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My Thoughts on Pythagorean Triples: Derivation and Numerical Analysis

The Pythagorean Triple Theorem

Define **P** = { (a,b,c) | $a^2 + b^2 = c^2$ forms a primitive Pythagorean triple }

- ⇨ Suppose $a \wedge b \ mod\ 2 = 0$ (a and b are $even$)
  - $\therefore c \ mod\ 2 = 0 \ \wedge a^2 + b^2 = c^2 \ has\ a\ common\ factor\ of\ 2$
  - $\therefore a \ \wedge b \ cannot\ be\ even, for\ they\ would\ have\ a\ common\ factor$
- ⇨ Suppose $a \wedge b \ mod\ 2 = 1$ (a and b are odd)
  - Define $a = 2x + 1 \ \wedge b = 2y + 1 \ \wedge c = 2z$
    - All odds can be written as an even +1 and all evens can be in for form of 2a.
  - $\therefore 4x^2 + 4x + 4y^2 + 4y + 2 = 4z^2$
  - $\therefore 2x^2 + 2x + 2y^2 + 2y + 1 = 2z^2$
  - $\therefore odd = even$
  - $contradition$
- ⇨ $\therefore Lemma\ 2.1(a, b, c) \in \mathrm{P} \ \longrightarrow a \vee b \ mod\ 2 = 1 \ \wedge \ (a \wedge b) mod 2 \neq 1$ (a or b must of a different parody)
- ⇨ $Define\ a^2 + b^2 = c^2, a\ mod\ 2 = 1 \wedge b\ mod\ 2 = \ 0 \ \wedge$ $a, b, c\ have\ no\ common\ factors$
- ⇨ $a^2 = c^2 - b^2 = (c - b)(c + b)$
  - $suppose\ d\ is\ a\ common\ factor\ of\ (c - b) \wedge (c + b)$
  - $d \mid c + b + c - b \ \wedge \ \ d \mid c + b - c + b$ (principle of division)
  - $d \mid 2c \ \wedge \ d \mid 2c$ (simplification)
  - $Contraidition$
  - $\therefore \ c \ \wedge b\ have\ no\ common\ factors$
- ⇨ $\therefore (c - b) \wedge (c + b)\ have\ no\ common\ factors$
- ⇨ $define\ c + b = s^2 \wedge c - b = t^2, s > t \geq 1$
  - $c = t^2 + b \ \wedge c = s^2 - b \ \wedge \ b = c - s^2 \ \wedge \ b = -c + \ s^2$
- ⇨ $c = \frac{(t^2 + s^2)}{2} \ \wedge \ b = \frac{(s^2 - t^2)}{2}$
- ⇨ $a = \sqrt{(c - b)(c + b)} = st$
- ⇨ $(st)^2 + \left(\frac{(s^2 - t^2)}{2}\right)^2 = s^2t^2 + \frac{(s^4 - 2s^2t^2 + t^4)}{4} = \left(\frac{(s^2 + t^2)}{2}\right)^2$ (the results are a tautology)
- ⇨ $\frac{(s^2 - t^2)}{2} \wedge \ (st)\ \ are\ of\ a\ different\ parody$
- ⇨ This means $\gcd(a, b) = \ 1.$
- ⇨ $\therefore a, b, c\ share\ no\ common\ factors$

$$a = (st), b = \frac{(s^2 - \ t^2)}{2}, c = \frac{(s^2 + \ t^2)}{2}$$

Q.E.D.

2.1) The properties of $a \wedge b$ in Pythagorean triples

Proof:

$\forall (a,b,c) \in \mathrm{P}, \mathrm{a} \oplus b \; mod \; 3 = 0$

- ⇨ $a \wedge b \; mod \; 3 \neq 0$ (lemma 2.1)
- ⇨ Suppose $a \; mod \; 3 \neq 0 \wedge b \; mod \; 2 \neq 0$
  - $a = 3k + 1 \vee 3k + 2 \wedge b = 3j + 1 \vee 3j + 2$
  - $a^2 = 9k^2 + 6k + 1 \vee 9k^2 + 12k + 4 \wedge b^2 = 9j^2 + 6j + 1 \vee 9j^2 + 12j + 4$
  - $\therefore a^2 mod \; 3 = 1 \wedge b^2 mod \; 3 = 1$
  - $\therefore (a^2 + b^2) \; mod \; 3 = 2$
  - $contradiction$ (as been shown, all square numbers are either multiples of 3 or have a remainder of one.)
    - $let \; x = \mathrm{N}$
      - $x = 3k \vee 3k + 1 \vee 3k + 2$
      - $x^2 = 9k^2 \vee 9k^2 + 6k + 1 \vee 9k^2 \vee 9k^2 + 12k + 4$
      - $x^2 \; mod \; 3 = 0,1$
  - $\therefore a \oplus b \; mod \; 3 = 0$

$\therefore a \; or \; b \; in \; a \; triple \; must \; be \; a \; multiple \; of \; three$

Q.E.D.

Proof:

$\forall (a,b,c) \in \mathrm{P}, \mathrm{a} \oplus b \oplus c \; mod \; 5 = 0$

- ⇨ $a \wedge b \; mod \; 5 \neq 0$ (lemma 2.1)
- ⇨ Suppose $a \oplus b \oplus c \; mod \; 5 \neq 0$
  - $Let \; x = a \oplus b \oplus c$
  - $x = 5k \vee 5k + 1 \vee 5k + 2 \vee 5k + 3 \vee 5k + 4$
  - $x^2 = 25k^2 \vee 25k^2 + 10k + 1 \vee 25k^2 + 20k + 4 \vee 25k^2 + 30k + 9 \vee 25k^2 + 40k + 16$
  - $\therefore x^2 \; mod \; 5 = 0, \pm 1$
  - $\therefore (a^2 + b^2) \; mod \; 5 = 0,1,4,2$
  - $(a^2 + b^2) \; mod \; 5 = 1,2 \longrightarrow$ Contradition!
  - $\therefore (a^2 + b^2) \; mod \; 5 = 0, \pm 1$ (definition of squared number)
- ⇨ $\therefore a \oplus b \oplus c \; mod \; 5 \neq 0$

$\therefore a \; or \; b \; or \; c \; in \; a \; triple \; must \; be \; a \; multiple \; of \; five$

Q.E.D.

Note that these proofs are relatively similar, and I believe another similar one for the values of a,b,c to be a multiple of 4 to be another proof of near identical structure.

