Mathematics 1A

ITMTA1-B44

Applications of Differentiation

With

Amakan Elisha Agoni
Amakan.agoni@EDUVOS.com

Lecture 3
Week 5

4 Applications of Differentiation

4.1 Maximum and Minimum Values

Maximum and Minimum Values

Some of the most important applications of differential calculus are *optimization problems*, in which we are required to find the optimal (best) way of doing something.

These problems can be reduced to finding the maximum or minimum values of a function.

Let’s first explain exactly what we mean by maximum and minimum values.

Absolute and Local Extreme Values

Absolute and Local Extreme Values

We see that the highest point on the graph of the function *f* shown in Figure 1 is the point (3, 5).

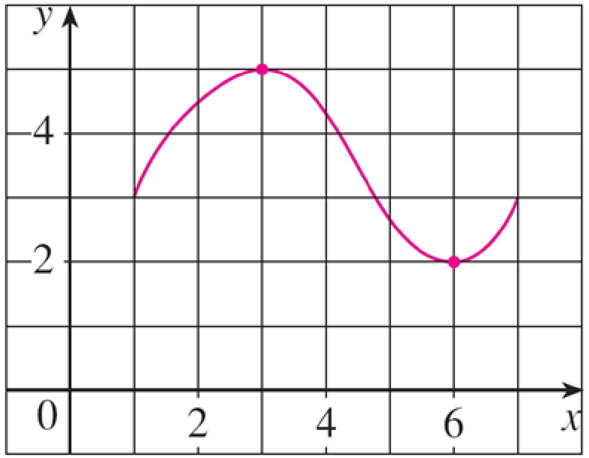


Figure 1

In other words, the largest value of *f* is $f(3) = 5$. Likewise, the smallest value is $f(6) = 2$.

Absolute and Local Extreme Values

We say that $f(3) = 5$ is the *absolute maximum* of f and $f(6) = 2$ is the *absolute minimum*. In general, we use the following definition.

1 Definition Let c be a number in the domain D of a function f. Then $f(c)$ is the

- **absolute maximum** value of f on D if $f(c) \geq f(x)$ for all x in D.
- **absolute minimum** value of f on D if $f(c) \leq f(x)$ for all x in D.

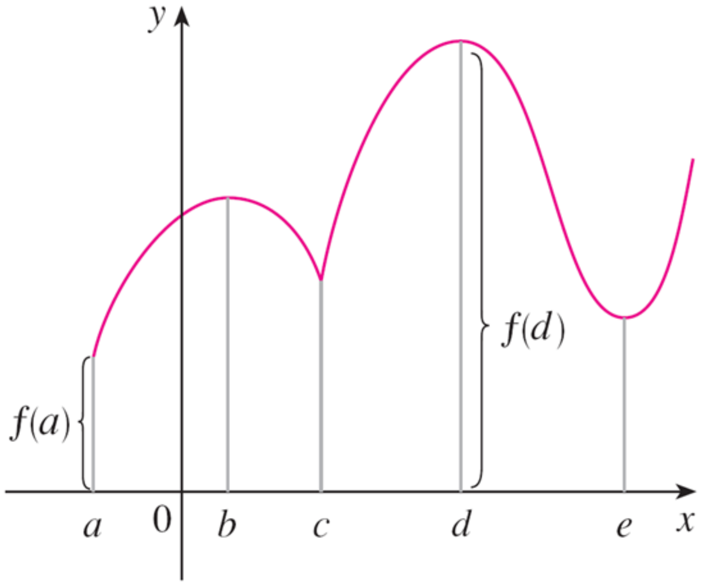
An absolute maximum or minimum is sometimes called a **global** maximum or minimum. The maximum and minimum values of f are called **extreme values** of f.

Absolute and Local Extreme Values

Figure 2 shows the graph of a function *f* with absolute maximum at *d* and absolute minimum at *a*.

Note that (d, $f(d)$) is the highest point on the graph and (a, $f(a)$) is the lowest point.

