Mathematics 1A

ITMTA1-B44

Limits and Derivatives

With

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Lecture 6
Week 2

2 Limits and Derivatives

2.3

Calculating Limits Using the Limit Laws

Properties of Limits

Properties of Limits

Limit Laws

Suppose that c is a constant and the limits $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist. Then

Sum Law

1. The limit of a sum is the sum of the limits.

$$\lim_{x\to a}\left[f(x)+g(x)\right]=\lim_{x\to a} f(x)+\lim_{x\to a} g(x)$$

Difference Law

2. The limit of a difference is the difference of the limits.

$$\lim_{x\to a}\left[f(x)-g(x)\right]=\lim_{x\to a} f(x)-\lim_{x\to a} g(x)$$

Constant Multiple Law

3. The limit of a constant times a function is the constant times the limit of the function.

$$\lim_{x\to a}\left[cf(x)\right]=c\lim_{x\to a} f(x)$$

Properties of Limits

Product Law

4. The limit of a product is the product of the limits.

$$\lim_{x \to a} \left[f(x) g(x) \right] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$$

Quotient Law

5. The limit of a quotient is the quotient of the limits (provided that the limit of the denominator is not 0).

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} \quad \text{if } \lim_{x \to a} g(x) \neq 0$$

Example 1

Use the Limit Laws and the graphs of *f* and *g* in Figure 1 to evaluate the following limits, if they exist.

(a) $\lim_{x \to -2} [f(x) + 5g(x)]$ (b) $\lim_{x \to 1} [f(x)g(x)]$ (c) $\lim_{x \to 2} \frac{f(x)}{g(x)}$

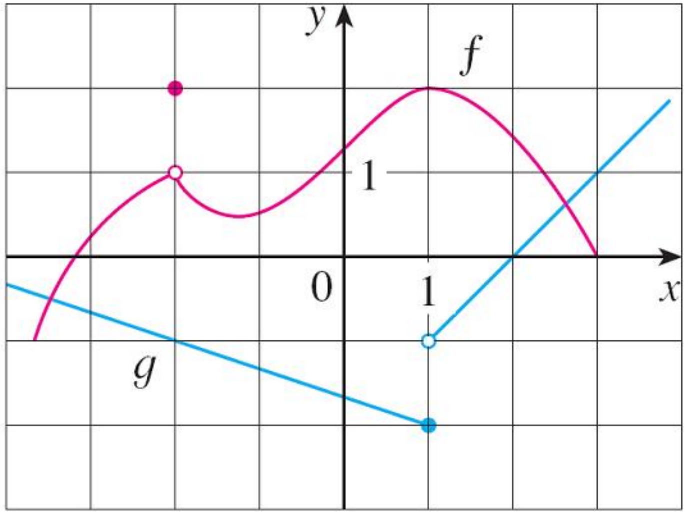
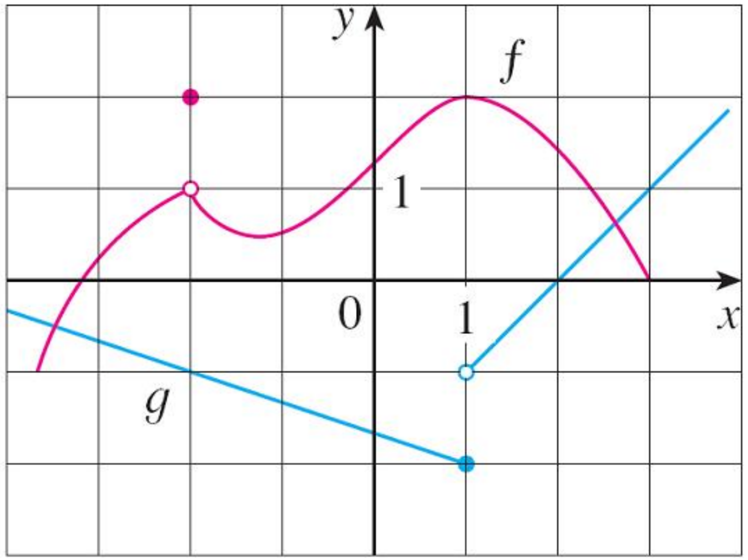


Figure 1

Example 1(a) – Solution

From the graphs of *f* and *g* we see that

$$\lim_{x\to-2} f(x) = 1 \quad \text{and} \quad \lim_{x\to-2} g(x) = -1$$



Therefore we have

$$\lim_{x\to-2}\left[f(x)+5g(x)\right] = \lim_{x\to-2} f(x) + \lim_{x\to-2}\left[5g(x)\right]$$ (by Limit Law 1)

$$= \lim_{x\to-2} f(x) + 5\lim_{x\to-2} g(x)$$ (by Limit Law 3)

