Mathematics 1A

ITMTA1-B44

Limits and Derivatives

With

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Lecture 4
Week 2

2 Limits and Derivatives

2.2 The Limit of a Function

Finding Limits Numerically and Graphically

Finding Limits Numerically and Graphically

Let's investigate the behavior of the function *f* defined by $f(x) = \frac{(x-1)}{(x^2-1)}$ for values of *x* near 1.

The following table gives values of *f*(*x*) for values of *x* close to 1 but not equal to 1.

$x < 1$	$f(x)$	$x > 1$	$f(x)$
0.5	0.666667	1.5	0.400000
0.9	0.526316	1.1	0.476190
0.99	0.502513	1.01	0.497512
0.999	0.500250	1.001	0.499750
0.9999	0.500025	1.0001	0.499975

1	0.5	1	0.5

Finding Limits Numerically and Graphically

From the table and the graph of *f* shown in Figure 1 we see that the closer *x* is to 1 (on either side of 1), the closer *f*(*x*) is to 0.5.

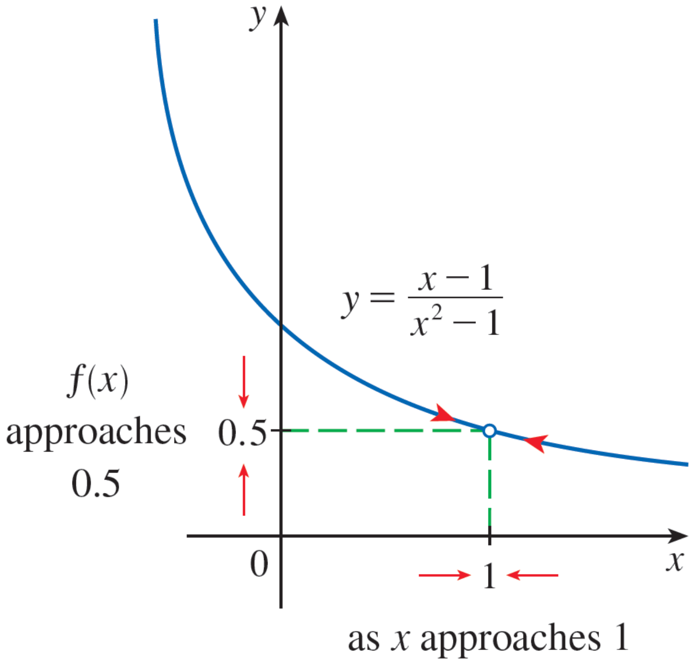


Figure 1

Finding Limits Numerically and Graphically

In fact, it appears that we can make the values of *f*(*x*) as close as we like to 0.5 by taking *x* sufficiently close to 1.

We express this by saying "the limit of the function $f(x) = \frac{(x-1)}{(x^2-1)}$ as *x* approaches 1 is equal to 0.5."

The notation for this is

$$\lim_{x \to 1} \frac{x-1}{x^2-1} = 0.5$$

Finding Limits Numerically and Graphically

1 Intuitive Definition of a Limit

Suppose *f*(*x*) is defined when *x* is near the number *a*. (This means chat *f* is defined on some open interval that contains ***a***, except possibly at ***a*** itself.) Then we write

$$\lim_{x \to a} f(x) = L$$

and say "the limit of *f*(*x*), as *x* approaches *a*, equals *L*"

if we can make the values of *f*(*x*) arbitrarily close to *L* (as close to *L* as we like) by restricting *x* to be sufficiently close to ***a*** (on either side of *a*) but not equal to *a*.

This says that the values of *f*(*x*) approach *L* as *x* approaches *a*. In other words, the values of *f*(*x*) tend to get closer and closer to the number *L* as *x* gets closer and closer to the number ***a*** (from either side of *a*) but $x \neq a$.