The Fundamental Theorem of Arithmetic

Define:

$\mathrm{K} = \{ p \in \mathrm{Z} \mid p \; is \; prime \} \wedge \mathrm{A} = \{ x \in \mathrm{Z} \mid x \text{ can be written as a product of primes} \}$

$Let \; x \in \mathrm{Z} \wedge$ x is the smallest number $\notin \mathrm{A}$
$x = mn, 1 < m, x < n$
$m \wedge n$ must be a product of primes as x is the smallest number that cannot be written as such a product
$\therefore contradition, m \wedge n \in A \rightarrow x \in A$

The Fundamental Theorem of Arithmetic:

$$\therefore n = (-1)^{\varepsilon(n)} \prod_{p} p^{a(p)}, n \in \mathrm{Z}$$

where $n > 0 \rightarrow \varepsilon(n) = 0 \wedge n < 0 \rightarrow \varepsilon(n) = 1$

Q.E.D.

2.3ab) On the analyzed patters of Pythagorean triples

What odd number can be a in a primitive Pythagorean triple?

Any number can be the a value of a triple because:

$$a = (st), b = \frac{(\mathrm{s}^2 - \mathrm{t}^2)}{2}, c = \frac{(\mathrm{s}^2 + \mathrm{t}^2)}{2}$$
$$let\ t = 1\ \wedge\ s = a$$

Which even number b can appear in a primitive Pythagorean triple?

b is of the following form: $b = \frac{(\mathrm{s}^2 - \mathrm{t}^2)}{2}$

$$\because s \wedge t\ mod\ 2 = 1, let\ s = 2n + 1 \wedge t = 2m + 1$$
$$\therefore b = \frac{(4n^2 + 4n + 1 - 4m^2 - 4m - 1)}{2} = 2n^2 + 2n - 2m^2 - 2m$$
$$= 2n(n + 1) - 2m(m + 1)$$
$$n(n + 1)\ mod\ 2 = 0 \wedge m(m + 1)\ mod\ 2 = 0 \because the\ parity\ of\ x \wedge x + 1\ are\ different$$
$$\therefore\ 2n(n + 1) \wedge\ 2m(m + 1)\ mod\ 4 = 0$$
$$\therefore x\ \in N, x\ \equiv 0\ mod\ 4\ can\ be\ b$$

2.3c) On the value of C

One justly can raise the following imperative question:
What number c can appear in a primitive Pythagorean triple?

Let us begin to answer that question:

$\text{note, as proven: } c \text{ } mod \text{ } 2 = 1$

What if the sum of squares was a set closed under multiplication? As in:

$$\left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right) = \left(\sum_{i=1}^{n} c_i^2\right)$$

Furthermore, let us not limit the values of a, b, c to any small set, but rather state $a, b, c \in C$ for reasons that will be advantageous later.

This would indeed would obviously yield many benefits as to the patters and nature of c in a triple.

Let us set out to prove that conjecture:

This is merely a special case of Langarange's, which is as follows.

$$\left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right) - \left(\sum_{k=1}^{n} a_k b_k\right)^2 = \sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\left(a_{ib_j} - a_{jb_i}\right)^2 = \frac{1}{2}\sum_{j=1}^{n-1}\sum_{j=1, j\neq i}^{n}\left(a_{ib_j} - a_{jb_i}\right)^2$$

or in a form most useful to our case:

$$\left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right) = \sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\left(a_{ib_j} - a_{jb_i}\right)^2 + \left(\sum_{k=1}^{n} a_k b_k\right)^2$$

$We\ can\ confirm\ this\ like\ so$:

$Let\ us\ expand\ the\ \left(a_{ib_j} - a_{jb_i}\right)^2\ term$

$a_i^2 b_j^2 + \ + a_j^2 b_i^2 - 2a_i b_i a_j b_j$

$$\therefore\ the\ right\ side\ becomes\ \frac{1}{2}\sum_{j=1}^{n-1}\sum_{j=1}^{n} a_i^2 b_j^2 + \frac{1}{2}\sum_{j=1}^{n-1}\sum_{j=1}^{n} a_j^2 b_i^2 - \sum_{j=1}^{n-1}\sum_{j=1}^{n} a_i b_i a_j b_j$$

also let us note the following property of sums

$$\left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right) = \left(\sum_{i=1}^{n}\sum_{j=1}^{n} a_i^2\, a_j^2\right)$$

the right side can now be rewritten as:

$$\left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right) - \left(\sum_{k=1}^{n} a_k b_k\right)^2$$

We have attained that the right side does indeed equal the left. Langarange's theorem is indeed true. A more rigorous proof can be done as well.

Formal Proof of Langarange's Theorem:

$$\left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right) = \left(\sum_{i=1}^{n}\sum_{j=1}^{n} a_i^2\, a_j^2\right) = \sum_{k=1}^{n} a_k^2 b_k^2 + \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} a_i^2\, a_j^2 + \sum_{j=1}^{n-1}\sum_{i=j+1}^{n} a_i^2\, a_j^2$$

(the product of the sum of squares is equal to the sum of the sum of the product of those squares (imagine this like a two dimensional array, a grid, or a matrix) which is equal to the sum of those products along a diagonal plus the jagged two dimensional array or triangular grids left over)

$$\left(\sum_{k=1}^{n} a_k b_k\right)^2 = \sum_{k=1}^{n} a_k^2 b_k^2 + 2\sum_{j=1}^{n-1}\sum_{i=j+1}^{n} a_i a_j b_i b_j$$

the sum of a symmetric square is equal to the sum of its diagonal and the two jagged arrays or triangular grids left over.

Therefore, when added together, we achieve langarange's theorem.

$$\sum_{k=1}^{n} a_k^2 b_k^2 + \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} a_i^2\, a_j^2 + \sum_{j=1}^{n-1}\sum_{i=j+1}^{n} a_i^2\, a_j^2 - \sum_{k=1}^{n} a_k^2 b_k^2 + 2\sum_{j=1}^{n-1}\sum_{i=j+1}^{n} a_i a_j b_i b_j =$$
$$\sum_{j=1}^{n-1}\sum_{j=1}^{n} a_i^2 b_j^2 + \sum_{j=1}^{n-1}\sum_{j=1}^{n} a_j^2 b_i^2 - 2\sum_{j=1}^{n-1}\sum_{j=1}^{n} a_i b_i a_j b_j$$

Q.E.D.

with this said, we have come across a very important fact. This can be shown by an example as follows:

$$(a^2 + b^2)(c^2 + d^2) = (ac - bd)^2 + (ad + bc)^2 = (ac + bd)^2 + (ad - bc)^2$$

In other words, the set of all sums of two squares is closed under multiplication. Because of this fact, we know that all triple c^2 values will be of this form. Let us now return to the idea we discussed prior, what number can really be c?

