COT 2000
Foundations of Computing

Summer 2024

Lecture 6 – part 1

Exam 1 – 05/31/24

Lab 3

Homework 2 - Due: 06/07/24

Lecture 6 – part 2

Review

Review

- Simplifying statement forms
- Conditional statements
- Conditional truth table
- Logical equivalences involving conditionals

Logical Equivalences

Theorem 2.1.1 Logical Equivalences

Given any statement variables p, q, and r, a tautology $\mathbf{t}$ and a contradiction $\mathbf{c}$, the following logical equivalences hold.

1.	*Commutative laws:*	$p \wedge q \equiv q \wedge p$	$p \vee q \equiv q \vee p$
2.	*Associative laws:*	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	$(p \vee q) \vee r \equiv p \vee (q \vee r)$
3.	*Distributive laws:*	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
4.	*Identity laws:*	$p \wedge \mathbf{t} \equiv p$	$p \vee \mathbf{c} \equiv p$
5.	*Negation laws:*	$p \vee \sim p \equiv \mathbf{t}$	$p \wedge \sim p \equiv \mathbf{c}$
6.	*Double negative law:*	$\sim(\sim p) \equiv p$	
7.	*Idempotent laws:*	$p \wedge p \equiv p$	$p \vee p \equiv p$
8.	*Universal bound laws:*	$p \vee \mathbf{t} \equiv \mathbf{t}$	$p \wedge \mathbf{c} \equiv \mathbf{c}$
9.	*De Morgan's laws:*	$\sim(p \wedge q) \equiv \sim p \vee \sim q$	$\sim(p \vee q) \equiv \sim p \wedge \sim q$
10.	*Absorption laws:*	$p \vee (p \wedge q) \equiv p$	$p \wedge (p \vee q) \equiv p$
11.	*Negations of* $\mathbf{t}$ *and* $\mathbf{c}$*:*	$\sim\mathbf{t} \equiv \mathbf{c}$	$\sim\mathbf{c} \equiv \mathbf{t}$

$\mathbf{P \land Q \land \sim R}$ $\mathbf{\sim (P \land R) \land Q \land P}$

$= (\sim P \lor \sim R) \land Q \land P$ (De Morgan's Law)

$= ((\sim P \lor \sim R) \land P) \land Q$ (Commutative and associative Law)

$= ((\sim P \land P) \lor (\sim R \land P)) \land Q$ (Distributive Law)

$= (\mathbf{c} \lor (\sim R \land P)) \land Q$ (Contradiction: $\sim P \land P$)

$= (\sim R \land P) \land Q$ (Identity Law)

$\Rightarrow \mathbf{P \land Q \land \sim R}$ (Commutative Law)

$p \to q$

• Definition

If p and q are statement variables, the **conditional** of q by p is "If p then q" or "p implies q" and is denoted $p \to q$. It is false when p is true and q is false; otherwise it is true. We call p the **hypothesis** (or **antecedent**) of the conditional and q the **conclusion** (or **consequent**).

A conditional statement that is true by virtue of the fact that its hypothesis is false is often called **vacuously true** or **true by default.** Thus the statement "If you show up for work Monday morning, then you will get the job" is vacuously true if you do not show up for work Monday morning. In general, when the "if" part of an if-then statement is false, the statement as a whole is said to be true, regardless of whether the conclusion is true or false.

Conditional truth table

If you show up for work Monday morning, then you will get the job

p	*q*	*p* → *q*
T	T	T
T	**F**	**F**
F	T	T
F	F	T

You show up for work Monday morning (True) and you get the job (True). p→q is True.
You show up for work Monday morning (True) but you don't get the job (False). p→q is False.
You don't show up for work Monday morning (False) but you get the job (True). p→q is True.
You don't show up for work Monday morning (False) and you don't get the job (False). p→q is True.

Logical equivalences Involving →

Exercise: Find the truth tables for each equivalence

$$p \to q \equiv \neg p \vee q \quad \text{(Conditional Identity)} \tag{1}$$

$$p \to q \equiv \neg q \to \neg p \quad \text{(Contrapositive)} \tag{2}$$

$$p \vee q \to r \equiv (p \to r) \wedge (q \to r) \quad \text{(Distributive Law of Implication)} \tag{3}$$

$$(p \wedge q) \to r \equiv p \to (q \to r) \quad \text{(Exportation)} \tag{4}$$

$$\neg(p \to q) \equiv p \wedge \neg q \quad \text{(Reduction)} \tag{5}$$

$$q \to p \quad \text{(Converse of } p \to q) \tag{6}$$

$$\neg p \to \neg q \quad \text{(Inverse of } p \to q) \tag{7}$$

Lecture 6 – part 3

More on conditional statements

Conditional Identity

- Representation of If-Then As Or.

$$p \to q \equiv {\sim}p \lor q$$

If you do not get to work on time, **then** you are fired.

Either you get to work on time, **or** you are fired.

p is: You do not get to work on time.

q is: You are fired.