Finding Limits Numerically and Graphically

An alternative notation for

$$\lim_{x \to a} f(x) = L$$

is

$$f(x) \to L \text{ as } x \to a$$

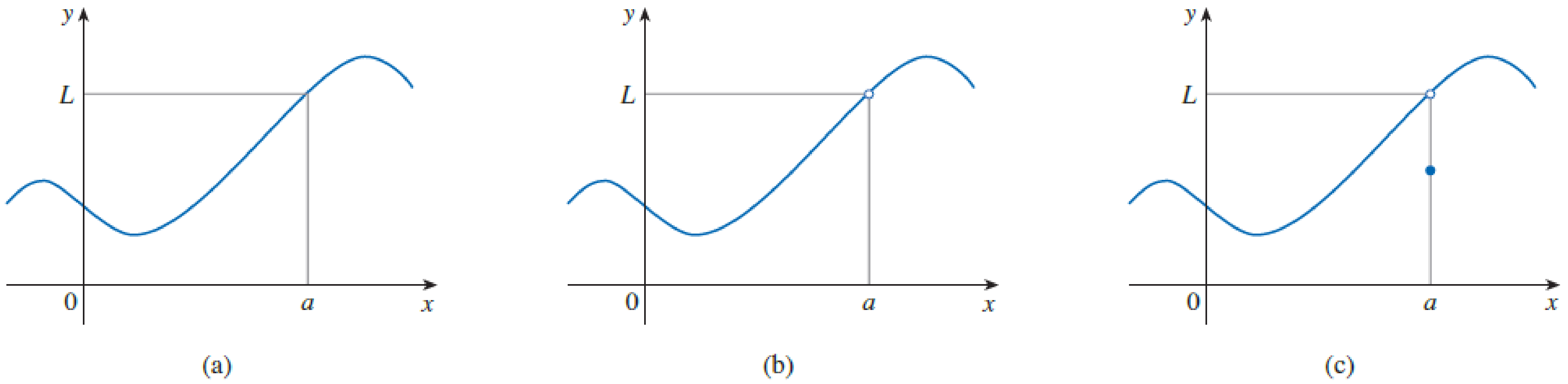
which is usually read "*f*(*x*) approaches *L* as *x* approaches *a*."

Notice the phrase "but *x* not equal to *a*" in the definition of limit. This means that in finding the limit of *f*(*x*) as *x* approaches *a*, we never consider *x* = *a*. In fact, *f*(*x*) need not even be defined when *x* = *a*. The only thing that matters is how *f* is defined *near a*.

Finding Limits Numerically and Graphically

Figure 2 shows the graphs of three functions. Note that in part (b), *f*(*a*) is not defined and in part (c), *f*(*a*) ≠ *L*.

But in each case, regardless of what happens at *a*, it is true that $\lim_{x \to a} f(x) = L$.



$\lim_{x \to a} f(x) = L$ in all three cases

Figure 2

One-Sided Limits

One-Sided Limits

The Heaviside function H is defined by

$$H(t) = \begin{cases} 0 & \text{if} \quad t < 0 \\ 1 & \text{if} \quad t \geq 0 \end{cases}$$

As t approaches 0 from the left, $H(t)$ approaches 0. As t approaches 0 from the right, $H(t)$ approaches 1.

We indicate this situation symbolically by writing

$$\lim_{t \to 0^-} H(t) = 0 \quad \text{and} \quad \lim_{t \to 0^+} H(t) = 1$$

and we call these *one-sided limits*.

One-Sided Limits

2 Intuitive Definition of One-Sided Limits We write

$$\lim_{x \to a^-} f(x) = L$$

and say the **left-hand limit** of *f*(*x*) as *x* approaches *a* [or the limit of *f*(*x*) as *x* approaches *a from the left*] is equal to *L* if we can make the values of *f*(*x*) arbitrarily close to *L* by restricting *x* to be sufficiently close to *a* with *x less than a*.

