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Mathematics 1A
ITMTA1-B44
Derivatives

With

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Lecture 3
Week 4

3 Differentiation Rules

3.2 The Product and Quotient Rules

The Product Rule

The Product Rule

The correct formula was discovered by Leibniz and is called the Product Rule.

If you have 2 separate functions given as $u = f(x)$ and $v = g(x)$, *and they* are both positive differentiable functions. Then we can interpret the product rule of uv as:

The Product Rule If f and g are both differentiable, then

$$\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)]$$

Or simply as:

$$\frac{d}{dx}[uv] = uv' + vu'$$

The Product Rule

In words, the Product Rule says that *the derivative of a product of two functions is the first function times the derivative of the second function plus the second function times the derivative of the first function.*

Example 1

(a) If $f(x) = xe^x$, find $f'(x), then\ solve\ for$ find $f''(x)\ and\ f''(0)$.

Solution:

(a) By the Product Rule, we have

$$
\begin{aligned}
f'(x) &= \frac{d}{dx}(xe^x) \\
&= x\frac{d}{dx}(e^x) + e^x\frac{d}{dx}(x) \\
&= xe^x + e^x \cdot 1
\end{aligned}
$$

$$f'(x) = (x+1)e^x$$

Example 1 – Solution

(b) Using the Product Rule a second time, we get

$$
\begin{aligned}
f''(x) &= \frac{d}{dx}[(x+1)e^x] \\
&= (x+1)\frac{d}{dx}(e^x) + e^x\frac{d}{dx}(x+1) \\
&\quad (x+1)e^x + e^x \cdot 1 \\
&= xe^x + e^x + e^x \\
&\quad = (x+2)e^x
\end{aligned}
$$

Examples on Product Rule

Given the following functions, $find\ f''(-2)$

1. $f(x) = x^2(3x+5)$

$f'(x) = g(x).h'(x) + g'(x).h(x)$

$f'(x) = x^2(3) + 2x(3x+5)$

$f'(x) = 9x^2 + 10x$

$f''(x) = 18\,x + 10$

$f''(-2) = -26$

2. $f(x) = (3x^2 - 7)(x^2 + 2x)$

$f'(x) = (3x^2 - 7)(2\mathrm{x} + 2) + (6\mathrm{x})(x^2 + 2x)$

$f'(x) = 6x^3 + 6x^2 - 14x - 14 + 6x^3 + 12x^2$

$f'(x) = 12x^3 + 18x^2 - 14x - 14$

$f''(x) = 36x^2 + 36x - 14$

$f''(-2) = 58$

Examples on Product Rule

Given the function, $find\ f''(1)$

3. $f(x) = \frac{1}{\sqrt{x}}(5x^2 - 4x)$

$$f(x) = x^{-1/2}(5x^2 - 4x)$$

$$f'(x) = \frac{1}{\sqrt{x}}(10x - 4) + \left[-\frac{1}{2}\right]x^{-3/2}(5x^2 - 4x)$$

$$f'(x) = \frac{1}{\sqrt{x}}(10x - 4) - \frac{1}{2\sqrt{x^3}}(5x^2 - 4x)$$

$$f''(x) = \left[\frac{1}{\sqrt{x}}(10) - \frac{1}{2\sqrt{x^3}}(10x - 4)\right] - \left[\frac{1}{2\sqrt{x^3}}(10x - 4) - \frac{3}{4\sqrt{x^5}}(5x^2 - 4x)\right]$$

$$f''(x) = \left[\frac{10}{\sqrt{x}} - \frac{(10x-4)}{2\sqrt{x^3}}\right] - \left[\frac{(10x-4)}{2\sqrt{x^3}} - \frac{3(5x^2-4x)}{4\sqrt{x^5}}\right]$$

$$f''(1) = 10 - 3 - 3 + \frac{3}{4} = \mathbf{4.75}$$

Examples on Product Rule

Given the function, $find\ f''(1)$

3. $f(x) = \frac{1}{\sqrt{x}}(5x^2 - 4x)$

$$f(x) = x^{-1/2}\,(5x^2 - 4x)$$

$$f'(x) = \frac{1}{\sqrt{x}}(10x - 4) + \left[-\frac{1}{2}\right] x^{-3/2}\,(5x^2 - 4x)$$

