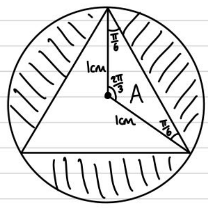
Interview 2 Answers

Question 1)

1) a.



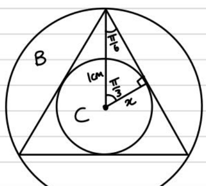
Area of triangle A:

$$\text{area} = \tfrac{1}{2}ab\sin C = \tfrac{1}{2}\cdot 1\cdot 1\cdot \sin\left(\tfrac{2\pi}{3}\right) = \frac{\sqrt{3}}{4}$$

Area of shaded region:

$$\text{area} = \text{area of circle} - 3(\text{area of A}) = \pi r^2 - 3\left(\tfrac{\sqrt{3}}{4}\right) = \pi\cdot 1^2 - 3\left(\tfrac{\sqrt{3}}{4}\right) = \boxed{\pi - \frac{3\sqrt{3}}{4}}$$

b.



Calculating x:

$$\sin\frac{\pi}{6} = \frac{x}{1} \quad \therefore x = \tfrac{1}{2}$$

Area proportion of circle C compared to circle B:

$$\text{proportion} = \frac{\text{area of C}}{\text{area of B}} = \frac{\pi r_C^2}{\pi r_B^2} = \frac{x^2}{1^2} = \left(\tfrac{1}{2}\right)^2 = \tfrac{1}{4}$$

Infinite geometric series:

$a = \pi - \frac{3\sqrt{3}}{4}$ $\quad r = \frac{1}{4}$

$$\text{total area} = \frac{a}{1-r} = \frac{\pi - \frac{3\sqrt{3}}{4}}{1-\frac{1}{4}} = \frac{\pi - \frac{3\sqrt{3}}{4}}{\frac{3}{4}} = \boxed{\frac{4\pi}{3} - \sqrt{3}}$$

Question 2)

2) a. Yes, with the following construction:

$$ab \xrightarrow{1} ab^2 \xrightarrow{1} ab^4 \xrightarrow{1} ab^8 \xrightarrow{2} ab^5$$

b. No, lets think about how to construct ab from ab^3 by applying these rules in <u>reverse</u>:
If we start with a multiple of 3 bs (ab^3) and can only add 3bs and half the number of bs, we will always have a multiple of 3 bs (halving a multiple of 3 results in a number that is also a multiple of 3).
Hence, we can never reach ab, which does not have a multiple of 3 bs.

c. Given a string $ab^n = ab^{2^k - 3m}$, we can construct it in the following manner:

$$\underbrace{ab \xrightarrow{1} ab^2 \xrightarrow{1} \cdots \xrightarrow{1} ab^{2^k}}_{k \text{ times}} \underbrace{\xrightarrow{2} ab^{2^k - 3} \xrightarrow{2} \cdots \xrightarrow{2} ab^{2^k - 3m}}_{m \text{ times}}$$

d. Let us apply induction over the base case (ab) and inductive cases (construction rules $ax \xrightarrow{1} axx$ and $abbbx \xrightarrow{2} ax$)

Base case:

$$u = ab = ab^{2^0 - 3(0)} = ab^{2^k - 3m} \text{ where } k = m = 0 \ (\geq 0)$$

Inductive hypothesis:

Assume a string u takes the form $ab^{2^k - 3m}$ where $k, m \geq 0$

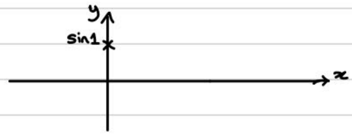
Inductive cases:

$$u \xrightarrow{1} u' : ab^{2^k - 3m} \xrightarrow{1} ab^{2^k - 3m} b^{2^k - 3m} = ab^{2(2^k - 3m)} = ab^{2^{(k+1)} - 3(2m)} \text{ where } (k+1), (2m) \geq 0$$

$$u \xrightarrow{2} u' : ab^{2^k - 3m - 3} b^3 \xrightarrow{2} ab^{2^k - 3m - 3} = ab^{2^k - 3(m+1)} \text{ where } k, (m+1) \geq 0$$

Question 3)

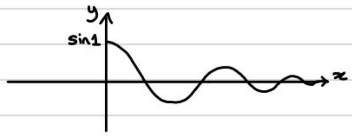
3) $y = \dfrac{\sin(e^x)}{e^x}$



at $x=0$: $y = \dfrac{\sin(e^0)}{e^0} = \sin 1$

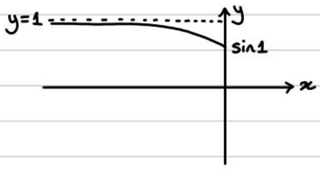
as $x \to +\infty$:

- $e^x \to +\infty$
- $\sin(e^x)$ oscillates faster and faster between $[-1,1]$
- $y = \dfrac{\sin(e^x)}{e^x} \to 0$ (oscillating positive and negative)



as $x \to -\infty$

- $e^x \to 0$
- $\sin(e^x) \approx e^x$ (small angle approximation)
- $y = \dfrac{\sin(e^x)}{e^x} \to \dfrac{e^x}{e^x} = 1$



So $y = \dfrac{\sin(e^x)}{e^x}$ looks like: