

Mathematics 1A ITMTA1-B44

Derivatives



With

Amakan Elisha Agoni Amakan.agoni@EDUVOS.com

Lecture 3 Week 4

3 Differentiation Rules



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3.2

The Product and Quotient Rules

The Product Rule

The Product Rule

The correct formula was discovered by Leibniz and is called the Product Rule.

If you have 2 separate functions given as u = f(x) and v = g(x), and they are both positive differentiable functions. Then we can interpret the product rule of uv as:

The Product Rule If f and g are both differentiable, then

$$\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)]$$

Or simply as:

$$\frac{d}{dx}[uv] = uv' + vu'$$

The Product Rule

In words, the Product Rule says that the derivative of a product of two functions is the first function times the derivative of the second function plus the second function times the derivative of the first function.

Example 1

(a) If $f(x) = xe^x$, find f'(x), then solve for find f''(x) and f''(0).

Solution:

(a) By the Product Rule, we have

$$f'(x) = \frac{d}{dx}(xe^{x})$$

$$= x\frac{d}{dx}(e^{x}) + e^{x}\frac{d}{dx}(x)$$

$$= xe^{x} + e^{x} \cdot 1$$

$$f'(x) = (x+1)e^{x}$$

Example 1 – Solution

(b) Using the Product Rule a second time, we get

$$f''(x) = \frac{d}{dx}[(x+1)e^x]$$

$$= (x+1)\frac{d}{dx}(e^x) + e^x\frac{d}{dx}(x+1)$$

$$= (x+1)e^x + e^x \cdot 1$$

$$= xe^x + e^x + e^x$$

$$= (x+2)e^x$$

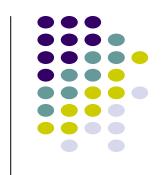
Given the following functions, find f''(-2)

1.
$$f(x) = x^{2}(3x + 5)$$

 $f'(x) = g(x).h'(x) + g'(x).h(x)$
 $f'(x) = x^{2}(3) + 2x(3x + 5)$
 $f'(x) = 9x^{2} + 10x$
 $f''(x) = 18x + 10$
 $f''(-2) = -26$

2.
$$f(x) = (3x^2 - 7)(x^2 + 2x)$$

 $f'(x) = (3x^2 - 7)(2x + 2) + (6x)(x^2 + 2x)$
 $f'(x) = 6x^3 + 6x^2 - 14x - 14 + 6x^3 + 12x^2$
 $f'(x) = 12x^3 + 18x^2 - 14x - 14$
 $f''(x) = 36x^2 + 36x - 14$
 $f''(-2) = 58$



Given the function, find f''(1)

3.
$$f(x) = \frac{1}{\sqrt{x}}(5x^2 - 4x)$$

$$f(x) = x^{-1/2} (5x^2 - 4x)$$

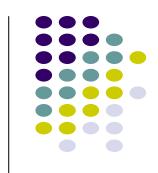
$$f'(x) = \frac{1}{\sqrt{x}}(10x - 4) + \left[-\frac{1}{2}\right]x^{-3/2}(5x^2 - 4x)$$

$$f'(x) = \frac{1}{\sqrt{x}}(10x - 4) - \frac{1}{2\sqrt{x^3}}(5x^2 - 4x)$$

$$f''(x) = \left[\frac{1}{\sqrt{x}}(10) - \frac{1}{2\sqrt{x^3}}(10x - 4)\right] - \left[\frac{1}{2\sqrt{x^3}}(10x - 4) - \frac{3}{4\sqrt{x^5}}(5x^2 - 4x)\right]$$

$$f''(x) = \left[\frac{10}{\sqrt{x}} - \frac{(10x - 4)}{2\sqrt{x^3}}\right] - \left[\frac{(10x - 4)}{2\sqrt{x^3}} - \frac{3(5x^2 - 4x)}{4\sqrt{x^5}}\right]$$

$$f''(1) = 10 - 3 - 3 + \frac{3}{4} = 4.75$$



Given the function, find f''(1)

3.
$$f(x) = \frac{1}{\sqrt{x}}(5x^2 - 4x)$$

$$f'(x) = x^{-1/2}(5x^2 - 4x)$$

$$f'(x) = \frac{1}{\sqrt{x}}(10x - 4) + \left[-\frac{1}{2}\right]x^{-3/2}(5x^2 - 4x)$$