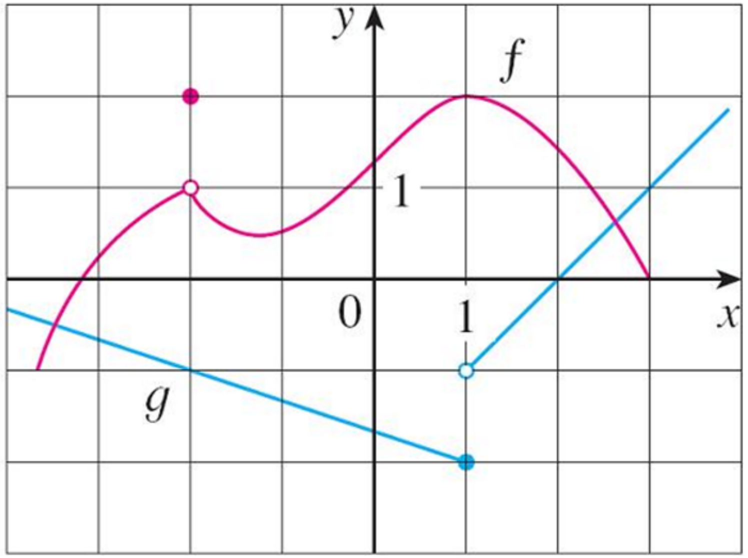
$$= 1 + 5(-1)$$

$$= -4$$

Example 1(b) – Solution

We see that $\lim_{x\to 1} f(x) = 2$. But $\lim_{x\to 1} g(x)$ does not exist because the left and right limits are different:

$$\lim_{x\to 1^-} g(x) = -2 \qquad \lim_{x\to 1^+} g(x) = -1$$



So we can't use Law 4 for the desired limit.

But we *can* use Law 4 for the one-sided limits:

$$\lim_{x\to 1^-}\left[f(x)g(x)\right] = \lim_{x\to 1^-} f(x)\cdot \lim_{x\to 1^-} g(x) = 2\cdot(-2) = -4$$

$$\lim_{x\to 1^+}\left[f(x)g(x)\right] = \lim_{x\to 1^+} f(x)\cdot \lim_{x\to 1^+} g(x) = 2\cdot(-1) = -2$$

The left and right limits aren't equal, so $\lim_{x\to 1}\left[f(x)g(x)\right]$ does not exist.

Example 1(c) – Solution

The graphs show that

$$\lim_{x \to 2} f(x) \approx 1.4 \text{ and } \lim_{x \to 2} g(x) = 0$$

Because the limit of the denominator is 0, we can't use Law 5.

The given limit does not exist because the denominator approaches 0 while the numerator approaches a nonzero number.

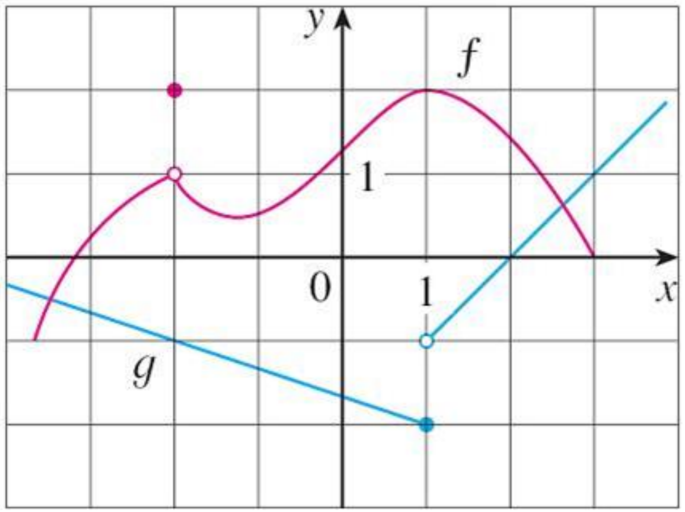


Figure 1

Example

Evaluate the following limits and justify each step.