In Figure 2, if we consider only values of *x* near *b* [for instance, if we restrict our attention to the interval (a, c)], then $f(b)$ is the largest of those values of $f(x)$ and is called a *local maximum value* of *f*.



Abs min $f(a)$, abs max $f(d)$, loc min $f(c)$, $f(e)$, loc max $f(b)$, $f(d)$

Figure 2

Absolute and Local Extreme Values

Likewise, $f(c)$ is called a *local minimum value* of f because $f(c) \leq f(x)$ for x near c [in the interval (b, d), for instance].

The function f also has a local minimum at e. In general, we have the following definition.

2 Definition The number $f(c)$ is a

- **local maximum** value of f if $f(c) \geq f(x)$ when x is near c.
- **local minimum** value of f if $f(c) \leq f(x)$ when x is near c.

In Definition 2 (and elsewhere), if we say that something is true **near** c, we mean that it is true on some open interval containing c.

Absolute and Local Extreme Values

For instance, in Figure 3 we see that $f(4) = 5$ is a local minimum because it's the smallest value of f on the interval I.

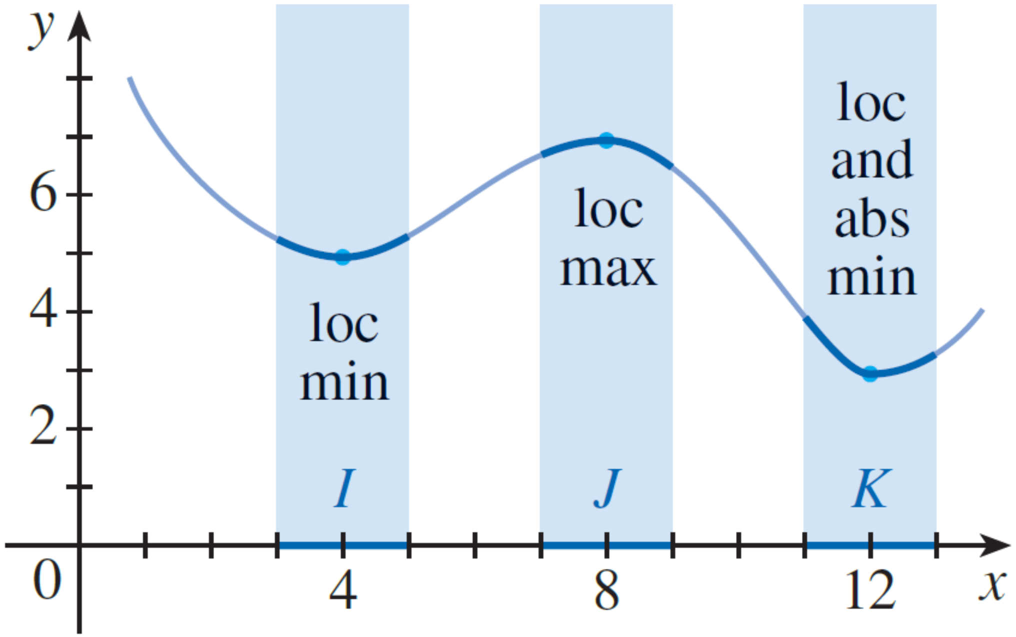


Figure 3

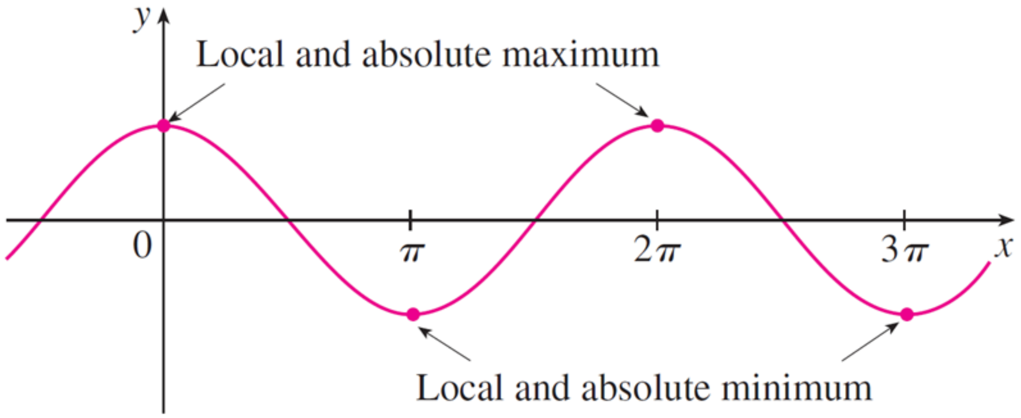
f(12) is the absolute minimum because $f(12)$ takes the smallest value when x is near 12 (in the interval K, for instance).

