Mathematics 1A

ITMTA1-B44

Limits and Derivatives 2

With

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Lecture 10
Week 3

2 Limits and Derivatives

2.7 Derivatives and Rates of Change

Derivatives and Rates of Change

This special type of limit is called a *derivative* and we will see that it can be interpreted as a rate of change in any of the natural or social sciences or engineering.

Tangents

Tangents (1 of 8)

If a curve C has equation $y = f(x)$ and we want to find the tangent line to C at the point $P(a, f(a))$, then we consider a nearby point $Q(x, f(x))$, where $x \neq a$, and compute the slope of the secant line PQ:

$$m_{PQ} = \frac{f(x) - f(a)}{x - a}$$

Then we let Q approach P along the curve C by letting x approach a.

Tangents

1 Definition

The **tangent line** to the curve $y = f(x)$ at the point $P(a, f(a))$ is the line through P with slope

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

provided that this limit exists.

Example 1

Find an equation of the tangent line to the parabola $y = x^2$ at the point *P*(1, 1).

Solution:

Here we have *a* = 1 and $f(x) = x^2,$ so the slope is

$$m = \lim_{x \to 1} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1} \frac{x^2 - 1}{x - 1}$$

$$= \lim_{x \to 1} \frac{(x - 1)(x + 1)}{x - 1}$$

Example 1 – Solution

$$\begin{aligned} &= \lim_{x \to 1}(x + 1) \\ &= 1 + 1 \\ &= 2 \end{aligned}$$

Using the point-slope form of the equation of a line, we find that an equation of the tangent line at (1, 1) is

$$y - 1 = 2(x - 1) \quad \text{or} \quad y = 2x - 1$$

Tangents (6 of 8)

- We sometimes refer to the slope of the tangent line to a curve at a point as the **slope of the curve** at the point.
- The idea is that if we zoom in far enough toward the point, the curve looks almost like a straight line.

- The more we zoom in, the more the parabola looks like a line.
- In other words, the curve becomes almost indistinguishable from its tangent line.
- There is another expression for the slope of a tangent line that is sometimes easier to use.

Tangents (7 of 8)

If $h = x - a$, then $x = a + h$ and so the slope of the secant line PQ is

$$m_{PQ} = \frac{f(a+h) - f(a)}{h}$$

(See Figure 3 where the case $h > 0$ is illustrated and Q is to the right of P. If it happened that $h < 0$, however, Q would be to the left of P.)

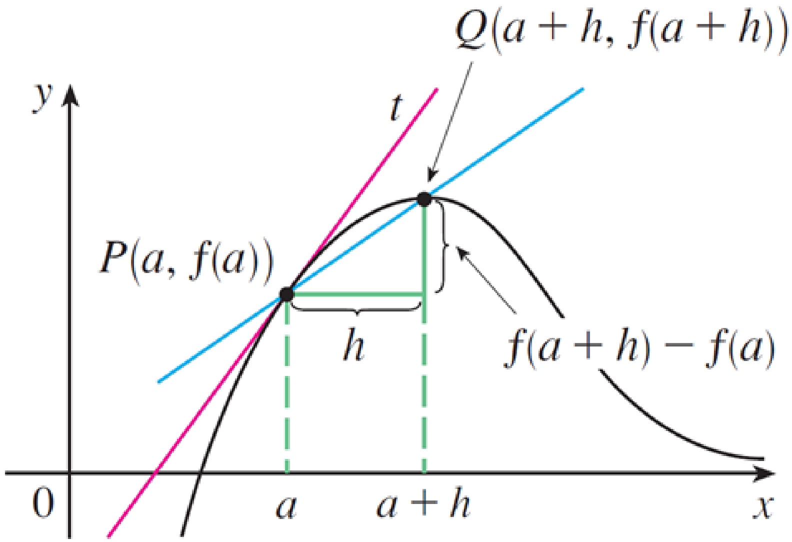


Figure 3

Velocities

Velocities (1 of 4)

In general, suppose an object moves along a straight line according to an equation of motion $s = f(t)$, where s is the displacement (directed distance) of the object from the origin at time t.

The function f that describes the motion is called the **position function** of the object.

In the time interval from $t = a$ to $t = a + h$ the change in position is $f(a + h) - f(a)$.

Velocities (2 of 4)

See Figure 5.

