Mathematics 1A

ITMTA1-B44

Functions and Models

With

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Lecture 1
Week 2

1.3 New Functions from Old Functions

Transformations of Functions

Transformations of Functions (1 of 8)

- By applying certain transformations to the graph of a given function we can obtain the graphs of related functions.
- This will give us the ability to sketch the graphs of many functions quickly by hand. It will also enable us to write equations for given graphs.
- Let’s first consider **translations** of graphs.
- If c is a positive number, then the graph of $y = f(x) + c$ is just the graph of $y = f(x)$ shifted upward a distance of c units (because each y-coordinate is increased by the same number c).

Transformations of Functions (2 of 8)

Likewise, if $g(x) = f(x - c)$, where $c > 0$, then the value of g at x is the same as the value of f at $x - c$ (c units to the left of x).

- $y = f(x) + c$, shift the graph of $y = f(x)$ a distance c units upward
- $y = f(x) - c$, shift the graph of $y = f(x)$ a distance c units downward
- $y = f(x - c)$, shift the graph of $y = f(x)$ a distance c units to the right
- $y = f(x + c)$, shift the graph of $y = f(x)$ a distance c units to the left

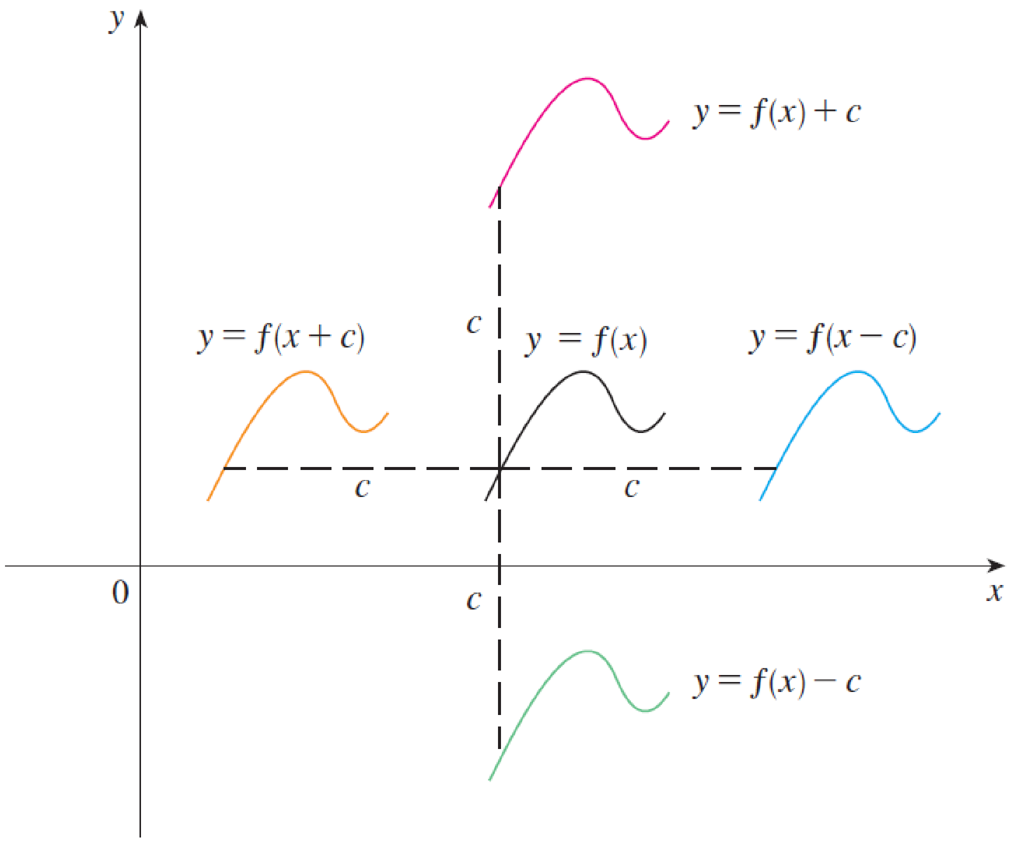


Figure 1 Translating the graph of *f*

Transformations of Functions (4 of 8)

The graph of *y* = −*f*(*x*) is the graph of *y* = *f*(*x*) reflected about the *x*-axis because the point (*x*, *y*) is replaced by the point (*x*, −*y*).

