Eduvos
Your Education. Your Future.

Mathematics 1A
ITMTA1-B44
Optimization

With

Amakan Elisha Agoni
Amakan.agoni@EDUVOS.com

Lecture 2
Week 7

5 Integrals

5.2 The Definite Integral

Evaluating Definite Integrals

Example 3

Evaluate

$$\int_0^3 (x^3 - 6x)dx.$$

Solution:

We have $f(x) = x^3 - 6x$, $a = 0$, $b = 3$,

$$\int_0^3 (x^3 - 6x)dx = \left[\frac{x^4}{4} - 3x^2\right]_0^3$$

$$\left[\frac{x^4}{4} - 3x^2\right]_0^3 = \left[\frac{(3)^4}{4} - 3(3)^2\right] - \left[\frac{(0)^4}{4} - 3(0)^2\right]$$

$$= \left[\frac{81}{4} - 27\right] - [0] = -\frac{27}{4}$$

The Midpoint Rule

The Midpoint Rule

We often choose the sample point x_i^* to be the right endpoint of the *i*th subinterval because it is convenient for computing the limit.

But if the purpose is to find an *approximation* to an integral, it is usually better to choose x_i^* to be the midpoint of the interval, which we denote by $\bar{x}_i$.

The Midpoint Rule

Any Riemann sum is an approximation to an integral, but if we use midpoints we get the following approximation.

Midpoint Rule

$$\int_a^b f(x)dx \approx \sum_{i=1}^{n} f(\overline{x}_i)\Delta x = \Delta x[f(\overline{x}_1) + \cdots + f(\bar{x}_n)]$$

where

$$\Delta x = \frac{b-a}{n}$$

and

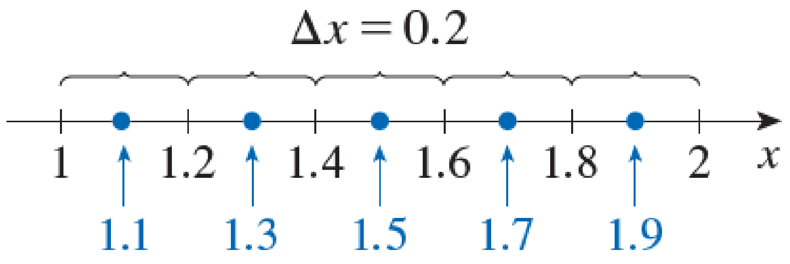
$$\bar{x}_i = \frac{1}{2}(x_{i-1} + x_i) = \text{midpoint of } [x_{i-1}, x_i]$$

Example 6

Use the Midpoint Rule with *n* = 5 to approximate $\int_1^2 \frac{1}{x}\,dx$.

Solution:

The endpoints of the five subintervals are 1, 1.2, 1.4, 1.6, 1.8, and 2.0, so the midpoints are 1.1, 1.3, 1.5, 1.7, and 1.9. (See Figure 11.)



The endpoints and midpoints of the subintervals used in Example 6

Figure 11

The width of the subintervals is $\Delta x = \frac{(2-1)}{5} = \frac{1}{5}$, so the Midpoint Rule gives

$$\int_1^2 \frac{1}{x}\,dx \approx \Delta x[f(1.1) + f(1.3) + f(1.5) + f(1.7) + f(1.9)]$$

Example 6 – Solution

$$= \frac{1}{5}\left(\frac{1}{1.1} + \frac{1}{1.3} + \frac{1}{1.5} + \frac{1}{1.7} + \frac{1}{1.9}\right)$$
$$\approx 0.691908$$

Since $f(x) = \frac{1}{x} > 0 \text{ for } 1 \leq x \leq 2,$

the integral represents an area, and the approximation given by the Midpoint Rule is the sum of the areas of the rectangles shown in Figure 12.

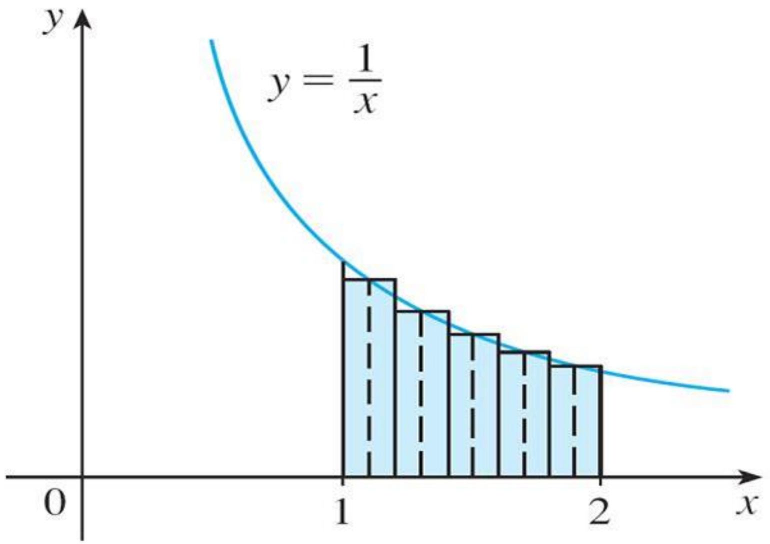


Figure 12

Exercise

Use the Midpoint Rule with $n = 5$ to approximate $\int_0^2 (x^2 - x\,)dx$.

