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The Squeeze Theorem Applied to Useful Trig Limits

Suggested Prerequesites: The Squeeze Theorem, An Introduction to Trig

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There are several useful trigonometric limits that are necessary for evaluating the derivatives of trigonometric functions. Let's start by stating some (hopefully) obvious limits:

$$\lim_{x \to 0} \sin(x) = 0$$
$$\lim_{x \to 0} \cos(x) = 1$$
$$\lim_{x \to 0} \tan(x) = 0.$$

Since each of the above functions is continuous at $x = 0$, the value of the limit at $x = 0$ is the value of the function at $x = 0$; this follows from the definition of limits.

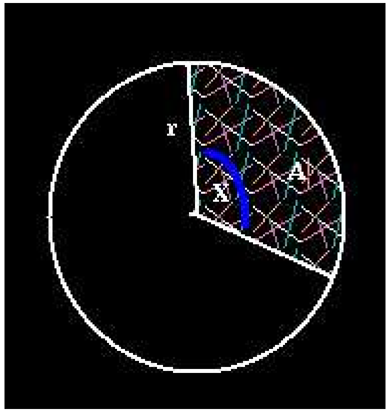
In order to evaluate the derivatives of sine and cosine we need to evaluate

$$\lim_{x \to 0} \frac{\sin(x)}{x} \qquad \text{and} \qquad \lim_{x \to 0} \frac{1 - \cos(x)}{x}.$$

In order to find these limits, we will need the following theorem of geometry:

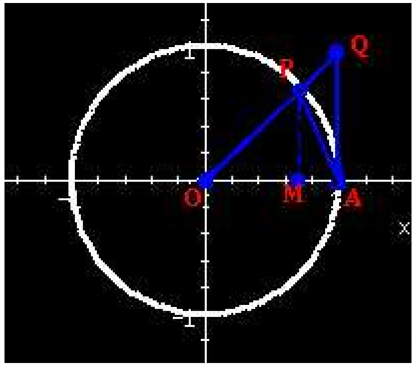
> If x is the measure of the central angle of a circle of radius r, then the area A of the sector determined by x is

$$A = r^2x/2$$



Let's start by looking at

$$\lim_{x \to 0} \frac{\sin(x)}{x}$$



If

$$0 \le x \le \frac{\pi}{2},$$

we have the situation in the figure to the left. Assume the circle is a unit circle, parameterized by
$x = \cos t$, $y = \sin t$
(for the rest of this page, the arguments of the trig functions will be denoted by t instead of x, in an attempt to reduce confusion with the cartesian coordinate).

If A_1 is the area of the triangle AOP, A_2 is the area of the circular sector AOP, and A_3 is the area of the triangle AOQ,

$$A_1 < A_2 < A_3.$$

The area of a triangle is equal to one-half of the product of the base times the height. Using this well-known result, and the above theorem for the area of a sector of a circle (with t as the central angle), we obtain:

$$
\begin{aligned}
A_1 &= \frac{1}{2}\left(\overline{OA}\right)\left(\overline{MP}\right) = \frac{1}{2}\sin(t) \\
A_2 &= \frac{1}{2}t \\
A_3 &= \frac{1}{2}\left(\overline{OA}\right)\left(\overline{AQ}\right) = \frac{1}{2}\tan(t).
\end{aligned}
$$

It follows that

$$\sin(t) < t < \tan(t),$$

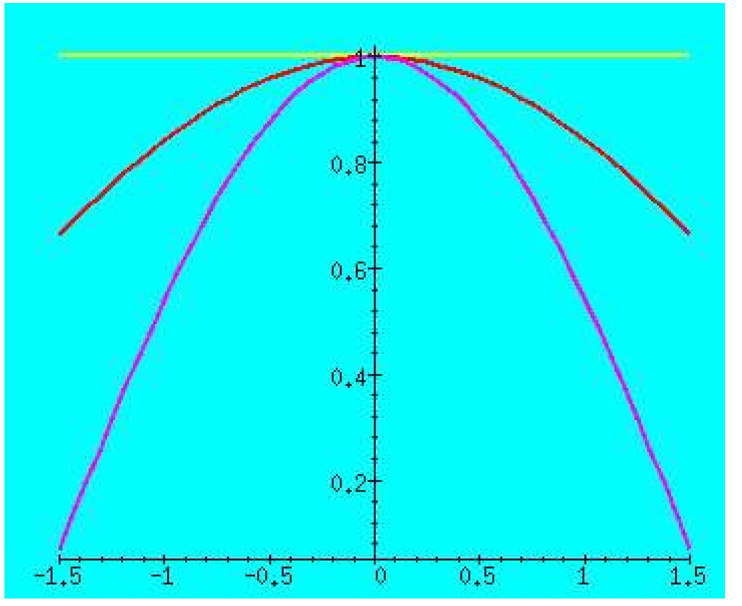
and hence

$$1 < \frac{t}{\sin(t)} < \frac{1}{\cos(t)},$$

which is equivalent to

$$1 > \frac{\sin(t)}{t} > \cos(t).$$

These three functions are easily plotted; the yellow line is the plot of the constant function 1, the magenta is the cosine, and the red is $\sin(t)/t$.



From the Squeeze Theorem, it follows that

$$\lim_{t \to 0} \frac{\sin(t)}{t} = 1.$$

To find

$$\lim_{t \to 0} \frac{1 - \cos(t)}{t},$$

we do some algebraic manipulations and trigonometric reductions:

$$\begin{aligned}
\frac{1-\cos(t)}{t} &= \left(\frac{1-\cos(t)}{t}\right)\left(\frac{1+\cos(t)}{1+\cos(t)}\right) \\
&= \frac{1-\cos^2(t)}{t\,(1+\cos(t))} \\
&= \frac{\sin^2(t)}{t\,(1+\cos(t))}
\end{aligned}$$

Therefore, it follows that

$$\lim_{t\to 0}\frac{1-\cos(t)}{t} = \lim_{t\to 0}\frac{\sin^2(t)}{t\,(1+\cos(t))} = \lim_{t\to 0}\left(\frac{\sin(t)}{t}\right)\left(\frac{\sin(t)}{(1+\cos(t))}\right) = 0.$$

To summarize the results of this page:

$$\begin{aligned}
\lim_{t\to 0}\frac{\sin(t)}{t} &= 1 \\
\lim_{t\to 0}\frac{1-\cos(t)}{t} &= 0.
\end{aligned}$$

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