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CSCI 3412

HW 2

Q1-2) (5 points) Prove the following statement using mathematical induction:

$$\textbf{a)}\ \forall n \in N, \sum_{k=1}^{n} k\,(k+1) = \frac{n(n+1)(n+2)}{3}$$

For n = 1 (Best Case):

- $\sum_{k=1}^{1} k(k+1) = 1(1+1) = 1 * 2 = 2$
- $\frac{1(1+1)(1+2)}{3} = \frac{1*2*3}{3} = \frac{6}{3} = 2$
- $Case\ holds\ true$

Inductive Hypothesis:

- $\sum_{k=1}^{n} k(k+1) = \frac{n(n+1)(n+2)}{3}$

Inductive Step:

- $\sum_{k=1}^{k+1} k(k+1) = \frac{k(k+1)(k+2)(k+3)}{3}$
- $\sum_{k=1}^{k+1} k(k+1) = \sum_{k=1}^{k} k(k+1) + (k+1)(k+2)$
- $\frac{kk+1)(k+2)}{3} + (k+1)(k+2)$
- $= (k+1)(k+2)\left(\frac{k}{3}+1\right) = (k+1)(k+2)\left(\frac{k+3}{3}\right) = \frac{(k+1)(k+2)(k+3)}{3}$
- $Proves\ True$

b) Show that n! > 3n for n ≥ 7

For n =7 (Best Case):

- $7! = 7 * 6 * 5 * 4 * 3 * 2 * 1 = 5040$
- 3(7) = 21
- 5040 > 21, base case holds

Inductive Hypothesis:

- Assume k! > 3k for some k ≥ 7

Inductive Step:

- Prove that $(k+1)! > 3(k+1)$

- $(k+1)! = (k+1) * k!$
- $(k+1) * k! > (k+1) * 3k$
- Since $k \geq 7, k+1 > 3$
- Thus, $(k+1) * 3k > 3(k+1)$

Q1 - 3) (5 points) Finding a counterfeit token

You are given 3^n identical-looking coins, where one of them is counterfeit and has a different weight (either lighter or heavier). Your task is to find the counterfeit coin using a balance scale, and you want to determine the minimum number of weighings required.

Prove by mathematical induction that the minimum number of weighings required to find the counterfeit coin among 3^n coins is n weighings

For n = 1 (Best Case):

  - $3^1 = 3$
- If we place one coin on the left side of the balance, one on the right, and leave one aside:
- If the balance tips, the counterfeit coin is in one of the two coins on the scale, and we can tell whether it is heavier or lighter.
- If the balance is equal, then the counterfeit coin is the one left out.
- This requires exactly 1 weighing, so the base case holds.

Inductive Hypothesis:

- Assume that for some $n = k$, the minimum number of weighings required to find the counterfeit coin among 3^k coins is exactly k.
- 3^k coins can be solved in k weighings

Inductive Step:

1. Given 3^{k+1} coins, we can divide them into three equal groups of 3^k coins each.
2. Then, we place one group on the left side of the balance and another group on the right side.
3. The balance can show us **three possible outcomes**:

- If the balance tilts, the counterfeit is in one of the two groups being weighed, and we know whether it is heavier or lighter.
- If the balance stays even, the counterfeit is in the third group.

4. After the first weighing, we are left with exactly 3^k coins, and we already assumed (by **inductive hypothesis**) that finding the counterfeit among 3^k coins requires *k* weighings.
5. Since we already did **one weighing**, we need *k* more weighings to complete the search.
6. Thus, the total number of weighings required is: (k + 1)