Interview 2 Hints

Question 1

Part a)

What information do we know? The circle's radius is 1, and the angles in an equilateral triangle are $\frac{2\pi}{3}$ (60°).

Try forming a smaller triangle by connecting the circle's centre to the triangle's vertices, and calculating the area of this smaller triangle.

From here you should be able to calculate the area of the full triangle and the circle, and hence the shaded area.

The formula for the area of a triangle given two side lengths (a and b) and the angle between these sides (C) is: Area $= \frac{1}{2}ab\sin C$.

Part b)

What is the proportion between the first inner grey area and the outer grey area (the one calculated in part 1a)?

$$\text{Proportion} = \frac{\text{First inner grey area}}{\text{Outer grey area}} = \frac{\pi r_{\text{inner}}^2}{\pi r_{\text{outer}}^2}$$

What does this tell us about the proportions for the second, third, etc. inner grey areas? What sort of series does this produce?

The infinite geometric series formula is: $\frac{a}{1-r}$ where a is the starting point and r is the ratio between consecutive elements.

Question 2

Part a)

What construction steps could lead us to this string? How many times do you need to double 1 and then subtract 3 to get 5?

Part b)

Think about applying the rules in reverse to get from ab^3 to ab. What do we know about these rules with respect to how many b's the string must contain? Will there always be a multiple of 3 b's, and if so can we reason that ab will never be reached?

Part c)

Think about what construction steps could lead to the string ab^{2^k-3m}. Firstly, what construction steps lead to the string ab^{2^k}? How many times would we need to apply Rule 1 to reach this string? From this, how would we then subtract 3 b's m number of times to reach ab^{2^k-3m}?

Part d)

This question requires us to think about induction more generally than just numbers.

In order to show that all constructable strings have the form ab^{2^k-3m}, it is sufficient to show that the starting string ab has this form, and that applying both Rules 1 and 2 to a string in this form produces a new string in this form.

Base case: $u = ab$. Can we show that ab fits the pattern ab^{2^k-3m}? It is sufficient to find values for k and m such that $2^k - 3m = 1$.

Inductive hypothesis: Assume a string u takes the form ab^{2^k-3m}, where $k, m \geqslant 0$.

Inductive cases:

Rule 1: $u \xrightarrow{1} u'$.

If $u = ab^{2^k-3m}$ then $u' = ab^{2^k-3m}b^{2^k-3m}$. Does u' take the right form?

Rule 2: $u \xrightarrow{2} u'$.

If $u = ab^{2^k-3m}$ then $u' = ab^{2^k-3m-3}$. Does u' take the right form?

Our inductive hypothesis assumes that all constructable strings take the form ab^{2^k-3m}. Given that the starting string takes this form, and that Rules 1 and 2 transform a constructable string in this form into a new constructable string in this form, all constructable strings must take this form.

Question 3

Consider what happens when $x = 0$, $x \to \infty^+$, and $x \to \infty^-$:

- $x = 0$: Simply plug $x = 0$ into the formula to get the y-intercept.
- $x \to \infty^+$: What happens to e^x and $\sin(e^x)$ as $x \to \infty^+$? Which of these values grows at a faster rate, are there any oscillations, and does the rate of these oscillations increase?
- $x \to \infty^-$: What happens to e^x as $x \to \infty^+$? Can we approximate $\sin(e^x)$ based off of this? Think small angle approximations ($\sin\theta \approx \theta$ as $\theta \to 0$).