$$f'(x) = \frac{1}{\sqrt{x}}(10x - 4) - \frac{1}{2\sqrt{x^3}}\,(5x^2 - 4x)$$

$$f'(x) = \left[10\sqrt{x} - \frac{4}{\sqrt{x}}\right] - \left[\frac{5\sqrt{x}}{2} - \frac{2}{\sqrt{x}}\right]$$

$$f'(x) = 10\sqrt{x} - \frac{4}{\sqrt{x}} - \frac{5\sqrt{x}}{2} + \frac{2}{\sqrt{x}}$$

$$f''(x) = \frac{5}{\sqrt{x}} + \frac{2}{\sqrt{x^3}} - \frac{5}{4\sqrt{x}} - \frac{1}{\sqrt{x^3}} = 5 + 2 - \frac{5}{4} - 1$$

$$f''(1) = \mathbf{4.75}$$

Examples on Product Rule

EXAMPLE 2 Differentiate the function $f(t) = \sqrt{t}\,(a + bt)$.

SOLUTION 1 Using the Product Rule, we have

$$f'(t) = \sqrt{t}\,\frac{d}{dt}(a + bt) + (a + bt)\frac{d}{dt}\left(\sqrt{t}\right)$$

$$= \sqrt{t} \cdot b + (a + bt) \cdot \tfrac{1}{2}t^{-1/2}$$

$$= b\sqrt{t} + \frac{a + bt}{2\sqrt{t}} = \frac{a + 3bt}{2\sqrt{t}}$$

SOLUTION 2 If we first use the laws of exponents to rewrite $f(t)$, then we can proceed directly without using the Product Rule.

$$f(t) = a\sqrt{t} + bt\sqrt{t} = at^{1/2} + bt^{3/2}$$

$$f'(t) = \tfrac{1}{2}at^{-1/2} + \tfrac{3}{2}bt^{1/2}$$

Examples on Product Rule

EXAMPLE 3 If $f(x) = \sqrt{x}\, g(x)$, where $g(4) = 2$ and $g'(4) = 3$, find $f'(4)$.

SOLUTION Applying the Product Rule, we get

$$f'(x) = \frac{d}{dx}\left[\sqrt{x}\, g(x)\right] = \sqrt{x}\, \frac{d}{dx}[g(x)] + g(x)\, \frac{d}{dx}\left[\sqrt{x}\right]$$

$$= \sqrt{x}\, g'(x) + g(x) \cdot \tfrac{1}{2} x^{-1/2}$$

$$= \sqrt{x}\, g'(x) + \frac{g(x)}{2\sqrt{x}}$$

So

$$f'(4) = \sqrt{4}\, g'(4) + \frac{g(4)}{2\sqrt{4}} = 2 \cdot 3 + \frac{2}{2 \cdot 2} = 6.5$$

The Quotient Rule

The Quotient Rule (1 of 4)

We find a rule for differentiating the quotient of two differentiable functions $u = f(x)$ and $v = g(x)$ in much the same way that we found the Product Rule.

If x, u, and v change by amounts Δx, Δu, and Δv, then the corresponding change in the quotient $\frac{u}{v}$ is

Or simply as:

$$\frac{d}{dx}\left[\frac{u}{v}\right] = \frac{vu' - uv'}{v^2}$$

The Quotient Rule (3 of 4)

The Quotient Rule If *f* and *g* are differentiable, then

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}[f(x)] - f(x)\frac{d}{dx}[g(x)]}{[g(x)]^2}$$

In words, the Quotient Rule says that the *derivative of a quotient is the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator, all divided by the square of the denominator*.

Example 4

Let $y = \dfrac{x^2 + x - 2}{x^3 + 6}$. Then

$$\begin{aligned}
y' &= \frac{(x^3+6)\frac{d}{dx}(x^2+x-2) - (x^2+x-2)\frac{d}{dx}(x^3+6)}{(x^3+6)^2} \\
&= \frac{(x^3+6)(2x+1) - (x^2+x-2)(3x^2)}{(x^3+6)^2} \\
&= \frac{(2x^4+x^3+12x+6) - (3x^4+3x^3-6x^2)}{(x^3+6)^2} \\
&= \frac{-x^4-2x^3+6x^2+12x+6}{(x^3+6)^2}
\end{aligned}$$