Conditional Identity $p \to q \equiv {\sim}p \lor q$

p	q	$p \to q$	${\sim}p$	${\sim}p \lor q$	$(p \to q) \equiv ({\sim}p \lor q)$
T	T				
T	F				
F	T				
F	F				

Conditional Identity $p \to q \equiv {\sim}p \vee q$

p	q	$p \to q$	${\sim}p$	${\sim}p \vee q$	$(p \to q) \equiv ({\sim}p \vee q)$
T	T	T	F	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T	T	T	T

The negation of a conditional statement

$$\boxed{\sim(p \to q) \equiv p \wedge \sim q}$$

$\sim(p \to q)$	$\equiv$	$\sim(\sim p \vee q)$	conditional identity
	$\equiv$	$\sim(\sim p) \wedge (\sim q)$	by De Morgan's laws
	$\equiv$	$p \wedge \sim q$	by the double negative law

Truth table

p	q	$p \rightarrow q$	$\sim(p \rightarrow q)$	$\sim q$	$p \wedge \sim q$
T	T				
T	F				
F	T				
F	F				

Truth table

p	q	$p \rightarrow q$	$\sim(p \rightarrow q)$	$\sim q$	$p \land \sim q$
T	T	T	F	F	F
T	F	F	T	T	T
F	T	T	F	F	F
F	F	T	F	T	F

The Contrapositive of a Conditional Statement

The contrapositive of $p \rightarrow q$ is $\sim q \rightarrow \sim p$

A conditional statement is logically equivalent to its contrapositive.

$$\mathbf{p \rightarrow q \equiv \sim q \rightarrow \sim p}$$

If Howard can swim across the lake, **then** Howard can swim to the island.

If Howard cannot swim to the island, **then** Howard cannot swim across the lake.

The Contrapositive of a Conditional Statement

$$p \rightarrow q \equiv \sim q \rightarrow \sim p$$

Can you derive the contrapositive using logical laws ?

$p \rightarrow q$	$\equiv$	$\sim p \vee q$	(Conditional Identity)
$\sim p \vee q$	$\equiv$	$q \vee \sim p$	(Conmutative)
$q \vee \sim p$	$\equiv$	$\sim q \rightarrow \sim p$	(Conditional Identity)

$$p \rightarrow q \equiv \sim p \vee q$$

$$q \rightarrow p \equiv \sim q \vee p$$

The Converse and Inverse of a Conditional Statement

• Definition

Suppose a conditional statement of the form "If p then q" is given.

1. The **converse** is "If q then p."
2. The **inverse** is "If $\sim p$ then $\sim q$."

Symbolically,

$$\text{The converse of } p \to q \text{ is } q \to p,$$

and

$$\text{The inverse of } p \to q \text{ is } \sim p \to \sim q.$$

Conditional & Converse

Conditional
$p \rightarrow q$

Not equivalent

Not equivalent

Inverse
$\sim p \rightarrow \sim q$

Equivalent
Why ?

Converse
$q \rightarrow p$

Truth Table

p	q	$\sim p$	$\sim q$	$p \to q$	$\sim q \to \sim p$ (Contrapositive)	$q \to p$ (Converse)	$\sim p \to \sim q$ (Inverse)
T	T						
T	F						
F	T						
F	F						

Truth Table

p	q	$\sim p$	$\sim q$	$p \to q$	$\sim q \to \sim p$ (Contrapositive)	$q \to p$ (Converse)	$\sim p \to \sim q$ (Inverse)
T	T	F	F	T	T	T	T
T	F	F	T	F	F	T	T
F	T	T	F	T	T	F	F
F	F	T	T	T	T	T	T

Lecture 6 – part 4

Biconditional

Biconditional $p \leftrightarrow q$

• Definition

Given statement variables p and q, the **biconditional of p and q** is "p if, and only if, q" and is denoted $p \leftrightarrow q$. It is true if both p and q have the same truth values and is false if p and q have opposite truth values. The words *if and only if* are sometimes abbreviated **iff.**

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Truth Table Showing that $p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$

p	q	$p \rightarrow q$	$q \rightarrow p$	$p \leftrightarrow q$	$(p \rightarrow q) \wedge (q \rightarrow p)$
T	T	T	T	T	T
T	F	F	T	F	F
F	T	T	F	F	F
F	F	T	T	T	T

$p \leftrightarrow q$ and $(p \rightarrow q) \wedge (q \rightarrow p)$
always have the same truth values,
so they are logically equivalent

Example: Write the following compound statement as two conditionals:

"This computer program is correct **if, and only if**, it produces correct answers for all possible sets of input data."

"If this program is correct, **then** it produces the correct answers for all possible sets of input data; **and if** this program produces the correct answers for all possible sets of input data, **then** it is correct."

Necessary and Sufficient Condition

- **Definition**

If r and s are statements:

r is a **sufficient condition** for s means "if r then s."

r is a **necessary condition** for s means "if not r then not s."

r is a necessary and sufficient condition for s means "r if, and only if, s."

$r \leftrightarrow s$

Lecture 6 – part 5

Logic Exercises 2 (up to 48)