y, \exists x \in X \text{ such that } F(x) = y.$$

When a function is onto, its range is equal to its co-domain.

Negation:

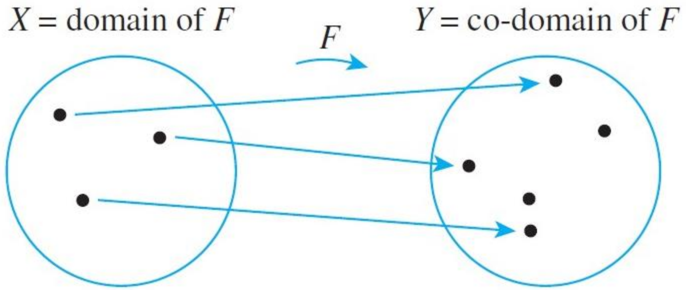
$$F\colon X \to Y \text{ is } \textit{not} \text{ onto} \quad \Leftrightarrow \quad \exists y \text{ in } Y \text{ such that } \forall x \in X,\ F(x) \neq y.$$

A function that is Onto



Each element y in Y equals $F(x)$ for at least one x in X.

A function that is not Onto

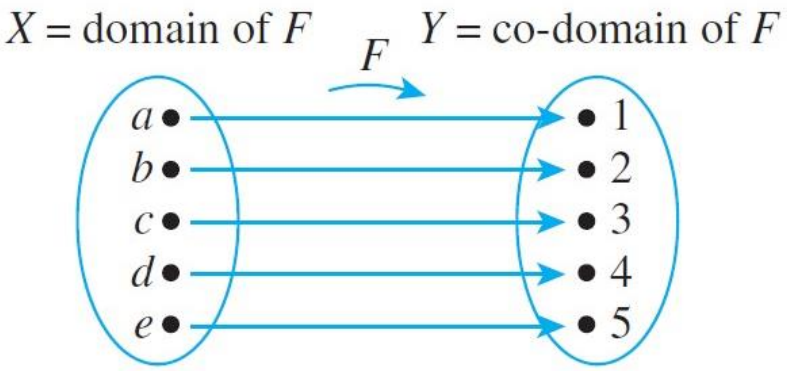


At least one element in Y does not equal $F(x)$ for any x in X.

One-to-One Correspondence

• Definition

A **one-to-one correspondence** (or **bijection**) from a set X to a set Y is a function $F: X \to Y$ that is both one-to-one and onto.



Inverse Functions

Suppose $F: X \to Y$ is a one-to-one correspondence; that is, suppose F is one-to-one and onto. Then there is a function $\boldsymbol{F^{-1}: Y \to X}$ that is defined as follows:

Given any element y in Y,

$$\boldsymbol{F^{-1}(y)} = \text{that unique element } x \text{ in } X \text{ such that } F(x) \text{ equals } y.$$

In other words,

$$F^{-1}(y) = x \quad \Leftrightarrow \quad y = F(x).$$

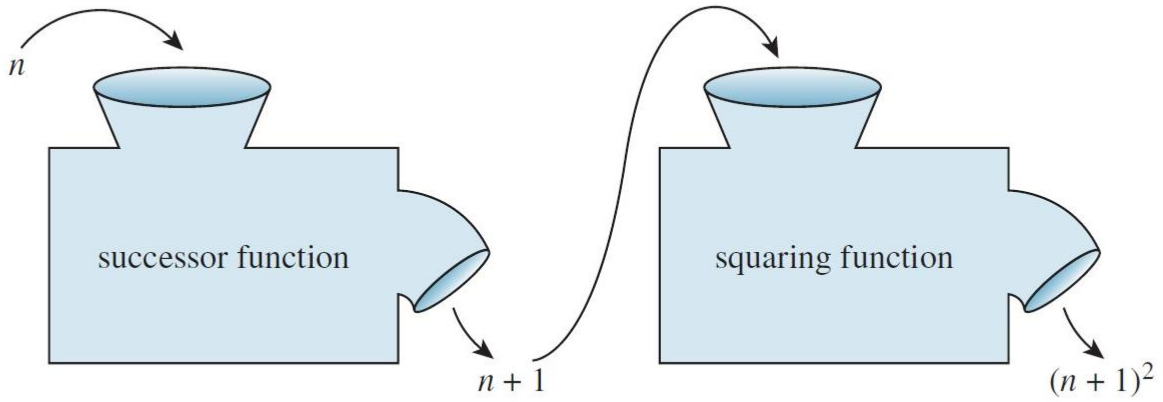
Composition of Functions

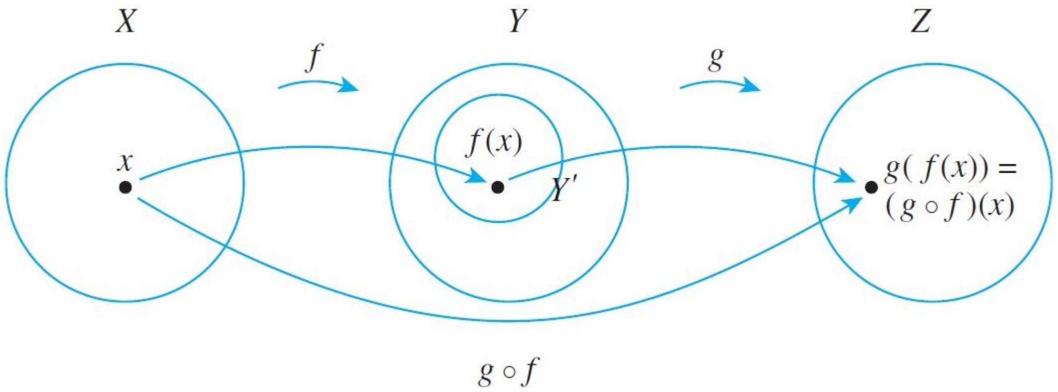
• Definition

Let $f: X \to Y'$ and $g: Y \to Z$ be functions with the property that the range of f is a subset of the domain of g. Define a new function $g \circ f: X \to Z$ as follows:

$$(g \circ f)(x) = g(f(x)) \quad \text{for all } x \in X,$$

where $g \circ f$ is read "g circle f" and $g(f(x))$ is read "g of f of x." The function $g \circ f$ is called the **composition of f and g.**





Lecture 15 – part 4

Function Exercises