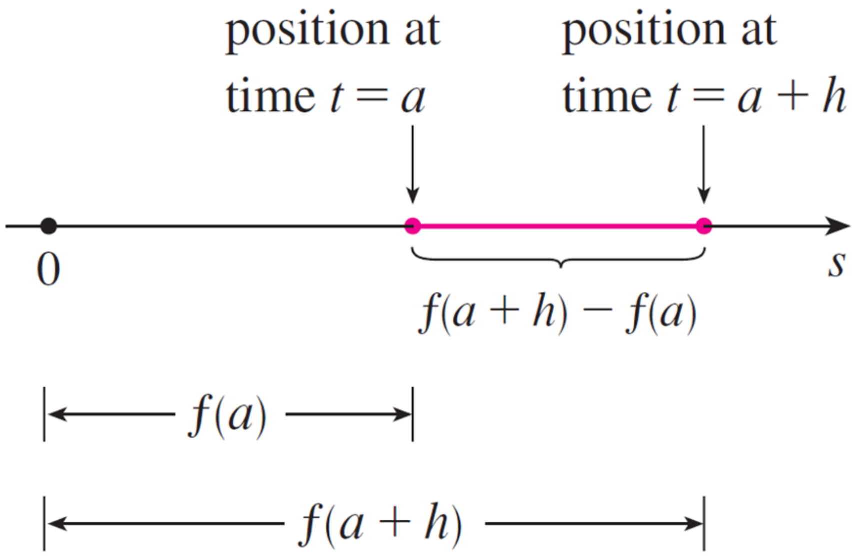


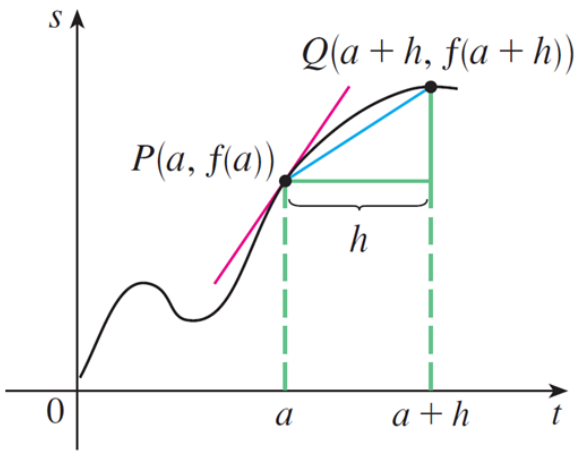
Figure 5

Velocities (3 of 4)

The average velocity over this time interval is

$$\text{average velocity} = \frac{\text{displacement}}{\text{time}} = \frac{f(a+h) - f(a)}{h}$$

which is the same as the slope of the secant line *PQ* in Figure 6.



$$m_{PQ} = \frac{f(a+h) - f(a)}{h}$$
$$= \text{average velocity}$$

Figure 6

Velocities (4 of 4)

Now suppose we compute the average velocities over shorter and shorter time intervals $[a, a + h]$.

In other words, we let h approach 0. As in the example of the falling ball, we define the **velocity** (or **instantaneous velocity**) $v(a)$ at time $t = a$ to be the limit of these average velocities:

3 Definition The **instantaneous velocity** of an object with position function $f(t)$ at time $t = a$ is

$$v(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

provided that this limit exists.

This means that the velocity at time $t = a$ is equal to the slope of the tangent line at P.

Example 3

Suppose that a ball is dropped from the upper observation deck of the CN Tower, 450 m above the ground.

(a) What is the velocity of the ball after 5 seconds?

(b) How fast is the ball traveling when it hits the ground?

Solution:

Since two different velocities are requested, it’s efficient to start by finding the velocity at a general time $t = a$.

Example 3 – Solution (1 of 4)

Using the equation of motion $s = f(t) = 4.9t^2$, we have

$$
\begin{aligned}
v(a) &= \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{4.9(a+h)^2 - 4.9a^2}{h} \\
&= \lim_{h \to 0} \frac{4.9(a^2 + 2ah + h^2 - a^2)}{h} = \lim_{h \to 0} \frac{4.9(2ah + h^2)}{h} \\
&= \lim_{h \to 0} \frac{4.9h(2a + h)}{h} = \lim_{h \to 0} 4.9(2a + h) = 9.8a
\end{aligned}
$$

Derivatives (1 of 4)

We have seen that the same type of limit arises in finding the slope of a tangent line (Equation 2) or the velocity of an object (Definition 3).

In fact, limits of the form

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

arise whenever we calculate a rate of change in any of the sciences or engineering, such as a rate of reaction in chemistry or a marginal cost in economics.

Since this type of limit occurs so widely, it is given a special name and notation.

Derivatives

4 Definition The **derivative of a function *f* at a number *a***, denoted by $f'(a)$, is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

if this limit exists.

If we write $x = a + h$, then we have $h = x - a$ and h approaches 0 if and only if x approaches a. Therefore an equivalent way of stating the definition of the derivative, as we saw in finding tangent lines, is

5
$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Example 4

Find the derivative of the function $f(x) = x^2 - 8x + 9$ at the numbers (a) 2 and (b) *a*.

Solution:

(a) From Definition 4 we have

$$
\begin{aligned}
f'(2) &= \lim_{h\to 0} \frac{f(2+h) - f(2)}{h} \\
&= \lim_{h\to 0} \frac{(2+h)^2 - 8(2+h) + 9 - (-3)}{h} \\
&= \lim_{h\to 0} \frac{4 + 4h + h^2 - 16 - 8h + 9 + 3}{h}
\end{aligned}
$$

$$= \lim_{x\to 0} \frac{h^2 - 4h}{h} = \mathbf{-4}$$

Example 5

EXAMPLE 4 Find f' if $f(x) = \dfrac{1-x}{2+x}$.

SOLUTION

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\dfrac{1-(x+h)}{2+(x+h)} - \dfrac{1-x}{2+x}}{h}$$

$$= \lim_{h\to 0} \frac{(1-x-h)(2+x) - (1-x)(2+x+h)}{h(2+x+h)(2+x)}$$

$$= \lim_{h\to 0} \frac{(2-x-2h-x^2-xh) - (2-x+h-x^2-xh)}{h(2+x+h)(2+x)}$$

$$= \lim_{h\to 0} \frac{-3h}{h(2+x+h)(2+x)} = \lim_{h\to 0} \frac{-3}{(2+x+h)(2+x)} = -\frac{3}{(2+x)^2}$$

Exercise

Using the definition of gradient (applying first principle) find the derivative of the following functions:

1. $f(x) = x^3 + \sqrt{x} + 2$
2. $f(x) = \frac{1}{\sqrt{x}}$
3. $f(x) = 2x + \sqrt{x} - 3$
4. $f(x) = 3x^3 - x$