Part 1

MATRICES AND LINEAR EQUATIONS

CHAPTER 1

SYSTEMS OF LINEAR EQUATIONS

1.1. Background

Topics: systems of linear equations; Gaussian elimination (Gauss' method), elementary row operations, leading variables, free variables, echelon form, matrix, augmented matrix, Gauss-Jordan reduction, reduced echelon form.

1.1.1. Definition. We will say that an operation (sometimes called *scaling*) which multiplies a row of a matrix (or an equation) by a nonzero constant is a ROW OPERATION OF TYPE I. An operation (sometimes called *swapping*) that interchanges two rows of a matrix (or two equations) is a ROW OPERATION OF TYPE II. And an operation (sometimes called *pivoting*) that adds a multiple of one row of a matrix to another row (or adds a multiple of one equation to another) is a ROW OPERATION OF TYPE III.

1.2. Exercises

(1) Suppose that L_1 and L_2 are lines in the plane, that the x-intercepts of L_1 and L_2 are 5 and -1, respectively, and that the respective y-intercepts are 5 and 1. Then L_1 and L_2 intersect at the point (___ , ___) .

(2) Consider the following system of equations.

$$\begin{cases} w + x + y + z = 6 \\ w \qquad + y + z = 4 \\ w \qquad + y \qquad = 2 \end{cases} \tag{$*$}$$

(a) List the leading variables ____________ .

(b) List the free variables ____________ .

(c) The general solution of $(*)$ (expressed in terms of the free variables) is

(________ , ___ , ___ , ___) .

(d) Suppose that a fourth equation $-2w + y = 5$ is included in the system $(*)$. What is the solution of the resulting system? Answer: (___ , ___ , ___ , ___).

(e) Suppose that instead of the equation in part (d), the equation $-2w - 2y = -3$ is included in the system $(*)$. Then what can you say about the solution(s) of the resulting system? Answer: __________________________ .

(3) Consider the following system of equations:

$$\begin{cases} x + \ \ y + \ \ z = 2 \\ x + 3y + 3z = 0 \\ x + 3y + 6z = 3 \end{cases} \tag{$*$}$$

(a) Use Gaussian elimination to put the augmented coefficient matrix into row echelon form. The result will be $\begin{bmatrix} 1 & 1 & 1 & a \\ 0 & 1 & 1 & b \\ 0 & 0 & 1 & c \end{bmatrix}$ where $a =$ _____ , $b =$ _____ , and $c =$ _____ .

(b) Use Gauss-Jordan reduction to put the augmented coefficient matrix in reduced row echelon form. The result will be $\begin{bmatrix} 1 & 0 & 0 & d \\ 0 & 1 & 0 & e \\ 0 & 0 & 1 & f \end{bmatrix}$ where $d =$ _____ , $e =$ _____ , and $f =$ _____ .

(c) The solutions of $(*)$ are $x =$ _____ , $y =$ _____ , and $z =$ _____ .

(4) Consider the following system of equations.

$$0.003000x + 59.14y = 59.17$$
$$5.291x - 6.130y = 46.78.$$

(a) Using only row operation III and back substitution find the exact solution of the system. Answer: $x =$ _____ , $y =$ _____ .

(b) Same as (a), but after performing each arithmetic operation round off your answer to four significant figures. Answer: $x =$ _________ , $y =$ _________.

(5) Find the values of k for which the system of equations

$$\begin{cases} x + ky = 1 \\ kx + y = 1 \end{cases}$$

has (a) no solution. Answer: ________________ .

(b) exactly one solution. Answer: ________________ .

(c) infinitely many solutions. Answer: ________________ .

(d) When there is exactly one solution, it is $x =$ ________ and $y =$ ________ .

(6) Consider the following two systems of equations.

$$\begin{cases} x + y + z = 6 \\ x + 2y + 2z = 11 \\ 2x + 3y - 4z = 3 \end{cases} \tag{1}$$

and

$$\begin{cases} x + y + z = 7 \\ x + 2y + 2z = 10 \\ 2x + 3y - 4z = 3 \end{cases} \tag{2}$$

Solve both systems simultaneously by applying Gauss-Jordan reduction to an appropriate 3×5 matrix.

(a) The resulting row echelon form of this 3×5 matrix is $\begin{bmatrix} & & & & \\ & & & & \\ & & & & \end{bmatrix}$.

(b) The resulting reduced row echelon form is $\begin{bmatrix} & & & & \\ & & & & \\ & & & & \end{bmatrix}$.

(c) The solution for (1) is (____ , ____ , ____) and the solution for (2) is (____ , ____ , ____) .

(7) Consider the following system of equations:

$$\begin{cases} x - y - 3z = 3 \\ 2x \qquad + z = 0 \\ \qquad 2y + 7z = c \end{cases}$$

(a) For what values of c does the system have a solution? Answer: $c =$ ____ .

(b) For the value of c you found in (a) describe the solution set geometrically as a subset of $\mathbb{R}^3$. Answer: ________________ .

(c) What does part (a) say about the planes $x - y - 3z = 3$, $2x + z = 0$, and $2y + 7z = 4$ in $\mathbb{R}^3$? Answer: ____________________________ .

(8) Consider the following system of linear equations (where $b_1, \dots, b_5$ are constants).

$$\begin{cases} u + 2v - w - 2x + 3y \phantom{{}-4z} = b_1 \\ x - y + 2z = b_2 \\ 2u + 4v - 2w - 4x + 7y - 4z = b_3 \\ -x + y - 2z = b_4 \\ 3u + 6v - 3w - 6x + 7y + 8z = b_5 \end{cases}$$

(a) In the process of Gaussian elimination the leading variables of this system are ________ and the free variables are ___________ .

(b) What condition(s) must the constants $b_1, \dots, b_5$ satisfy so that the system is consistent? Answer: ____ .

(c) Do the numbers $b_1 = 1$, $b_2 = -3$, $b_3 = 2$, $b_4 = b_5 = 3$ satisfy the condition(s) you listed in (b)? _______ . If so, find the general solution to the system as a function of the free variables. Answer:

$u =$ ______________________

$v =$ __________

$w =$ __________

$x =$ __________

$y =$ __________

$z =$ __________ .

(9) Consider the following homogeneous system of linear equations (where a and b are nonzero constants).

$$\begin{cases} x + 2y \phantom{{}+3z} = 0 \\ ax + 8y + 3z = 0 \\ by + 5z = 0 \end{cases}$$

(a) Find a value for a which will make it necessary during Gaussian elimination to interchange rows in the coefficient matrix. Answer: $a =$ ____ .

(b) Suppose that a does not have the value you found in part (a). Find a value for b so that the system has a nontrivial solution.
Answer: $b = \frac{c}{3} + \frac{d}{3}a$ where $c =$ _______ and $d =$ _______ .