As proven, the set of the sums of squares is indeed closed under multiplication for any sum of squares of the form $x^2 + y^2, x, y \in \mathfrak{C} \therefore 1$ is the identity and this set under this operation forms a loop under multiplication. We now know that all c values of a primitive Pythagorean triple can be formed as a product of the sums of squares. Therefore, all c values must be able to be written as the product of the sum of squares, and therefore must be either prime themselves of the form $x^2 + y^2$ or composite numbers of the form:

$$\prod_{i=1}^{n} a_i^2 + b_i^2$$

One can then note that all therefore all c values must be of that form according to the unique factorization of the naturals as proven with the fundamental theorem of arithmetic:

$$\prod_{1}^{n} p_n, \;\; p_n \; is\; a\; prime\; of\; the\; form\; x^2 + y^2$$

One then raises the question of what primes are of the form $x^2 + y^2$? It becomes clear by looking at the first several examples, 5, 13, 17, that these numbers are congruent to $1 \; mod \; 4$, but why?

Let us do a proof of existence. This proof will explain several concepts needed for itself along the way.

Proof:

$$p = a^2 + b^2 \;\; \rightarrow p \;\equiv 1 \; mod \; 4$$

Let p be prime $\equiv 1 \, mod \, 4$.
Suppose $p = x^2 + y^2$

This indeed means that this is the same as writing $p = (x + iy)(x - iy)$

Let us not that this form of factorization is known as factorization over the Gaussian integers, or complex numbers of the form:

$g = a + bi.$

The set of Gaussian integers will be referred to as $Z_{[i]}$

It is worth saying that it is well known that $Z_{[i]}$ form a unique factorization domain, a commutative ring where every non-zero non-unit element can be written as a product of prime elements uniquely. These properties of this ring will not be proven but rather just explained.

Let us note the following order of these rings:

Commutative rings ⊃ integral domains ⊃ integrally closed domains ⊃ unique factorization domains ⊃ principal ideal domains ⊃ Euclidean domains ⊃ fields ⊃ finite fields

Thus, some Gaussian integers act as units (e.g. 1, -1, i, and -i), some as primes (e.g. 1 + i), and the rest composite Gaussian integers, that can be written as a product of primes in that ring.

The question of how prime $p \in Z$ factors in $Z_{[i]}$ is indeed a difficult one.

Note the following:

$(p-1)! = -1 \, mod \, p$
$\because expanding \, (p-1)! \, yields \, 1 * 2 * 3 * \ldots * (p-1).$
Notice that all the numbers aside from $1 \wedge (p-1) \, have \, a \, modular \, inverse \, in \, (p-1)!$
$\therefore (p-1)! \, mod \, p = (p-1) mod \, p = -1 \, mod \, p$

Thus, observe that:

$(p-1)! \, mod \, p = 1 * 2 * 3 * \ldots * (4n) \; mod \, p$
$= (1 * 2 * \ldots * 2n)\big((2n+1) * \ldots * (4n)\big) \, mod \, p$
$= (1 * 2 * \ldots * 2n)\big((-2n) * \ldots * (-1)\big) \, mod \, p$

This is due to the facts that,

$4n \equiv\ -1\ mod\ p$
$2n+1\ \equiv\ -2n\ mod\ p$
$2n+2\ \equiv\ -2n+1\ mod\ p$

Thus,

$(1*2*\dots*2n)\big((-2n)*\dots*(-1)\big)\ mod\ p\ \equiv\ (1*2*\dots*2n)^2(-2)^2_n$
$\equiv\ (1*2*\dots*2n)^2$
$\equiv\ -1\ mod\ p$

suppose $m=(2n)!, M^2\ \equiv\ -1\ mod\ p$
$\therefore p \mid m^2+1$

$m^2+1\ factors\ in\ \ Z_{[i]}\ as\ (m-i)(m+i)$
$\therefore p$ is not prime in $Z_{[i]}$
$\therefore p=(a+bi)z,(a+bi),z\in Z_{[i]}\ (composite\ in\ Z_{[i]})$
$\therefore p=(a-bi)\acute{z}$
$\therefore p^2=norm(a+bi)norm(z)$
$p^2\ \exists! =\ p*p$

$$\therefore p=a^2+b^2$$

$\mathscr{Q.E.D.}$

And therefore, because the set of the sum of squares under multiplication is closed, all

$$c^2\ \in P[c^2]=\prod_p p_n,\ p_n\equiv 1\ mod\ 4$$

$\mathscr{Q.E.D.}$

2.4) The nature of C Stuck Triples

P_c $= \{ \left((a1, b1, c1), (a2, b2, c2)\right) \mid {a_1}^2 + {b_1}^2 = {a_2}^2 + {b_2}^2 = {c_1}^2 = {c_2}^2 \}$

As shown,

$$c^2 = \prod_{1}^{n} p_n, \ \ p_n \ is\ the\ nth\ prime\,.\ This\ prime\ must\ be\ congruent\ to\ 1\ mod\ 4$$

Let us use the factorization of Gaussian integers and the facts derived above to come to a conclusion. For example,

$(a + bi)(a - bi) = a^2 + b^2$
$let\ \ c = pqr, p, q, r\ are\ prime$
$p = (a + bi)(a - bi)$
$q = (c + di)(c - di)$
$r = (e + fi)(e - fi)$
$(a + bi)(c + di)(e + fi) = g + hi$
$(a + bi)(c + di)(e - fi) = j + ki$
$(a + bi)(c + di)(e + fi)(a - bi)(c - di)(e - fi) = (g + hi)(g - hi) = g^2 + h^2 = pqr$
$(a + bi)(c + di)(e + fi)(a - bi)(c - di)(e - fi) = (j + ki)(j - ki) = j^2 + ki^2 = pqr$

$$\therefore pqr = g^2 + h^2 = j^2 + k^2$$

This both affirms and employs the theorems described earlier. Here is an example given our findings.

$let\ c^2 = 5 * 13 * 17 = 1105 = (2 - i)(2 + i)(3 + 2i)(3 - 2i)(4 + i)(4 - i)$

now, as has been shown: $(2 + i)(3 + 2i)(4 + i) = 9 + 32i$

Let us test:
$9^2 + 32^2 = 1105$
$33^2 + 4^2 = 1105$
$23^2 + 24^2 = 1105$

We have found a c stuck triple. Not only have we found a c stuck triple, we have found 3 triples with the same c value.

I believe that to be rather remarkable. Now, is the time to take the ideals we have combined and discovered, and have a computer process them with precision and speed. Our conclusion holds true for all all values of the form $x^2 = y^2 + z^2$, not only primes of this form. Therefore, for the purposes of computation let us not discriminate the c^2 value based on its composition of primes, as we know it can be written as a product of primes $\equiv 1\ mod\ 4$. Note that this does indeed we will miss some c stuck triples, but we will be generating however many we wish nonetheless.

