The Spectrum of Resolvable Bose Triple Systems

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Abstract. A classical construction of Bose produces a Steiner triple system of order $3n$ from a symmetric, idempotent latin square of order n whenever n is odd. In an application to access-balancing in storage systems, these Bose triple systems play a central role. A natural question arises: For which orders v does there exist a resolvable Bose triple system? In this paper, we show that the necessary conditions are also sufficient; namely, whenever $v \equiv 9 \pmod{18}$.

1 Introduction

A *Steiner system* with parameters t, k, and v, or $S(t, k, v)$, is a pair $(V, \mathcal{B})$, where V is a set of v points and $\mathcal{B}$ is a set of k-subsets (*blocks*) of V such that every t-subset of V occurs in exactly one block. A Steiner system $S(2, 3, v)$ is a *Steiner triple system* of *order* v, or STS(v); see [5] for background on STSs. A necessary and sufficient condition for the existence of an STS(v) is that $v \equiv 1, 3 \pmod 6$ [10]; such an order v is *admissible.*

A *parallel class* of an $S(t, k, v)$, $D = (V, \mathcal{B})$, is a subset of disjoint blocks from $\mathcal{B}$ whose union is V. A partition of $\mathcal{B}$ into parallel classes is a *resolution*, and a Steiner system is *resolvable* if it admits a resolution. The study of resolvable block designs is one of the central pursuits of design theory [9]. An $\mathrm{S}(2, 3, v)$ with a resolution is a *Kirkman triple system* of *order* v, or KTS(v), named after Reverend Kirkman [11]. A KTS(v) exists if and only if $v \equiv 3 \pmod 6$ [12, 14].

Our goal is to establish that a large class of Steiner triple systems, which we call Bose-averaging triple systems (see Section 2), are resolvable. One motivation arises out of the demand for access balancing in storage systems. Given their balance properties, block designs have served as a basis for many systems, such as distributed storage systems [7, 16], systems for batch coding [17], and multiserver private information retrieval systems [8]. Dau and Milenkovic [6] propose ranking the popularity of the files making up such a storage system to improve its access balancing: Formally this amounts to equipping an $S(t, k, v)$ $(V, \mathcal{B})$ with a *point labelling*; that is, a bijection $\mathsf{rk} : V \to \{0, \dots, v-1\}$. A Steiner system equipped with a point labelling is a *point-labelled Steiner system.* Given a point-labelled $S(t, k, v)$ $(D = (V, \mathcal{B}), \mathsf{rk})$, the *block sum* $\mathrm{sum}(B)$ is $\sum_{x \in B} \mathsf{rk}(x)$ when $B \in \mathcal{B}$. Metrics for D to address the access-balancing problem are defined in [6] and investigated in [3]. One such metric is the *minimum sum* or *MinSum*, $\mathrm{MinSum}(D, \mathsf{rk})$, defined as $\min_{B \in \mathcal{B}} \mathrm{sum}(B)$, and the problem of maximizing this metric for a particular Steiner system is nontrivial. In [6] the

Bose-averaging triple system of order $v \equiv 3 \pmod 6$ is point-labelled to achieve MinSum v, the largest MinSum of any point-labelled Steiner triple system. Producing $S(2,4,v)$s with large MinSum appears to be more challenging. There one might hope to use the $3v+1$ construction [15] that combines an ingredient $S(2,4,v)$ and a KTS$(2v+1)$ to produce an $S(2,4,3v+1)$. For this purpose one wants a resolvable MinSum-optimal STS$(2v+1)$, hence our interest in *resolvable* Bose-averaging triple systems. Brummond [2], also motivated by access balancing in storage systems, established that every Bose-averaging triple system of order 3^k, $k \geq 1$, is resolvable. We show that every resolvable Bose triple system of order v has $v \equiv 9 \pmod{18}$ and that the Bose-averaging triple system of order $v \equiv 9 \pmod{18}$ is resolvable.

2 Preliminaries

2.1 Latin Squares and Quasigroups

A *latin square* of *order* n is an $n \times n$ array $L = (L_{x,y})$ in which every row is a permutation of an n-set S (the *symbol set* of L) and every column is a permutation of S. Although rows and columns could be indexed by different sets of size n, henceforth we take the index sets for rows, columns, and symbols to be the symbol set S. Any two such latin squares L and L' with symbol sets S and S', respectively, are *isomorphic* if there exists a bijection $\phi : S \to S'$ such that $\phi(L_{x,y}) = L'_{\phi(x),\phi(y)}$. An ordered triple $(i, j, L_{i,j})$ is a *cell* of L whose row is i, column is j, and *entry* is $L_{i,j}$. If $c = (x, y, z)$ is a cell of L, denote by $c_{\{\}}$ the set $\{x, y, z\}$; if C is a set of cells of L, define $C_{\{\}} = \{c_{\{\}} : c \in C\}$.

Let Q be a finite set of size n, and let $\circ$ be a binary operation on Q. The pair $(Q, \circ)$ is a *quasigroup of order* n provided that it satisfies (1) For every $x, y \in Q$, the equation $x \circ z = y$ has a unique solution $z \in Q$, and (2) For every $x, y \in Q$, the equation $z \circ x = y$ has a unique solution $z \in Q$. The *operation table* of $(Q, \circ)$ is the $|Q| \times |Q|$ array $A = (A_{x,y})$, where $A_{x,y} = x \circ y$. A quasigroup $(Q, \circ)$ is *idempotent* if $x \circ x = x$ for all $x \in Q$, and *symmetric* if $x \circ y = y \circ x$ for all $x, y \in Q$. A symmetric and idempotent quasigroup of order n exists if and only if n is odd [18]. Quasigroups and latin squares are related, in that $(Q, \circ)$ is a quasigroup if and only if its operation table is a latin square [4].

Let L be an idempotent, symmetric latin square L of order n with symbol set S so that $n = |S| \equiv 0 \pmod 3$. A *partial latin square parallel class* (PLSPC) of L is a set $\mathcal{P}$ of cells of L so that whenever $c, c' \in \mathcal{P}$ and $c \neq c'$, we have $c_{\{\}} \cap c'_{\{\}} = \emptyset$. A *latin square parallel class* (LSPC) of L is a PLSPC $\mathcal{P}$ with $|\mathcal{P}| = n/3$; for an LSPC $\mathcal{P}$, $\bigcup_{c \in \mathcal{P}} c_{\{\}} = S$.

Let $S = \{0, \ldots, n-1\}$. For each $k \in S$, the *k-diagonal* of L is the set of cells $\{(i, i+k, L_{i,i+k}) : 0 \leq i < n-k\}$ and the *−k-diagonal* of L is the set of cells $\{(i+k, i, L_{i,i+k}) : 0 \leq i < n-k\}$; in turn the *k-diagonal pair* of L, $k \in S$, is the union of its k- and $-(n-k)$-diagonals. $C_k(L)$ denotes the (set of cells of) the k-diagonal pair of L, and k is the *index* of $C_k(L)$. When L is clear from the context, we abbreviate this to C_k.

The *averaging latin square* $B = (B_{i,j})$ of order n, n odd, is the latin square with symbol set $S = \{0, \ldots, n-1\}$ such that for all $x, y \in S$, $B_{x,y} = \frac{n+1}{2}(x+y) \pmod{n}$. B is thus the operation table of the quasigroup $Q_B = ([0, n-1], \circ)$, where $x \circ y = \frac{n+1}{2}(x+y) \pmod{n}$, and Q_B is both symmetric and idempotent [18].

2.2 The Bose Construction

The Bose triple systems are built via the *Bose construction* [1]; we follow the presentation in [18]. Let $L = (L_{i,j})$ be the operation table of a symmetric idempotent quasigroup of order n having symbol set $S = [0, n-1]$, and put $V = S \times \mathbb{Z}_3$. For every $x \in [0, n-1]$ define the block $A_x = \{(x,0), (x,1), (x,2)\}$. Then for every pair of distinct elements $x, y \in [0, n-1]$, define a block $C_{x,y,i} = \{(x,i), (y,i), (L_{x,y}, i+1 \pmod 3)\}$. Set

$$\mathcal{B} = \{A_x : x \in [0, n-1]\} \cup \{C_{x,y,i} : x, y \in S, x < y, i \in \mathbb{Z}_3\}.$$

Then $(V, \mathcal{B})$ is an STS$(3n)$, a *Bose triple system* of order $3n$. Because a symmetric idempotent quasigroup of order n exists precisely when n is odd, a Bose triple system of order v exists precisely when $v \equiv 3 \pmod 6$. If L is the averaging latin square, $(V, \mathcal{B})$ is the *Bose-averaging triple system* of order $3n$. A set of cells $\mathcal{C}$ of L (where L is not necessarily the averaging latin square of order n) is a *Bose-generating set of* L if

$$\{C_{x,y,i} : x, y \in S, x < y, i \in \mathbb{Z}_3\} = \{(x,i), (y,i), (L_{x,y}, i+1 \pmod 3)) : (x, y, L_{x,y}) \in \mathcal{C}\}.$$

In other words, $\mathcal{C}$ generates all of the type $C_{x,y,i}$ blocks of $\mathcal{B}$. A *Bose resolution of* L *from* $\mathcal{C}$ is either (1) a partition of $\mathcal{C}$ into LSPCs or (2) a partition of $\mathcal{C}$ into LSPCs *and* a set $P = \{P_1, \ldots, P_p\}$ of $p \leq n$ proper PLSPCs for which there exists a partition $\pi = \{\pi_1, \ldots, \pi_q\}$ of $[0, n-1]$ into q sets and a bijection $\rho : P \to \pi$ such that $P_{i_{\{\}}} \cup \rho(P_i)$ is a partition of $[0, n-1]$ for all $i \in [1, p]$.

Now we establish that a Bose resolution of the operation table of any symmetric idempotent quasigroup induces a resolution of the Bose triple system constructed from that quasigroup.

Theorem 1. *Let L be the operation table of a symmetric idempotent quasigroup of order n and D the Bose triple system of order $3n$ constructed from B. Then any Bose resolution of L induces a resolution of D.*

Proof. Let $C_{x,y,i}$ and A_x denote the two block types of D, as defined in the Bose construction. Suppose we have a Bose resolution of L of the first kind, consisting of LSPCs $\{\mathcal{L}_1, \ldots, \mathcal{L}_{(3n-3)/2}\}$. Then

$$\mathcal{P}_j = \bigcup_{(x,y,z) \in \mathcal{L}_j, i \in \mathbb{Z}_3} C_{x,y,i}$$

is a parallel class of D for all $j \in [1, (3n-3)/2]$. The set $\mathcal{A} = \bigcup_{x \in [0, n-1]} A_x$ is a parallel class of D, and thus $\{\mathcal{P}_1, \ldots, \mathcal{P}_{(3n-3)/2}, \mathcal{A}\}$ is a resolution of D.