(c) Suppose that a does not have the value you found in part (a) and that $b = 100$. Suppose further that a is chosen so that the solution to the system is *not* unique. The general solution to the system (in terms of the free variable) is $\left(\frac{1}{\alpha}\, z \,, -\frac{1}{\beta}\, z \,,\, z \right)$ where $\alpha =$ ______ and $\beta =$ ______ .

1.3. Problems

(1) Give a geometric description of a single linear equation in three variables.

Then give a geometric description of the solution set of a system of 3 linear equations in 3 variables if the system

(a) is inconsistent.
(b) is consistent and has no free variables.
(c) is consistent and has exactly one free variable.
(d) is consistent and has two free variables.

(2) Consider the following system of equations:

$$\begin{cases} -m_1x + y = b_1 \\ -m_2x + y = b_2 \end{cases} \tag{$*$}$$

(a) Prove that if $m_1 \neq m_2$, then $(*)$ has exactly one solution. What is it?
(b) Suppose that $m_1 = m_2$. Then under what conditions will $(*)$ be consistent?
(c) Restate the results of (a) and (b) in geometrical language.

1.4. Answers to Odd-Numbered Exercises

(1) 2, 3

(3)
 (a) 2, -1, 1
 (b) 3, -2, 1
 (c) 3, -2, 1

(5)
 (a) $k = -1$
 (b) $k \neq -1, 1$
 (c) $k = 1$
 (d) $\dfrac{1}{k+1}$, $\dfrac{1}{k+1}$

(7)
 (a) -6
 (b) a line
 (c) They have no points in common.

(9)
 (a) 4
 (b) 40, -10
 (c) 10, 20

CHAPTER 2

ARITHMETIC OF MATRICES

2.1. Background

Topics: addition, scalar multiplication, and multiplication of matrices, inverse of a nonsingular matrix.

2.1.1. Definition. Two square matrices A and B of the same size are said to COMMUTE if $AB = BA$.

2.1.2. Definition. If A and B are square matrices of the same size, then the COMMUTATOR (or LIE BRACKET) of A and B, denoted by $[A, B]$, is defined by

$$[A, B] = AB - BA\,.$$

2.1.3. Notation. If A is an $m \times n$ matrix (that is, a matrix with m rows and n columns), then the element in the i^{th} row and the j^{th} column is denoted by a_{ij}. The matrix A itself may be denoted by $\big[a_{ij}\big]_{i=1}^{m}{}_{j=1}^{n}$ or, more simply, by $[a_{ij}]$. In light of this notation it is reasonable to refer to the index i in the expression a_{ij} as the ROW INDEX and to call j the COLUMN INDEX. When we speak of the "value of a matrix A at (i, j)," we mean the entry in the i^{th} row and j^{th} column of A. Thus, for example,

$$A = \begin{bmatrix} 1 & 4 \\ 3 & -2 \\ 7 & 0 \\ 5 & -1 \end{bmatrix}$$

is a 4×2 matrix and $a_{31} = 7$.

2.1.4. Definition. A matrix $A = [a_{ij}]$ is UPPER TRIANGULAR if $a_{ij} = 0$ whenever $i > j$.

2.1.5. Definition. The TRACE of a square matrix A, denoted by $\operatorname{tr} A$, is the sum of the diagonal entries of the matrix. That is, if $A = [a_{ij}]$ is an $n \times n$ matrix, then

$$\operatorname{tr} A := \sum_{j=1}^{n} a_{jj}.$$

2.1.6. Definition. The TRANSPOSE of an $n \times n$ matrix $A = \big[a_{ij}\big]$ is the matrix $A^t = \big[a_{ji}\big]$ obtained by interchanging the rows and columns of A. The matrix A is SYMMETRIC if $A^t = A$.

2.1.7. Proposition. *If A is an $m \times n$ matrix and B is an $n \times p$ matrix, then $(AB)^t = B^t A^t$.*

2.2. Exercises

(1) Let $A = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 3 & 1 & -1 \\ 2 & 4 & 0 & 3 \\ -3 & 1 & -1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 3 & -1 \\ 0 & -2 \\ 4 & 1 \end{bmatrix}$, and $C = \begin{bmatrix} 3 & -2 & 0 & 5 \\ 1 & 0 & -3 & 4 \end{bmatrix}$.

(a) Does the matrix $D = ABC$ exist? ______ If so, then $d_{34} =$ ______ .
(b) Does the matrix $E = BAC$ exist? ______ If so, then $e_{22} =$ ______ .
(c) Does the matrix $F = BCA$ exist? ______ If so, then $f_{43} =$ ______ .
(d) Does the matrix $G = ACB$ exist? ______ If so, then $g_{31} =$ ______ .
(e) Does the matrix $H = CAB$ exist? ______ If so, then $h_{21} =$ ______ .
(f) Does the matrix $J = CBA$ exist? ______ If so, then $j_{13} =$ ______ .

(2) Let $A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, and $C = AB$. Evaluate the following.

(a) $A^{37} = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}$ (b) $B^{63} = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}$

(c) $B^{138} = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}$ (d) $C^{42} = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}$

Note: If M is a matrix M^p is the product of p copies of M.

(3) Let $A = \begin{bmatrix} 1 & 1/3 \\ c & d \end{bmatrix}$. Find numbers c and d such that $A^2 = -I$.
Answer: $c =$ ______ and $d =$ ______ .

(4) Let A and B be symmetric $n \times n$-matrices. Then $[A, B] = [B, X]$, where $X =$ ______ .

(5) Let A, B, and C be $n \times n$ matrices. Then $[A, B]C + B[A, C] = [X, Y]$, where $X =$ _____ and $Y =$ ________ .

(6) Let $A = \begin{bmatrix} 1 & 1/3 \\ c & d \end{bmatrix}$. Find numbers c and d such that $A^2 = 0$. Answer: $c =$ ______ and $d =$ ______ .

(7) Consider the matrix $\begin{bmatrix} 1 & 3 & 2 \\ a & 6 & 2 \\ 0 & 9 & 5 \end{bmatrix}$ where a is a real number.

(a) For what value of a will a row interchange be required during Gaussian elimination? Answer: $a =$ _____ .
(b) For what value of a is the matrix singular? Answer: $a =$ ______ .

(8) Let $A = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 3 & 1 & -1 \\ 2 & 4 & 0 & 3 \\ -3 & 1 & -1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 3 & -1 \\ 0 & -2 \\ 4 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 3 & -2 & 0 & 5 \\ 1 & 0 & -3 & 4 \end{bmatrix}$, and
$M = 3A^3 - 5(BC)^2$. Then $m_{14} =$ ____________ and $m_{41} =$ ____________ .

(9) If A is an $n \times n$ matrix and it satisfies the equation $A^3 - 4A^2 + 3A - 5I_n = 0$, then A is nonsingular

and its inverse is ______________________________ .