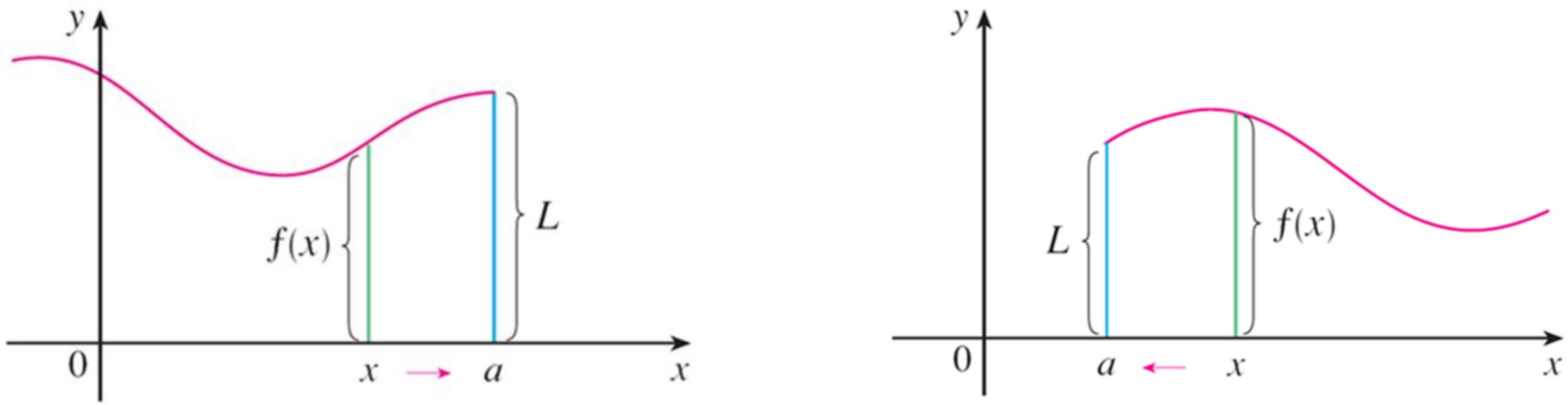
Also,

$$\lim_{x \to a^+} f(x) = L$$

and say that the **right-hand limit** of *f*(*x*) as *x* approaches *a* [or the limit of *f*(*x*) as *x* approaches *a from the right*] is equal to *L* if we can make the values of *f*(*x*) arbitrarily close to *L* by restricting *x* to be sufficiently close to *a* with *x greater than a*.

One-Sided Limits

For instance, the notation $x \to 5^-$ means that we consider only $x < 5$, and $x \to 5^+$ means that we consider only $x > 5$. Definition 2 is illustrated in Figure 6.



(a) $\lim_{x \to a^-} f(x) = L$ (b) $\lim_{x \to a^+} f(x) = L$

Figure 6

Example 4

The graph of a function *g* is shown in Figure 7. Use the graph to state the values (if they exist) of the following:

(a) $\lim_{x \to 2^-} g(x)$ (b) $\lim_{x \to 2^+} g(x)$ (c) $\lim_{x \to 2} g(x)$ (d) $\lim_{x \to 5^-} g(x)$ (e) $\lim_{x \to 5^+} g(x)$ (f) $\lim_{x \to 5} g(x)$

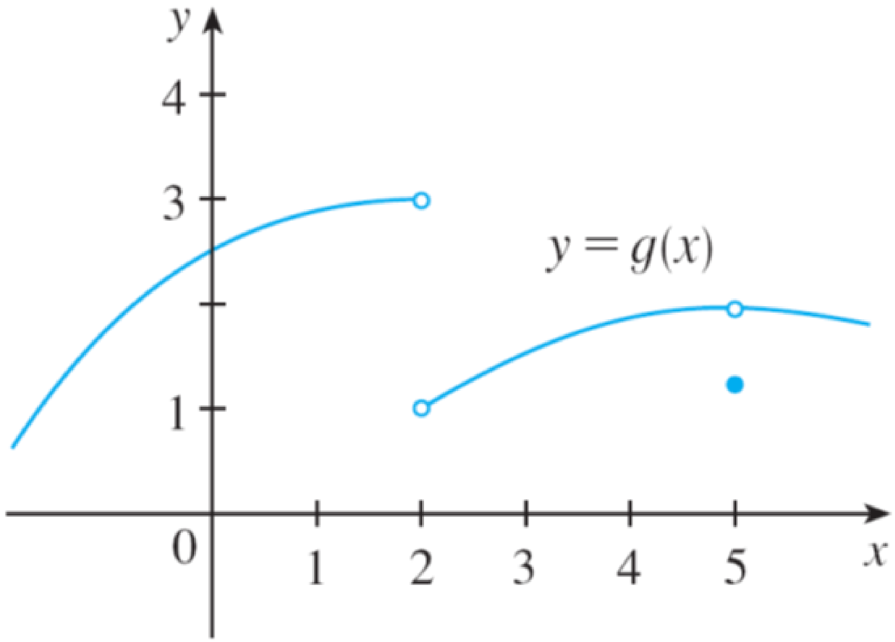


Figure 7

Example 4

Looking at the graph we see that the values of $g(x)$ approach 3 as x approaches 2 from the left, but they approach 1 as x approaches 2 from the right.

Therefore

(a) $\lim_{x \to 2^-} g(x) = 3$

(b) $\lim_{x \to 2^+} g(x) = 1$

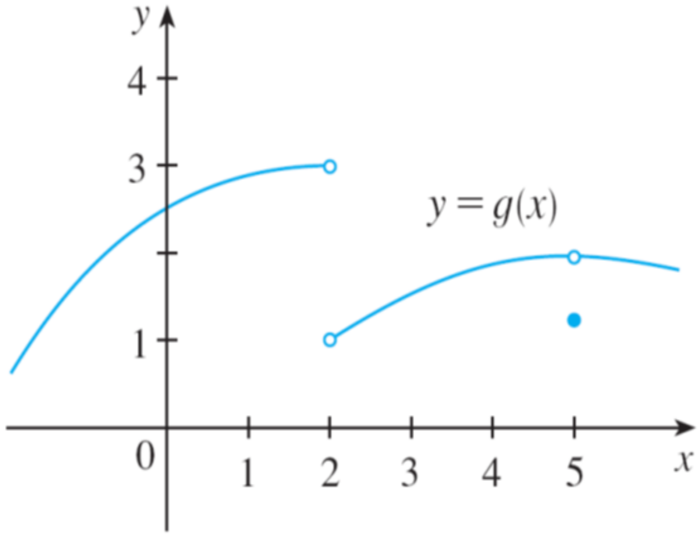


Figure 7

(c) Since the left and right limits are different, we conclude that the limit does not exist. $\lim_{x \to 2} g(x)$

(d) $\lim_{x \to 5^-} g(x) = 2$

(e) $\lim_{x \to 5^+} g(x) = 2$

(f) This time the left and right limits are the same and so, by (3), we have

$\lim_{x \to 5} g(x) = 2$

Despite this fact, notice that $g(5) \neq 2$.

