COT 2000
Foundations of Computing

Summer 2024

Lecture 10 – part 1

Lab 5

Homework 3 - Due: 06/14/24

Exam 2 – 06/21/24

Lecture 10 – part 2

Review

Review

- Predicates, Domain, Truth set
- Universal and Existential Quantifiers $\forall$, $\exists$
- Universal conditional statement
- Negation
- Contrapositive, Converse, Inverse with quantifiers
- The rule of universal instantiation
- Arguments with quantified statements
- Fallacies with quantified statements

Predicates

Predicate	**Predicate Variable**	**Statements**	
$P(x)$: "$x^2 > x$"	x	$P(2)$: "$2^2 > 2$"	$P(1/2)$: "$1/2^2 > 2$"
		True	False

Domain

$x \in \mathbb{R}$

Truth set of a Predicate

Let $P(x)$: "n is a factor of 8. The domain of n is the set of all positive integers"

$$\text{Truth Set} = \{x \in \mathbb{Z}^+ \mid P(x)\} = \{1, 2, 4, 8\}$$

Review

Universal Quantifier Statement **Counterexample**

$$\forall x \in \mathbb{R},\ x^2 > x.$$ False $x = \frac{1}{2}$

Existential Quantifier Statement **Example**

$$\exists x \in \mathbb{Z}^+,\ x^2 = x.$$ True $x = 1$

Universal Conditional Statement

$$\forall x \in \mathbb{R},\ (x > 2 \rightarrow x^2 > 4).$$ True

Negation

$$\sim (\forall x \in D, Q(x)) \ \equiv\ \exists x \in D, \sim Q(x).$$

$$\sim (\exists x \in D, P(x)) \ \equiv\ \forall x \in D, \sim P(x).$$

$$\sim (\forall x \in D, P(x) \rightarrow Q(x)) \ \equiv\ \exists x \text{ such that } (P(x) \wedge \sim Q(x)).$$

Valid Argument Forms

Modus Ponens	$p \to q$ p $\therefore q$	**Elimination**	**a.** $p \vee q$ $\sim q$ $\therefore p$ **b.** $p \vee q$ $\sim p$ $\therefore q$
Modus Tollens	$p \to q$ $\sim q$ $\therefore \sim p$	**Transitivity**	$p \to q$ $q \to r$ $\therefore p \to r$
Generalization	**a.** p $\therefore p \vee q$ **b.** q $\therefore p \vee q$	**Proof by Division into Cases**	$p \vee q$ $p \to r$ $q \to r$ $\therefore r$
Specialization	**a.** $p \wedge q$ $\therefore p$ **b.** $p \wedge q$ $\therefore q$		
Conjunction	p q $\therefore p \wedge q$	**Contradiction Rule**	$\sim p \to c$ $\therefore p$

Variants of Universal Conditional Statements

• Definition

Consider a statement of the form: $\forall x \in D$, if $P(x)$ then $Q(x)$.

1. Its **contrapositive** is the statement: $\forall x \in D$, if $\sim Q(x)$ then $\sim P(x)$.
2. Its **converse** is the statement: $\forall x \in D$, if $Q(x)$ then $P(x)$.
3. Its **inverse** is the statement: $\forall x \in D$, if $\sim P(x)$ then $\sim Q(x)$.

The rule of universal instantiation

If some property is true of *everything* in a set, then it is true of *any particular* thing in the set.

The validity of this argument form follows immediately from the definition of truth values for a universal statement.

All men are mortal.
Socrates is a man.
∴ Socrates is mortal.

Universal **instantiation** is the fundamental tool of deductive reasoning.

Universal Modus Ponens

$p \to q$
p
$\therefore q$

The rule of universal instantiation can be combined with modus ponens to obtain the valid form of argument called **universal modus ponens**.

Universal Modus Ponens

Formal Version	*Informal Version*
$\forall x$, if $P(x)$ then $Q(x)$.	If x makes $P(x)$ true, then x makes $Q(x)$ true.
$P(a)$ for a particular a.	a makes $P(x)$ true.
$\therefore Q(a)$.	$\therefore$ a makes $Q(x)$ true.

Universal Modus Tollens

$p \to q$
$\sim q$
$\therefore \sim p$

Universal modus tollens is the heart of **proof of contradiction**, which is one of the most important methods of mathematical argument.

Universal Modus Tollens

Formal Version	*Informal Version*
$\forall x$, if $P(x)$ then $Q(x)$.	If x makes $P(x)$ true, then x makes $Q(x)$ true.
$\sim Q(a)$, for a particular a.	a does not make $Q(x)$ true.
$\therefore \sim P(a)$.	$\therefore$ a does not make $P(x)$ true.