(10) Let A, B, and C be $n \times n$ matrices. Then $[[A,B],C]+[[B,C],A]+[[C,A],B]=X$, where

$$X = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}.$$

(11) Let A, B, and C be $n \times n$ matrices. Then $[A,C]+[B,C]=[X,Y]$, where $X =$ ________ and
$Y =$ ____ .

(12) Find the inverse of $\begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{4} & 1 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & 1 & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 1 \end{bmatrix}$. Answer: $\begin{bmatrix} & & & \\ & & & \\ & & & \\ & & & \end{bmatrix}$.

(13) The matrix

$$H = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} \end{bmatrix}$$

is the 4×4 HILBERT MATRIX. Use Gauss-Jordan elimination to compute $K = H^{-1}$. Then K_{44} is (exactly) __________ . Now, create a new matrix H' by replacing each entry in H by its approximation to 3 decimal places. (For example, replace $\frac{1}{6}$ by 0.167.) Use Gauss-Jordan elimination again to find the inverse K' of H'. Then K'_{44} is _______________ .

(14) Suppose that A and B are symmetric $n \times n$ matrices. In this exercise we prove that AB is symmetric if and only if A commutes with B. Below are portions of the proof. Fill in the missing steps and the missing reasons. Choose reasons from the following list.

(H1) Hypothesis that A and B are symmetric.
(H2) Hypothesis that AB is symmetric.
(H3) Hypothesis that A commutes with B.
(D1) Definition of *commutes*.
(D2) Definition of *symmetric*.
(T) Proposition 2.1.7.

PROOF. Suppose that AB is symmetric. Then

$AB =$ ______ (reason: (H2) and _____)
$= B^tA^t$ (reason: _____)
$=$ ______ (reason: (D2) and _____)

So A commutes with B (reason: _____).

Conversely, suppose that A commutes with B. Then

$(AB)^t =$ ______ (reason: (T))
$= BA$ (reason: _____ and _____)
$=$ ______ (reason: _____ and _____)

Thus AB is symmetric (reason: _______). □

2.3. Problems

(1) Let A be a square matrix. Prove that if A^2 is invertible, then so is A.
Hint. Our assumption is that there exists a matrix B such that
$$A^2B = BA^2 = I\,.$$
We want to show that there exists a matrix C such that
$$AC = CA = I\,.$$
Now to start with, you ought to find it fairly easy to show that there are matrices L and R such that
$$LA = AR = I\,. \tag{$*$}$$
A matrix L is a LEFT INVERSE of the matrix A if $LA = I$; and R is a RIGHT INVERSE of A if $AR = I$. Thus the problem boils down to determining whether A can have a left inverse and a right inverse that are *different*. (Clearly, if it turns out that they must be the same, then the C we are seeking is their common value.) So try to prove that if $(*)$ holds, then $L = R$.

(2) Anton speaks French and German; Geraldine speaks English, French and Italian; James speaks English, Italian, and Spanish; Lauren speaks all the languages the others speak except French; and no one speaks any other language. Make a matrix $A = \begin{bmatrix} a_{ij} \end{bmatrix}$ with rows representing the four people mentioned and columns representing the languages they speak. Put $a_{ij} = 1$ if person i speaks language j and $a_{ij} = 0$ otherwise. Explain the significance of the matrices AA^t and A^tA.

(3) Portland Fast Foods (PFF), which produces 138 food products all made from 87 basic ingredients, wants to set up a simple data structure from which they can quickly extract answers to the following questions:
   (a) How many ingredients does a given product contain?
   (b) A given pair of ingredients are used together in how many products?
   (c) How many ingredients do two given products have in common?
   (d) In how many products is a given ingredient used?

   In particular, PFF wants to set up a single table in such a way that:
   (i) the answer to any of the above questions can be extracted easily and quickly (matrix arithmetic permitted, of course); and
   (ii) if one of the 87 ingredients is added to or deleted from a product, only a single entry in the table needs to be changed.

   Is this possible? Explain.

(4) Prove proposition 2.1.7.

(5) Let A and B be 2×2 matrices.
   (a) Prove that if the trace of A is 0, then A^2 is a scalar multiple of the identity matrix.
   (b) Prove that the square of the commutator of A and B commutes with every 2×2 matrix C. *Hint.* What can you say about the trace of $[A, B]$?
   (c) Prove that the commutator of A and B can never be a nonzero multiple of the identity matrix.

(6) The matrices that represent rotations of the xy-plane are

$$A(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

(a) Let $\mathbf{x}$ be the vector $(-1, 1)$, $\theta = 3\pi/4$, and $\mathbf{y}$ be $A(\theta)$ acting on $\mathbf{x}$ (that is, $\mathbf{y} = A(\theta)\mathbf{x}^t$). Make a sketch showing $\mathbf{x}$, $\mathbf{y}$, and θ.

(b) Verify that $A(\theta_1)A(\theta_2) = A(\theta_1 + \theta_2)$. Discuss what this means geometrically.

(c) What is the product of $A(\theta)$ times $A(-\theta)$? Discuss what this means geometrically.

(d) Two sheets of graph paper are attached at the origin and rotated in such a way that the point $(1, 0)$ on the upper sheet lies directly over the point $(-5/13, 12/13)$ on the lower sheet. What point on the lower sheet lies directly below $(6, 4)$ on the upper one?

(7) Let

$$A = \begin{bmatrix} 0 & a & a^2 & a^3 & a^4 \\ 0 & 0 & a & a^2 & a^3 \\ 0 & 0 & 0 & a & a^2 \\ 0 & 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The goal of this problem is to develop a “calculus” for the matrix A. To start, recall (or look up) the power series expansion for $\dfrac{1}{1-x}$. Now see if this formula works for the matrix A by first computing $(I - A)^{-1}$ directly and then computing the power series expansion substituting A for x. (Explain why there are no convergence difficulties for the series when we use this particular matrix A.) Next try to define $\ln(I + A)$ and e^A by means of appropriate series. Do you get what you expect when you compute $e^{\ln(I+A)}$? Do formulas like $e^A e^A = e^{2A}$ hold? What about other familiar properties of the exponential and logarithmic functions?

Try some trigonometry with A. Use series to define sin, cos, tan, arctan, and so on. Do things like $\tan(\arctan(A))$ produce the expected results? Check some of the more obvious trigonometric identities. (What do you get for $\sin^2 A + \cos^2 A - I$? Is $\cos(2A)$ the same as $\cos^2 A - \sin^2 A$?)

A relationship between the exponential and trigonometric functions is given by the famous formula $e^{ix} = \cos x + i \sin x$. Does this hold for A?

Do you think there are other matrices for which the same results might hold? Which ones?

(8) (a) Give an example of two symmetric matrices whose product is not symmetric.
Hint. Matrices containing only 0’s and 1’s will suffice.

(b) Now suppose that A and B are symmetric $n \times n$ matrices. Prove that AB is symmetric if and only if A commutes with B.

Hint. To prove that a statement P holds “if and only if” a statement Q holds you must first show that P implies Q and then show that Q implies P. In the current problem, there are 4 conditions to be considered:

(i) $A^t = A$ (A is symmetric),
(ii) $B^t = B$ (B is symmetric),
(iii) $(AB)^t = AB$ (AB is symmetric), and
(iv) $AB = BA$ (A commutes with B).

Recall also the fact given in
(v) theorem 2.1.7.

The first task is to derive (iv) from (i), (ii), (iii), and (v). Then try to derive (iii) from (i), (ii), (iv), and (v).

2.4. Answers to Odd-Numbered Exercises

(1) (a) yes, 142
    (b) no, –
    (c) yes, -45
    (d) no, –
    (e) yes, -37
    (f) no, –

(3) -6, -1

(5) A, BC

(7) (a) 2
    (b) -4

(9) $\dfrac{1}{5}(A^2 - 4A + 3I_n)$

(11) $A + B$, C

(13) 2800, -1329.909

CHAPTER 3

ELEMENTARY MATRICES; DETERMINANTS

3.1. Background

Topics: elementary (reduction) matrices, determinants.

The following definition says that we often regard the effect of multiplying a matrix M on the left by another matrix A as the *action of A on M*.

3.1.1. Definition. We say that the matrix A ACTS ON the matrix M to produce the matrix N if $N = AM$. For example the matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ acts on any 2×2 matrix by interchanging (swapping) its rows because $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$.

3.1.2. Notation. We adopt the following notation for elementary matrices which implement type I row operations. Let A be a matrix having n rows. For any real number $r \neq 0$ denote by $M_j(r)$ the $n \times n$ matrix which acts on A by multiplying its j^{th} row by r. (See exercise 1.)

3.1.3. Notation. We use the following notation for elementary matrices which implement type II row operations. (See definition 1.1.1.) Let A be a matrix having n rows. Denote by P_{ij} the $n \times n$ matrix which acts on A by interchanging its i^{th} and j^{th} rows. (See exercise 2.)

3.1.4. Notation. And we use the following notation for elementary matrices which implement type III row operations. (See definition 1.1.1.) Let A be a matrix having n rows. For any real number r denote by $E_{ij}(r)$ the $n \times n$ matrix which acts on A by adding r times the j^{th} row of A to the i^{th} row. (See exercise 3.)

3.1.5. Definition. If a matrix B can be produced from a matrix A by a sequence of elementary row operations, then A and B are ROW EQUIVALENT.

Some Facts about Determinants

3.1.6. Proposition. *Let $n \in \mathbb{N}$ and $\mathbf{M}_{n\times n}$ be the collection of all $n \times n$ matrices. There is exactly one function*

$$\det\colon \mathbf{M}_{n\times n} \to \mathbb{R}\colon A \mapsto \det A$$

which satisfies

(a) $\det I_n = 1$.
(b) *If $A \in \mathbf{M}_{n\times n}$ and A' is the matrix obtained by interchanging two rows of A, then* $\det A' = -\det A$.
(c) *If $A \in \mathbf{M}_{n\times n}$, $c \in \mathbb{R}$, and A' is the matrix obtained by multiplying each element in one row of A by the number c, then* $\det A' = c \det A$.
(d) *If $A \in \mathbf{M}_{n\times n}$, $c \in \mathbb{R}$, and A' is the matrix obtained from A by multiplying one row of A by c and adding it to another row of A (that is, choose i and j between 1 and n with $i \neq j$ and replace a_{jk} by $a_{jk} + ca_{ik}$ for $1 \leq k \leq n$), then* $\det A' = \det A$.

3.1.7. Definition. The unique function $\det\colon \mathbf{M}_{n\times n} \to \mathbb{R}$ described above is the $n \times n$ DETERMINANT FUNCTION.

3.1.8. Proposition. *If $A = [a]$ for $a \in \mathbb{R}$ (that is, if $A \in \mathbf{M}_{1\times 1}$), then $\det A = a$; if $A \in \mathbf{M}_{2\times 2}$, then*

$$\det A = a_{11}a_{22} - a_{12}a_{21}\,.$$

3.1.9. Proposition. *If $A, B \in \mathbf{M}_{n\times n}$, then $\det(AB) = (\det A)(\det B)$.*

3.1.10. Proposition. *If $A \in \mathbf{M}_{n\times n}$, then $\det A^t = \det A$. (An obvious corollary of this: in conditions* (b), (c), *and* (d) *of proposition 3.1.6 the word "columns" may be substituted for the word "rows".)*

3.1.11. Definition. Let A be an $n \times n$ matrix. The MINOR of the element a_{jk}, denoted by M_{jk}, is the determinant of the $(n-1)\times(n-1)$ matrix which results from the deletion of the j^{th} row and k^{th} column of A. The COFACTOR of the element a_{jk}, denoted by C_{jk} is defined by

$$C_{jk} := (-1)^{j+k} M_{jk}.$$

3.1.12. Proposition. *If $A \in \mathbf{M}_{n\times n}$ and $1 \le j \le n$, then*

$$\det A = \sum_{k=1}^{n} a_{jk}C_{jk}.$$

This is the (LAPLACE) EXPANSION *of the determinant along the j^{th} row.*

In light of 3.1.10, it is clear that expansion along columns works as well as expansion along rows. That is,

$$\det A = \sum_{j=1}^{n} a_{jk}C_{jk}$$

for any k between 1 and n. This is the (LAPLACE) EXPANSION of the determinant along the k^{th} column.

3.1.13. Proposition. *An $n \times n$ matrix A is invertible if and only if $\det A \neq 0$. If A is invertible, then*

$$A^{-1} = (\det A)^{-1} C^{\,t}$$

where $C = \big[C_{jk}\big]$ is the matrix of cofactors of elements of A.

3.2. Exercises

(1) Let A be a matrix with 4 rows. The matrix $M_3(4)$ which multiplies the 3^{rd} row of A by 4 is $\begin{bmatrix} \quad & \quad & \quad & \quad \\ \\ \\ \\ \end{bmatrix}$. (See 3.1.2.)

(2) Let A be a matrix with 4 rows. The matrix P_{24} which interchanges the 2^{nd} and 4^{th} rows of A is $\begin{bmatrix} \quad & \quad & \quad & \quad \\ \\ \\ \\ \end{bmatrix}$. (See 3.1.3.)

(3) Let A be a matrix with 4 rows. The matrix $E_{23}(-2)$ which adds -2 times the 3^{rd} row of A to the 2^{nd} row is $\begin{bmatrix} \quad & \quad & \quad & \quad \\ \\ \\ \\ \end{bmatrix}$. (See 3.1.4.)