Now suppose we have a Bose resolution of L of the second kind, and let $\mathcal{L} = \{\mathcal{L}_1, \dots, \mathcal{L}_p\}$ denote its set of LSPCs and $P = \{P_1, \dots, P_q\}$ its set of all proper PLSPCs having a partition $\pi = \{\pi_1, \dots, \pi_q\}$ of $[0, n-1]$ and a bijection $\rho : P \to \pi$ such that $P_{i_{\{\}}} \cup \rho(P_i)$ is a partition of $[0, n-1]$ for all $i \in [1, q]$. Then both

$$\mathcal{P}_j = \bigcup_{(x,y,z)\in\mathcal{L}_j, i\in\mathbb{Z}_3} C_{x,y,i}$$

and

$$\mathcal{P}'_k = \left(\bigcup_{(x,y,z)\in P_j, i\in\mathbb{Z}_3} C_{x,y,i} \right) \bigcup \left(\bigcup_{x\in\rho(P_j)} A_x \right)$$

are parallel classes of D for all $j \in [1, p]$ and all $k \in [1, q]$ and thus

$$\mathcal{R} = \left(\bigcup_{j\in[1,p]} \mathcal{P}_j \right) \bigcup \left(\bigcup_{k\in[1,q]} \mathcal{P}'_k \right)$$

is a resolution of D. □

Not every operation table of a symmetric idempotent quasigroup admits a Bose resolution.

Theorem 2. *If a Bose triple system of order* $3n$ *is resolvable, then* $n \equiv 3 \pmod 6$.

Proof. Suppose that $\mathcal{P}$ is a parallel class of the Bose triple system $\mathcal{D} = (V, \mathcal{B})$ on $V = [0, n-1] \times \mathbb{Z}_3$. There are four possible configurations of a block $B \in \mathcal{B}$ depending on the second coordinates of its points:

1. $B = \{(x, 0), (y, 0), (z, 1)\}$,
2. $B = \{(x, 1), (y, 1), (z, 2)\}$,
3. $B = \{(x, 2), (y, 2), (z, 0)\}$, or
4. $B = \{(x, 0), (y, 1), (z, 2)\}$;

represent the number of each type in $\mathcal{P}$ as A, B, C, and D, respectively. Then we have:

$$\begin{cases} A + B + C + D = n \\ 2A + C + D = n \\ A + 2B + D = n \\ B + 2C + D = n, \end{cases}$$

where the first equation counts the triples in $\mathcal{P}$ and the rest count the points in $[0, n-1] \times \{i\}$ for $i \in \mathbb{Z}_3$. Solve this system to get $A = B = C = \frac{n-D}{3}$; it follows that $D \equiv n \pmod 3$.

When $n \equiv 1 \pmod 6$, $D \geq 1$ and thus $\mathcal{D}$ has at most n disjoint parallel classes. But $\mathcal{D}$ requires $\frac{3n-1}{2}$ disjoint parallel classes to be resolvable, and thus no resolution exists. When $n \equiv 5 \pmod 6$, $D \geq 2$ and thus $\mathcal{D}$ would have at most $\frac{n-1}{2}$ disjoint parallel classes. □

Our aim is to prove that the necessary condition of Theorem 2 is sufficient for the class of Bose-averaging triple systems.

3 Constructing Bose Resolutions

Let B be the averaging latin square of order $n \equiv 3 \pmod 6$. While myriad Bose-generating sets of B exist, we henceforth construct a Bose resolution of B *exclusively* from the Bose-generating set of B given by the (disjoint) union of the even-index diagonal pairs of B; that is,

$$E = \bigcup_{k \in \{2,4,6,\ldots,n-1\}} C_k(B).$$

E is the *even* Bose-generating set of B.

3.1 Averaging Latin Square Properties

To construct Bose resolutions, we employ several properties of the averaging latin square B. Henceforth, unless otherwise stated, n denotes the order of the averaging latin square B and C_k denotes the cells of the k-diagonal pair of B.

Consider a cell $c = (x, y, B_{x,y}) \in C_k(B)$. The *next adjacent cell of* c, denoted $c \oplus_n 1$, is the cell $(x+1, y+1, B_{x+1,y+1}) \in C_k(B)$, with coordinate-wise addition performed modulo n. Extend $\oplus_n$ to all $\alpha \in \mathbb{N}$, defining $c \oplus_n 0 = c$ and $c \oplus_n \alpha = (c \oplus_n 1) \oplus_n (\alpha - 1)$ if $\alpha \geq 2$. We call $\oplus_n$ the *diagonal pair traversal operator* for B. When the averaging latin square is clear from the context, we simply write $\oplus$. Visually, traversing $C_k(B)$ in full in this fashion starting from $c_0 = (0, k, \ell) \in C_k(B)$ (i.e., traversing the sequence of cells $(c_0, c_0 \oplus 1, c_0 \oplus 2, \ldots, c_0 \oplus n-1)$) amounts to running southeast down the k-diagonal, then running southeast down the $-(n-k)$-diagonal.

Property 1. Let $c = (x, y, z)$ be a cell in the averaging latin square B of order n. Then $c \oplus_n i = (x + i \pmod n, y + i \pmod n, z + i \pmod n)$ for all $i \in \mathbb{Z}$.

Proof. Suppose that $\hat{c} = (0, k, \ell) \in C_k(B)$ such that $c, \hat{c} \in C_k$. Then $\ell \equiv k \cdot \frac{n+1}{2} \pmod n$. For all $i \in [0, n-1]$,

$$\begin{aligned} \frac{n+1}{2}(i + (k+i)) &\equiv \frac{n+1}{2}(k + 2i) \pmod n \\ &\equiv k\frac{n+1}{2} + (n+1)i \pmod n \\ &\equiv \ell + i \pmod n, \end{aligned}$$

and hence $\hat{c} \oplus_n i = (i \pmod n, k + i \pmod n, \ell + i \pmod n) \in C_k(B)$. Thus,

$$\begin{aligned} c \oplus_n i = (\hat{c} \oplus_n x) \oplus_n i = \hat{c} \oplus_n (x+i) &= (x+i, k+x+i, \ell+x+i) \\ &= (x+i, y+i, z+i), \end{aligned}$$

where the last equality holds because $\hat{c} + x = c$, since by definition only one cell of C_k can have row coordinate equal to x. $\square$

Property 2. The set of rows, the set of columns, and the set of symbols of $C_k(B)$ are each equal to $[0, n-1]$.

Proof. Follows immediately from Property 1. □

Property 3. Given $(0, k, \ell) \in C_k(B)$, k even, then $\ell = k/2$.

Proof. $\ell \equiv k \cdot \frac{n+1}{2} \pmod n \iff \ell \equiv k/2(n+1) \pmod n \iff \ell \equiv k/2 \pmod n$. As $k/2 < n$, the result follows.

Property 4. Let $n \equiv 3 \pmod 6$ and $c \in C_k(B)$. Suppose that $k \in [1, n-1]$ is even. Then if $k \equiv 0 \pmod 6$, the elements of c are pairwise equivalent modulo 3; otherwise, the elements of c are pairwise inequivalent modulo 3.

Proof. By Property 3, $(0, k, \ell) \in C_k$ has the putative property. Hence, by Property 1, so does any $c \in C_k$. □

3.2 Recursive construction for orders $n \equiv 9 \pmod{18}$

Henceforth, we assume that all arithmetic is performed modulo n.

Lemma 1. *If $n \equiv 3 \pmod 6$ and k is even with $k \not\equiv 0 \pmod 3$, then C_k can be partitioned into three LSPCs.*

Proof. By Property 3, we may write $c = (0, 2d\gamma/3, d\gamma/3)$ such that $2d\gamma/3 \equiv k \pmod n$, $\gamma \mid n$ (so that $0 \le \gamma < n$), $3 \mid \gamma$, and $d \not\equiv 0 \pmod 3$. Consider the subset of cells of C_k:

$$\mathcal{B} = \bigcup_{i \in [0, \gamma/3-1]} \{(i, 2d\gamma/3 + i, d\gamma/3 + i)\}.$$

By Property 4, the three coordinates of each cell in $\mathcal{B}$ are pairwise inequivalent modulo 3. Further, each coordinate of $(i, 2d\gamma/3 + i, d\gamma/3 + i) \in \mathcal{B}$ is congruent to i modulo $\gamma/3$. Thus, since $\gamma \equiv 0 \pmod 3$, for $c = (i, 2d\gamma/3 + i, d\gamma/3 + i) \in \mathcal{B}$, if

$$\mathcal{P}_c = \bigcup_{i \in [0, n/\gamma - 1]} c \oplus i\gamma,$$

then $\mathcal{P}_{c_{\{\}}}$ is a partition of $R_\alpha = \{x \in [0, n-1] : x \equiv i \pmod{\gamma/3}\}$. Hence,

$$\mathcal{L}_1 = \bigcup_{c \in \mathcal{B}} \bigcup_{i \in [0, n/\gamma - 1]} c \oplus i\gamma, \tag{2}$$

is an LSPC. The set of symbols of the cells of $\mathcal{L}_1$ is

$$\bigcup_{i \in [0, \gamma/3-1]} \{d\gamma/3 + i, d\gamma/3 + \gamma + i, \ldots, d\gamma/3 + n - \gamma + i\}.$$

Hence,

$$\mathcal{L}_2 = \bigcup_{c \in \mathcal{L}_1} c \oplus \gamma/3, \text{ and}$$
$$\mathcal{L}_3 = \bigcup_{c \in \mathcal{L}_1} c \oplus 2\gamma/3$$

are LSPCs such that the sets of symbols of the cells of $\mathcal{L}_1$, $\mathcal{L}_2$, and $\mathcal{L}_3$ are pairwise disjoint. Thus, by Property 2, the $\mathcal{L}_i$'s partition C_k. □

Theorem 3. *If the averaging latin square of order n has a Bose resolution whenever $n \equiv 3, 15 \pmod{18}$, then the averaging latin square of order n has a Bose resolution whenever $n \equiv 9 \pmod{18}$.*

Proof. The recursion proceeds as follows. For even k such that $k \not\equiv 0 \pmod 3$, partition the cells of $C_k(B)$ into three LSPCs by applying Lemma 1. Now the remaining cells of the even Bose-generating set of B:

$$\bigcup_{j \equiv 0 \pmod 6, j>0} C_j,$$

partition into three row-column-symbol disjoint latin subsquares of B: L_0, L_1, and L_2, such that the symbol set, row index set, and column index set of L_i are each equal to $\{x \in [0, n-1] : x \equiv i \pmod 3\}$, and each L_i is isomorphic to the averaging latin square of order $n/3$. If $n/3 \equiv 3, 15 \pmod{18}$, merge the putative Bose resolutions of these three subsquares to obtain a Bose resolution for B. Otherwise, $n/3 \equiv 9 \pmod{18}$, so recurse on each of the subsquares. □

3.3 Constructions for orders $n \equiv 3, 15 \pmod{18}$

We employ more elaborate methods to organize the diagonal pairs C_k, $k > 0$, for which $k \equiv 0 \pmod 6$ into LSPCs. For any set of integers S, define $S + j = \{s + j : s \in S\}$. Further, we must stress at the outset the subtle but important difference between the $=$ and $\equiv$ operators in the context of modular arithmetic. When we say, for example, that $x = y \pmod n$, we mean that x is *equal* to the *least nonnegative residue of y modulo n*. Conversely, when we say that $x \equiv y \pmod n$, we mean that $x \in \{\dots, y' - n, y', n + y', 2n + y', \dots\}$, where y' is the least nonnegative residue of y modulo n.