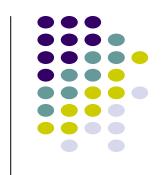
$$f'(x) = \frac{1}{\sqrt{x}}(10x - 4) - \frac{1}{2\sqrt{x^3}}(5x^2 - 4x)$$

$$f'(x) = \left[10\sqrt{x} - \frac{4}{\sqrt{x}}\right] - \left[\frac{5\sqrt{x}}{2} - \frac{2}{\sqrt{x}}\right]$$

$$f'(x) = 10\sqrt{x} - \frac{4}{\sqrt{x}} - \frac{5\sqrt{x}}{2} + \frac{2}{\sqrt{x}}$$

$$f''(x) = \frac{5}{\sqrt{x}} + \frac{2}{\sqrt{x^3}} - \frac{5}{4\sqrt{x}} - \frac{1}{\sqrt{x^3}} = 5 + 2 - \frac{5}{4} - 1$$

$$f'''(1) = 4.75$$





EXAMPLE 2 Differentiate the function $f(t) = \sqrt{t} (a + bt)$.

SOLUTION 1 Using the Product Rule, we have

$$f'(t) = \sqrt{t} \frac{d}{dt} (a + bt) + (a + bt) \frac{d}{dt} (\sqrt{t})$$
$$= \sqrt{t} \cdot b + (a + bt) \cdot \frac{1}{2} t^{-1/2}$$
$$= b\sqrt{t} + \frac{a + bt}{2\sqrt{t}} = \frac{a + 3bt}{2\sqrt{t}}$$

SOLUTION 2 If we first use the laws of exponents to rewrite f(t), then we can proceed directly without using the Product Rule.

$$f(t) = a\sqrt{t} + bt\sqrt{t} = at^{1/2} + bt^{3/2}$$
$$f'(t) = \frac{1}{2}at^{-1/2} + \frac{3}{2}bt^{1/2}$$



EXAMPLE 3 If
$$f(x) = \sqrt{x} g(x)$$
, where $g(4) = 2$ and $g'(4) = 3$, find $f'(4)$.

SOLUTION Applying the Product Rule, we get

$$f'(x) = \frac{d}{dx} \left[\sqrt{x} \ g(x) \right] = \sqrt{x} \ \frac{d}{dx} \left[g(x) \right] + g(x) \frac{d}{dx} \left[\sqrt{x} \right]$$
$$= \sqrt{x} \ g'(x) + g(x) \cdot \frac{1}{2} x^{-1/2}$$
$$= \sqrt{x} \ g'(x) + \frac{g(x)}{2\sqrt{x}}$$

So
$$f'(4) = \sqrt{4} g'(4) + \frac{g(4)}{2\sqrt{4}} = 2 \cdot 3 + \frac{2}{2 \cdot 2} = 6.5$$

The Quotient Rule

The Quotient Rule (1 of 4)

We find a rule for differentiating the quotient of two differentiable functions u = f(x) and v = g(x) in much the same way that we found the Product Rule.

If x, u, and v change by amounts Δx , Δu , and Δv , then the corresponding change in the quotient $\frac{1}{v}$ is

Or simply as:

$$\frac{d}{dx} \left[\frac{u}{v} \right] = \frac{vu' - uv'}{v^2}$$

The Quotient Rule (3 of 4)

The Quotient Rule If f and g are differentiable, then

$$\frac{d}{dx} \left[\frac{f(x)}{g(x)} \right] = \frac{g(x) \frac{d}{dx} [f(x)] - f(x) \frac{d}{dx} [g(x)]}{[g(x)]^2}$$

In words, the Quotient Rule says that the *derivative* of a quotient is the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator, all divided by the square of the denominator.

Example 4

Let
$$y = \frac{x^2 + x - 2}{x^3 + 6}$$
. Then
$$y' = \frac{(x^3 + 6)\frac{d}{dx}(x^2 + x - 2) - (x^2 + x - 2)\frac{d}{dx}(x^3 + 6)}{(x^3 + 6)^2}$$

$$= \frac{(x^3 + 6)(2x + 1) - (x^2 + x - 2)(3x^2)}{(x^3 + 6)^2}$$

$$= \frac{(2x^4 + x^3 + 12x + 6) - (3x^4 + 3x^3 - 6x^2)}{(x^3 + 6)^2}$$

$$= \frac{-x^4 - 2x^3 + 6x^2 + 12x + 6}{(x^3 + 6)^2}$$

Table of Differentiation Formulas

$$\frac{d}{dx}(c) = 0 \qquad \frac{d}{dx}(x^n) = nx^{n-1} \qquad \frac{d}{dx}(e^x) = e^x$$

$$(cf)' = cf' \qquad (f+g)' = f'+g' \qquad (f-g)' = f'-g'$$

$$(fg)' = fg' + gf' \qquad \left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}$$