- $y = cf(x)$, stretch the graph of $y = f(x)$ vertically by a factor of c
- $y = (1/c)f(x)$, shrink the graph of $y = f(x)$ vertically by a factor of c
- $y = f(cx)$, shrink the graph of $y = f(x)$ horizontally by a factor of c
- $y = f\left(\frac{x}{c}\right)$, stretch the graph of $y = f(x)$ horizontally by a factor of c
- $y = -f(x)$, reflect the graph of $y = f(x)$ about the $x-$axis
- $y = f(-x)$, reflect the graph of $y = f(x)$ about the $y-$axis

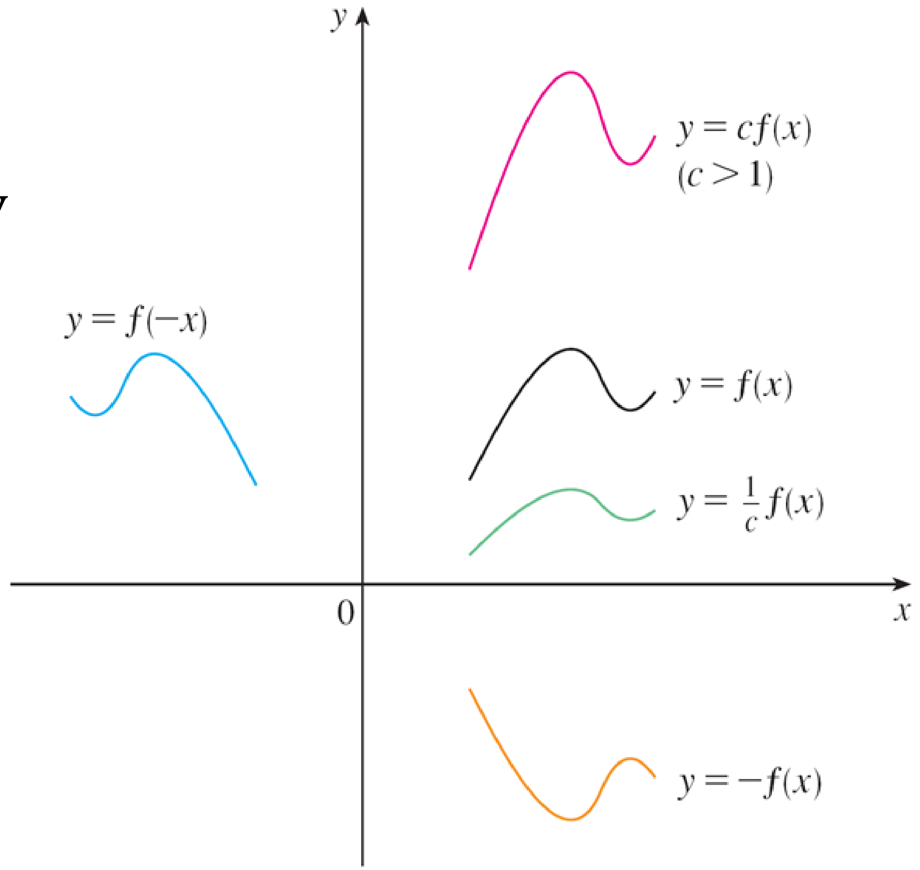


Figure 2 Stretching and reflecting the graph of *f*

Transformations of Functions (6 of 8)

Figure 3 illustrates these stretching transformations when applied to the cosine function with c = 2.

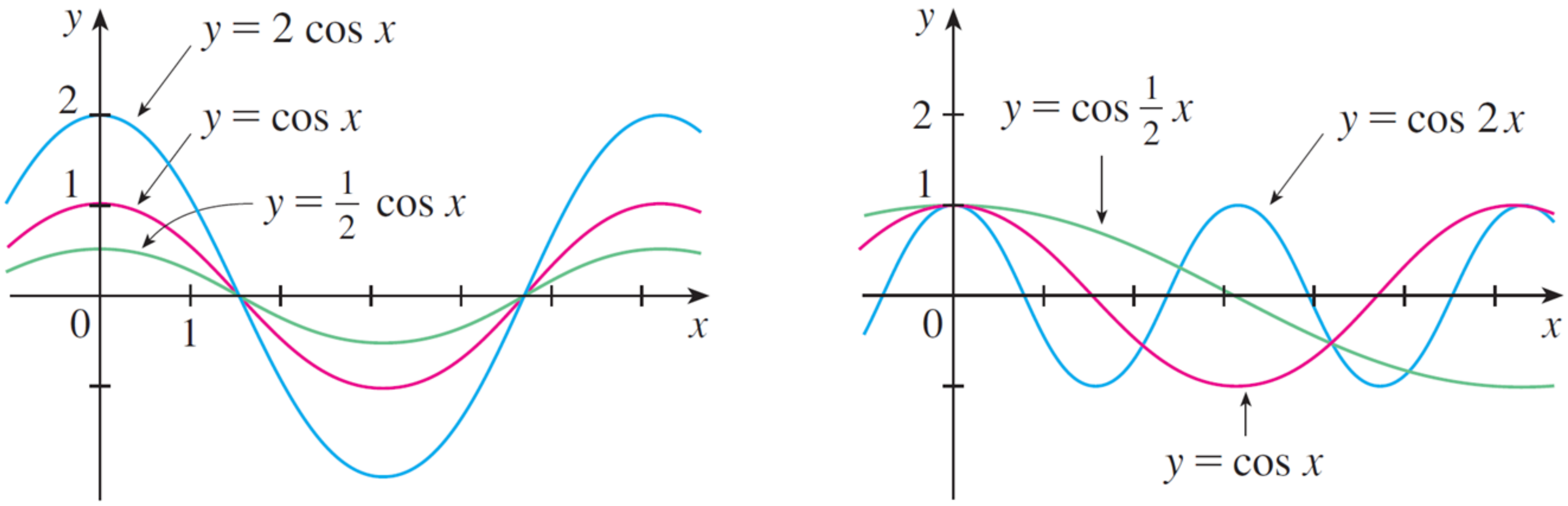
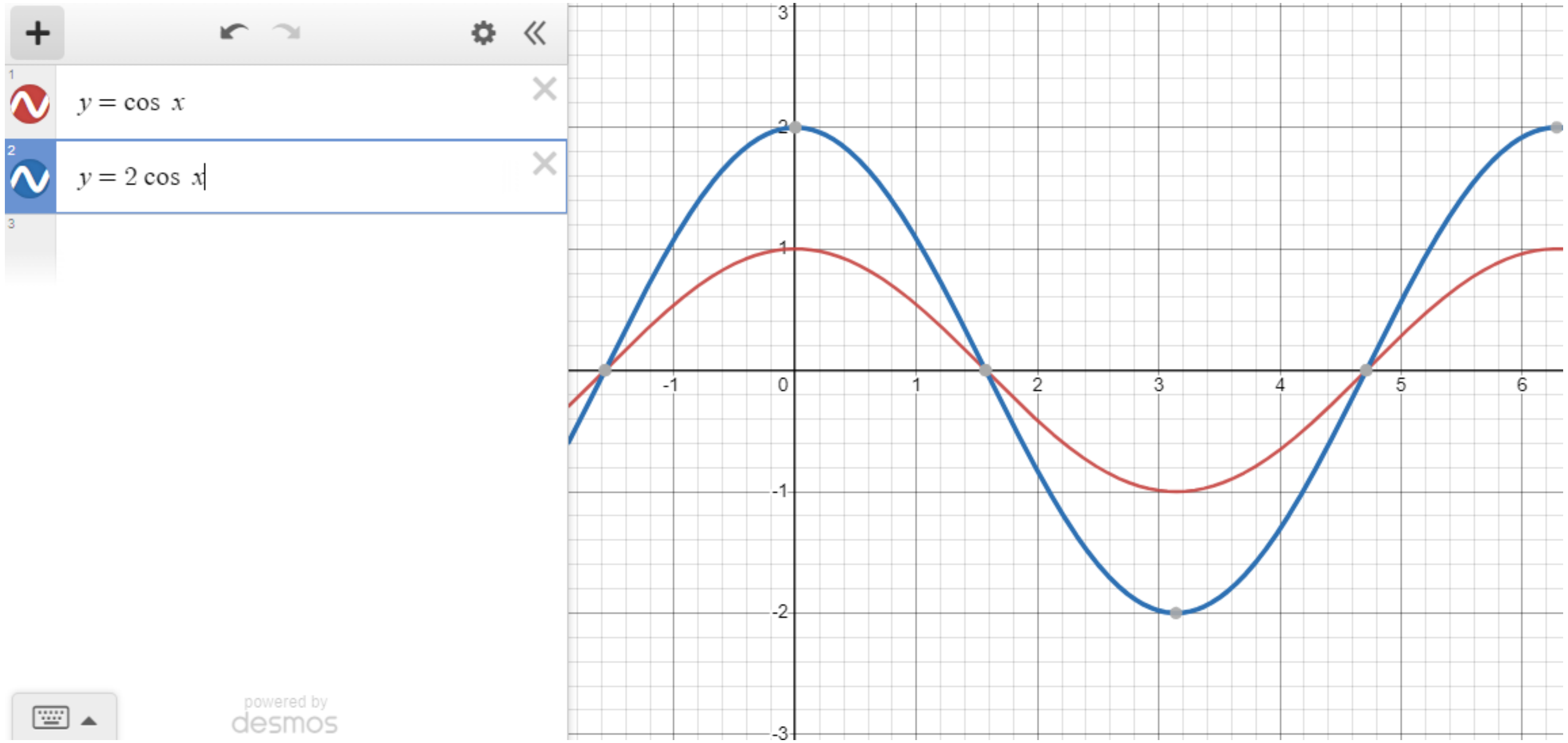


Figure 3

Example 1

- Plot the graph of $y = 2 \cos x$ by multiplying the y-coordinate of each point on the graph of $y = \cos x$ by 2.
- This means that the graph of $y = \cos x$ gets stretched vertically by a factor of 2.



Example 2

Given the graph of $y = \sqrt{x}$, use transformations to graph

$y = \sqrt{x} - 2$, $y = \sqrt{x - 2}$, $y = -\sqrt{x}$, $y = 2\sqrt{x}$, and $y = \sqrt{-x}$.

Solution:

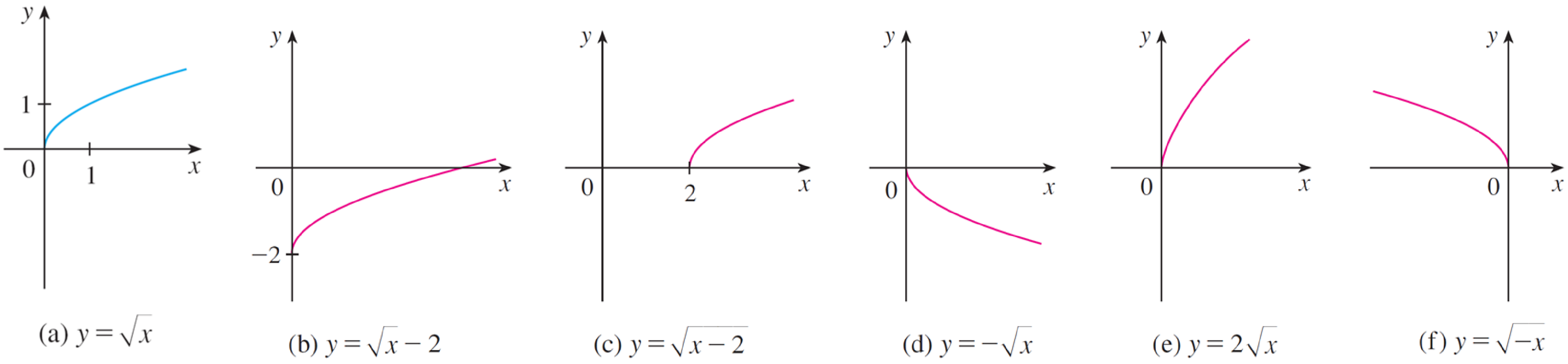


Figure 4

Combinations of Functions

Combinations of Functions (1 of 6)

Two functions *f* and *g* can be combined to form new functions *f* + *g*, *f* − *g*, *fg*, and $\frac{f}{g}$ in a manner similar to the way we add, subtract, multiply, and divide real numbers.