Remember,

$$(a^2+b^2)(c^2+d^2) = (ac-bd)^2+(ad+bc)^2 = (ac+bd)^2+(ad-bc)^2$$

A program to implement this fact is now trivial to write in Java:

```java
public class CStuckTriples1 extends ConsoleProgram {

    public void run() { printManyCStuckSTriples(1000); }

    public double[][] getCStuckTriple(double[][] validTriples) {
        double[] a = validTriples[0]; double[] b = validTriples[1]; double[] c = validTriples[2];

        double newC2 = c[0]*c[0]*c[1]*c[1]; double newA1 = a[0]*a[1] + b[0]*b[1]; double newB1 =
        a[0]*b[1] - a[1]*b[0]; double newA2 = a[0]*a[1] - b[0]*b[1]; double newB2 = a[0]*b[1] +
        a[1]*b[0];

        double[][] triples = new double[3][2];
        triples[0][0] = Math.abs(newA1); triples[0][1] = Math.abs(newA2);
        triples[1][0] = Math.abs(newB1);triples[1][1] = Math.abs(newB2);
        triples[2][0] = triples[2][1] = Math.sqrt(newC2);
        return triples;
    }

    public void printManyCStuckSTriples(int timeInMillisecondsDesired) {
        double s = 3; double t = 1; long startTime = System.currentTimeMillis();
        while((System.currentTimeMillis()-startTime)<timeInMillisecondsDesired) {
            ArrayList<Double> a = new ArrayList<Double>();
            ArrayList<Double> b = new ArrayList<Double>();
            ArrayList<Double> c = new ArrayList<Double>();
            for(int i=0;i<2;i++) {
                double[] triple = getTriples(s,t);
                a.add(triple[0]);
                b.add(triple[1]);
                c.add(triple[2]);
                s+=2;
            }
            Double[] aValues = new Double[a.size()];
            Double[] bValues = new Double[b.size()];
            Double[] cValues = new Double[c.size()];
            a.toArray(aValues);
            b.toArray(bValues);
            c.toArray(cValues);
            double[] av = new double[a.size()];
            double[] bv = new double[b.size()];
            double[] cv = new double[c.size()];
            for(int i=0;i<aValues.length;i++) {
                av[i] = aValues[i];
                bv[i] = bValues[i];
                cv[i] = cValues[i];
            }
            double[][] result = {av,bv,cv};
            double[][] cStuck = getCStuckTriple(result);
            println("a values: " + cStuck[0][0] + "," + cStuck[0][1] + " " +
                "b values: " + cStuck[1][0] + "," + cStuck[1][1] + " " +
                "c value: " + cStuck[2][0]);
        }
    }

    public double[] getTriples(double s, double t) {
        if (t >= 1 && s > t) { double[] a = {s*t, (s*s - t*t) / 2, (s*s + t*t) / 2};
            return a;
        } else { double[] a = {-1,-1,-1};
            return a;
        }
    }
}
```

The resulting output would be as follows:

```
a values: 63.0,33.0 b values: 16.0,56.0 c value: 65.0
a values: 1023.0,897.0 b values: 64.0,496.0 c value: 1025.0
a values: 5183.0,4897.0 b values: 144.0,1704.0 c value: 5185.0
a values: 16383.0,15873.0 b values: 256.0,4064.0 c value: 16385.0
a values: 39999.0,39201.0 b values: 400.0,7960.0 c value: 40001.0
a values: 82943.0,81793.0 b values: 576.0,13776.0 c value: 82945.0
a values: 153663.0,152097.0 b values: 784.0,21896.0 c value: 153665.0
a values: 262143.0,260097.0 b values: 1024.0,32704.0 c value: 262145.0
a values: 419903.0,417313.0 b values: 1296.0,46584.0 c value: 419905.0
a values: 639999.0,636801.0 b values: 1600.0,63920.0 c value: 640001.0
a values: 937023.0,933153.0 b values: 1936.0,85096.0 c value: 937025.0
a values: 1327103.0,1322497.0 b values: 2304.0,110496.0 c value: 1327105.0
a values: 1827903.0,1822497.0 b values: 2704.0,140504.0 c value: 1827905.0
a values: 2458623.0,2452353.0 b values: 3136.0,175504.0 c value: 2458625.0
a values: 3239999.0,3232801.0 b values: 3600.0,215880.0 c value: 3240001.0
a values: 4194303.0,4186113.0 b values: 4096.0,262016.0 c value: 4194305.0
a values: 5345343.0,5336097.0 b values: 4624.0,314296.0 c value: 5345345.0
a values: 6718463.0,6708097.0 b values: 5184.0,373104.0 c value: 6718465.0
a values: 8340543.0,8328993.0 b values: 5776.0,438824.0 c value: 8340545.0
a values: 1.0239999E7,1.0227201E7 b values: 6400.0,511840.0 c value: 1.0240001E7
a values: 1.2446783E7,1.2432673E7 b values: 7056.0,592536.0 c value: 1.2446785E7
a values: 1.4992383E7,1.4976897E7 b values: 7744.0,681296.0 c value: 1.4992385E7
a values: 1.7909823E7,1.7892897E7 b values: 8464.0,778504.0 c value: 1.7909825E7
a values: 2.1233663E7,2.1215233E7 b values: 9216.0,884544.0 c value: 2.1233665E7
a values: 2.4999999E7,2.4980001E7 b values: 10000.0,999800.0 c value: 2.5000001E7
a values: 2.9246463E7,2.9224833E7 b values: 10816.0,1124656.0 c value: 2.9246465E7
a values: 3.4012223E7,3.3988897E7 b values: 11664.0,1259496.0 c value: 3.4012225E7
a values: 3.9337983E7,3.9312897E7 b values: 12544.0,1404704.0 c value: 3.9337985E7
a values: 4.5265983E7,4.5239073E7 b values: 13456.0,1560664.0 c value: 4.5265985E7
a values: 5.1839999E7,5.1811201E7 b values: 14400.0,1727760.0 c value: 5.1840001E7
a values: 5.9105343E7,5.9074593E7 b values: 15376.0,1906376.0 c value: 5.9105345E7
a values: 6.7108863E7,6.7076097E7 b values: 16384.0,2096896.0 c value: 6.7108865E7
a values: 7.5898943E7,7.5864097E7 b values: 17424.0,2299704.0 c value: 7.5898945E7
a values: 8.5525503E7,8.5488513E7 b values: 18496.0,2515184.0 c value: 8.5525505E7
a values: 9.6039999E7,9.6000801E7 b values: 19600.0,2743720.0 c value: 9.6040001E7
a values: 1.07495423E8,1.07453953E8 b values: 20736.0,2985696.0 c value: 1.07495425E8
a values: 1.19946303E8,1.19902497E8 b values: 21904.0,3241496.0 c value: 1.19946305E8
a values: 1.33448703E8,1.33402497E8 b values: 23104.0,3511504.0 c value: 1.33448705E8
a values: 1.48060223E8,1.48011553E8 b values: 24336.0,3796104.0 c value: 1.48060225E8
a values: 1.63839999E8,1.63788801E8 b values: 25600.0,4095680.0 c value: 1.63840001E8
a values: 1.80848703E8,1.80794913E8 b values: 26896.0,4410616.0 c value: 1.80848705E8
a values: 1.99148543E8,1.99092097E8 b values: 28224.0,4741296.0 c value: 1.99148545E8
a values: 2.18803263E8,2.18744097E8 b values: 29584.0,5088104.0 c value: 2.18803265E8
a values: 2.39878143E8,2.39816193E8 b values: 30976.0,5451424.0 c value: 2.39878145E8
a values: 2.62439999E8,2.62375201E8 b values: 32400.0,5831640.0 c value: 2.62440001E8
a values: 2.86557183E8,2.86489473E8 b values: 33856.0,6229136.0 c value: 2.86557185E8
a values: 3.12299583E8,3.12228897E8 b values: 35344.0,6644296.0 c value: 3.12299585E8
a values: 3.39738623E8,3.39664897E8 b values: 36864.0,7077504.0 c value: 3.39738625E8
a values: 3.68947263E8,3.68870433E8 b values: 38416.0,7529144.0 c value: 3.68947265E8
a values: 3.99999999E8,3.99920001E8 b values: 40000.0,7999600.0 c value: 4.00000001E8
a values: 4.32972863E8,4.32889633E8 b values: 41616.0,8489256.0 c value: 4.32972865E8
a values: 4.67943423E8,4.67856897E8 b values: 43264.0,8998496.0 c value: 4.67943425E8
a values: 5.04990783E8,5.04900897E8 b values: 44944.0,9527704.0 c value: 5.04990785E8
a values: 5.44195583E8,5.44102273E8 b values: 46656.0,1.0077264E7 c value: 5.44195585E8
a values: 5.85639999E8,5.85543201E8 b values: 48400.0,1.064756E7 c value: 5.85640001E8
a values: 6.29407743E8,6.29307393E8 b values: 50176.0,1.1238976E7 c value: 6.29407745E8
a values: 6.75584063E8,6.75480097E8 b values: 51984.0,1.1851896E7 c value: 6.75584065E8
a values: 7.24255743E8,7.24148097E8 b values: 53824.0,1.2486704E7 c value: 7.24255745E8
a values: 7.75511103E8,7.75399713E8 b values: 55696.0,1.3143784E7 c value: 7.75511105E8
a values: 8.29439999E8,8.29324801E8 b values: 57600.0,1.382352E7 c value: 8.29440001E8
a values: 8.86133823E8,8.86014753E8 b values: 59536.0,1.4526296E7 c value: 8.86133825E8
a values: 9.45685503E8,9.45562497E8 b values: 61504.0,1.5252496E7 c value: 9.45685505E8
a values: 1.008189503E9,1.008062497E9 b values: 63504.0,1.6002504E7 c value: 1.008189505E9
```