(a) $\lim_{x \to 5} (2x^2 - 3x + 4)$ (b) $\lim_{x \to -2} \dfrac{x^3 + 2x^2 - 1}{5 - 3x}$

SOLUTION

(a)
$$\begin{aligned}\lim_{x \to 5} (2x^2 - 3x + 4) &= \lim_{x \to 5} (2x^2) - \lim_{x \to 5} (3x) + \lim_{x \to 5} 4 && \text{(by Laws 2 and 1)}\\ &= 2 \lim_{x \to 5} x^2 - 3 \lim_{x \to 5} x + \lim_{x \to 5} 4 && \text{(by 3)}\\ &= 2(5^2) - 3(5) + 4 && \text{(by 9, 8, and 7)}\\ &= 39\end{aligned}$$

Example

(b) $\lim\limits_{x \to -2} \dfrac{x^3 + 2x^2 - 1}{5 - 3x}$

$$\lim_{x \to -2} \frac{x^3 + 2x^2 - 1}{5 - 3x} = \frac{\lim\limits_{x \to -2} (x^3 + 2x^2 - 1)}{\lim\limits_{x \to -2} (5 - 3x)} \qquad \text{(by Law 5)}$$

$$= \frac{\lim\limits_{x \to -2} x^3 + 2 \lim\limits_{x \to -2} x^2 - \lim\limits_{x \to -2} 1}{\lim\limits_{x \to -2} 5 - 3 \lim\limits_{x \to -2} x} \qquad \text{(by 1, 2, and 3)}$$

$$= \frac{(-2)^3 + 2(-2)^2 - 1}{5 - 3(-2)} \qquad \text{(by 9, 8, and 7)}$$

$$= -\frac{1}{11}$$

Evaluating Limits by Direct Substitution

Evaluating Limits by Direct Substitution

Direct Substitution Property

If *f* is a polynomial or a rational function and ***a*** is in the domain of ***f***, then

$$\lim_{x \to a} f(x) = f(a)$$

Functions that have the Direct Substitution Property are called *continuous at* ***a***.

Example 3

Find

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1}.$$

Solution:

Let $f(x) = (x^2 - 1)/(x - 1)$. We can't find the limit by substituting $x = 1$ because $f(1)$ isn't defined. Nor can we apply the Quotient Law, because the limit of the denominator is 0.

Instead, we need to do some preliminary algebra. We factor the numerator as a difference of squares:

$$\frac{x^2 - 1}{x - 1} = \frac{(x-1)(x+1)}{x - 1}$$

Example 3 – Solution

The numerator and denominator have a common factor of $x - 1$. When we take the limit as x approaches 1, we have $x \neq 1$ and so $x - 1 \neq 0$.

Therefore we can cancel the common factor, $x - 1$, and then compute the limit by direct substitution as follows:

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1} \frac{(x-1)(x+1)}{x-1}$$

$$= \lim_{x \to 1}(x+1) = 1 + 1 = 2$$

Rationalization

- In algebra, rationalization refers to the process of eliminating radicals (square roots, cube roots, etc.) from the denominator of a fraction.
- This is done to simplify expressions and make them easier to work with.
- The general idea is to multiply both the numerator and denominator by an appropriate expression to remove the radical from the denominator.

Rationalization

Here's a basic example:

Original Fraction: $\frac{1}{\sqrt{2}}$

To rationalize the denominator, you can multiply both the numerator and denominator by the conjugate of the denominator (the same expression but with the opposite sign between the terms):

$$\frac{1}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{1 \cdot \sqrt{2}}{\sqrt{2} \cdot \sqrt{2}} = \frac{\sqrt{2}}{2}$$

The result, $\frac{\sqrt{2}}{2}$, is a rationalized form of the original fraction because it no longer contains a radical in the denominator.

Exercise

Evaluate the following limit analytically. Show all your calculations:

1. $\lim\limits_{x \to 0} \frac{\sqrt{x+9}-3}{x}$

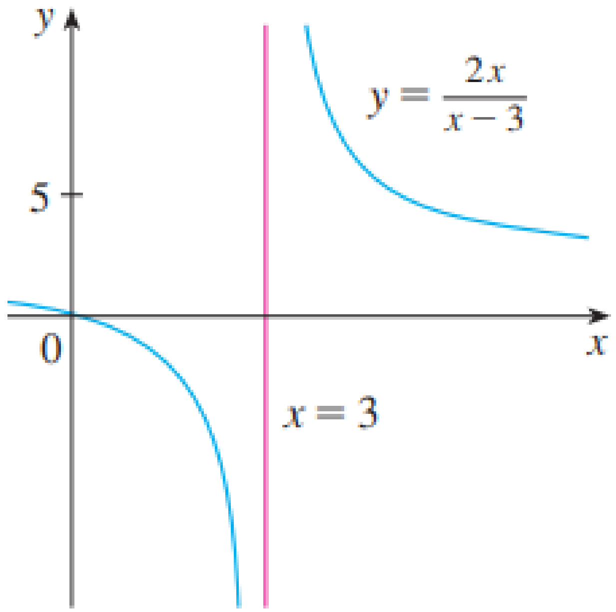
$\boldsymbol{Ans: \frac{1}{6}}$

2. $\lim\limits_{x \to 0} \frac{5-\sqrt{25+x}}{x}$

$\boldsymbol{Ans: -\frac{1}{10}}$

Example 8

Find $\lim_{x \to 3^+} \dfrac{2x}{x-3}$ and $\lim_{x \to 3^-} \dfrac{2x}{x-3}$.