Lemma 2. *Suppose that B is the averaging latin square of order $n \equiv 3, 15 \pmod{18}$. If $k \equiv 0 \pmod 6$, $k > 0$, $(0, k, \ell) \in C_k(B)$, $\gamma = \gcd(k, n)$, and $m' \equiv \lfloor n/(3\gamma) \rfloor^{-1} \cdot \ell/\gamma \pmod{n/\gamma}$, then $gcd(m', n/\gamma) = 1$.*

Proof. We first compute $\lfloor n/(3\gamma) \rfloor^{-1}$ modulo n/γ:

$$\begin{aligned}
& \lfloor n/(3\gamma) \rfloor x \equiv 1 \pmod{n/\gamma} \\
\iff\ & x(n/\gamma - r)/3 \equiv 1 \pmod{n/\gamma}, \text{ where } r \in \{1, 2\} \\
\iff\ & x(n/\gamma - r) \equiv 3 \pmod{n/\gamma} \\
\iff\ & -xr \equiv 3 \pmod{n/\gamma} \\
\iff\ & x \equiv -3r^{-1} \pmod{n/\gamma}.
\end{aligned}$$

Thus, $m' \equiv -3r^{-1} \cdot \ell/\gamma \pmod{n/\gamma}$. Now $\gcd(3, n/k) = 1$ and if $r = 1$, then $\gcd(r^{-1}, \gamma) = 1$. If $r = 2$, then $r^{-1} \equiv \frac{1}{2}(n/\gamma + 1) \pmod{n/\gamma}$ and since $\gcd(n/\gamma, n/\gamma + 1) = 1$, again $\gcd(r^{-1}, n/k) = 1$. Finally, as k is even,

$$\gcd(\ell/\gamma, n/\gamma) = \gcd(k/(2\gamma), n/\gamma) = 1,$$

implying $\gcd(m', n/\gamma) = 1$, as desired. □

Lemma 2 allows us to construct a class of PLSPCs.

Lemma 3. *Suppose that B is the averaging latin square of order $n \equiv 3, 15 \pmod{18}$, and that $C_k(B)$ is a diagonal pair with cell $c = (0, k, \ell)$, $k \equiv 0 \pmod{6}$. Put $\gamma = \gcd(k, n)$. Then for $m \equiv 3m' \pmod{3n/\gamma}$, where $m' \equiv \ell/\gamma \lfloor n/(3\gamma) \rfloor^{-1} \pmod{n/\gamma}$, $d \not\equiv 0 \pmod{3}$, $j \in \mathbb{Z}_3$, and $\alpha \in [0, \gamma/3 - 1]$, the set*

$$P_{k,d,j,\alpha} = \bigcup_{h \in [0, \lfloor n/(3\gamma) \rfloor - 1], i \in \mathbb{Z}_3} c \oplus (\alpha + \gamma/3(hm + id) + j\ell) \tag{3}$$

is a PLSPC of size $3\lfloor n/(3\gamma) \rfloor$ such that for distinct $j, j' \in \mathbb{Z}_3$, $P_{k,d,j,\alpha} \cap P_{k,d,j',\alpha} = \emptyset$.

Proof. As k is even, $\ell = k/2$ and hence $k, \ell \equiv 0 \pmod{\gamma/3}$; thus, for all $\alpha \in [0, \gamma/3 - 1]$, the union of the rows, columns, and symbols comprising the cells of

$$\bigcup_{h \in [0, 3n/\gamma - 1]} c \oplus (\alpha + h\gamma/3)$$

is precisely the elements of $[0, n-1]$ congruent to α modulo $\gamma/3$. It suffices, therefore, to prove the result for $P_{k,d,j,0}$. We claim that the set $P'_j \subset P_{k,d,j,0}$ given by

$$P'_j = \bigcup_{h \in [0, \lfloor n/(3\gamma) \rfloor - 1]} c \oplus (hm\gamma/3 + j\ell)$$

is a PLSPC. To establish this, consider a solution m' to the system of equations:

$$\begin{cases} \ell/\gamma \equiv m' \lfloor n/(3\gamma) \rfloor \pmod{n/\gamma} & \text{(4a)} \\ k/\gamma \equiv 2m' \lfloor n/(3\gamma) \rfloor \pmod{n/\gamma} & \text{(4b)} \end{cases}$$

with the constraint that $\gcd(m', n/\gamma) = 1$, so that m' generates $\mathbb{Z}_{n/\gamma}$. Then putting $m \equiv 3m' \pmod{3n/\gamma}$, the multiples of $m\gamma/3$ modulo n are pairwise distinct and thus if

$$S_\ell = \{\lfloor n/(3\gamma) \rfloor m\gamma/3 = \ell, (\lfloor n/(3\gamma) \rfloor + 1)m\gamma/3, \ldots, (2\lfloor n/(3\gamma) \rfloor - 1)m\gamma/3\},$$

then $S_\ell + j$ gives the set of symbols of the cells of P'_j, while the remaining multiples of $m\gamma/3$ modulo n, plus j give the rows and columns. The system (4) appears to be overdetermined, but subtracting the first equation from the second, and noting that $\ell \equiv k - \ell \pmod{n}$ by Property 3, $\ell/\gamma \equiv k/\gamma - \ell/\gamma \equiv m'\lfloor n/(3\gamma)\rfloor \pmod{n/\gamma}$; applying Lemma 2 gives the desired m'. Hence, P'_j is a PLSPC as claimed and thus because $\gamma d/3 \not\equiv 0 \pmod{3}$, then by Property 4, $P_{k,d,j,0}$ is also a PLSPC.

Finally, suppose that there exists some $(x_1, x_2, x_3) \in P_{k,d,j,0} \cap P_{k,d,j',0}$, with $j, j' \in \{0, 1, 2\}$ and $j > j'$, and suppose without loss of generality that $x_3 \equiv 0 \pmod{3}$. Then there exist $(y_1, y_2, y_3), (z_1, z_2, z_3) \in P'_0$ (and hence $y_3, z_3 \in S_\ell$) such that $x_3 \equiv y_3 + j\ell \equiv z_3 + j'\ell \pmod{n}$. Because $(j - j')\ell \equiv z_3 - y_3 \pmod{n}$, either

$$\begin{aligned} z_3 - y_3 &\equiv \lfloor n/(3\gamma)\rfloor m\gamma/3 \pmod{n}, \text{ or} \\ z_3 - y_3 &\equiv 2\lfloor n/(3\gamma)\rfloor m\gamma/3 \pmod{n}. \end{aligned}$$

But neither $\lfloor n/(3\gamma)\rfloor m\gamma/3$ nor $2\lfloor n/(3\gamma)\rfloor m\gamma/3$ is a possible difference of any two elements of S_ℓ. □

Given $n \equiv 3, 15 \pmod{18}$, even $k, \beta \in [1, n-1]$, and a PLSPC $P \subset C_k$, a β-*completing set* for P is a set $S \subset C_\beta$ such that $P \cup S$ is an LSPC. More generally, given $\hat{c} = (0, k, \ell) \in C_k$, $\gamma = \gcd(k, n)$, $\alpha \in [0, \gamma/3 - 1]$, and

$$U_\alpha = \bigcup_{h \in [0, 3n/\gamma - 1]} (\hat{c} \oplus (\alpha + h\gamma/3)),$$

a PLSPC $P \subset U_\alpha$ is (α, β)-*completable* if there exists some subset $S \subset C_\beta$, $|S| = n/\gamma - |P|$, such that $P' = P \cup S$ is a PLSPC satisfying

$$\bigcup_{c \in P'} c_{\{\}} = \{x \in [0, n-1] : x \equiv \alpha \pmod{\gamma/3}\}.$$

Such a set S is an (α, β)-*completing set* for P.

The proof of Lemma 4 describes a method, completely determined by the single parameter $d \not\equiv 0 \pmod{3}$ in the lemma statement, of partitioning any C_k, $k \equiv 0 \pmod{6}$, together with an LSPC of C_β, for some suitable even $\beta \in [1, n-1] \setminus \{k\}$, into four LSPCs. This procedure is the *standard method of handling* C_k *with parameter* d, and C_β the *completing diagonal pair* for C_k.

Lemma 4. *Suppose that* $n \equiv 3, 15 \pmod{18}$ *with* $n > 15$, $k \equiv 0 \pmod{6}$, $k > 0$, $\gamma = \gcd(k, n)$, $3n/\gamma > 15$, $\hat{c} = (0, k, \ell) \in C_k$, *and fix* $d \not\equiv 0 \pmod{3}$, *in turn setting* $\beta = 2d\gamma/3 \pmod{n}$ *if* $2d\gamma/3 \pmod{n}$ *is even and* $\beta = -2d\gamma/3 \pmod{n}$ *otherwise (thus ensuring that* β *is even in either case). Then there exists a partition* π_k *of* C_k *into four PLSPCs and a partition* π_β *into four PLSPCs of some LSPC* $\mathcal{L}$ *of* C_β *of type (2) of Lemma 1, such that for each* $S_k \in \pi_k$, *there exists a unique* $S_\beta \in \pi_\beta$ *such that* S_β *is a* β*-completing set for* S_k.

Proof. For each $\alpha \in [0, \gamma/3 - 1]$ and $j \in \mathbb{Z}_3$, form the PLSPC $P_{k,d,j,\alpha}$ as in (3). Set

$$U_\alpha = \bigcup_{h \in [0, 3n/\gamma - 1]} (\hat{c} \oplus (\alpha + h\gamma/3))$$

and $N_\alpha = U_\alpha \setminus \bigcup_{j\in\mathbb{Z}_3} P_{k,d,j,\alpha}$. For $j \in \mathbb{Z}_3$ define $\mathcal{Y}_j = \bigcup_{c\in P_{k,d,j,\alpha}} c_{\{\}}$, and set $Y = \bigcup_{j\in\mathbb{Z}_3} P_{k,d,j,\alpha}$ and $R_\alpha = \{x \in [0, n-1] : x \equiv \alpha \pmod{\gamma/3}\}$. Then $\mathcal{N}_j = R_\alpha \setminus \mathcal{Y}_j$ partitions into three sets $\mathcal{N}_{j,0}, \mathcal{N}_{j,1}$, and $\mathcal{N}_{j,2}$ such that $|\mathcal{N}_{j,0}| = |\mathcal{N}_{j,1}| = |\mathcal{N}_{j,2}|$ and $\mathcal{N}_{j,i}$ is the set of all elements of $\mathcal{N}_j$ congruent to i modulo 3. Suppose that $x \in \mathcal{N}_{j,\alpha \,(\mathrm{mod}\ 3)}$; we claim that $x + hd\gamma/3 \pmod{n} \in \mathcal{N}_{j,\alpha+hd\gamma/3\,(\mathrm{mod}\ 3)}$ for all $h \in \{1, 2\}$. Suppose instead that $x + h\beta/2 \pmod{n} \in \mathcal{Y}_j$ for some $h \in \{1, 2\}$; then there exists some $i \in [0, \lfloor n/(3\gamma)\rfloor - 1]$ such that the cell $\hat{c} \oplus (\alpha + \gamma/3(hd + im) + j\ell)$ has $x + hd\gamma/3 \pmod{n}$ as one of its coordinates. But then the cell $c = \hat{c} \oplus (\alpha + im\gamma/3 + j\ell)$ satisfies $c \in P_{k,d,j,\alpha}$ and has x as one of its coordinates, implying that $x \in \mathcal{Y}_j$, a contradiction. Now if $\beta = 2d\gamma/3 \pmod{n}$, then by Property 3, $(0, 2d\gamma/3, d\gamma/3) \in C_\beta$ and thus by Property 1

$$S_{\alpha,j} = \{(x, x + 2d\gamma/3, x + d\gamma/3) : x \in \mathcal{N}_{j,\alpha\,(\mathrm{mod}\ 3)}\}$$

is an (α, β)-completing set for $P_{k,d,j,\alpha}$. Otherwise, if $\beta = -2d\gamma/3 \pmod{n}$, then by Property 4, $(0, -2d\gamma/3, -d\gamma/3) \in C_\beta$ and thus by Property 1,

$$S_{\alpha,j} = \{(x + 2d\gamma/3, x, x + d\gamma/3) : x \in \mathcal{N}_{j,\alpha\,(\mathrm{mod}\ 3)}\}$$

is an (α, β)-completing set for $P_{k,d,j,\alpha}$.