In fact $f(12) = 3$ is both a local minimum and the absolute minimum.

Similarly, $f(8) = 7$ is a local maximum, but not the absolute maximum because f takes larger values near 1.

Example 2

The function $f(x) = \cos x$ takes on its (local and absolute) maximum value of 1 infinitely many times, because $\cos 2n\pi = 1$ for any integer n and $-1 \le \cos x \le 1$ for all x. (See Figure 5.)



y = cos *x*

Figure 5

Likewise, $\cos(2n + 1)\pi = -1$ is its minimum value, where n is any integer.

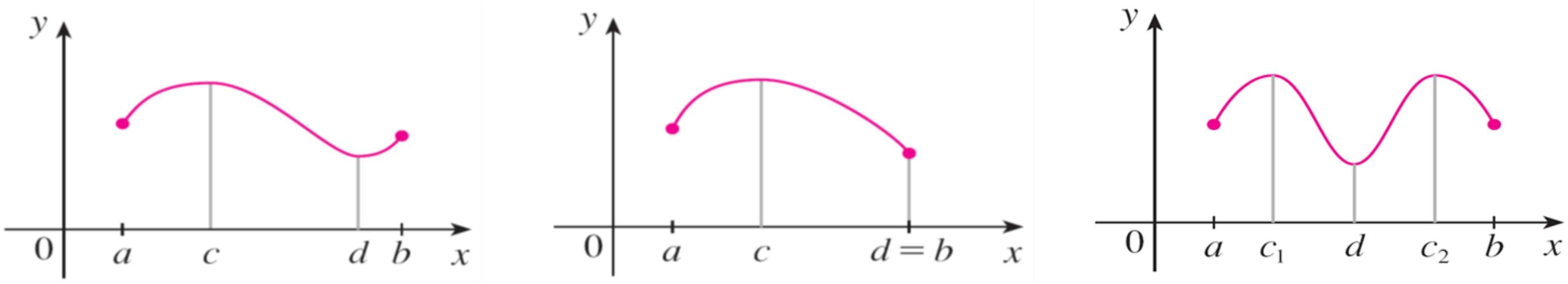
Absolute and Local Extreme Values

The following theorem gives conditions under which a function is guaranteed to possess extreme values.

3 The Extreme Value Theorem If f is continuous on a closed interval $[a, b]$, then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers c and d in $[a, b]$.

Absolute and Local Extreme Values

The Extreme Value Theorem is illustrated in Figure 8.



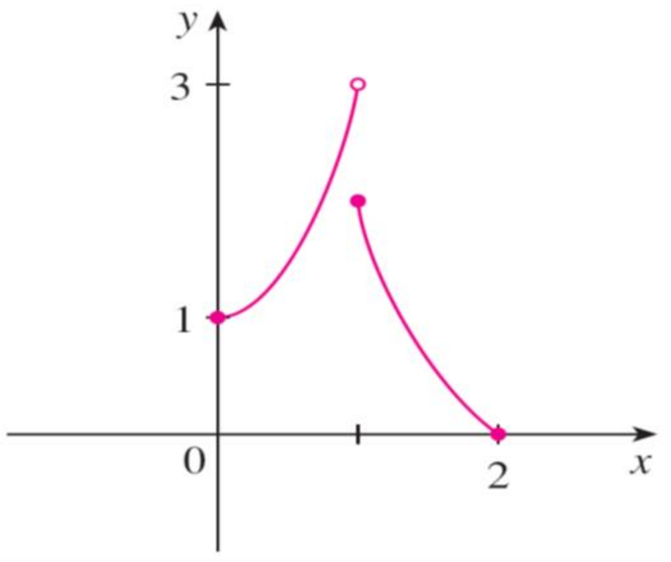
Functions continuous on a closed interval always attain extreme values.

