Mathematics 1A

ITMTA1-B44

Limits and Derivatives 2

With

Amakan Elisha Agoni
Amakan.agoni@EDUVOS.com

Lecture 11
Week 3

Rates of Change

Rates of Change

Suppose y is a quantity that depends on another quantity x. Thus y is a function of x and we write $y = f(x)$.

If x changes from x_1 to x_2, then the change in x (also called the **increment** of x) is

$$\Delta x = x_2 - x_1$$

and the corresponding change in y is

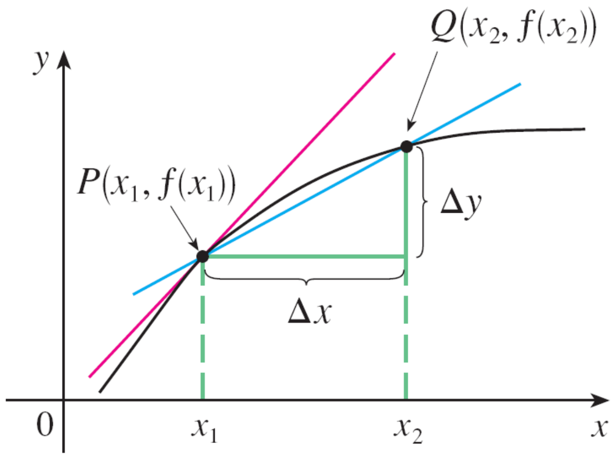
$$\Delta y = f(x_2) - f(x_1)$$

Rates of Change

The difference quotient

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

is called the **average rate of change of *y* with respect to *x*** over the interval [x_1, x_2] and can be interpreted as the slope of the secant line *PQ* in Figure 8.



average rate of change $= m_{PQ}$

instantaneous rate of change $=$ slope of tangent at P

Figure 8

Rates of Change

By analogy with velocity, we consider the average rate of change over smaller and smaller intervals by letting x_2 approach x_1 and therefore letting Δx approach 0.

The limit of these average rates of change is called the (**instantaneous**) **rate of change of *y* with respect to *x*** at $x = x_1$, which (as in the case of velocity) is interpreted as the slope of the tangent to the curve $y = f(x)$ at $P(x_1, f(x_1))$:

6 $$\text{instantaneous rate of change} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{x_2 \to x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

We recognize this limit as being the derivative $f'(x_1)$.

Example 7

A manufacturer produces bolts of a fabric with a fixed width. The cost of producing x meters length of this fabric is $C = f(x)$ Rand.

(a) What is the meaning of the derivative $f'(x)$? What are its units?

(b) In practical terms, what does it mean to say that $f'(1000) = 9$?

(c) Which do you think is greater, $f'(50)$ or $f'(500)$?

Example 7(a) – Solution

The derivative $f'(x)$ is the instantaneous rate of change of *C* with respect to *x*; that is, $f'(x)$ means the rate of change of the production cost with respect to the lenth of fabric produced.

Because

$$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta c}{\Delta x}$$

the units for $f'(x)$ are the same as the units for the difference quotient $\frac{\Delta C}{\Delta x}$.

Since ΔC is measured in Rand and Δx in meters, it follows that the units for $f'(x)$ are Rand per meter.

Example 7(b) – Solution

(b) In practical terms, what does it mean to say that $f'(x)$? What are its units?

The statement that $f'(1000) = 9$ means that, after 1000 yards of fabric have been manufactured, the rate at which the production cost is increasing is R9/meter. (When x = 1000, C is increasing 9 times as fast as x.)

Since Δx = 1 is small compared with x = 1000, we could use the approximation

$$f'(1000) \approx \frac{\Delta C}{\Delta x} = \frac{\Delta C}{1} = \Delta C$$

and say that the cost of manufacturing the 1000th yard (or the 1001st) is about \$9.

Example 7(c) – Solution

The rate at which the production cost is increasing (per yard) is probably lower when $x = 500$ than when $x = 50$ (the cost of making the 500th yard is less than the cost of the 50th yard) because of economies of scale. (The manufacturer makes more efficient use of the fixed costs of production.)

So

$$f'(50) > f'(500)$$

Example 8

When Eduvos reduced their annual fees to R35 000 per year for IT degree, the number of application received was 2000 and rising at the rate of 100 more application for each R500 taken off the fee amount.

a. Use K(x) to state the number of application received when x amount of rands taken off the annual fee. Use equations involving K and and K′(x) to represent the scenario above.

b. Estimate the number of applications that will be received if R6000 more is taken of the annual fee.

Solution:

$$K(x) = 2000 + \frac{100(35000 - x)}{500}$$

Example 9

Compute the average rate of change of $f(x) = x^2 - \frac{1}{x}$ on the interval $[2,\ 4]$.

The **average rate of change of *y* with respect to *x*** over the interval $[x_1, x_2]$

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

Solution

We can start by computing the function values at each endpoint of the interval.

$$\begin{aligned} f(2) &= 2^2 - \tfrac{1}{2} \\ &= 4 - \tfrac{1}{2} \\ &= \tfrac{7}{2} \end{aligned} \qquad \begin{aligned} f(4) &= 4^2 - \tfrac{1}{4} \\ &= 16 - \tfrac{1}{4} \\ &= \tfrac{63}{4} \end{aligned}$$

Now we compute the average rate of change.

$$\begin{aligned} \text{Average rate of change} &= \frac{f(4)-f(2)}{4-2} \\ &= \frac{\frac{63}{4}-\frac{7}{2}}{4-2} \\ &= \frac{\frac{49}{4}}{2} \\ &= \frac{49}{8} \end{aligned}$$