Next we derive an (α, β)-completing set S_{N_α} for N_α. We treat two cases.
Case 1: $3n/\gamma \equiv 3 \pmod{18}$. Then $n/\gamma \equiv 1 \pmod{6}$, and so

$$\begin{aligned}|Y| &= 9\lfloor n/(3\gamma)\rfloor \\ &= 9(n/\gamma - 1)/3 \\ &= 3n/\gamma - 3.\end{aligned}$$

Thus, $|N_\alpha| = 3$. Moreover, since $d\gamma/3 \not\equiv 0 \pmod{3}$ and $m \equiv 0 \pmod{3}$, we may write $N_\alpha = \{c_0, c_1, c_2\}$ such that the row, column, and symbol of cell c_i are equivalent to i modulo 3 for $i \in \mathbb{Z}_3$. Hence, N_α is a PLSPC. Additionally, we claim that $c_{i+\alpha\,(\mathrm{mod}\ 3)} = c_{\alpha\,(\mathrm{mod}\ 3)} \oplus id\gamma/3$ for $i \in \{1, 2\}$. For suppose to the contrary that $c_{\alpha\,(\mathrm{mod}\ 3)} \oplus id\gamma/3 \in Y$ for some $i \in \{1, 2\}$. Then there must exist some $h \in [0, \lfloor 3n/\gamma\rfloor - 1]$ and $j \in \mathbb{Z}_3$ such that

$$c_{\alpha\,(\mathrm{mod}\ 3)} \oplus id\gamma/3 = \hat{c} \oplus (\alpha + \gamma/3(hm + id) + j\ell).$$

But then $c_{\alpha\,(\mathrm{mod}\ 3)} \in Y$, since $c_{\alpha\,(\mathrm{mod}\ 3)} = \hat{c} \oplus (\alpha + hm\gamma/3 + j\ell)$, a contradiction. Hence, setting

$$T_{\alpha,3} = \{(x, x + 2d\gamma/3, x + d\gamma/3) : x \in {c_{\alpha\,(\mathrm{mod}\ 3)}}_{\{\}}\}$$

when $\beta = 2d\gamma/3 \pmod{n}$ and

$$T_{\alpha,3} = \{(x + 2d\gamma/3, x, x + d\gamma/3) : x \in {c_{\alpha\,(\mathrm{mod}\ 3)}}_{\{\}}\}$$

otherwise, then in either case, $\bigcup_{c\in T_{\alpha,3}} c_{\{\}} = \bigcup_{c'\in N_\alpha} c'_{\{\}}$.

Case 2: $3n/\gamma \equiv 15 \pmod{18}$. Then $n/\gamma \equiv 5 \pmod{6}$, and so

$$\begin{aligned}|Y| &= 9\lfloor n/(3\gamma)\rfloor \\ &= 9(n/\gamma - 2)/3 \\ &= 3n/\gamma - 6.\end{aligned}$$

Thus, $|N_\alpha| = 6$. Since $d\gamma/3 \not\equiv 0 \pmod{3}$ and $m \equiv 0 \pmod{3}$, we may write

$$N_\alpha = \{c_{0,0}, c_{0,1}, c_{1,0}, c_{1,1}, c_{2,0}, c_{2,1}\}$$

such that the row, column, and symbol of the cell $c_{i,j}$ are equivalent to i modulo 3 for all $i \in \mathbb{Z}_3$ and $j \in \{0,1\}$. Using the same argument as in the previous case, without loss of generality $c_{\alpha+i \pmod{3},j} = c_{\alpha \pmod{3},j} \oplus id\gamma/3$ for all $i \in \{1,2\}$, $j \in \{0,1\}$. In turn, setting

$$T_{\alpha,15} = \{(y, y + 2d\gamma/3, y + d\gamma/3) : y \in c_{\alpha \pmod{3},0_{\{\}}} \cup c_{\alpha \pmod{3},1_{\{\}}}\}$$

when $\beta = 2d\gamma/3 \pmod{n}$ and

$$T_{\alpha,15} = \{(y + 2d\gamma/3, y, y + d\gamma/3) : y \in c_{\alpha \pmod{3},0_{\{\}}} \cup c_{\alpha \pmod{3},1_{\{\}}}\}$$

otherwise, then in either case $\bigcup_{c \in T_{\alpha,15}} c_{\{\}} = \bigcup_{c' \in N_\alpha} c'_{\{\}}$.

R_α has size $3n/\gamma$, and since $\gamma/3 \not\equiv 0 \pmod{3}$, it partitions into three sets $R_{\alpha,0}, R_{\alpha,1}$, and $R_{\alpha,2}$, such that $|R_{\alpha,0}| = |R_{\alpha,1}| = |R_{\alpha,2}|$ and $R_{\alpha,i}$ is the set of all elements of R_α congruent to i modulo 3. Thus, setting

$$\mathcal{P}_\alpha = \{(x, x + 2d\gamma/3, x + d\gamma/3) : x \in R_{\alpha,\alpha \pmod{3}}\}$$

when $\beta = 2d\gamma/3 \pmod{n}$ and

$$\mathcal{P}_\alpha = \{(x + 2d\gamma/3, x, x + d\gamma/3) : x \in R_{\alpha,\alpha \pmod{3}}\}$$

otherwise, then in either case $\mathcal{P}_\alpha$ is a PLSPC such that $\mathcal{P}_\alpha \subset C_\beta$ and $\bigcup_{c \in \mathcal{P}_\alpha} \{c_{\{\}}\}$ is a partition of R_α. Hence, $S_{N_\alpha} = \mathcal{P}_\alpha \setminus T_{\alpha,3}$ or $S_{N_\alpha} = \mathcal{P}_\alpha \setminus T_{\alpha,15}$ is an (α,β)-completing set for N_α, when $n \equiv 3 \pmod{18}$ or $n \equiv 15 \pmod{18}$, respectively.

Define $\mathcal{Y} = \bigcup_{c \in N_\alpha} c_{\{\}}$ and $\mathcal{N} = R_\alpha \setminus \mathcal{Y}$. By Lemma 3, $|\mathcal{Y}_j| = 9\lfloor n/(3\gamma)\rfloor$ and if $3n/\gamma \equiv 3 \pmod{18}$, then $|\mathcal{Y}| = 9$; on the other hand, if $n \equiv 15 \pmod{18}$, then $|\mathcal{Y}| = 18$. In either case, $|\mathcal{Y}_j| + |\mathcal{Y}| > 3n/\gamma$ and thus by the pigeonhole principle, $R_\alpha = \mathcal{Y}_j \cup \mathcal{Y}$, so that $\mathcal{N}_j \cap \mathcal{N} = \emptyset$ for all $j \in \mathbb{Z}_3$. Now suppose there exists some $x \in R_\alpha$ such that $x \notin \mathcal{N}_0 \cup \mathcal{N}_1 \cup \mathcal{N}_2 \cup \mathcal{N}$. Then $x \in \mathcal{Y}_0 \cap \mathcal{Y}_1 \cap \mathcal{Y}_2 \cap \mathcal{Y}$, which is impossible since by Property 2, x can only occur in at most three cells of C_k. Hence, R_α admits a partition into sets $\mathcal{N}_0, \mathcal{N}_1, \mathcal{N}_2$, and $\mathcal{N}$, and thus, since for each $j \in \mathbb{Z}_3$ the symbol of each cell of $S_{\alpha,j}$ belongs to $R_{\alpha,\alpha+d\gamma/3 \pmod{3}}$, it must be that $S_{\alpha,0} \cup S_{\alpha,1} \cup S_{\alpha,2} \cup S_{N_\alpha} = \mathcal{P}_\alpha$.

We claim that $\mathcal{L} = \bigcup_{\alpha \in [0,\gamma/3-1]} \mathcal{P}_\alpha$ is an LSPC of C_β of type (2) of Lemma 1. As the cells of $\mathcal{P}_\alpha$, treated as 3-sets, partition R_α, $\mathcal{L}$ is an LSPC. Suppose that $z \in [0, \gamma/3 - 1]$. Then for each $j \in [0, n/\gamma - 1]$, since $\gamma \equiv 0 \pmod{3}$,

$z + j\gamma \equiv z \pmod 3$; this and the fact that $z + j\gamma \equiv z \pmod{\gamma/3}$ imply that $z + j\gamma \in R_{z,z \pmod 3}$. Thus, $d\gamma/3 + z + j\gamma$ is a symbol of some cell of $\mathcal{P}_z$.

Finally, putting

$$\pi_k = \bigcup_{j \in \mathbb{Z}_3} \left\{ \bigcup_{\alpha \in [0,\gamma/3-1]} P_{k,d,j,\alpha} \right\} \cup \left\{ \bigcup_{\alpha \in [0,\gamma/3-1]} N_\alpha \right\}$$

and

$$\pi_\beta = \bigcup_{j \in \mathbb{Z}_3} \left\{ \bigcup_{\alpha \in [0,\gamma/3-1]} S_{\alpha,j} \right\} \cup \left\{ \bigcup_{\alpha \in [0,\gamma/3-1]} S_{N_\alpha} \right\}$$

gives us the desired pair of partitions, where $\bigcup_{\alpha \in [0,\gamma/3-1]} S_{\alpha,j} \in \pi_\beta$ is a β-completing set for $\bigcup_{\alpha \in [0,\gamma/3-1]} P_{k,d,j,\alpha} \in \pi_k$ for each $j \in \mathbb{Z}_3$ and $\bigcup_{\alpha \in [0,\gamma/3-1]} S_{N_\alpha} \in \pi_\beta$ is a β-completing set for $\bigcup_{\alpha \in [0,\gamma/3-1]} N_\alpha \in \pi_k$. □

The only kind of diagonal pair C_k of even index with $k \not\equiv 0 \pmod 3$ that is not handled by the standard method is one satisfying $3n/\gcd(k,n) = 15$; we now treat this exception.