Figure 8

Note that an extreme value can be taken on more than once.

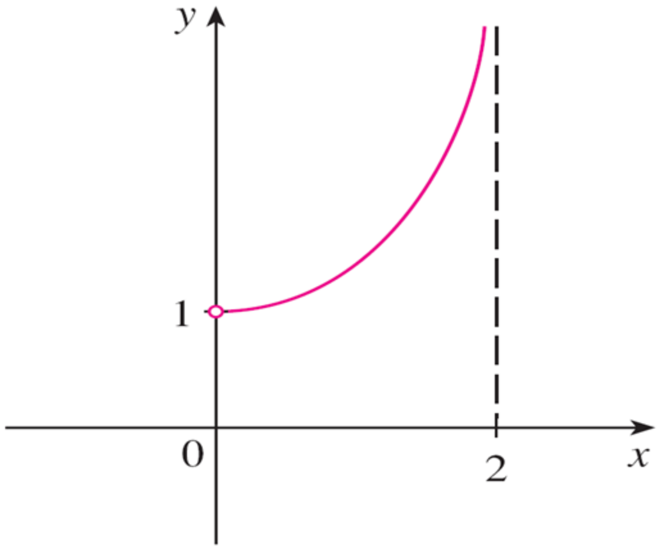
Absolute and Local Extreme Values

Figures 9 and 10 show that a function need not possess extreme values if either hypothesis (continuity or closed interval) is omitted from the Extreme Value Theorem.



This function has minimum value $f(2) = 0$, but no maximum value.

Figure 9



This continuous function g has no maximum or minimum.

Figure 10

Critical Numbers and the Closed Interval Method

Critical Numbers and the Closed Interval Method

The Extreme Value Theorem says that a continuous function on a closed interval has a maximum value and a minimum value, but it does not tell us how to find these extreme values.

Figure 11 shows the graph of a function *f* with a local maximum at *c* and a local minimum at *d*.

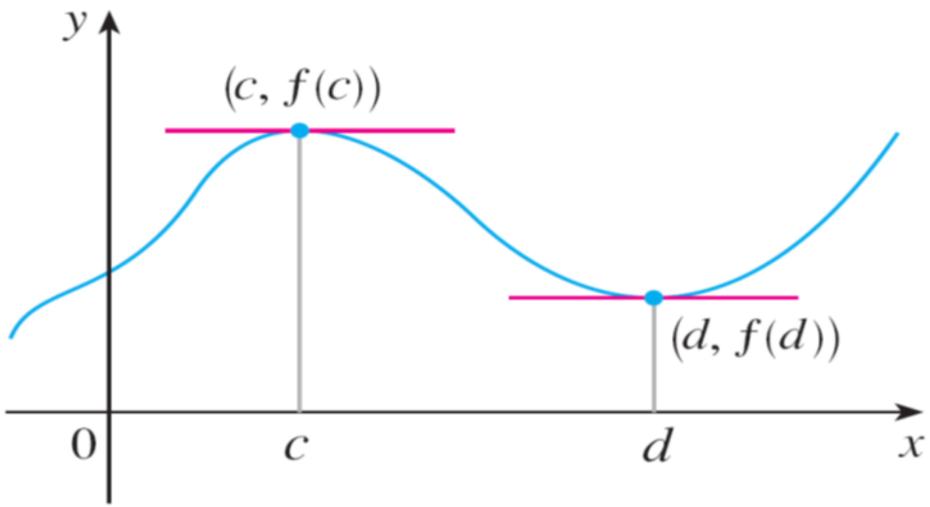


Figure 11

Critical Numbers and the Closed Interval Method (2 of 5)

It appears that at the maximum and minimum points the tangent lines are horizontal and therefore each has slope 0.