(4) Let A be the 4×4 elementary matrix $E_{43}(-6)$. Then $A^{11} = \begin{bmatrix} \quad & \quad & \quad & \quad \\ \\ \\ \\ \end{bmatrix}$ and $A^{-9} = \begin{bmatrix} \quad & \quad & \quad & \quad \\ \\ \\ \\ \end{bmatrix}$.

(5) Let B be the elementary 4×4 matrix P_{24}. Then $B^{-9} = \begin{bmatrix} \quad & \quad & \quad & \quad \\ \\ \\ \\ \end{bmatrix}$ and $B^{10} = \begin{bmatrix} \quad & \quad & \quad & \quad \\ \\ \\ \\ \end{bmatrix}$.

(6) Let C be the elementary 4×4 matrix $M_3(-2)$. Then $C^4 = \begin{bmatrix} \quad & \quad & \quad & \quad \\ \\ \\ \\ \end{bmatrix}$ and $C^{-3} = \begin{bmatrix} \quad & \quad & \quad & \quad \\ \\ \\ \\ \end{bmatrix}$.

(7) Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 1 \\ -2 & 1 & 0 \\ -1 & 2 & -3 \end{bmatrix}$ and $B = P_{23}E_{34}(-2)M_3(-2)E_{42}(1)P_{14}A$. Then $b_{23} =$ ______ and $b_{32} =$ ______ .

(8) We apply Gaussian elimination (using type III elementary row operations only) to put a 4×4 matrix A into upper triangular form. The result is

$$E_{43}(\tfrac{5}{2})E_{42}(2)E_{31}(1)E_{21}(-2)A = \begin{bmatrix} 1 & 2 & -2 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & 0 & 10 \end{bmatrix}.$$

Then the determinant of A is ________ .

(9) The system of equations:

$$\begin{cases} \quad\;\; 2y + 3z = 7 \\ \;\; x + \; y - \; z = -2 \\ -x + \; y - 5z = 0 \end{cases}$$

is solved by applying Gauss-Jordan reduction to the augmented coefficient matrix $A = \begin{bmatrix} 0 & 2 & 3 & 7 \\ 1 & 1 & -1 & -2 \\ -1 & 1 & -5 & 0 \end{bmatrix}$. Give the names of the elementary 3×3 matrices $X_1, \dots, X_8$ which implement the following reduction.

$$A \xrightarrow{X_1} \begin{bmatrix} 1 & 1 & -1 & -2 \\ 0 & 2 & 3 & 7 \\ -1 & 1 & -5 & 0 \end{bmatrix} \xrightarrow{X_2} \begin{bmatrix} 1 & 1 & -1 & -2 \\ 0 & 2 & 3 & 7 \\ 0 & 2 & -6 & -2 \end{bmatrix} \xrightarrow{X_3} \begin{bmatrix} 1 & 1 & -1 & -2 \\ 0 & 2 & 3 & 7 \\ 0 & 0 & -9 & -9 \end{bmatrix}$$

$$\xrightarrow{X_4} \begin{bmatrix} 1 & 1 & -1 & -2 \\ 0 & 2 & 3 & 7 \\ 0 & 0 & 1 & 1 \end{bmatrix} \xrightarrow{X_5} \begin{bmatrix} 1 & 1 & -1 & -2 \\ 0 & 2 & 0 & 4 \\ 0 & 0 & 1 & 1 \end{bmatrix} \xrightarrow{X_6} \begin{bmatrix} 1 & 1 & -1 & -2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\xrightarrow{X_7} \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix} \xrightarrow{X_8} \begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

Answer: $X_1 =$ __________ , $X_2 =$__________ , $X_3 =$__________ , $X_4 =$__________ ,

$X_5 =$__________ , $X_6 =$__________ , $X_7 =$__________ , $X_8 =$__________ .

(10) Solve the following equation for x:

$$\det \begin{bmatrix} 3 & -4 & 7 & 0 & 6 & -2 \\ 2 & 0 & 1 & 8 & 0 & 0 \\ 3 & 4 & -8 & 3 & 1 & 2 \\ 27 & 6 & 5 & 0 & 0 & 3 \\ 3 & x & 0 & 2 & 1 & -1 \\ 1 & 0 & -1 & 3 & 4 & 0 \end{bmatrix} = 0. \qquad \text{Answer: } x = \underline{\qquad}.$$

(11) Let $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 4 \\ 1 & 2 & 3 \end{bmatrix}$. Find A^{-1} using the technique of augmenting A by the identity matrix I and performing Gauss-Jordan reduction on the augmented matrix. The reduction can be accomplished by the application of five elementary 3×3 matrices. Find elementary matrices X_1, X_2, and X_3 such that $A^{-1} = X_3 E_{13}(-3) X_2 M_2(1/2) X_1 I$.

(a) The required matrices are $X_1 = P_{1i}$ where $i =$ _____ , $X_2 = E_{jk}(-2)$ where $j =$ _____ and $k =$ _____ , and $X_3 = E_{12}(r)$ where $r =$ ______ .

(b) And then $A^{-1} = \begin{bmatrix} \qquad\qquad & & \\ & & \\ & & \end{bmatrix}$.

(12) $\det \begin{bmatrix} 1 & t & t^2 & t^3 \\ t & 1 & t & t^2 \\ t^2 & t & 1 & t \\ t^3 & t^2 & t & 1 \end{bmatrix} = (1 - a(t))^p$ where $a(t) =$ ________ and $p =$ _____ .

(13) Evaluate each of the following determinants.

(a) $\det \begin{bmatrix} 6 & 9 & 39 & 49 \\ 5 & 7 & 32 & 37 \\ 3 & 4 & 4 & 4 \\ 1 & 1 & 1 & 1 \end{bmatrix} = $ ______ .

(b) $\det \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & -1 & 2 & 0 \\ 2 & -1 & 3 & 1 \\ 4 & 17 & 0 & -5 \end{bmatrix} = $ ______ .

(c) $\det \begin{bmatrix} 13 & 3 & -8 & 6 \\ 0 & 0 & -4 & 0 \\ 1 & 0 & 7 & -2 \\ 3 & 0 & 2 & 0 \end{bmatrix} = $ ______ .

(14) Let M be the matrix $\begin{bmatrix} 5 & 4 & -2 & 3 \\ 5 & 7 & -1 & 8 \\ 5 & 7 & 6 & 10 \\ 5 & 7 & 1 & 9 \end{bmatrix}$.

(a) The determinant of M can be expressed as the constant 5 times the determinant of the single 3×3 matrix $\begin{bmatrix} 3 & 1 & 5 \\ 3 & & \\ 3 & & \end{bmatrix}$.

(b) The determinant of this 3×3 matrix can be expressed as the constant 3 times the determinant of the single 2×2 matrix $\begin{bmatrix} 7 & 2 \\ 2 & \end{bmatrix}$.

(c) The determinant of this 2×2 matrix is _____ .