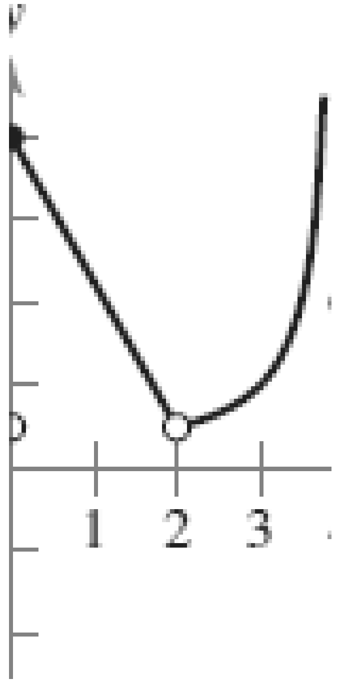


$$\lim_{x \to 3^+} \frac{2x}{x-3} = \infty$$

$$\lim_{x \to 3^-} \frac{2x}{x-3} = -\infty$$

Exercises

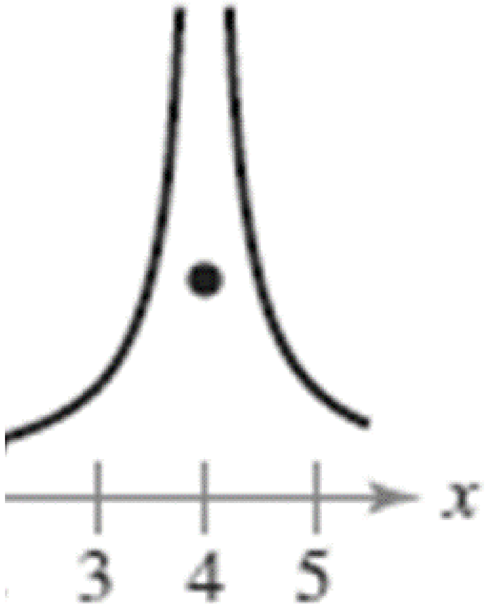
Find the limit $\lim_{x \to 2} f(x)$



$$\lim_{x \to 2} f(x) \approx 0.5$$

Exercises

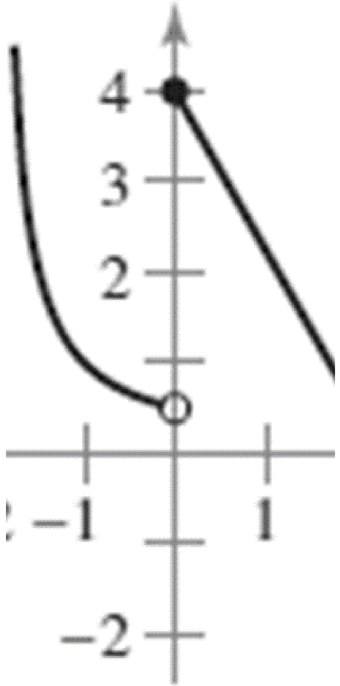
Find the limit $\lim_{x \to 4} f(x)$



$$\lim_{x \to 4} f(x) = \infty$$

Exercises

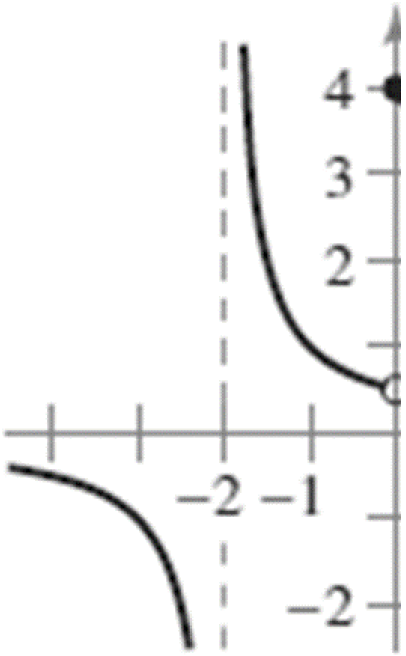
Find the limit $\lim_{x \to 0} f(x)$



$\lim_{x \to 0} f(x) =$ DNE

Exercises

Find the limit $\lim_{x \to -2} f(x)$



$\lim_{x \to -2} f(x) =$ DNE