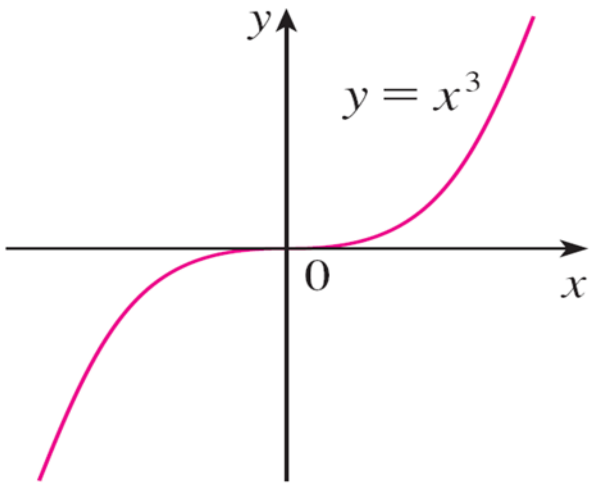
We know that the derivative is the slope of the tangent line, so it appears that $f'(c) = 0$ and $f'(d) = 0$. The following theorem says that this is always true for differentiable functions.

4 Fermat's Theorem If f has a local maximum or minimum at c, and if $f'(c)$ exists, then $f'(c) = 0$.

Example 5

If $f(x) = x^3$, then $f'(x) = 3x^2$, so $f'(0) = 0$.

But f has no maximum or minimum at 0, as you can see from its graph in Figure 12.



If $f(x) = x^3$, then $f'(0) = 0$ but f has no maximum or minimum.

Figure 12

Example 5

The fact that $f'(0) = 0$ simply means that the curve $y = x^3$ has a horizontal tangent at (0, 0).

Instead of having a maximum or minimum at (0, 0), the curve crosses its horizontal tangent there.

Critical Numbers and the Closed Interval Method

To find an absolute maximum or minimum of a continuous function on a closed interval, we note that either it is local or it occurs at an endpoint of the interval.

Thus the following three-step procedure always works.

The Closed Interval Method To find the *absolute* maximum and minimum values of a continuous function f on a closed interval $[a, b]$:

1. Find the values of f at the critical numbers of f in (a, b).

2. Find the values of f at the endpoints of the interval.

3. The largest of the values from Steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

Examples

Find all the critical points of the following functions $f(x) = x^{3/5}(4 - x)$

SOLUTION The Product Rule gives

$$f'(x) = x^{3/5}(-1) + (4 - x)\left(\tfrac{3}{5}x^{-2/5}\right) = -x^{3/5} + \frac{3(4 - x)}{5x^{2/5}}$$

$$= \frac{-5x + 3(4 - x)}{5x^{2/5}} = \frac{12 - 8x}{5x^{2/5}}$$

[The same result could be obtained by first writing $f(x) = 4x^{3/5} - x^{8/5}$.] Therefore $f'(x) = 0$ if $12 - 8x = 0$, that is, $x = \frac{3}{2}$, and $f'(x)$ does not exist when $x = 0$. Thus the critical numbers are $\frac{3}{2}$ and 0.

Examples

Find all the critical points of the following functions, the maximum and minimum points, if they exist.

$$f(x) = x^3 - 3x^2 + 1 \qquad -\frac{1}{2} \leq x \leq 4$$

SOLUTION Since f is continuous on $\left[-\frac{1}{2}, 4\right]$, we can use the Closed Interval Method:

$$f(x) = x^3 - 3x^2 + 1$$

$$f'(x) = 3x^2 - 6x = 3x(x - 2)$$

Examples

Since $f'(x)$ exists for all x, the only critical numbers of f occur when $f'(x) = 0$, that is, $x = 0$ or $x = 2$. Notice that each of these critical numbers lies in the interval $\left(-\frac{1}{2}, 4\right)$. The values of f at these critical numbers are