2.5) The properties of b in Pythagorean triples and their relation to Triangular Numbers

Let us note the first several Pythagorean triples. 0, 1, 3, 6, 10, 15, 21, 28, 36, 45, 55, 66, 78, 91, 105, 120, 136, 153, 171, 190, 210, 231.

Let us note some patterns:

$T_5 = 15$

∴ (11, 60, 61) has a b value that matches 4 T_5

$T_6 = 21$

∴ (13, 84, 85) has a b value that matches $4T_6$

$T_7 = 28$

∴ (15, 112, 113) has a b value that matches $4T_7$

Let us recall the Pythagorean Triple Theorem:

$$a = (st), b = \frac{(s^2 - t^2)}{2}, c = \frac{(s^2 + t^2)}{2}$$

let t = 1
let s $= 2n + 1$, as $s \bmod 2 = 1$

$$\frac{(s^2 - t^2)}{2} = \frac{(2n+1)^2 - 1}{2} = \frac{4n^2 + 4n}{2} = 2n^2 + 2n = 4\left(\frac{n^2 + n}{2}\right) = 4T_n$$

$$\therefore a = 2n + 1, b = 4T_n, c = \frac{(2n+1)^2 + 1}{2}$$

Q.E.D.

The following program written in Java employs the methods derived above to numerically generate these "FourBTriangularTriples" that are some $(a, b, c) \in \mathrm{P},\ \mathrm{b} = 4T_n$

The output of the run method in this class is as follows. All be values are of the form $4T_n$:

```java
import acm.program.*;

public class FourBTriangularTriples extends ConsoleProgram
{

    public void run()
    {
        long i = 1;
        long startTime = System.currentTimeMillis();
        while((System.currentTimeMillis()-startTime)<1000) {
            long[] triple = generateFourBTriangularTriple(i);
            println("a value: " + triple[0] + " " + "b value: " +
            triple[1] + " " + "c value: " + triple[2]);
            i++;
        }
    }

    public long[] generateFourBTriangularTriple(long n) {
        long a = 2*n+1;
        long b = 4*generateTriangele(n);
        long c = ((2*n+1)*(2*n+1) + 1)/2;
        return new long[]{a,b,c};
    }

    public long generateTriangele(long n) {
        return (n*n + n)/2;
    }


}
```

Below is an example output of the function:

a value: 3 b value: 4 c value: 5
a value: 5 b value: 12 c value: 13
a value: 7 b value: 24 c value: 25
a value: 9 b value: 40 c value: 41
a value: 11 b value: 60 c value: 61
a value: 13 b value: 84 c value: 85
a value: 15 b value: 112 c value: 113
a value: 17 b value: 144 c value: 145
a value: 19 b value: 180 c value: 181
a value: 21 b value: 220 c value: 221
a value: 23 b value: 264 c value: 265
a value: 25 b value: 312 c value: 313
a value: 27 b value: 364 c value: 365
a value: 29 b value: 420 c value: 421
a value: 31 b value: 480 c value: 481
a value: 33 b value: 544 c value: 545
a value: 35 b value: 612 c value: 613
a value: 37 b value: 684 c value: 685
a value: 39 b value: 760 c value: 761
a value: 41 b value: 840 c value: 841
a value: 43 b value: 924 c value: 925
a value: 45 b value: 1012 c value: 1013
a value: 47 b value: 1104 c value: 1105
a value: 49 b value: 1200 c value: 1201
a value: 51 b value: 1300 c value: 1301
a value: 53 b value: 1404 c value: 1405
a value: 55 b value: 1512 c value: 1513
a value: 57 b value: 1624 c value: 1625
a value: 59 b value: 1740 c value: 1741
a value: 61 b value: 1860 c value: 1861
a value: 63 b value: 1984 c value: 1985
a value: 65 b value: 2112 c value: 2113
a value: 67 b value: 2244 c value: 2245
a value: 69 b value: 2380 c value: 2381
a value: 71 b value: 2520 c value: 2521
a value: 73 b value: 2664 c value: 2665
a value: 75 b value: 2812 c value: 2813
a value: 77 b value: 2964 c value: 2965
a value: 79 b value: 3120 c value: 3121
a value: 81 b value: 3280 c value: 3281
a value: 83 b value: 3444 c value: 3445
a value: 85 b value: 3612 c value: 3613
a value: 87 b value: 3784 c value: 3785
a value: 89 b value: 3960 c value: 3961
a value: 91 b value: 4140 c value: 4141
a value: 93 b value: 4324 c value: 4325
a value: 95 b value: 4512 c value: 4513
a value: 97 b value: 4704 c value: 4705
a value: 99 b value: 4900 c value: 4901
a value: 101 b value: 5100 c value: 5101
a value: 103 b value: 5304 c value: 5305
a value: 105 b value: 5512 c value: 5513
a value: 107 b value: 5724 c value: 5725
a value: 109 b value: 5940 c value: 5941
a value: 111 b value: 6160 c value: 6161
a value: 113 b value: 6384 c value: 6385
a value: 115 b value: 6612 c value: 6613
a value: 117 b value: 6844 c value: 6845
a value: 119 b value: 7080 c value: 7081
a value: 121 b value: 7320 c value: 7321
a value: 123 b value: 7564 c value: 7565
a value: 125 b value: 7812 c value: 7813
a value: 127 b value: 8064 c value: 8065
a value: 129 b value: 8320 c value: 8321
a value: 131 b value: 8580 c value: 8581
a value: 133 b value: 8844 c value: 8845
a value: 135 b value: 9112 c value: 9113
a value: 137 b value: 9384 c value: 9385
a value: 139 b value: 9660 c value: 9661
a value: 141 b value: 9940 c value: 9941
a value: 143 b value: 10224 c value: 10225
a value: 145 b value: 10512 c value: 10513
a value: 147 b value: 10804 c value: 10805
a value: 149 b value: 11100 c value: 11101
a value: 151 b value: 11400 c value: 11401
a value: 153 b value: 11704 c value: 11705
a value: 155 b value: 12012 c value: 12013
a value: 157 b value: 12324 c value: 12325
a value: 159 b value: 12640 c value: 12641
a value: 161 b value: 12960 c value: 12961
a value: 163 b value: 13284 c value: 13285
a value: 165 b value: 13612 c value: 13613
a value: 167 b value: 13944 c value: 13945
a value: 169 b value: 14280 c value: 14281
a value: 171 b value: 14620 c value: 14621
a value: 173 b value: 14964 c value: 14965
a value: 175 b value: 15312 c value: 15313
a value: 177 b value: 15664 c value: 15665
a value: 179 b value: 16020 c value: 16021
a value: 181 b value: 16380 c value: 16381
a value: 183 b value: 16744 c value: 16745
a value: 185 b value: 17112 c value: 17113
a value: 187 b value: 17484 c value: 17485
a value: 189 b value: 17860 c value: 17861
a value: 191 b value: 18240 c value: 18241
a value: 193 b value: 18624 c value: 18625
a value: 195 b value: 19012 c value: 19013
a value: 197 b value: 19404 c value: 19405
a value: 199 b value: 19800 c value: 19801
a value: 201 b value: 20200 c value: 20201
a value: 203 b value: 20604 c value: 20605
a value: 205 b value: 21012 c value: 21013
a value: 207 b value: 21424 c value: 21425
a value: 209 b value: 21840 c value: 21841
a value: 211 b value: 22260 c value: 22261
a value: 213 b value: 22684 c value: 22685
a value: 215 b value: 23112 c value: 23113
a value: 217 b value: 23544 c value: 23545
a value: 219 b value: 23980 c value: 23981
a value: 221 b value: 24420 c value: 24421
a value: 223 b value: 24864 c value: 24865
a value: 225 b value: 25312 c value: 25313
a value: 227 b value: 25764 c value: 25765
a value: 229 b value: 26220 c value: 26221
a value: 231 b value: 26680 c value: 26681
a value: 233 b value: 27144 c value: 27145
a value: 235 b value: 27612 c value: 27613
a value: 237 b value: 28084 c value: 28085
a value: 239 b value: 28560 c value: 28561
a value: 241 b value: 29040 c value: 29041
a value: 243 b value: 29524 c value: 29525
a value: 245 b value: 30012 c value: 30013
a value: 247 b value: 30504 c value: 30505
a value: 249 b value: 31000 c value: 31001
a value: 251 b value: 31500 c value: 31501
a value: 253 b value: 32004 c value: 32005
a value: 261 b value: 34060 c value: 34061
a value: 263 b value: 34584 c value: 34585
a value: 265 b value: 35112 c value: 35113
a value: 267 b value: 35644 c value: 35645
a value: 269 b value: 36180 c value: 36181
a value: 271 b value: 36720 c value: 36721
a value: 273 b value: 37264 c value: 37265
a value: 275 b value: 37812 c value: 37813
a value: 277 b value: 38364 c value: 38365
a value: 279 b value: 38920 c value: 38921
a value: 281 b value: 39480 c value: 39481
a value: 283 b value: 40044 c value: 40045
a value: 285 b value: 40612 c value: 40613
a value: 287 b value: 41184 c value: 41185
a value: 289 b value: 41760 c value: 41761
a value: 291 b value: 42340 c value: 42341
a value: 293 b value: 42924 c value: 42925
a value: 295 b value: 43512 c value: 43513
a value: 297 b value: 44104 c value: 44105
a value: 299 b value: 44700 c value: 44701
a value: 301 b value: 45300 c value: 45301
a value: 303 b value: 45904 c value: 45905
a value: 305 b value: 46512 c value: 46513
a value: 307 b value: 47124 c value: 47125
a value: 309 b value: 47740 c value: 47741
a value: 311 b value: 48360 c value: 48361
a value: 313 b value: 48984 c value: 48985
a value: 315 b value: 49612 c value: 49613
a value: 317 b value: 50244 c value: 50245
a value: 319 b value: 50880 c value: 50881
a value: 321 b value: 51520 c value: 51521
a value: 323 b value: 52164 c value: 52165
a value: 325 b value: 52812 c value: 52813
a value: 327 b value: 53464 c value: 53465
a value: 329 b value: 54120 c value: 54121
a value: 331 b value: 54780 c value: 54781
a value: 333 b value: 55444 c value: 55445
a value: 335 b value: 56112 c value: 56113
a value: 337 b value: 56784 c value: 56785
a value: 339 b value: 57460 c value: 57461
a value: 341 b value: 58140 c value: 58141
a value: 343 b value: 58824 c value: 58825
a value: 345 b value: 59512 c value: 59513
a value: 347 b value: 60204 c value: 60205
a value: 349 b value: 60900 c value: 60901
a value: 351 b value: 61600 c value: 61601
a value: 353 b value: 62304 c value: 62305
a value: 355 b value: 63012 c value: 63013