(d) Thus the determinant of M is _____ .

(15) Find the determinant of the matrix $\begin{bmatrix} 1 & 2 & 5 & 7 & 10 \\ 1 & 2 & 3 & 6 & 7 \\ 1 & 1 & 3 & 5 & 5 \\ 1 & 1 & 2 & 4 & 5 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$. Answer: ______ .

(16) Find the determinants of the following matrices.

$$A = \begin{bmatrix} -73 & 78 & 24 \\ 92 & 66 & 25 \\ -80 & 37 & 10 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} -73 & 78 & 24 \\ 92 & 66 & 25 \\ -80 & 37 & 10.01 \end{bmatrix}.$$

Hint. Use a calculator (thoughtfully). Answer: $\det A =$ _______ and $\det B =$ _______ .

(17) Find the determinant of the following matrix.

$$\begin{bmatrix} 283 & 5 & \pi & 347.86 \times 10^{15^{83}} \\ 3136 & 56 & 5 & \cos(2.7402) \\ 6776 & 121 & 11 & 5 \\ 2464 & 44 & 4 & 2 \end{bmatrix}.$$

Hint. Do not use a calculator. Answer: _______ .

(18) Let $A = \begin{bmatrix} 0 & -\frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ 1 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$. We find A^{-1} using elementary row operations to convert the 4×8 matrix $\left[\, A \,\vdots\, I_4 \,\right]$ to the matrix $\left[\, I_4 \,\vdots\, A^{-1} \,\right]$.

Give the names of the elementary 4×4 matrices $X_1, \dots, X_{11}$ which implement the following Gauss-Jordan reduction and fill in the missing matrix entries.

$$\begin{bmatrix} 0 & -\frac{1}{2} & 0 & \frac{1}{2} & \vdots & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & \vdots & 0 & 1 & 0 & 0 \\ \frac{1}{2} & 0 & -\frac{1}{2} & 0 & \vdots & 0 & 0 & 1 & 0 \\ 1 & 0 & \frac{1}{2} & \frac{1}{2} & \vdots & 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{X_1} \begin{bmatrix} 1 & 0 & \frac{1}{2} & \frac{1}{2} & \vdots & & & & \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & \vdots & & & & \\ \frac{1}{2} & 0 & -\frac{1}{2} & 0 & \vdots & & & & \\ 0 & -\frac{1}{2} & 0 & \frac{1}{2} & \vdots & & & & \end{bmatrix}$$

$$\xrightarrow{X_2} \begin{bmatrix} 1 & 0 & \frac{1}{2} & \frac{1}{2} & \vdots & & & & \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & \vdots & & & & \\ 0 & 0 & -\frac{3}{4} & -\frac{1}{4} & \vdots & & & & \\ 0 & -\frac{1}{2} & 0 & \frac{1}{2} & \vdots & & & & \end{bmatrix} \xrightarrow{X_3} \begin{bmatrix} 1 & 0 & \frac{1}{2} & \frac{1}{2} & \vdots & & & & \\ 0 & -\frac{1}{2} & 0 & \frac{1}{2} & \vdots & & & & \\ 0 & 0 & -\frac{3}{4} & -\frac{1}{4} & \vdots & & & & \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & \vdots & & & & \end{bmatrix}$$

$$\xrightarrow{X_4} \begin{bmatrix} 1 & 0 & \frac{1}{2} & \frac{1}{2} & \vdots & & & & \\ 0 & 1 & 0 & -1 & \vdots & & & & \\ 0 & 0 & -\frac{3}{4} & -\frac{1}{4} & \vdots & & & & \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & \vdots & & & & \end{bmatrix} \xrightarrow{X_5} \begin{bmatrix} 1 & 0 & \frac{1}{2} & \frac{1}{2} & \vdots & & & & \\ 0 & 1 & 0 & -1 & \vdots & & & & \\ 0 & 0 & 1 & \frac{1}{3} & \vdots & & & & \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & \vdots & & & & \end{bmatrix}$$

$$\xrightarrow{X_6} \begin{bmatrix} 1 & 0 & \frac{1}{2} & \frac{1}{2} & \vdots & & & & \\ 0 & 1 & 0 & -1 & \vdots & & & & \\ 0 & 0 & 1 & \frac{1}{3} & \vdots & & & & \\ 0 & 0 & 0 & \frac{1}{3} & \vdots & & & & \end{bmatrix} \xrightarrow{X_7} \begin{bmatrix} 1 & 0 & \frac{1}{2} & \frac{1}{2} & \vdots & & & & \\ 0 & 1 & 0 & -1 & \vdots & & & & \\ 0 & 0 & 1 & 0 & \vdots & & & & \\ 0 & 0 & 0 & \frac{1}{3} & \vdots & & & & \end{bmatrix}$$

$$\xrightarrow{X_8} \begin{bmatrix} 1 & 0 & \frac{1}{2} & \frac{1}{2} & \vdots & & & & \\ 0 & 1 & 0 & -1 & \vdots & & & & \\ 0 & 0 & 1 & 0 & \vdots & & & & \\ 0 & 0 & 0 & 1 & \vdots & & & & \end{bmatrix} \xrightarrow{X_9} \begin{bmatrix} 1 & 0 & \frac{1}{2} & \frac{1}{2} & \vdots & & & & \\ 0 & 1 & 0 & 0 & \vdots & & & & \\ 0 & 0 & 1 & 0 & \vdots & & & & \\ 0 & 0 & 0 & 1 & \vdots & & & & \end{bmatrix}$$

$$\xrightarrow{X_{10}} \begin{bmatrix} 1 & 0 & \frac{1}{2} & 0 & \vdots & & & & \\ 0 & 1 & 0 & 0 & \vdots & & & & \\ 0 & 0 & 1 & 0 & \vdots & & & & \\ 0 & 0 & 0 & 1 & \vdots & & & & \end{bmatrix} \xrightarrow{X_{11}} \begin{bmatrix} 1 & 0 & 0 & 0 & \vdots & & & & \\ 0 & 1 & 0 & 0 & \vdots & & & & \\ 0 & 0 & 1 & 0 & \vdots & & & & \\ 0 & 0 & 0 & 1 & \vdots & & & & \end{bmatrix}$$

Answer: $X_1 =$ __________ , $X_2 =$__________ , $X_3 =$__________ , $X_4 =$__________ ,

$X_5 =$__________ , $X_6 =$__________ , $X_7 =$__________ , $X_8 =$__________ .

$X_9 =$__________ , $X_{10} =$__________ , $X_{11} =$__________ .

(19) Suppose that A is a square matrix with determinant 7. Then

(a) $\det(P_{24}A) =$ _____ .

(b) $\det(E_{23}(-4)A) =$ _____ .

(c) $\det(M_3(2)A) =$ _____ .