Examples on Quotient Rule

1.
$$f(x) = \frac{x^2}{x+5} = \frac{g(x)}{h(x)}$$

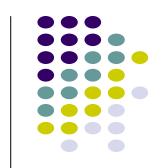
$$f'(x) = \frac{[h(x)g'(x)] - [g(x)h'(x)]}{[h(x)]^2}$$

$$f'(x) = \frac{[(x+5)2x] - [x^2(1)]}{[(x+5)]^2}$$

$$= \frac{x^2 + 10x}{x^2 + 10x + 25}$$

2.
$$f(x) = \frac{\sqrt{x}}{x+5}$$

$$f'(x) = \frac{\left[(x+5)\frac{1}{2\sqrt{x}} \right] - \left[\sqrt{x}(1) \right]}{\left[(x+5) \right]^2}$$



Activities Quotient Rule

1. If
$$f(x) = \frac{\sqrt{x}}{x^2}$$
 find $f'(2)$

$$f'(x) = \frac{\left[\frac{x^2}{2\sqrt{x}}\right] - \left[2x\sqrt{x}\right]}{x^4}$$

$$=\frac{\frac{x^2 - (2x\sqrt{x})(2\sqrt{x})}{2\sqrt{x}}}{\frac{2\sqrt{x}}{x^4}}$$

$$=\frac{\frac{x^2 - 4x^2}{2\sqrt{x}}}{\frac{2\sqrt{x}}{x^4}} = \frac{-3x^2}{2x^4\sqrt{x}}$$

$$f'(x) = \frac{-3}{2x^2 \cdot x^{1/2}} = \frac{-3}{2x^{5/2}} = \frac{-3}{2\sqrt{x^5}}$$

2. If
$$f(x) = \frac{5x^3}{x-10}$$
 find $f'(1)$

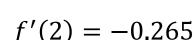
$$f'(x) = \frac{[(x-10)(15x^2) - [5x^3(1)]]}{(x-10)^2}$$

$$= \frac{15x^3 - 150x^2 - 5x^3}{(x-10)^2}$$

$$= \frac{10x^3 - 150x^2}{(x-10)^2}$$

$$f'(x) = \frac{10x^2(x-15)}{(x-10)^2}$$

$$f'(1) = -1.728$$



Activities Quotient Rule

3. If
$$f(x) = \frac{7x^2 - 4}{x^3 - 6}$$
 find $f''(-2)$



Exercises

3-26 Differentiate.

3.
$$f(x) = (x^3 + 2x)e^x$$

5.
$$y = \frac{x}{e^x}$$

7.
$$g(x) = \frac{1+2x}{3-4x}$$

$$9. \ H(u) = \left(u - \sqrt{u}\right)\left(u + \sqrt{u}\right)$$

10.
$$J(v) = (v^3 - 2v)(v^{-4} + v^{-2})$$

11.
$$F(y) = \left(\frac{1}{y^2} - \frac{3}{y^4}\right)(y + 5y^3)$$

4.
$$g(x) = \sqrt{x} e^x$$

6.
$$y = \frac{e^x}{1 - e^x}$$

8.
$$G(x) = \frac{x^2 - 2}{2x + 1}$$

Exercises

12.
$$f(z) = (1 - e^z)(z + e^z)$$

13.
$$y = \frac{x^3}{1 - x^2}$$

15.
$$y = \frac{t^2 + 2}{t^4 - 3t^2 + 1}$$

17.
$$y = e^p (p + p\sqrt{p})$$

19.
$$y = \frac{v^3 - 2v\sqrt{v}}{v}$$

21.
$$f(t) = \frac{2t}{2 + \sqrt{t}}$$

14.
$$y = \frac{x+1}{x^3+x-2}$$

16.
$$y = \frac{t}{(t-1)^2}$$

18.
$$y = \frac{1}{s + ke^s}$$

20.
$$z = w^{3/2}(w + ce^w)$$

22.
$$g(t) = \frac{t - \sqrt{t}}{t^{1/3}}$$