Construction 1. We construct a Bose resolution of the second kind for the averaging latin square B of order 15. The first part of the resolution is the collection of LSPCs:

$$\begin{aligned}
&\{(3,0,9),(1,7,4),(5,2,11),(12,14,13),(6,10,8)\}\\
&\{(9,6,0),(7,13,10),(11,8,2),(3,5,4),(12,1,14)\}\\
&\{(0,12,6),(2,14,8),(13,4,1),(9,11,10),(3,7,5)\}\\
&\{(3,6,12),(4,10,7),(8,5,14),(0,2,1),(9,13,11)\}\\
&\{(10,1,13),(12,9,3),(14,11,5),(6,8,7),(0,4,2)\}\\
&\{(1,3,2),(4,6,5),(7,9,8),(10,12,11),(13,0,14)\}\\
&\{(2,4,3),(5,7,6),(8,10,9),(11,13,12),(14,1,0)\},\ \text{and}\\
&\{(2,6,4),(5,9,7),(8,12,10),(11,0,13),(14,3,1)\}.
\end{aligned}$$

To obtain the second part, partition each of $C_{14}(B)$, $C_{10}(B)$, and $C_8(B)$ into LSPCs by applying Lemma 1 to get the remaining LSPCs of the resolution. As for the third part, the remaining set of cells of B above the main diagonal partitions into five PLSPCs, each of size four. We now present these PLSPCs $\hat{P}_i$, adjoining to each a 3-set $\hat{T}_i$ of the points not covered by the corresponding

PLSPC, such that the collection of these 3-sets gives a partition of $[0, 14]$:

$$\hat{P}_1 = \{(3,9,6),(7,4,13),(5,11,8),(10,14,12)\} \text{ and } \hat{T}_1 = \{0,1,2\};$$
$$\hat{P}_2 = \{(6,12,9),(10,7,1),(8,14,11),(13,2,0)\} \text{ and } \hat{T}_2 = \{3,4,5\};$$
$$\hat{P}_3 = \{(9,0,12),(11,2,14),(13,10,4),(1,5,3)\} \text{ and } \hat{T}_3 = \{6,7,8\};$$
$$\hat{P}_4 = \{(1,13,7),(12,3,0),(14,5,2),(4,8,6)\} \text{ and } \hat{T}_4 = \{9,10,11\};$$
$$\hat{P}_5 = \{(0,6,3),(4,1,10),(2,8,5),(7,11,9)\} \text{ and } \hat{T}_5 = \{12,13,14\}.$$

This completes the construction.

For any integer $x > 1$, write its prime factorization as $x = p_1 \cdot p_2 \cdot \cdots \cdot p_q$, and define the multiset $\mathsf{pf}(x) = \{p_1, \ldots, p_q\}$. If $x \in \{0, 1\}$, define $\mathsf{pf}(x) = \emptyset$. For any $n \equiv 3, 15 \pmod{18}$ and $m \equiv 0 \pmod 3$ such that $\mathsf{pf}(m/3) \subset \mathsf{pf}(n/3)$, let B_m and B_n be the averaging latin squares of orders m and n, respectively. (If $m = 0$, then B_m is the empty latin square). The *diagonal pair indices of B_m in B_n*, denoted $D_m(B_n)$, is the empty set if $m = 0$ and otherwise the set

$$\{2n/m, 4n/m, \ldots, (m-1)n/m\}$$

of $(m-1)/2$ diagonal pair indices of B_n. For each $\alpha \in [0, n/m-1]$, define the map $\varphi_\alpha : [0, m-1] \to \{\alpha, \alpha + n/m, \alpha + 2n/m, \ldots, \alpha + (m-1)n/m\}$, where $\varphi_\alpha(i) = in/m + \alpha$. Extend φ_α naturally to 3-tuples by applying the map to each component; i.e., if (x, y, z) satifies $x, y, z \in [0, m-1]$, then $\varphi_\alpha((x, y, z)) = (\varphi_\alpha(x), \varphi_\alpha(y), \varphi_\alpha(z))$. Further, for any $S \subset \{(x, y, z) : x, y, z \in [0, m-1]\}$, define $\varphi_\alpha(S) = \{\varphi_\alpha((x, y, z)) : (x, y, z) \in S\}$.

Lemma 5. *Let B_n be the averaging latin square of order $n \equiv 3, 15 \pmod{18}$, and B_m the averaging latin square of order m, $m \equiv 3 \pmod 6$, such that $\mathsf{pf}(m/3) \subset \mathsf{pf}(n/3)$. Then for each diagonal pair $C_{k'}(B_m)$, k' even, $C_{k'n/m}(B_n)$ partitions into n/m isomorphic copies of $C_{k'}(B_m)$, with the n/m isomorphisms being $\{\varphi_\alpha : \alpha \in [0, n/m-1]\}$.*

Proof. For even $k' \in [1, m-1]$ and $\alpha \in [0, n/m-1]$, define

$$U_{k',\alpha} = \bigcup_{i \in [0,m-1]} c_{n,k'n/m} \oplus_n (\alpha + in/m),$$

where $c_{n,k'n/m} = (0, k'n/m, \ell) \in C_{k'n/m}(B_n)$. Moreover, let $c_{m,k'} = (0, k', \ell') \in C_{k'}(B_m)$. For $i \in [0, m-1]$,

$$\begin{aligned}
\varphi_\alpha(c_{m,k'} \oplus_m i) &= \varphi_\alpha((i, k'+i, k'/2+i)) \\
&= (in/m + \alpha, n/m(k'+i) + \alpha, n/m(k'/2+i) + \alpha) \\
&= c_{n,k'n/m} \oplus_n (\alpha + in/m),
\end{aligned}$$

where the last equality obtains by Property 1. Thus, $\varphi_\alpha(C_{k'}(B_m)) = U_{k',\alpha}$, so that the φ_α's are the desired isomorphisms. □

Corollary 1. *Let B be the averaging latin square of order $n \equiv 3, 15 \pmod{18}$ such that $n > 15$ and $n = 15t$, and let B_{15} be the averaging latin square of order 15. Let $D_{15}(B)$ denote the diagonal pair indices of B_{15} in B, so that*

$$U = \bigcup_{j \in D_{15}(B)} C_j(B) = \bigcup_{i \in \{2,4,\ldots,14\}} C_{it}(B).$$

Then there exists a partition of U into a set $\mathcal{L} = \{L_1, \ldots, L_{17}\}$ of LSPCs, and a set $\mathcal{P} = \{P_1, \ldots, P_5\}$ of PLSPCs of size $4n/15$ for which there exists (1) a partition $\mathcal{T} = \{T_1, \ldots, T_{n/3}\}$ of $[0, n-1]$ into triples and (2 a partition $\mathcal{T}' = \{T_1', \ldots, T_5'\}$ of $\mathcal{T}$ into sets (of triples) of size $n/15$ such that for all $i \in [1, 5]$,

$$\bigcup_{c \in P_i} c_{\{\}} \cup \bigcup_{T \in T_i'} T = [0, n-1].$$

Proof. For each $\alpha \in [0, t-1]$, set

$$U_\alpha = \bigcup_{c \in \{(0, it, B_{0,it}) : i \in \{2,4,\ldots,14\}\}, j \in [0,14]} c \oplus (\alpha + jt).$$

Then by Lemma 5,

$$\varphi_\alpha \left(\bigcup_{k \in \{2,4,\ldots,14\}} C_k(B_{15}) \right) = U_\alpha.$$

By Property 3, $it \equiv B_{0,it} \equiv 0 \pmod{t}$. Therefore, the union of the cells (treated as 3-sets) of U_α is

$$R_\alpha = \{x \equiv \alpha \ (\mathrm{mod}\ t) : x \in [0, n-1]\}.$$

Let $\hat{\mathcal{L}} = \{\hat{L}_1, \ldots, \hat{L}_{17}\}$, $\hat{\mathcal{P}} = \{\hat{P}_1, \ldots, \hat{P}_5\}$, and $\hat{\mathcal{T}} = \{\hat{T}_1, \ldots, \hat{T}_5\}$ be the 17 LSPCs, the 5 PLSPCs (each of size 4), and the 5 triples (which partition [0,14]), respectively, that comprise the Bose resolution of B_{15} given in Construction 1.

Let us now partition U. For each $i \in [1, 17]$, put

$$L_i = \bigcup_{\alpha \in [0, t-1]} \varphi_\alpha(\hat{L}_i).$$

The union of the cells of each $\varphi_\alpha(\hat{L}_i)$ is equal to R_α, and thus by Lemma 5 L_i is an LSPC of B. For each $j \in [1, 5]$, put

$$P_j = \bigcup_{\alpha \in [0, t-1]} \varphi_\alpha(\hat{P}_j), \text{ and}$$

$$T_j' = \bigcup_{\alpha \in [0, t-1]} \varphi_\alpha(\hat{T}_j).$$

As

$$\bigcup_{c \in \varphi_\alpha(\hat{P}_j)} c_{\{\}} \cup \varphi_\alpha(\hat{T}_j) = R_\alpha,$$

the result follows. □

Define $K = \{k \in [1, n-1] : k \equiv 0 \pmod 6\}$. Given distinct diagonal pairs $C_k, C_{k'}$, with $k, k' \in K$, suppose we apply the standard method to C_k and $C_{k'}$ with parameters d and d', respectively, such that $d \cdot \gcd(k, n) \equiv d' \cdot \gcd(k', n) \pmod n$. Then C_k and $C_{k'}$ share the same completing diagonal pair; in fact, the same LSPC of that completing diagonal pair is used by the standard method to complete the four PLSPCs of C_k and the four PLSPCs of $C_{k'}$ to LSPCs. This is a *collision between* C_k *and* $C_{k'}$. To produce a Bose resolution for B_n when $5 \nmid n$, we wish to apply the standard method to each diagonal pair of $S = \{C_k : k \in K\}$ such that there is no collision between any two $C_k, C_{k'} \in S$. To produce a Bose resolution for B_n when $5 \mid n$, we wish to apply the standard method to each diagonal pair of $S = \{C_k : k \in K \setminus D_{15(B_n)}\}$ such that there is no collision between any two $C_k, C_{k'} \in S$.

In order to avoid collisions, we first characterize the set of all completing diagonal pairs C_β for C_k that could possibly result from applying the standard method to handle C_k with parameter $d \not\equiv 0 \pmod 3$. Let $\mu_k = \gcd(k, n)/3$, and recall from the statement of Lemma 4 that when the standard method is applied with parameter $d \not\equiv 0 \pmod 3$ to handle C_k, the resulting completing diagonal pair for C_k is C_β, where $\beta = 2d\mu_k \pmod n$ when $2d\mu_k \pmod n$ is even and $\beta = -2d\mu_k \pmod n$ otherwise.

Lemma 6. *Let* $n \equiv 3, 15 \pmod{18}$ *and* $k \in [1, n-1]$ *with* $k \equiv 0 \pmod 6$*. Then setting* $\mu_k = \gcd(k, n)/3$*, let*

$$\begin{aligned}
E_k &= \{2d\mu_k \pmod n : d \not\equiv 0 \pmod 3, 2d\mu_k \pmod n \text{ is even}\},\\
O_k &= \{-2d\mu_k \pmod n : d \not\equiv 0 \pmod 3, 2d\mu_k \pmod n \text{ is odd}\}, \text{ and}\\
\mathfrak{C}_k &= \{d\mu_k : d \in [2, n/\mu_k - 1], d \text{ even, and } d \not\equiv 0 \pmod 3\}.
\end{aligned}$$

Then $\mathfrak{C}_k = E_k \cup O_k$*.*

Proof. First, suppose that $x \in \mathfrak{C}_k$; say $x = d_x\mu_k$ with $d_x \in [2, n/\mu_k - 1]$, d_x even, and $d_x \not\equiv 0 \pmod 3$. Then we may write $x = 2(d_x/2)\mu_k$, where $d_x/2 \not\equiv 0 \pmod 3$ and x is even. Hence $x \in E_k \cup O_k$.