Solution:

The endpoints of the five subintervals are 0, 0.4, 0.8, 1.2, 1.6, and 2, so the midpoints are 0.2, 0.6, 1, 1.4, and 1.8. (See Figure 11.)

The width of the subintervals is $\Delta x = \dfrac{(2-0)}{5} = \dfrac{2}{5}$, so the Midpoint Rule gives

$$\int_0^2 (x^2 - x\,)dx \approx \Delta x[f(0.2) + f(0.6) + f(1) + f(1.4) + f(1.8)]$$

$$\int_0^2 (x^2 - x\,)dx \approx \frac{16}{25} = \mathbf{0.64}$$

Properties of the Definite Integral

Properties of the Definite Integral

Properties of the Integral

1. $\int_a^b c \ dx = c(b-a)$, where c is any constant

2. $\int_a^b [f(x)+g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$

3. $\int_a^b cf(x)dx = c\int_a^b f(x)dx$, where c is any constant

4. $\int_a^b [f(x)-g(x)]dx = \int_a^b f(x)dx - \int_a^b g(x)dx$

Example 7

Use the properties of integrals to evaluate $\int_0^1 (4+3x^2)dx$.

Solution:

Using Properties 2 and 3 of integrals, we have

$$\begin{aligned} \int_0^1 (4+3x^2)dx &= \int_0^1 4dx + \int_0^1 3x^2dx \\ &= \int_0^1 4dx + 3\int_0^1 x^2dx \end{aligned}$$

We know from Property 1 that $\int_0^1 4dx = 4(1-0) = 4$

and we have found that $\int_0^1 x^2dx = \frac{1}{3}$.

So

$$\begin{aligned} \int_0^1 (4+3x^2)dx &= \int_0^1 4dx + 3\int_0^1 x^2dx \\ &= 4 + 3\cdot\frac{1}{3} \\ &= 5 \end{aligned}$$

Example 3

Evaluate

$$\int_0^3 (x^3 - 6x)dx.$$

Solution:

We have $f(x) = x^3 - 6x$, $a = 0$, $b = 3$,

$$\int_0^3 (x^3 - 6x)dx = \left[\frac{x^4}{4} - 3x^2\right]_0^3$$

$$\left[\frac{x^4}{4} - 3x^2\right]_0^3 = \left[\frac{(3)^4}{4} - 3(3)^2\right] - \left[\frac{(0)^4}{4} - 3(0)^2\right]$$

$$= \left[\frac{81}{4} - 27\right] - [0] = -\frac{27}{4}$$

Exercises

Using Riemann's Left and Right Sum, determine the area $\int_0^3 (x^3 - 6x)dx$ over 6 intervals.

$$\Delta x = \frac{b-a}{n} = \frac{3-0}{6} = \frac{3}{6} = 0.5$$

So, the right endpoints are given by: $x_i = a + i\Delta x$

x_0=0 x_1=0.5 x_2=1 x_3=1.5 x_4=2.0 x_5=2.5 x_6=3

Riemann Left sum

$$A = L_n = \Delta x[f(x_0) + f(x_1) + f(x_2) + f(x_3) + f(x_4) + f(x_5)]$$
$$A = L_n = 0.5[f(0) + f(0.5) + f(1) + f(1.5) + f(2) + f(2.5)]$$

$$A = L_n = -8.4375$$

Riemann Right sum

$$A = R_n = \Delta x[f(x_1) + f(x_2) + f(x_3) + f(x_4) + f(x_5) + f(x_6)]$$
$$A = R_n = 0.5[f(0.5) + f(1) + f(1.5) + f(2) + f(2.5) + f(3)]$$

$$A = R_n = -3.9375$$

Exercise

1. Use the properties of integrals to evaluate $\int_{-1}^{1} (x^2 - 2x)dx$.

Answer = $\mathbf{\frac{2}{3}}$

2. Evaluate the Right and left Riemann sums $\int_{-1}^{1} (x^2 - 2x)dx$ over 4 intervals.

$\boldsymbol{R_n = -\frac{1}{4} = -0.25}$

$\boldsymbol{L_n = 1.75}$

3. Evaluate the mid-point sum $\int_{-1}^{1} (x^2 - 2x)dx$ over 5 intervals.

$\boldsymbol{S_n = \frac{16}{25} = 0.64}$