2.6) Triples of the form $\mathrm{a} + 2 = \mathrm{c}$

It is known from the Pythagorean Triples Theorem that $a = (st), c = \frac{(\mathrm{s}^2 + \mathrm{t}^2)}{2}$

- ⇨ For triples of the form $\mathrm{a} + 2 = \mathrm{c}$, $\frac{(\mathrm{s}^2 + \mathrm{t}^2)}{2} - st = 2$
  - ○ $\mathrm{s}^2 + \mathrm{t}^2 - 2st = 4$
  - ○ $(s - t)^2 = 4$
  - ○ $s - t = 2$, note that $s > t$ *according to the Pythagorean triple theorem*
- ⇨ $\therefore a = st = t(t + 2) = t^2 + 2t$
- ⇨ $\therefore b = \frac{(\mathrm{s}^2 - \mathrm{t}^2)}{2} = \frac{((\mathrm{t}+2)^2 - \mathrm{t}^2)}{2} = \frac{(\mathrm{t}^2 + 4t + 4 - \mathrm{t}^2)}{2} = 2t + 2$
- ⇨ $\therefore c = \frac{(\mathrm{s}^2 + \mathrm{t}^2)}{2} = \frac{((\mathrm{t}+2)^2 + \mathrm{t}^2)}{2} = \frac{(\mathrm{t}^2 + 4t + 4 + \mathrm{t}^2)}{2} = t^2 + 2t + 2$

Q.E.D.

The following program written in Java employs the methods derived above to numerically generate these "CIsAPlustTwoTriples" that are some $(a, b, c) \in \mathrm{P},\ \mathrm{c} = 2 + \mathrm{a}$

```java
import acm.program.*;

public class CIsAPlusTwoTriples extends ConsoleProgram
{

    public void run()
    {
        long i = 1;
        long startTime = System.currentTimeMillis();
        while((System.currentTimeMillis()-startTime)<1000) {
            long[] triple = generateFourBTriangularTriple(i);
            println("a value: " + triple[0] + " " + "b value: " +
            triple[1] + " " + "c value: " + triple[2]);
            i++;
        }
    }

    public long[] generateFourBTriangularTriple(long n) {
        long a = n*n + 2*n;
        long b = 2*n+2;
        long c = a+2;
        return new long[]{a,b,c};
    }

}
```