3.3. Problems

(1) For this problem assume that we know the following: If X is an $m \times m$ matrix, if Y is an $m \times n$ matrix and if $\mathbf{0}$ and $\mathbf{I}$ are zero and identity matrices of appropriate sizes, then $\det \begin{bmatrix} X & Y \\ \mathbf{0} & \mathbf{I} \end{bmatrix} = \det X$.

Let A be an $m \times n$ matrix and B be an $n \times m$ matrix. Prove carefully that

$$\det \begin{bmatrix} \mathbf{0} & A \\ -B & \mathbf{I} \end{bmatrix} = \det AB\,.$$

Hint. Consider the product $\begin{bmatrix} \mathbf{0} & A \\ -B & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ B & \mathbf{I} \end{bmatrix}$.

(2) Let A and B be $n \times n$-matrices. Your good friend Fred R. Dimm believes that

$$\det \begin{bmatrix} A & B \\ B & A \end{bmatrix} = \det(A+B)\det(A-B).$$

He offers the following argument to support this claim:

$$\begin{aligned} \det \begin{bmatrix} A & B \\ B & A \end{bmatrix} &= \det(A^2 - B^2) \\ &= \det[(A+B)(A-B)] \\ &= \det(A+B)\det(A-B)\,. \end{aligned}$$

(a) Comment (helpfully) on his "proof". In particular, explain carefully why each of the three steps in his "proof" is correct or incorrect. (That is, provide a proof or a counterexample to each step.)

(b) Is the result he is trying to prove actually true?

Hint: Consider the product $\begin{bmatrix} I & B \\ 0 & A-B \end{bmatrix} \begin{bmatrix} A+B & 0 \\ 0 & I \end{bmatrix}$.

(3) Let x be a fixed real number which is not an integer multiple of π. For each natural number n let $A_n = \begin{bmatrix} a_{jk} \end{bmatrix}$ be the $n \times n$-matrix defined by

$$a_{jk} = \begin{cases} 0, & \text{for } |j-k| > 1 \\ 1, & \text{for } |j-k| = 1 \\ 2\cos x, & \text{for } j = k. \end{cases}$$

Show that $\det A_n = \dfrac{\sin(n+1)x}{\sin x}$. *Hint.* For each integer n let $D_n = \det A_n$ and prove that

$$D_{n+2} - 2D_{n+1}\cos x + D_n = 0.$$

(Use mathematical induction.)

3.4. Answers to Odd-Numbered Exercises

(1) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(3) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(5) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(7) -8, -1

(9) P_{12}, $E_{31}(1)$, $E_{32}(-1)$, $M_3(-\frac{1}{9})$, $E_{23}(-3)$, $M_2(\frac{1}{2})$, $E_{13}(1)$, $E_{12}(-1)$

(11) (a) 3, 2, 3, -2
(b) $\begin{bmatrix} 1 & -1 & 1 \\ -2 & \frac{1}{2} & 0 \\ 1 & 0 & 0 \end{bmatrix}$

(13) 100, 0, -72

(15) -10

(17) 6

(19) (a) -7
(b) 7
(c) 14

CHAPTER 4

VECTOR GEOMETRY IN $\mathbb{R}^n$

4.1. Background

Topics: inner (dot) products, cross products, lines and planes in 3-space, norm of a vector, angle between vectors.

4.1.1. Notation. There are many more or less standard notations for the inner product (or dot product) of two vectors $\mathbf{x}$ and $\mathbf{y}$. The two that we will use interchangeably in these exercises are $\mathbf{x} \cdot \mathbf{y}$ and $\langle \mathbf{x}, \mathbf{y} \rangle$.

4.1.2. Definition. If $\mathbf{x}$ is a vector in $\mathbb{R}^n$, then the NORM (or LENGTH) of $\mathbf{x}$ is defined by

$$\|\mathbf{x}\| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}\,.$$

4.1.3. Definition. Let $\mathbf{x}$ and $\mathbf{y}$ be nonzero vectors in $\mathbb{R}^n$. Then $\measuredangle(\mathbf{x}, \mathbf{y})$, the ANGLE between $\mathbf{x}$ and $\mathbf{y}$, is defined by

$$\measuredangle(\mathbf{x}, \mathbf{y}) = \arccos \frac{\langle \mathbf{x}, \mathbf{y} \rangle}{\|\mathbf{x}\|\,\|\mathbf{y}\|}$$

4.1.4. Theorem (Cauchy-Schwarz inequality)**.** *If $\mathbf{x}$ and $\mathbf{y}$ are vectors in $\mathbb{R}^n$, then*

$$|\langle \mathbf{x}, \mathbf{y} \rangle| \leq \|\mathbf{x}\|\,\|\mathbf{y}\|\,.$$

(We will often refer to this just as the *Schwarz inequality.*)

4.1.5. Definition. If $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{y} = (y_1, y_2, y_3)$ are vectors in $\mathbb{R}^3$, then their CROSS PRODUCT, denoted by $\mathbf{x} \times \mathbf{y}$, is the vector $(x_2y_3 - x_3y_2, x_3y_1 - x_1y_3, x_1y_2 - x_2y_1)$.

4.2. Exercises

(1) The angle between the vectors $(1, 0, -1, 3)$ and $(1, \sqrt{3}, 3, -3)$ in $\mathbb{R}^4$ is $a\pi$ where $a =$ ________ .

(2) Find the angle θ between the vectors $\mathbf{x} = (3, -1, 1, 0, 2, 1)$ and $\mathbf{y} = (2, -1, 0, \sqrt{2}, 2, 1)$ in $\mathbb{R}^6$. Answer: $\theta =$ _________ .

(3) If $a_1, \dots, a_n > 0$, then

$$\left(\sum_{j=1}^{n} a_j\right)\left(\sum_{k=1}^{n} \frac{1}{a_k}\right) \geq n^2.$$

The proof of this is obvious from the *Cauchy-Schwarz inequality* when we choose the vectors $\mathbf{x}$ and $\mathbf{y}$ as follows:

$\mathbf{x} =$ ______________________ and $\mathbf{y} =$ ______________________ .

(4) Find all real numbers α such that the angle between the vectors $2\mathbf{i} + 2\mathbf{j} + (\alpha - 2)\mathbf{k}$ and $2\mathbf{i} + (\alpha - 2)\mathbf{j} + 2\mathbf{k}$ is $\frac{\pi}{3}$. Answer: $\alpha =$ _____ and ______ .

(5) Which of the angles (if any) of triangle ABC, with $A = (1, -2, 0)$, $B = (2, 1, -2)$, and $C = (6, -1, -3)$, is a right angle? Answer: the angle at vertex _____ .

(6) The hydrogen atoms of a methane molecule (CH_4) are located at $(0, 0, 0)$, $(1, 1, 0)$, $(0, 1, 1)$, and $(1, 0, 1)$ while the carbon atom is at $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. Find the cosine of the angle θ between two rays starting at the carbon atom and going to different hydrogen atoms.

