

## Interview 2

### Question 1

- a) Pictured below is a circle with radius 1cm inscribing an equilateral triangle.

What is the area of the shaded region?



- b) Assume that this pattern repeats indefinitely (each circle inscribes an equilateral triangle, and each equilateral triangle inscribes a circle).

What is the area of the shaded region?



## Question 2

In this question, a *string* is defined as a sequence of symbols  $a$  and  $b$ . For example,  $ab$ ,  $abbbb$ , and  $baabbababba$  are all valid strings.

We define *constructable strings* as strings that can be constructed according to the following rules:

The string  $ab$  is a constructable string. New constructable strings can be formed by taking an existing constructable string and applying one of the following rules (where  $x$  represents any arbitrary string):

- Rule 1:  $ax \xrightarrow{1} axx$  (If  $ax$  is a constructable string, then so is  $axx$ )
- Rule 2:  $abbbx \xrightarrow{2} ax$  (If  $abbbx$  is a constructable string, then so is  $ax$ )

For example, if  $ab$  is a constructable string, then  $abb$  is also a constructable string by applying Rule 1 (where  $x = b$ ). This is written as  $ab \xrightarrow{1} abb$ .

Since we have shown that  $abb$  is a constructable string, then  $abbbb$  is also a constructable string ( $abb \xrightarrow{1} abbbb$ , where  $x = bb$ ). The full derivation would look like:  $ab \xrightarrow{1} abb \xrightarrow{1} abbbb$ .

If  $abbbb$  is a constructable string, then we can apply Rule 2 to show that  $ab$  is a constructable string by applying Rule 2 ( $abbbb \xrightarrow{2} ab$ ). This is a redundant operation, as our definition already defines  $ab$  as a constructable string.

From now on we will use the following shorthand:  $a^2 = aa$ ,  $a^3b = aaab$ , etc.

Answer the following questions:

- a) Is the string  $ab^5$  constructable? If not, informally justify why there is no way to construct this string.
- b) Is the string  $ab^3$  constructable? If not, informally justify why there is no way to construct this string.
- c) Prove that all strings  $u = ab^n$  are constructable, where  $n = 2^k - 3m \geq 0$  for some integers  $k, m \geq 0$ .
- d) Prove (by induction) that if a string  $u$  can be constructed, then  $u = ab^n$  where  $n = 2^k - 3m \geq 0$  for some for some integers  $k, m \geq 0$ .

Hint: Instead of induction over numbers, can you do induction over the provided rules? Using the string  $ab$  as the base case, and Rules 1 and 2 as the inductive cases.

### Question 3

Sketch the graph:

$$y = \frac{\sin e^x}{e^x}$$

Pay close attention to:

- The coordinates of the y-axis intercept
- The function's behaviour as  $x$  approaches  $\pm\infty$
- The equations of the function's asymptote(s).