Definition Given two functions *f* and *g*, the **sum**, **difference**, **product**, and **quotient** functions are defined by

$$(f+g)(x)=f(x)+g(x) \quad (f-g)(x)=f(x)-g(x)$$

$$(fg)(x)=f(x)g(x) \quad \left(\frac{f}{g}\right)(x)=\frac{f(x)}{g(x)}$$

If the domain of *f* is *A* and the domain of *g* is *B*, then the domain of *f* + *g* is the intersection $A \cap B$ because both *f*(*x*) and *g*(*x*) have to be defined.

Combinations of Functions (2 of 6)

For example, the domain of $f(x)=\sqrt{x}$ is $A=[0,\ \infty)$ and the domain of $g(x)=\sqrt{2-x}$ is $B=(-\infty,\ 2]$, so the domain of $(f+g)(x)=\sqrt{x}+\sqrt{2-x}$ is $A\cap B=[0,\ 2]$.

The domain of *fg* is $A\cap B$, but we can’t divide by 0 and so the domain of $\frac{f}{g}$ is $\{x\in A\cap B \mid g(x)\neq 0\}$.

For instance, if $f(x)=x^2$ and $g(x) = x - 1$, then the domain of the rational function $\left(\frac{f}{g}\right)(x)=\frac{x^2}{(x-1)}$ is $\{x \mid x\neq 1\}$, or $(-\infty,\ 1)\cup(1,\ \infty)$.

Examples

Given $k(x) = 3x + 2$ and $g(x) = 4 - 5x$, find

i)$(k - g)(x)$

$$(k - g)(x) = [(3x + 2) - (4 - 5x)] = 3x + 2 - 4 + 5x = 8x - 2$$

ii) $(k + g)(x) = [(3x + 2) + (4 - 5x)] = 3x + 2 + 4 - 5x = 6 - 2x$

iii) $(k.g)(x) = (3x + 2)(4 - 5x) = 12x - 15x2 + 8 - 10x$

$$= 15x2 + 2x + 8$$

iv) $(\frac{k}{g})x = \frac{3x+2}{4-5x}$

Combinations of Functions (3 of 6)

There is another way of combining two functions to obtain a new function. For example, suppose that $y = f(u) = \sqrt{u}$ and $u = g(x) = x^2 + 1$.

Since *y* is a function of *u* and *u* is, in turn, a function of *x*, it follows that *y* is ultimately a function of *x*. We compute this by substitution:

$$y = f(u) = f(g(x)) = f(x^2 + 1) = \sqrt{x^2 + 1}$$

The procedure is called *composition* because the new function is *composed* of the two given functions *f* and *g*.

Combinations of Functions (4 of 6)

In general, given any two functions *f* and *g*, we start with a number *x* in the domain of *g* and calculate *g*(*x*). If this number *g*(*x*) is in the domain of *f*, then we can calculate the value of *f*(*g*(*x*)).

The result is a new function *h*(*x*) = *f*(*g*(*x*)) obtained by substituting *g* into *f*. It is called the *composition* (or *composite*) of *f* and *g* and is denoted by $f \circ g$ ("f circle g").

Definition Given two functions *f* and *g*, the **composite function** $f \circ g$ (also called the **composition** of *f* and *g*) is defined by

$$(f \circ g)(x) = f\big(g(x)\big)$$

Example 6

If $f(x) = x^2$ and $g(x) = x - 3$, find the composite functions $f \circ g$ and $g \circ f$.

Solution:

We have

$$(f \circ g)(x) = f(g(x)) = f(x-3) = (x-3)^2$$
$$(g \circ f)(x) = g(f(x)) = g(x^2) = x^2 - 3$$

Combinations of Functions (6 of 6)

We know that, the notation $f \circ g$ means that the function g is applied first and then f is applied second. In Example 6, $f \circ g$ is the function that *first* subtracts 3 and *then* squares; $g \circ f$ is the function that *first* squares and *then* subtracts 3.

It is possible to take the composition of three or more functions. For instance, the composite function $f \circ g \circ h$ is found by first applying h, then g, and then f as follows:

$$(f \circ g \circ h)(x) = f\left(g\left(h(x)\right)\right)$$

Exercise

If $f(x) = x^3 - x + 5 \; and \; g(x) = 3x - 2, find\; the\; composite\; function\; h(x)$ $where\; h(x) = f \circ g$ and $\mathrm{k(x)} where\; k(x) = g \circ f$.

Simplify your answers and hence find $h(-10) and\; k(\sqrt{2})$.