Table of Differentiation Formulas

$$\frac{d}{dx}(c)=0 \qquad \frac{d}{dx}\left(x^n\right)=nx^{n-1} \qquad \frac{d}{dx}\left(e^x\right)=e^x$$

$$(cf)'=cf' \qquad (f+g)'=f'+g' \qquad (f-g)'=f'-g'$$

$$(fg)'=fg'+gf' \qquad \left(\frac{f}{g}\right)'=\frac{gf'-fg'}{g^2}$$

Examples on Quotient Rule

1. $f(x) = \frac{x^2}{x+5} = \frac{g(x)}{h(x)}$

$$f'(x) = \frac{[h(x)g'(x)] - [g(x)\,h'(x)]}{[h(x)]^2}$$

$$f'(x) = \frac{[(x+5)2x] - [x^2(1)]}{[(x+5)]^2}$$

$$= \frac{x^2 + 10x}{x^2 + 10x + 25}$$

2. $f(x) = \frac{\sqrt{x}}{x+5}$

$$f'(x) = \frac{[(x+5)\frac{1}{2\sqrt{x}}] - [\sqrt{x}(1)]}{[(x+5)]^2}$$

Activities Quotient Rule

1. If $f(x) = \frac{\sqrt{x}}{x^2}$ $find\ f'(2)$

$$f'(x) = \frac{[\frac{x^2}{2\sqrt{x}}] - [2x\sqrt{x}]}{x^4}$$

$$= \frac{\frac{x^2 - (2x\sqrt{x})(2\sqrt{x})}{2\sqrt{x}}}{x^4}$$

$$= \frac{\frac{x^2 - 4x^2}{2\sqrt{x}}}{x^4} = \frac{-3x^2}{2x^4\sqrt{x}}$$

$$f'(x) = \frac{-3}{2x^2.x^{1/2}} = \frac{\mathbf{-3}}{\mathbf{2x^{5/2}}} = \frac{\mathbf{-3}}{\mathbf{2\sqrt{x^5}}}$$

$$f'(2) = -0.265$$

2. If $f(x) = \frac{5x^3}{x-10}$ $find\ f'(1)$

$$f'(x) = \frac{[(x-10)(15x^2) - [5x^3(1)]}{(x-10)^2}$$

$$= \frac{15x^3 - 150x^2 - 5x^3}{(x-10)^2}$$

$$= \frac{10x^3 - 150x^2}{(x-10)^2}$$

$$f'(x) = \frac{\mathbf{10x^2(x-15)}}{\mathbf{(x-10)^2}}$$

$$\boldsymbol{f'(1) = -1.728}$$

Activities Quotient Rule

3. If $f(x) = \frac{7x^2-4}{x^3-6}$ $\quad find\ f''(-2)$

Exercises

3–26 Differentiate.

3. $f(x) = (x^3 + 2x)e^x$

4. $g(x) = \sqrt{x}\, e^x$

5. $y = \dfrac{x}{e^x}$

6. $y = \dfrac{e^x}{1 - e^x}$

7. $g(x) = \dfrac{1 + 2x}{3 - 4x}$

8. $G(x) = \dfrac{x^2 - 2}{2x + 1}$

9. $H(u) = \left(u - \sqrt{u}\right)\left(u + \sqrt{u}\right)$

10. $J(v) = (v^3 - 2v)(v^{-4} + v^{-2})$

11. $F(y) = \left(\dfrac{1}{y^2} - \dfrac{3}{y^4}\right)(y + 5y^3)$

Exercises

12. $f(z) = (1 - e^z)(z + e^z)$

13. $y = \dfrac{x^3}{1 - x^2}$

14. $y = \dfrac{x + 1}{x^3 + x - 2}$

15. $y = \dfrac{t^2 + 2}{t^4 - 3t^2 + 1}$

16. $y = \dfrac{t}{(t - 1)^2}$

17. $y = e^p\left(p + p\sqrt{p}\right)$

18. $y = \dfrac{1}{s + ke^s}$

19. $y = \dfrac{v^3 - 2v\sqrt{v}}{v}$

20. $z = w^{3/2}(w + ce^w)$

21. $f(t) = \dfrac{2t}{2 + \sqrt{t}}$

22. $g(t) = \dfrac{t - \sqrt{t}}{t^{1/3}}$