Second, suppose that $x \in E_k \cup O_k$. Then $x \in \langle \mu_k \rangle \leq \mathbb{Z}_n$. If $x \in E_k$, then we may write $x = 2d_x\mu_k \pmod n$, where x is even and $d_x \not\equiv 0 \pmod 3$. Suppose to the contrary that $x \notin \mathfrak{C}_k$, so that $x = d'_x\mu_k$ with $d'_x \in [0, n/\mu_k - 1]$ such that d'_x is odd or $d'_x \equiv 0 \pmod 3$. But if d'_x is odd, then x is odd, a contradiction. If $d'_x \equiv 0 \pmod 3$, then since $x = 2d_x\mu_k - in$ for some nonnegative i, and since $3 \nmid 2d_x\mu_k$ and $3 \mid n$, then $3 \nmid x$, a contradiction. If $x \in O_k$, then we may write $x = -2d_x\mu_k \pmod n$, where x is even and $d_x \not\equiv 0 \pmod 3$. Suppose to the contrary that $x \notin \mathfrak{C}_k$, so that $x = d'_x\mu_k$ with $d'_x \in [0, n/\mu_k - 1]$ such that d'_x

is odd or $d'_x \equiv 0 \pmod 3$. But if d'_x is odd, then x is odd, a contradiction. If $d'_x \equiv 0 \pmod 3$, then since $x = in - 2d_x\mu_k$ for some nonnegative i, and since $3 \nmid 2d_x\mu_k$ and $3 \mid n$, then $3 \nmid x$, a contradiction. Hence, $x \in \mathfrak{C}_k$. $\square$

By Lemma 6, then,

$$\mathfrak{C}_k = \{d\mu_k : d \in [2, n/\mu_k - 1], d \text{ even, and } d \not\equiv 0 \,(\mathrm{mod}\ 3)\}.$$

gives the set of (indexes of) all possible completing diagonal pairs for handling C_k via the standard method.

Second, we make use of the well-known Hall's marriage theorem. A *transversal* for a set family $\mathcal{M}$ over universe U is an injective function $\rho : \mathcal{M} \to U$ such that $\rho(S) \in S$ for all $S \in \mathcal{M}$, and $\mathcal{M}$ satisfies the *marriage condition* if for each submultiset $\mathcal{H} \subseteq \mathcal{M}$, $|\mathcal{H}| \leq \left|\bigcup_{H\in\mathcal{H}} H\right|$.

Theorem 4 (Hall's marriage theorem [13]). *Suppose that $\mathcal{M}$ is a multiset of finite subsets of some universe U. Then $\mathcal{M}$ has a transversal if and only if it satisfies the marriage condition.*

Let $K = \{k \in [1, n-1] : k \equiv 0 \pmod 6\}$ and for each $k \in K$, write $\mu_k = \gcd(k, n)/3$. For any integer m such that $3 \mid m$ and $\mathsf{pf}(m) \subset \mathsf{pf}(n)$ with B_m the averaging latin square of order m, we remind the reader that $D_m(B_n)$ is the set of diagonal pair indices of B_m in B_n. Let

$$F_k^{-D_m(B_n)} = \mathfrak{C}_k \setminus D_m(B_n);$$

by Lemma 6, $F_k^{-D_m(B_n)}$ gives precisely the set of indices of all possible completing diagonal pairs, excluding those indexed by $D_m(B_n)$, which we may use in the standard method of handling C_k. Accordingly, we call $F_k^{-D_m(B_n)}$ the *completing candidate set for k with forbidden diagonal pairs $D_m(B_n)$*. The *family of all completing candidate sets with forbidden diagonal pairs $D_m(B_n)$* is the set family $\mathcal{F}^{-D_m(B_n)} = \bigcup_{k\in K\setminus D_m(B_n)}\{F_k^{-D_m(B_n)}\}$. If $D_m(B_n) = \emptyset$, we simply write F_k to denote the completing candidate set for k and $\mathcal{F}$ to denote the family of all completing candidate sets (with no forbidden diagonal pairs).

For any $\mathcal{H} = \{F_{k_1}, \ldots, F_{k_q}\} \subseteq \mathcal{F}$, the *multiple closure of $\mathcal{H}$ over K minus $D_m(B_n)$*, denoted $\mathsf{mcl}_{K\setminus D_m(B_n)}(\mathcal{H})$, is the subset of $\mathcal{F}^{-D_m(B_n)}$ given by

$$\mathsf{mcl}_{K\setminus D_m(B_n)}(\mathcal{H}) = \{F_k^{-D_m(B_n)} \in \mathcal{F}^{-D_m(B_n)} : \mu_{k_i} \mid \mu_k \text{ for some } i \in [1, q]\},$$

or equivalently,

$$\mathsf{mcl}_{K\setminus D_m(B_n)}(\mathcal{H}) = \bigcup_{j\in[1,q]} \{F_{6i\mu_{k_j}}^{-D_m(B_n)} : i \in [1, \lfloor n/(6\mu_{k_j})\rfloor], 6i\mu_{k_j} \notin D_m(B_n)\}. \tag{5}$$

In words, $\mathsf{mcl}_{K\setminus D_m(B_n)}(\mathcal{H})$ is the set of all completing candidate sets of $\mathcal{F}^{-D_m(B_n)}$ that are subscripted by a multiple of μ_{k_j} in $K\setminus D_m(B_n)$ for each $j \in [1, m]$. If $\mathcal{H} =$

$\{F_k\}$, we write $\mathsf{mcl}_{K\setminus D_m(B_n)}(F_k)$. $\mathcal{H} = \{F_{k_1}, \ldots, F_{k_q}\} \subseteq \mathcal{F}$ is *multiple-closed over K minus $D_m(B_n)$* if $\mathsf{mcl}_{K\setminus D_m(B_n)}(\mathcal{H}) = \{F_{k_1}^{-D_m(B_n)}, \ldots, F_{k_m}^{-D_m(B_n)}\}$.

Given a multiset S and any element $x \in S$, the *multiplicity of x in S* is the number of times that x occurs in S. For any two multisets S_1 and S_2, the *multiset union of S_1 and S_2*, denoted $S_1 \uplus S_2$, is the multiset for which each element has multiplicity equal to the maximum of its multiplicities in S_1 and S_2.

Henceforth, we assume that $n \equiv 3, 15 \pmod{18}$, $m \equiv 0 \pmod 3$, $\mathsf{pf}(m/3) \subset \mathsf{pf}(n/3)$, B_n is the averaging latin square of order n, $K = \{k \in [1, n-1] : k \equiv 0 \pmod 6\}$, and $\mu_k = \gcd(k, n)/3$ for $k \in K$.

Lemma 7. *Suppose that $x, y > 1$. Then over $\mathbb{Z}$, the intersection of the set of multiples of x and the set of multiples of y is equal to the set of multiples of the product of $\mathsf{pf}(x) \uplus \mathsf{pf}(y)$.*

Proof. This is a routine verification. □

Lemma 8. *For any divisor d of n satisfying $3 \nmid d$, $\lfloor n/(6d) \rfloor = (n - 3d)/(6d)$.*

Proof. Because $n/6 = (n-3)/6 + 1/2$, where $(n-3)/6$ is an integer,

$$\begin{aligned} \lfloor n/(6d) \rfloor &= \lfloor (n/d - 3)/6 + 1/2 \rfloor \\ &= (n/d - 3)/6 \\ &= n/(6d) - 3/6 \\ &= (n - 3d)/(6d). \end{aligned}$$

□

Lemma 9. *For $k \in K$, $\bigcup_{F_j^{-D_m(B_n)} \in \mathsf{mcl}_{K\setminus D_m(B_n)}(F_k)} F_j^{-D_m(B_n)} = F_k^{-D_m(B_n)}$.*

Proof. If $F_j^{-D_m(B_n)} \in \mathsf{mcl}_{K\setminus D_m(B_n)}(F_k)$, by definition $\mu_k \mid \mu_j$, and thus $F_j^{-D_m(B_n)} \subseteq F_k^{-D_m(B_n)}$.

Lemma 10. *For $k \in K$, suppose that $F_k \in \mathcal{F}$ is the completing candidate set for k. Then*

$$2 \cdot |\mathsf{mcl}_K(F_k)| + 1 = \left| \bigcup_{F_j \in \mathsf{mcl}_K(F_k)} F_j \right|.$$

Proof. By Lemma 9 $|\bigcup_{F_j \in \mathsf{mcl}_K(F_k)} F_j| = |F_k| = n/(3\mu_k)$. By (5),

$$\mathsf{mcl}_K(F_k) = \{F_{6i\mu_k} : i \in [1, \lfloor n/(6\mu_{k_j}) \rfloor]\}.$$

Hence, by Lemma 8, $|\mathsf{mcl}_K(F_k)| = (n - 3\mu_k)/(6\mu_k)$. □

Lemma 11. *For any pair of completing candidate sets $F_j, F_k \in \mathcal{F}$ such that $\mu_j, \mu_k > 1$ and $\mathsf{pf}(\mu_k) \uplus \mathsf{pf}(\mu_j) \neq \mathsf{pf}(n/3)$, there exists some $\ell \in K$ for which*

$$\mathsf{mcl}_{K \setminus D_m(B_n)}(F_j) \cap \mathsf{mcl}_{K \setminus D_m(B_n)}(F_k) = \mathsf{mcl}_{K \setminus D_m(B_n)}(F_\ell),$$

such that μ_ℓ is the product of the elements of $\mathsf{pf}(\mu_k) \uplus \mathsf{pf}(\mu_j)$. Conversely, if $\mathsf{pf}(\mu_k) \uplus \mathsf{pf}(\mu_j) = \mathsf{pf}(n/3)$, then

$$\mathsf{mcl}_{K \setminus D_m(B_n)}(F_j) \cap \mathsf{mcl}_{K \setminus D_m(B_n)}(F_k) = \emptyset.$$

Proof. Let p be the product of the elements of $\mathsf{pf}(\mu_j) \uplus \mathsf{pf}(\mu_k) \neq \mathsf{pf}(n/3)$. Then p is sufficiently small that there must exist some $\ell \in K$ such that $p = \mu_\ell = \gcd(\ell, n)/3$. Now

$$\begin{aligned}
&\mathsf{mcl}_{K \setminus D_m(B_n)}(F_j) \cap \mathsf{mcl}_{K \setminus D_m(B_n)}(F_k) \\
&= \{F_{6i\mu_j}^{-D_m(B_n)} : i \in [1, \left\lfloor \frac{n}{6\mu_j} \right\rfloor], 6i\mu_j \notin D_m(B_n)\} \cap \{F_{6i\mu_k}^{-D_m(B_n)} : i \in [1, \left\lfloor \frac{n}{6\mu_k} \right\rfloor], 6i\mu_k \notin D_m(B_n)\} \\
&= \{F_{6i\mu_\ell}^{-D_m(B_n)} : i \in [1, \min\left(\frac{\mu_j}{\mu_\ell}\left\lfloor \frac{n}{6\mu_j} \right\rfloor, \frac{\mu_k}{\mu_\ell}\left\lfloor \frac{n}{6\mu_k} \right\rfloor\right)], 6i\mu_\ell \notin D_m(B_n)\} \\
&= \{F_{6i\mu_\ell}^{-D_m(B_n)} : i \in [1, \min\left(\frac{\mu_j}{\mu_\ell} \cdot \frac{n-3\mu_j}{6\mu_j}, \frac{\mu_k}{\mu_\ell} \cdot \frac{n-3\mu_k}{6\mu_k}\right)], 6i\mu_\ell \notin D_m(B_n)\} \\
&= \{F_{6i\mu_\ell}^{-D_m(B_n)} : i \in [1, \min\left(\frac{n-3\mu_j}{6\mu_\ell}, \frac{n-3\mu_k}{6\mu_\ell}\right)], 6i\mu_\ell \notin D_m(B_n)\} \\
&= \{F_{6i\mu_\ell}^{-D_m(B_n)} : i \in [1, \frac{n-3\mu_\ell}{6\mu_\ell}], 6i\mu_\ell \notin D_m(B_n)\} \\
&= \mathsf{mcl}_{K \setminus D_m(B_n)}(F_\ell),
\end{aligned}$$

where:

1. The first equality follows from (5).
2. The second equality follows from Lemma 7.
3. The third equality follows from Lemma 8.
4. The fifth equality follows from the fact that $\frac{n-3\mu_\ell}{6\mu_\ell} \leq \min\left(\frac{n-3\mu_j}{6\mu_\ell}, \frac{n-3\mu_k}{6\mu_\ell}\right)$ and the fact that $\{6i\mu_\ell : i \in [1, \frac{n-3\mu_\ell}{6\mu_\ell}]\}$ is the minimum-sized set that yields every multiple of μ_ℓ in K.
5. The sixth equality follows from (5) and Lemma 8.