Answer: $\cos\theta =$ _____ .

(7) If a, b, c, d, e, $f \in \mathbb{R}$, then

$$|ad + be + cf| \leq \sqrt{a^2 + b^2 + c^2}\sqrt{d^2 + e^2 + f^2}.$$

The proof of this inequality is obvious since this is just the *Cauchy-Schwarz inequality* where $x =$ (_____, _____ , _____) and $y =$ (_____, _____ , _____) .

(8) The volume of the parallelepiped generated by the three vectors $\mathbf{i} + 2\mathbf{j} - \mathbf{k}$, $\mathbf{j} + \mathbf{k}$, and $3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ is ____ .

(9) The equations of the line containing the points $(3, -1, 4)$ and $(7, 9, 10)$ are

$$\frac{x-3}{2} = \frac{y-j}{b} = \frac{z-k}{c}$$

where $b =$ _____ , $c =$ _____ , $j =$ _____ , and $k =$ _____ .

(10) The equations of the line containing the points $(5, 2, -1)$ and $(9, -4, 1)$ are

$$\frac{x-h}{a} = \frac{y-2}{-3} = \frac{z-k}{c}$$

where $a =$ _____ , $c =$ _____ , $h =$ _____ , and $k =$ _____ .

(11) Find the equations of the line containing the point $(1, 0, -1)$ which is parallel to the line $\dfrac{x-4}{2} = \dfrac{2y-3}{5} = \dfrac{3z-7}{6}$.

Answer: $\dfrac{x-h}{a} = \dfrac{y-j}{b} = \dfrac{z+1}{4}$ where $a =$ _____ , $b =$ _____ , $h =$ _____ , and $j =$ _____ .

(12) The equation of the plane containing the points $(0, -1, 1)$, $(1, 0, 2)$, and $(3, 0, 1)$ is $x + by + cz = d$ where $b =$ _____ , $c =$ _____ , and $d =$ _____ .

(13) The equation of the plane which passes through the points $(0, -1, -1)$, $(5, 0, 1)$, and $(4, -1, 0)$ is $ax + by + cz = 1$ where $a =$ _____ , $b =$ _____ , and $c =$ _____ .

(14) The angle between the planes $4x + 4z - 16 = 0$ and $-2x + 2y - 13 = 0$ is $\dfrac{a}{b}\pi$ where $a =$ _____ and $b =$ _____ .

(15) Suppose that $\mathbf{u} \in \mathbb{R}^3$ is a vector which lies in the first quadrant of the xy-plane and has length 3 and that $\mathbf{v} \in \mathbb{R}^3$ is a vector that lies along the positive z-axis and has length 5. Then
(a) $\|\mathbf{u} \times \mathbf{v}\| =$ ____ ;
(b) the x-coordinate of $\mathbf{u} \times \mathbf{v}$ is _____ 0 (choose $<$, $>$, or $=$);
(c) the y-coordinate of $\mathbf{u} \times \mathbf{v}$ is _____ 0 (choose $<$, $>$, or $=$); and
(d) the z-coordinate of $\mathbf{u} \times \mathbf{v}$ is _____ 0 (choose $<$, $>$, or $=$).

(16) Suppose that $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{R}^7$ both of length $2\sqrt{2}$ and that the length of $\mathbf{u} - \mathbf{v}$ is also $2\sqrt{2}$. Then $\|\mathbf{u} + \mathbf{v}\| =$ _______ and the angle between $\mathbf{u}$ and $\mathbf{v}$ is _______ .

4.3. Problems

(1) Show that if a, b, $c > 0$, then $\left(\frac{1}{2}a + \frac{1}{3}b + \frac{1}{6}c\right)^2 \le \frac{1}{2}a^2 + \frac{1}{3}b^2 + \frac{1}{6}c^2$.

(2) Show that if $a_1, \dots, a_n, w_1, \dots, w_n > 0$ and $\sum_{k=1}^n w_k = 1$, then

$$\left(\sum_{k=1}^n a_k w_k\right)^2 \le \sum_{k=1}^n {a_k}^2 w_k.$$

(3) Prove that if $(a_1, a_2, \dots)$ is a sequence of real numbers such that the series $\displaystyle\sum_{k=1}^\infty {a_k}^2$ converges, then the series $\displaystyle\sum_{k=1}^\infty \frac{1}{k}\, a_k$ converges absolutely.

You may find the following steps helpful in organizing your solution.

(i) First of all, make sure that you recall the difference between a sequence of numbers $(c_1, c_2, \dots)$ and an infinite series $\displaystyle\sum_{k=1}^\infty c_k$.

(ii) The key to this problem is an important theorem from third term Calculus:

A nondecreasing sequence of real numbers converges if and only if it is bounded. $(*)$

(Make sure that you know the meanings of all the terms used here.)

(iii) The hypothesis of the result we are trying to prove is that the series $\displaystyle\sum_{k=1}^\infty {a_k}^2$ converges. What, exactly, does this mean?

(iv) For each natural number n let $b_n = \displaystyle\sum_{k=1}^n {a_k}^2$. Rephrase (iii) in terms of the sequence (b_n).

(v) Is the sequence (b_n) nondecreasing?

(vi) What, then, does $(*)$ say about the sequence (b_n)?

(vii) For each natural number n let $c_n = \displaystyle\sum_{k=1}^n \frac{1}{k^2}$. What do we know about the sequence (c_n) from third term Calculus? What does $(*)$ say about the sequence (c_n)?

(viii) The conclusion we are trying to prove is that the series $\displaystyle\sum_{k=1}^\infty \frac{1}{k} a_k$ converges absolutely. What does this mean?

(ix) For each natural number n let $s_n = \displaystyle\sum_{k=1}^n \frac{1}{k}|a_k|$. Rephrase (viii) in terms of the sequence (s_n).

(x) Explain how for each n we may regard the number s_n as the dot product of two vectors in $\mathbb{R}^n$.

(xi) Apply the *Cauchy-Schwarz inequality* to the dot product in (x). Use (vi) and (vii) to establish that the sequence (s_n) is bounded above.

(xii) Use $(*)$ one last time—keeping in mind what you said in (ix).

4.4. Answers to Odd-Numbered Exercises

(1) $\dfrac{3}{4}$

(3) $\left(\sqrt{a_1}, \sqrt{a_2}, \dots, \sqrt{a_n}\right)$, $\left(\frac{1}{\sqrt{a_1}}, \frac{1}{\sqrt{a_2}}, \dots, \frac{1}{\sqrt{a_n}}\right)$

(5) B

(7) a, b, c, d, e, f

(9) 5, 3, -1, 4

(11) 4, 5, 1, 0

(13) 1, 3, -4

(15)
- (a) 15
- (b) $>$
- (c) $<$
- (d) $=$