How Can a Limit Fail to Exist?

How Can a Limit Fail to Exist? (1 of 1)

We have seen that a limit fails to exist at a number a if the left- and right-hand limits are not equal (as in Example 4). The next example illustrate additional ways that a limit can fail to exist.

Example 5

Investigate $\lim_{x \to 0} \sin \frac{\pi}{x}$.

Solution:

Notice that the function $f(x) = \sin(\pi/x)$ is undefined at 0. Evaluating the function for some small values of *x*, we get

$$f(1) = \sin \pi = 0 \qquad f\left(\frac{1}{2}\right) = \sin 2\pi = 0$$
$$f\left(\frac{1}{3}\right) = \sin 3\pi = 0 \qquad f\left(\frac{1}{4}\right) = \sin 4\pi = 0$$
$$f(0.1) = \sin 10\pi = 0 \qquad f(0.01) = \sin 100\pi = 0$$

Similarly, $f(0.001) = f(0.0001) = 0$.

Example 5 – Solution

On the basis of this information we might be tempted to guess that the limit is 0, but this time our guess is wrong.

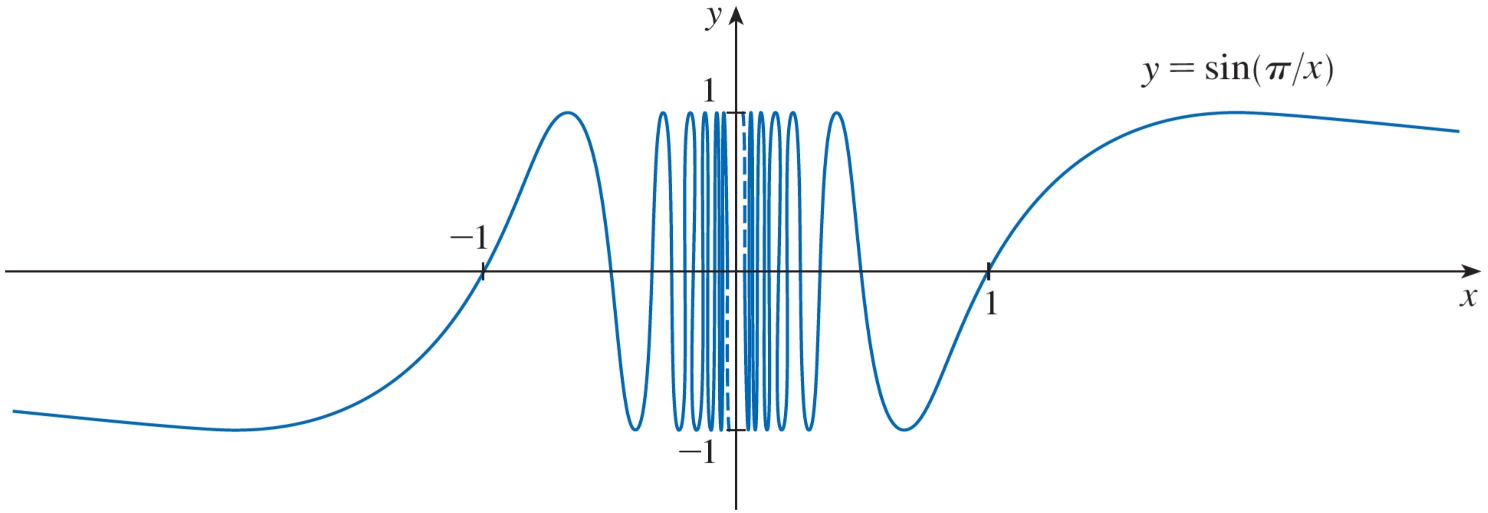


Figure 8

The dashed lines near the *y*-axis indicate that the values of $\sin(\pi/x)$

oscillate between 1 and −1 infinitely often as *x* approaches 0. Since the values of *f*(*x*) do not approach a fixed number as *x* approaches 0,

$$\lim_{x \to 0} \sin \frac{\pi}{x} \text{ does not exist}$$

Next Class…

Infinite Limits; Vertical Asymptotes