Conversely, by Lemma 7, if $\mathsf{pf}(\mu_k) \uplus \mathsf{pf}(\mu_j) = \mathsf{pf}(n/3)$, then over $\mathbb{Z}$, the intersection of the set of multiples of μ_j and the set of multiples of μ_k is the set of multiples of $n/3$. But the only multiples of $n/3$ in $[1, n-1]$ are $n/3$ and $2n/3$, neither of which belongs to K. □

Lemma 12. *For any pair of completing candidate sets $F_j^{-D_m(B_n)}, F_k^{-D_m(B_n)} \in \mathcal{F}^{-D_m(B_n)}$ with forbidden diagonal pairs $D_m(B_n)$ such that $\mu_j, \mu_k > 1$ and $\mathsf{pf}(\mu_k) \uplus \mathsf{pf}(\mu_j) \neq \mathsf{pf}(n/3)$, there exists some $\ell \in K$ for which*

$$F_j^{-D_m(B_n)} \cap F_k^{-D_m(B_n)} = F_\ell^{-D_m(B_n)}$$

such that μ_ℓ *is the product of the elements of* $\mathsf{pf}(\mu_k) \uplus \mathsf{pf}(\mu_j)$*. Conversely, if* $\mathsf{pf}(\mu_k) \uplus \mathsf{pf}(\mu_j) = \mathsf{pf}(n/3)$*, then*

$$F_j^{-D_m(B_n)} \cap F_k^{-D_m(B_n)} = \{2n/3\} \setminus D_m(B_n).$$

Proof. Let p be the product of the elements of $\mathsf{pf}(\mu_j) \uplus \mathsf{pf}(\mu_k) \neq \mathsf{pf}(n/3)$. Then p is sufficiently small that there must exist some $\ell \in K$ such that $p = \mu_\ell = \gcd(\ell, n)/3$. Now

$$\begin{aligned}
&F_j^{-D_m(B_n)} \cap F_k^{-D_m(B_n)} \\
&= \bigcap_{h \in \{j,k\}} \{d\mu_h : d \in [2, n/\mu_h - 1], d \text{ even}, d \not\equiv 0 \pmod 3\} \setminus D_m(B_n) \\
&= \{d\mu_\ell : d \in [2, n/\mu_\ell - 1], d \text{ even}, d \not\equiv 0 \pmod 3\} \setminus D_m(B_n) \\
&= F_\ell^{-D_m(B_n)},
\end{aligned}$$

where the second equality follows from Lemma 7.

Conversely, by Lemma 7, if $\mathsf{pf}(\mu_k) \uplus \mathsf{pf}(\mu_j) = \mathsf{pf}(n/3)$, then over $\mathbb{Z}$, the intersection of the set of multiples of μ_j and the set of multiples of μ_k is the set of multiples of $n/3$. But the only multiples of $n/3$ in $[1, n-1]$ are $n/3$ and $2n/3$, and of those two, only $2n/3$ is a valid index of a completing diagonal pair. $\square$

Lemma 13. *Suppose that* $D_m(B_n) \neq \emptyset$ *and, for* $k \in K$*, that* $F_k \in \mathcal{F}$ *is the completing candidate set for* k*. Then*

$$2 \cdot |\mathsf{mcl}_{K \setminus D_m(B_n)}(F_k)| = \left| \bigcup_{H \in \mathsf{mcl}_{K \setminus D_m(B_n)}(F_k)} H \right|.$$

Proof. By definition, $D_m(B_n) = \bigcup_{i \in \{2,4,\ldots,m-1\}} in/m$. Hence, $F_{6n/m} \in \mathcal{F}$ such that by (5), $\mathsf{mcl}_K(F_{6n/m}) = \{F_{6in/m} : i \in [1, \lfloor m/6 \rfloor]\} \subset \bigcup_{i \in D_m(B_n)} \{F_i\}$, implying

$$\mathsf{mcl}_K(F_k) \cap \bigcup_{i \in D_m(B_n)} \{F_i\} = \mathsf{mcl}_K(F_k) \cap \mathsf{mcl}_K(F_{6n/m}).$$

Suppose first that $\mathsf{pf}(\mu_k) \uplus \mathsf{pf}(\mu_{6n/m} = n/m) = n/3$; then by Lemma 11,

$$\mathsf{mcl}_K(F_k) \cap \mathsf{mcl}_K(F_{6n/m}) = \emptyset,$$

and thus

$$\mathsf{mcl}_K(F_k) = \mathsf{mcl}_{K \setminus D_m(B_n)}(F_k).$$

Further, by Lemmas 7 and 9,

$$\bigcup_{F_j \in \mathsf{mcl}_K(F_k)} F_j \cap D_m(B_n) = F_k \cap D_m(B_n) = \{2n/3\}.$$

Thus,

$$\left|\bigcup_{F_j \in \mathsf{mcl}_{K \setminus D_m(B_n)}(F_k)} F_j\right| = \left|\bigcup_{F_j \in \mathsf{mcl}_K(F_k)} F_j\right| - 1,$$

and the result then follows by Lemma 10.

Second, suppose that $\mathsf{pf}(\mu_k) \uplus \mathsf{pf}(n/m) \neq n/3$. Then by Lemma 11,

$$\mathsf{mcl}_K(F_k) \cap \mathsf{mcl}_K(F_{6n/m}) = \mathsf{mcl}_K(F_\ell),$$

where $F_\ell \in \mathcal{F}$ with μ_ℓ being the product of the elements of $\mathsf{pf}(\mu_k) \uplus \mathsf{pf}(n/m)$. By Lemma 9,

$$\bigcup_{F_j \in \mathsf{mcl}_K(F_k)} F_j \cap D_m(B_n) = F_k \cap D_m(B_n) = F_\ell.$$

Next the fact that

$$\mathsf{mcl}_{K \setminus D_m(B_n)}(F_k) = \{F_j \setminus D_m(B_n) : F_j \in \mathsf{mcl}_K(F_k)\} \setminus \bigcup_{F_i \in \mathcal{F}, i \in D_m(B_n) \cap K} \{F_i \setminus D_m(B_n)\},$$

implies both

$$\begin{aligned} |\mathsf{mcl}_{K \setminus D_m(B_n)}(F_k)| &= |\mathsf{mcl}_K(F_k)| - |\mathsf{mcl}_K(F_k) \cap \bigcup_{F_i \in \mathcal{F}, i \in D_m(B_n) \cap K} \{F_i\}| \\ &= |\mathsf{mcl}_K(F_k)| - |\mathsf{mcl}_K(F_\ell)|, \end{aligned}$$

and, together with Lemma 9,

$$\left|\bigcup_{H \in \mathsf{mcl}_{K \setminus D_m(B_n)}(F_k)} H\right| = F_k \setminus D_m(B_n) = |F_k| - |F_\ell|.$$

Applying Lemma 10 twice, we have that

$$\begin{aligned} &2|\mathsf{mcl}_K(F_k)| + 1 - (2|\mathsf{mcl}_K(F_\ell)| + 1) = |F_k| - |F_\ell| \\ \iff &2(|\mathsf{mcl}_K(F_k)| - |\mathsf{mcl}_K(F_\ell)|) = |F_k| - |F_\ell|, \end{aligned}$$

as desired. □

Lemma 14. *For any collection $\mathcal{H} = \{F_{k_1}, \ldots, F_{k_q}\} \subseteq \mathcal{F}$ of completing candidate sets for k,*

$$|\mathsf{mcl}_{K \setminus D_m(B_n)}(\mathcal{H})| \leq \left|\bigcup_{H \in \mathsf{mcl}_{K \setminus D_m(B_n)}(\mathcal{H})} H\right|.$$

Proof. First, suppose there exists some $i \in [1, q]$ such that $\mu_{k_i} = 1$. Then

$$\begin{aligned}|\mathsf{mcl}_{K \setminus D_m(B_n)}(\mathcal{H})| &\leq |K \setminus D_m(B_n)| \\ &= \lfloor (n-1)/6 \rfloor - \lfloor (m-1)/6 \rfloor \\ &\leq (n-m)/3 \\ &= |F_{k_i}^{-D_m(B_n)}| \leq \left| \bigcup_{H \in \mathsf{mcl}_{K \setminus D_m(B_n)}(\mathcal{H})} H \right| .\end{aligned}$$

Otherwise, we compute the sizes of $\mathsf{mcl}_{K \setminus D_m(B_n)}(\mathcal{H})$ and $\bigcup_{H \in \mathsf{mcl}_{K \setminus D_m(B_n)}(\mathcal{H})} H$ using the inclusion-exclusion principle:

$$\begin{aligned}&|\mathsf{mcl}_{K \setminus D_m(B_n)}(\mathcal{H})| \\ &= \left| \bigcup_{i=1}^{q} \mathsf{mcl}_{K \setminus D_m(B_n)}(F_{k_i}) \right| \\ &= \sum_{i=1}^{q} (-1)^{i+1} \left(\sum_{1 \leq h_1 < \cdots < h_i \leq q} |\mathsf{mcl}_{K \setminus D_m(B_n)}(F_{k_{h_1}}) \cap \cdots \cap \mathsf{mcl}_{K \setminus D_m(B_n)}(F_{k_{h_i}})| \right),\end{aligned} \tag{6}$$

and

$$\begin{aligned}\left| \bigcup_{H \in \mathsf{mcl}_{K \setminus D_m(B_n)}(\mathcal{H})} H \right| &= \left| \bigcup_{i=1}^{q} F_{k_i}^{-D_m(B_n)} \right| \qquad \text{(by Lemma 9)} \\ &= \sum_{i=1}^{q} (-1)^{i+1} \left(\sum_{1 \leq h_1 < \cdots < h_i \leq q} |F_{k_{h_1}}^{-D_m(B_n)} \cap \cdots \cap F_{k_{h_i}}^{-D_m(B_n)}| \right).\end{aligned} \tag{7}$$