a value: 3 b value: 4 c value: 5
a value: 8 b value: 6 c value: 10
a value: 15 b value: 8 c value: 17
a value: 24 b value: 10 c value: 26
a value: 35 b value: 12 c value: 37
a value: 48 b value: 14 c value: 50
a value: 63 b value: 16 c value: 65
a value: 80 b value: 18 c value: 82
a value: 99 b value: 20 c value: 101
a value: 120 b value: 22 c value: 122
a value: 143 b value: 24 c value: 145
a value: 168 b value: 26 c value: 170
a value: 195 b value: 28 c value: 197
a value: 224 b value: 30 c value: 226
a value: 255 b value: 32 c value: 257
a value: 288 b value: 34 c value: 290
a value: 323 b value: 36 c value: 325
a value: 360 b value: 38 c value: 362
a value: 399 b value: 40 c value: 401
a value: 440 b value: 42 c value: 442
a value: 483 b value: 44 c value: 485
a value: 528 b value: 46 c value: 530
a value: 575 b value: 48 c value: 577
a value: 624 b value: 50 c value: 626
a value: 675 b value: 52 c value: 677
a value: 728 b value: 54 c value: 730
a value: 783 b value: 56 c value: 785
a value: 840 b value: 58 c value: 842
a value: 899 b value: 60 c value: 901
a value: 960 b value: 62 c value: 962
a value: 1023 b value: 64 c value: 1025
a value: 1088 b value: 66 c value: 1090
a value: 1155 b value: 68 c value: 1157
a value: 1224 b value: 70 c value: 1226
a value: 1295 b value: 72 c value: 1297
a value: 1368 b value: 74 c value: 1370
a value: 1443 b value: 76 c value: 1445
a value: 1520 b value: 78 c value: 1522
a value: 1599 b value: 80 c value: 1601
a value: 1680 b value: 82 c value: 1682
a value: 1763 b value: 84 c value: 1765
a value: 1848 b value: 86 c value: 1850
a value: 1935 b value: 88 c value: 1937
a value: 2024 b value: 90 c value: 2026
a value: 2115 b value: 92 c value: 2117
a value: 2208 b value: 94 c value: 2210
a value: 2303 b value: 96 c value: 2305
a value: 2400 b value: 98 c value: 2402
a value: 2499 b value: 100 c value: 2501
a value: 2600 b value: 102 c value: 2602
a value: 2703 b value: 104 c value: 2705
a value: 2808 b value: 106 c value: 2810
a value: 2915 b value: 108 c value: 2917
a value: 3024 b value: 110 c value: 3026
a value: 3135 b value: 112 c value: 3137
a value: 3248 b value: 114 c value: 3250
a value: 3363 b value: 116 c value: 3365
a value: 3480 b value: 118 c value: 3482
a value: 3599 b value: 120 c value: 3601
a value: 3720 b value: 122 c value: 3722
a value: 3843 b value: 124 c value: 3845
a value: 3968 b value: 126 c value: 3970
a value: 4095 b value: 128 c value: 4097
a value: 4224 b value: 130 c value: 4226
a value: 4355 b value: 132 c value: 4357
a value: 4488 b value: 134 c value: 4490
a value: 4623 b value: 136 c value: 4625
a value: 4760 b value: 138 c value: 4762
a value: 4899 b value: 140 c value: 4901
a value: 5040 b value: 142 c value: 5042
a value: 5183 b value: 144 c value: 5185
a value: 5328 b value: 146 c value: 5330
a value: 5475 b value: 148 c value: 5477
a value: 5624 b value: 150 c value: 5626
a value: 5775 b value: 152 c value: 5777
a value: 5928 b value: 154 c value: 5930
a value: 6083 b value: 156 c value: 6085
a value: 6240 b value: 158 c value: 6242
a value: 6399 b value: 160 c value: 6401
a value: 6560 b value: 162 c value: 6562
a value: 6723 b value: 164 c value: 6725
a value: 6888 b value: 166 c value: 6890
a value: 7055 b value: 168 c value: 7057
a value: 7224 b value: 170 c value: 7226
a value: 7395 b value: 172 c value: 7397
a value: 7568 b value: 174 c value: 7570
a value: 7743 b value: 176 c value: 7745
a value: 7920 b value: 178 c value: 7922
a value: 8099 b value: 180 c value: 8101
a value: 8280 b value: 182 c value: 8282
a value: 8463 b value: 184 c value: 8465
a value: 8648 b value: 186 c value: 8650
a value: 8835 b value: 188 c value: 8837
a value: 9024 b value: 190 c value: 9026
a value: 9215 b value: 192 c value: 9217
a value: 9408 b value: 194 c value: 9410
a value: 9603 b value: 196 c value: 9605
a value: 9800 b value: 198 c value: 9802
a value: 9999 b value: 200 c value: 10001
a value: 10200 b value: 202 c value: 10202
a value: 10403 b value: 204 c value: 10405
a value: 10608 b value: 206 c value: 10610
a value: 10815 b value: 208 c value: 10817
a value: 11024 b value: 210 c value: 11026
a value: 11235 b value: 212 c value: 11237
a value: 11448 b value: 214 c value: 11450
a value: 11663 b value: 216 c value: 11665
a value: 11880 b value: 218 c value: 11882
a value: 12099 b value: 220 c value: 12101
a value: 12320 b value: 222 c value: 12322
a value: 12543 b value: 224 c value: 12545
a value: 12768 b value: 226 c value: 12770
a value: 12995 b value: 228 c value: 12997
a value: 13224 b value: 230 c value: 13226
a value: 13455 b value: 232 c value: 13457
a value: 13688 b value: 234 c value: 13690
a value: 13923 b value: 236 c value: 13925
a value: 14160 b value: 238 c value: 14162
a value: 14399 b value: 240 c value: 14401
a value: 14640 b value: 242 c value: 14642
a value: 14883 b value: 244 c value: 14885
a value: 15128 b value: 246 c value: 15130
a value: 15375 b value: 248 c value: 15377
a value: 15624 b value: 250 c value: 15626
a value: 15875 b value: 252 c value: 15877
a value: 16128 b value: 254 c value: 16130
a value: 16383 b value: 256 c value: 16385
a value: 16640 b value: 258 c value: 16642
a value: 16899 b value: 260 c value: 16901
a value: 17160 b value: 262 c value: 17162
a value: 17423 b value: 264 c value: 17425
a value: 17688 b value: 266 c value: 17690
a value: 17955 b value: 268 c value: 17957
a value: 18224 b value: 270 c value: 18226
a value: 18495 b value: 272 c value: 18497
a value: 18768 b value: 274 c value: 18770
a value: 19043 b value: 276 c value: 19045
a value: 19320 b value: 278 c value: 19322
a value: 19599 b value: 280 c value: 19601
a value: 19880 b value: 282 c value: 19882
a value: 20163 b value: 284 c value: 20165
a value: 20448 b value: 286 c value: 20450
a value: 20735 b value: 288 c value: 20737
a value: 21024 b value: 290 c value: 21026
a value: 21315 b value: 292 c value: 21317
a value: 21608 b value: 294 c value: 21610
a value: 21903 b value: 296 c value: 21905
a value: 22200 b value: 298 c value: 22202
a value: 22499 b value: 300 c value: 22501
a value: 22800 b value: 302 c value: 22802
a value: 23103 b value: 304 c value: 23105
a value: 23408 b value: 306 c value: 23410
a value: 23715 b value: 308 c value: 23717
a value: 24024 b value: 310 c value: 24026
a value: 24335 b value: 312 c value: 24337
a value: 24648 b value: 314 c value: 24650
a value: 24963 b value: 316 c value: 24965
a value: 25280 b value: 318 c value: 25282
a value: 25599 b value: 320 c value: 25601
a value: 25920 b value: 322 c value: 25922
a value: 26243 b value: 324 c value: 26245
a value: 26568 b value: 326 c value: 26570
a value: 26895 b value: 328 c value: 26897
a value: 27224 b value: 330 c value: 27226
a value: 27555 b value: 332 c value: 27557
a value: 27888 b value: 334 c value: 27890
a value: 28223 b value: 336 c value: 28225
a value: 28560 b value: 338 c value: 28562
a value: 28899 b value: 340 c value: 28901
a value: 29240 b value: 342 c value: 29242
a value: 29583 b value: 344 c value: 29585
a value: 29928 b value: 346 c value: 29930
a value: 30275 b value: 348 c value: 30277
a value: 30624 b value: 350 c value: 30626
a value: 30975 b value: 352 c value: 30977
a value: 31328 b value: 354 c value: 31330
a value: 31683 b value: 356 c value: 31685
a value: 32040 b value: 358 c value: 32042
a value: 32399 b value: 360 c value: 32401
a value: 32760 b value: 362 c value: 32762
a value: 33123 b value: 364 c value: 33125
a value: 33488 b value: 366 c value: 33490
a value: 33855 b value: 368 c value: 33857
a value: 34224 b value: 370 c value: 34226
a value: 34595 b value: 372 c value: 34597
a value: 34968 b value: 374 c value: 34970
a value: 35343 b value: 376 c value: 35345
a value: 35720 b value: 378 c value: 35722
a value: 36099 b value: 380 c value: 36101
a value: 36480 b value: 382 c value: 36482
a value: 36863 b value: 384 c value: 36865
a value: 37248 b value: 386 c value: 37250
a value: 37635 b value: 388 c value: 37637
a value: 38024 b value: 390 c value: 38026
a value: 38415 b value: 392 c value: 38417
a value: 38808 b value: 394 c value: 38810
a value: 39203 b value: 396 c value: 39205
a value: 39600 b value: 398 c value: 39602

2.7) Perfect squares of the form $2c - 2a$

Let us note that according to the Pythagorean triples theorem, $a = (st), c = \frac{(s^2 + t^2)}{2}$

$2c - 2a = (s^2 + t^2) - 2st = (s - t)^2$
$\therefore 2c - 2a \textit{ is a perfect square}$

$\mathcal{Q.E.D.}$

2.8) The sums of fractions of the form $\frac{1}{m} + \frac{1}{n}, m = n + 2$

Remember the Pythagorean triple theorem,

$$a = (st), b = \frac{(s^2 - t^2)}{2}, c = \frac{(s^2 + t^2)}{2}$$

$$\frac{1}{m} + \frac{1}{n}, m = n + 2 = \frac{2n + 2}{n^2 + 2n}$$
$$(2n + 2)^2 = 4n^2 + 8n + 4$$
$$(n^2 + 2n)^2 = n^4 + 4n^3 + 4n^2$$
$$(n^2 + 2n)^2 + (2n + 2)^2 = n^4 + 4n^3 + 8n^2 + 8n + 4 = (n^2 + 2n + 2)^2$$

$\therefore (2n + 2, n^2 + 2n, (n^2 + 2n + 2)^2) \in P$

$\mathcal{Q.E.D.}$