Fallacies

$$p \rightarrow q$$
$$q$$
$$\therefore p$$

Converse Error (Quantified Form)

Formal Version	*Informal Version*
$\forall x$, if $P(x)$ then $Q(x)$.	If x makes $P(x)$ true, then x makes $Q(x)$ true.
$Q(a)$ for a particular a.	a makes $Q(x)$ true.
$\therefore P(a)$. ← invalid conclusion	$\therefore a$ makes $P(x)$ true. ← invalid conclusion

$$p \rightarrow q$$
$$\sim p$$
$$\therefore \sim q$$

Inverse Error (Quantified Form)

Formal Version	*Informal Version*
$\forall x$, if $P(x)$ then $Q(x)$.	If x makes $P(x)$ true, then x makes $Q(x)$ true.
$\sim P(a)$, for a particular a.	a does not make $P(x)$ true.
$\therefore \sim Q(a)$. ← invalid conclusion	$\therefore a$ does not make $Q(x)$ true. ← invalid conclusion

Lecture 10 – part 3

Logic Exercises

True Values

Negation

p	$\sim p$
T	F
F	T

Conjunction

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Disjunction

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Exclusive OR

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

Conditional

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Biconditional

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Logical Equivalences

$\sim(p \wedge q)$ $\quad$ $\sim p \vee \sim q$ $\quad$ $\sim(p \wedge q) \equiv \sim p \vee \sim q$ De Morgan's Law

$p \vee (q \wedge r)$ $\quad$ $(p \vee q) \wedge (p \vee r)$ $\quad$ $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ Distributive Law

$p \rightarrow q \equiv \sim p \vee q$ Conditional Identity

$\sim(p \rightarrow q) \equiv p \wedge \sim q$ Negation of conditional

$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$ Biconditional identity

$\sim(p \oplus q) \equiv p \leftrightarrow q$

Logical Equivalences

Theorem 2.1.1 Logical Equivalences

Given any statement variables p, q, and r, a tautology $\mathbf{t}$ and a contradiction $\mathbf{c}$, the following logical equivalences hold.

1.	*Commutative laws:*	$p \wedge q \equiv q \wedge p$	$p \vee q \equiv q \vee p$
2.	*Associative laws:*	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	$(p \vee q) \vee r \equiv p \vee (q \vee r)$
3.	*Distributive laws:*	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
4.	*Identity laws:*	$p \wedge \mathbf{t} \equiv p$	$p \vee \mathbf{c} \equiv p$
5.	*Negation laws:*	$p \vee \sim p \equiv \mathbf{t}$	$p \wedge \sim p \equiv \mathbf{c}$
6.	*Double negative law:*	$\sim(\sim p) \equiv p$	
7.	*Idempotent laws:*	$p \wedge p \equiv p$	$p \vee p \equiv p$
8.	*Universal bound laws:*	$p \vee \mathbf{t} \equiv \mathbf{t}$	$p \wedge \mathbf{c} \equiv \mathbf{c}$
9.	*De Morgan's laws:*	$\sim(p \wedge q) \equiv \sim p \vee \sim q$	$\sim(p \vee q) \equiv \sim p \wedge \sim q$
10.	*Absorption laws:*	$p \vee (p \wedge q) \equiv p$	$p \wedge (p \vee q) \equiv p$
11.	*Negations of* $\mathbf{t}$ *and* $\mathbf{c}$:	$\sim \mathbf{t} \equiv \mathbf{c}$	$\sim \mathbf{c} \equiv \mathbf{t}$

Biconditional and Exclusive OR

$p \oplus q \equiv (p \vee q) \wedge \sim(p \wedge q)$	By definition
$(p \vee q) \wedge (\sim p \vee \sim q)$	Morgan's Law
$[(p \vee q) \wedge \sim p] \vee [(p \vee q) \wedge \sim q]$	Distributive Law
$[(p \wedge \sim p) \vee (q \wedge \sim p)] \vee [(p \wedge \sim q) \vee (q \wedge \sim q)]$	Distributive Law
$[c \vee (q \wedge \sim p)] \vee [(p \wedge \sim q) \vee c]$	Negation Law
$(q \wedge \sim p) \vee (p \wedge \sim q)$	Identity Law
$\boxed{p \oplus q \equiv (p \wedge \sim q) \vee (q \wedge \sim p)}$	Commutative Law

$$p \leftrightarrow q \equiv p \rightarrow q \wedge q \rightarrow p$$ By definition