For each $i \in [1, q]$ and for each i-subset $S = \{F_{k_{h_1}}, \ldots, F_{k_{h_i}}\} \subseteq \mathcal{H}$, either (C1) $\biguplus_{j \in [1,i]} \mathsf{pf}(\mu_{k_{h_j}}) \neq \mathsf{pf}(n/3)$, in which case we can "collapse" the corresponding inclusion-exclusion expressions:

$$\begin{aligned}&\mathsf{mcl}_{K \setminus D_m(B_n)}(F_{k_{h_1}}) \cap \cdots \cap \mathsf{mcl}_{K \setminus D_m(B_n)}(F_{k_{h_i}}), \text{ and} \\ &F_{k_{h_1}}^{-D_m(B_n)} \cap \cdots \cap F_{k_{h_i}}^{-D_m(B_n)}\end{aligned}$$

by (repeated) application of Lemma 11 and Lemma 12, respectively, to yield

$$\begin{aligned}&\bigcap_{j=1}^{i} \mathsf{mcl}_{K \setminus D_m(B_n)}(F_{k_{h_j}}) = \mathsf{mcl}_{K \setminus D_m(B_n)}(F_{\sigma(S)}), \text{ and} \\ &\bigcap_{j=1}^{i} F_{k_{h_j}}^{-D_m(B_n)} = F_{\sigma(S)}^{-D_m(B_n)},\end{aligned}$$

respectively, where $F_{\sigma(S)} \in \mathcal{F}$ is the completing candidate set for $\sigma(S)$ such that $\mu_{\sigma(S)}$ is the product of the elements of $\biguplus_{j\in[1,i]} \mathsf{pf}(\mu_{k_{h_j}})$; or (C2) $\biguplus_{j\in[1,i]} \mathsf{pf}(\mu_{k_{h_j}}) = \mathsf{pf}(n/3)$, in which case (repeated) application of Lemma 11 and Lemma 12, respectively, yields

$$\bigcap_{j=1}^{i} \mathsf{mcl}_{K\setminus D_m(B_n)}(F_{k_{h_j}}^{-D_m(B_n)}) = \emptyset, \text{ and}$$

$$\bigcap_{j=1}^{i} F_{k_{h_j}}^{-D_m(B_n)} = \{2n/3\} \setminus D_m(B_n),$$

respectively. Let $\mathcal{S}_1$ be the set of all subsets of $\mathcal{H}$ that satisfy (C1) and $\mathcal{S}_2$ the set of all subsets of $\mathcal{H}$ that satisfy (C2). Then there exists a map $\tau : \mathcal{S}_1 \cup \mathcal{S}_2 \to \{-1, 1\}$ (which exactly mirrors the way in which each inner summation of (6) and (7) is multiplied by $(-1)^{i+1}$) such that

$$|\mathsf{mcl}_{K\setminus D_m(B_n)}(\mathcal{H})| = \sum_{S\in\mathcal{S}_1} \tau(S)|\mathsf{mcl}_{K\setminus D_m(B_n)}(F_{\sigma(S)})| + \sum_{S\in\mathcal{S}_2} \tau(S)\cdot 0, \text{ and}$$

$$\left|\bigcup_{H\in\mathsf{mcl}_{K\setminus D_m(B_n)}(\mathcal{H})} H\right| = \sum_{S\in\mathcal{S}_1} \tau(S)|F_{\sigma(S)}^{-D_m(B_n)}| + \sum_{S\in\mathcal{S}_2} \tau(S)\cdot|\{2n/3\} \setminus D_m(B_n)|.$$

By Lemma 9, together with either Lemma 10 or 13 (depending on whether $D_m(B_n) = \emptyset$),

$$\sum_{S\in\mathcal{S}_1} \tau(S)|F_{\sigma(S)}^{-D_m(B_n)}| \geq 2 \cdot \sum_{S\in\mathcal{S}_1} \tau(S)|\mathsf{mcl}_{K\setminus D_m(B_n)}(F_{\sigma(S)})|.$$

Moreover, either $2n/3$ occurs exactly once in $\bigcup_{H\in\mathsf{mcl}_{K\setminus D_m(B_n)}(\mathcal{H})} H$ or not at all. Hence,

$$\sum_{S\in\mathcal{S}_2} \tau(S)\cdot|\{2n/3\} \setminus D_m(B_n)| = 1 \text{ or } 0,$$

and thus the desired result obtains. □

Corollary 2. *For any collection $\mathcal{H} \subseteq \mathcal{F}$ of completing candidate sets that is not multiple-closed over K minus $D_m(B_n)$, $|\mathcal{H}| \leq \left|\bigcup_{H\in\mathcal{H}} H\right|$.*

Proof. Suppose to the contrary that there exists some $\mathcal{H} \subseteq \mathcal{F}$ such that $|\mathcal{H}| > \left|\bigcup_{H\in\mathcal{H}} H\right|$. But $|\bigcup_{G\in\mathsf{mcl}_{K\setminus D_m(B_n)}(\mathcal{H})} G| = |\bigcup_{H\in\mathcal{H}} H|$; also $|\mathsf{mcl}_{K\setminus D_m(B_n)}(\mathcal{H})| \geq |\mathcal{H}|$. Thus, $|\mathsf{mcl}_{K\setminus D_m(B_n)}(\mathcal{H})| > |\bigcup_{G\in\mathsf{mcl}_{K\setminus D_m(B_n)}(\mathcal{H})} G|$, contradicting Lemma 14.

Theorem 5. *The family $\mathcal{F}^{-D_m(B_n)}$ of all completing candidate sets with forbidden diagonal pairs $D_m(B_n)$ satisfies the marriage condition.*

Proof. Any $\mathcal{H} \subseteq \mathcal{F}$ is either multiple-closed over $K \setminus D_m(B_n)$, or it isn't. Hence, the result follows from Lemma 14 and Corollary 2. $\square$

We now have all the machinery to construct Bose resolutions.

Theorem 6. *For all* $n \equiv 3, 15 \pmod{18}$ *with* $n > 15$*, the averaging latin square* B *of order* n *admits a Bose resolution.*

Proof. There are two cases to treat.

Case 1: $5 \nmid n$. Let $\mathcal{F} = \bigcup_{k \in K}\{F_k\}$ be the family of all completing candidate sets. Each $F_k \in \mathcal{F}$ gives the set of indices of all possible completing diagonal pairs for C_k that could result from applying the standard method to handle C_k. By Theorem 5, $\mathcal{F}$ meets the marriage condition, and so by Theorem 4, there exists a transversal $\rho : \mathcal{F} \to J$ used to prevent a collision between two diagonal pairs $C_k, C_{k'}$, where

$$J = \{\beta \in [2, n-1] : \beta \text{ is even and } \beta \not\equiv 0 \,(\mathrm{mod}\ 3)\}.$$

Fix $k \in K$, and apply the standard method with parameter $\rho(F_k)/2$ to handle C_k (so that the completing diagonal pair for C_k is $C_{\rho(F_k)}$), giving us a partition $\pi_k = \{\mathcal{P}_{k,0}, \mathcal{P}_{k,1}, \mathcal{P}_{k,2}, \mathcal{P}_{k,3}\}$ of C_k into four PLSPCs, a partition $\pi_{\rho(F_k)}$ into four PLSPCs of some LSPC $\mathcal{L}_{\rho(F_k),0}$ of $C_{\rho(F_k)}$ of type (2) of Lemma 1, and an injective map $f_k : \pi_k \to \pi_{\rho(F_k)}$ such that (the disjoint union) $\mathcal{P}_{k,i} \cup f_k(\mathcal{P}_{k,i})$ is an LSPC for $i \in [0, 3]$. Next, apply Lemma 1 to partition $C_{\rho(F_k)}$ into three LSPCs $\mathcal{L}_{\rho(F_k),0}, \mathcal{L}_{\rho(F_k),1}$, and $\mathcal{L}_{\rho(F_k),2}$. The desired Bose resolution of B is

$$\bigcup_{k \in K}\left(\bigcup_{i \in [0,3]}\{\mathcal{P}_{k,i} \cup f_k(\mathcal{P}_{k,i})\} \cup \{\mathcal{L}_{\rho(F_k),1}, \mathcal{L}_{\rho(F_k),2}\}\right),$$

together with any partition of each diagonal pair with index in $J \setminus \mathrm{range}(\rho)$ into three LSPCs, which may be obtained via Lemma 1.

Case 2: $n = 15t$. Let $D_{15}(B)$ be the set of diagonal pair indices of the averaging latin square of order 15 in B_n. Put $U = \bigcup_{i \in \{2,4,\ldots,14\}} C_{it}(B)$, and apply Corollary 1 to handle U, giving us the first part of the Bose resolution $\mathcal{R}$ for B. Next, let $\mathcal{F}^{-D_{15}(B)} = \bigcup_{k \in K \setminus D_{15}(B)} F_k^{-D_{15}(B)}$ be the family of all completing candidate sets with forbidden diagonal pairs $D_{15}(B)$. Each $F_k^{-D_{15}(B)} \in \mathcal{F}^{-D_{15}(B)}$ gives the set of indices of all possible completing diagonal pairs for C_k, excluding the diagonal pairs indexed by $D_{15}(B)$, that could result from applying the standard method to handle C_k. By Theorem 5, $\mathcal{F}^{-D_{15}(B)}$ satisfies the marriage condition, and thus by Theorem 4, there exists a transversal $\rho' : \mathcal{F}^{-D_{15}(B)} \to J \setminus D_{15}(B)$. To obtain the second part of $\mathcal{R}$, similar to case 1, apply the standard method to handle each of the diagonal pairs indexed by $K \setminus D_{15}(B)$, using ρ' to prevent a collision between any two such diagonal pairs. Finally, to obtain the third part of $\mathcal{R}$, apply Lemma 1 to partition each diagonal pair with index in $J \setminus (D_{15}(B) \cup \mathrm{range}(\rho'))$ into three LSPCs. $\square$

3.4 Main Result

Suppose that $n \equiv 3 \pmod 6$, which by Theorem 2 is necessarily the only possible order of an averaging latin square with a Bose resolution. Applying Construction 1 when $n = 15$, Theorem 3 when $n \equiv 9 \pmod{18}$, and Theorem 6 otherwise, we obtain a Bose resolution to which we apply Theorem 1 to obtain a resolution of the corresponding Bose-averaging triple system, thus establishing our main result.

Theorem 7. *Every resolvable Bose triple system of order v has $v \equiv 9 \pmod{18}$ and the Bose-averaging triple system of order $v \equiv 9 \pmod{18}$ is resolvable.*

4 Concluding Remarks

When $n = 3p$, with $p > 5$ prime, every diagonal pair in $K = \{k \in [1,(n-1)/2] : k \equiv 0 \pmod 3\}$ is coprime with p, and thus the members of the family $\mathcal{F}$ of all completing candidate sets are pairwise equal, each having maximum size $n/3$. Hence, $\binom{n/3}{(n-3)/6} \cdot ((n-3)/6)!$ gives the number of distinct transversals of $\mathcal{F}$, and consequently the number of Bose resolutions of the averaging latin square of order n. Each Bose resolution produces a distinct (but perhaps isomorphic) resolution of the Bose-averaging triple system of order $9p$. This flexibility may prove useful in selecting a Kirkman triple system that has further desired properties.

Finally, it appears that the spectrum of equitable weakly 3-chromatic Kirkman triple systems has not been studied, and that the class of Bose-averaging triple systems is the largest such class of Kirkman triple systems known up to this point. It is plausible that this spectrum is in fact all $v > 3$ with $v \equiv 3 \pmod 6$.

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