$$\equiv (\sim p \vee q) \wedge (\sim q \vee p) \quad (1)$$ Conditional identity

$$p \oplus q \equiv (p \wedge \sim q) \vee (q \wedge \sim p)$$

$$\sim(p \oplus q) \equiv \sim[(p \wedge \sim q) \vee (q \wedge \sim p)]$$

$$\equiv \sim(p \wedge \sim q) \wedge \sim(q \wedge \sim p)$$ De Morgan's Law

$$\equiv (\sim p \vee q) \wedge (\sim q \vee p) \quad (2)$$ De Morgan's Law

Since (1) ≡ (2), therefore

$$\boxed{p \leftrightarrow q \equiv \sim(p \oplus q)}$$

Exercise: Complete the logical equivalence

Solution:

$$\begin{aligned}(p\wedge \sim q) \vee (p \wedge q) &\equiv p \wedge (\sim q \vee q) && \text{by (a)} \\ &\equiv p \wedge (q \vee \sim q) && \text{by (b)} \\ &\equiv p \wedge t && \text{by (c)} \\ &\equiv p && \text{by (d)}\end{aligned}$$

(a) Distributive Law

(b) Commutative Law

(c) Negation Law

(d) Identity Law

Exercise: Simplify $(p \wedge \sim q) \vee p$ using logical equivalences

$$(p \wedge \sim q) \vee p \equiv p \vee (p \wedge \sim q)$$ Commutative

$$\equiv p$$ Absorption Law

$$(p \wedge \sim q) \vee p \equiv p \vee (p \wedge \sim q)$$ Commutative

$$\equiv (p \wedge t) \vee (p \wedge \sim q)$$ identity

$$\equiv p \wedge (t \vee \sim q)$$ Distributive

$$\equiv p \wedge t$$ Univ. Bound law

$$\equiv p$$ identity

Exercise: Rewrite $(p \vee \sim q \rightarrow r \vee q)$ only using $\wedge$ and $\sim$.

$$(p \vee \sim q) \rightarrow (r \vee q)$$

$$\sim(p \vee \sim q) \vee (r \vee q)$$

$$(\sim p \wedge q) \vee \sim\sim(r \vee q)$$

$$\sim\sim\left[(\sim p \wedge q) \vee \sim(\sim r \wedge \sim q)\right]$$

$$\sim\left[\sim(\sim p \wedge q) \wedge (\sim r \wedge \sim q)\right]$$

Exercise: Is this argument valid:

$p \lor q$
$p \to \neg q$
$p \to r$
$\therefore r$

Solution:

p	q	r	$\neg q$	$p \lor q$	$p \to \neg q$	$p \to r$	r	
T	T	T	F	T	F	T	T	
T	T	F	F	T	F	F	F	
T	F	T	T	T	T	T	T	← ✓
T	F	F	T	T	T	F	F	
F	T	T	F	T	T	T	T	← ✓
F	T	F	F	T	T	T	F	← X Invalid
F	F	T	T	F	T	T	T	
F	F	F	T	F	T	T	F	

Valid Argument Forms

Modus Ponens	$p \to q$ p $\therefore q$	**Elimination**	**a.** $p \lor q$ $\sim q$ $\therefore p$ **b.** $p \lor q$ $\sim p$ $\therefore q$
Modus Tollens	$p \to q$ $\sim q$ $\therefore \sim p$	**Transitivity**	$p \to q$ $q \to r$ $\therefore p \to r$
Generalization	**a.** p $\therefore p \lor q$ **b.** q $\therefore p \lor q$	**Proof by Division into Cases**	$p \lor q$ $p \to r$ $q \to r$ $\therefore r$
Specialization	**a.** $p \land q$ $\therefore p$ **b.** $p \land q$ $\therefore q$		
Conjunction	p q $\therefore p \land q$	**Contradiction Rule**	$\sim p \to c$ $\therefore p$

Find the conclusion :

(a) $A \vee B$
(b) $B \rightarrow (C \vee D)$
(c) $\sim C$
(d) $\sim D$

$\boxed{\therefore \quad A}$

Solution

c) $\sim C$
d) $\sim D$
$\therefore$ e) $\sim C \wedge \sim D \equiv \sim (C \vee D)$ Conjuction

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b) $B \rightarrow (C \vee D)$
e) $\sim (C \vee D)$
$\therefore$ f) $\sim B$ Modus Tollens

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a) $A \vee B$
f) $\sim B$ Elimination
$\therefore$ g) $\underbrace{A}_{\text{conclusion}}$

Exercises

70, 74, 76, 77, 82, 87, 91, 106, 108, 118, 121...