$$f(0) = 1 \qquad\qquad f(2) = -3$$

The values of f at the endpoints of the interval are

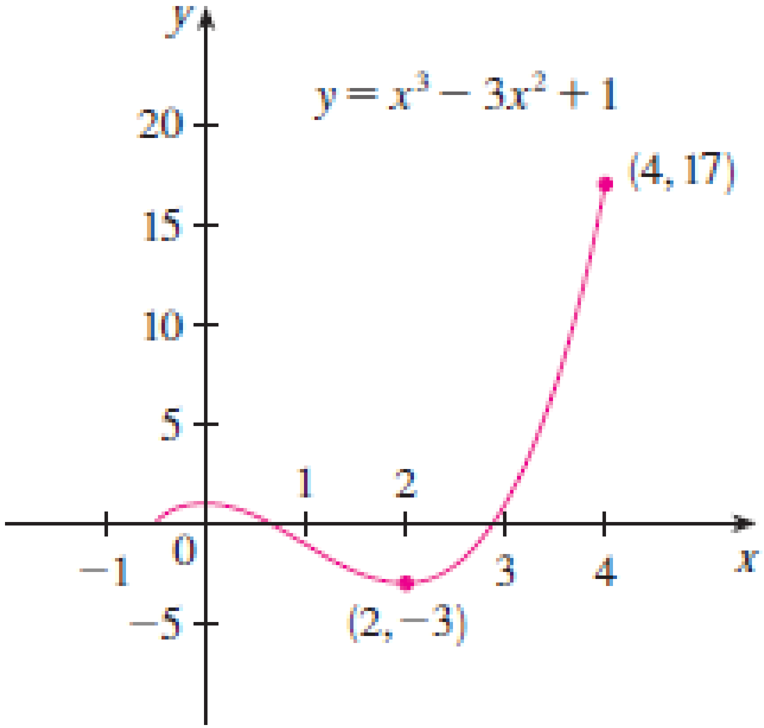
$$f\left(-\tfrac{1}{2}\right) = \tfrac{1}{8} \qquad\qquad f(4) = 17$$

Comparing these four numbers, we see that the absolute maximum value is $f(4) = 17$ and the absolute minimum value is $f(2) = -3$.

Examples

Note that in this example the absolute maximum occurs at an endpoint, whereas the absolute minimum occurs at a critical number. The graph of is sketched in Figure 14.

If you have a graphing calculator or a computer with graphing software, it is possible to estimate maximum and minimum values very easily. But, as the next example shows, calculus is needed to find the exact values.



Exercise

Find all the critical points of the following functions, the maximum and minimum points, if they exist.

1. $f(x) = x^3 + x^2 + 1 \qquad -1 \leq x \leq \frac{1}{2}$

Ans: Critical Points: $-\frac{2}{3}$ $\boldsymbol{and\ 0}$

$\boldsymbol{Absolute\ max = (0.5, 1.38)} \qquad \boldsymbol{Absolute\ min = (-1, 1) and\ (0, 1)}$

2. $f(x) = x\sqrt{4 - x} \qquad where\ x < 3.$

Ans: Critical Point(s): 2.67

$\boldsymbol{Absolute\ max = (2.67, 3.08)} \qquad \boldsymbol{Absolute\ min = n/a}$

Examples

EXAMPLE 10 The Hubble Space Telescope was deployed on April 24, 1990, by the space shuttle *Discovery*. A model for the velocity of the shuttle during this mission, from liftoff at $t = 0$ until the solid rocket boosters were jettisoned at $t = 126$ s, is given by

$$v(t) = 0.001302t^3 - 0.09029t^2 + 23.61t - 3.083$$

(in feet per second). Using this model, estimate the absolute maximum and minimum values of the *acceleration* of the shuttle between liftoff and the jettisoning of the boosters.

SOLUTION We are asked for the extreme values not of the given velocity function, but rather of the acceleration function. So we first need to differentiate to find the acceleration:

$$a(t) = v'(t) = \frac{d}{dt}(0.001302t^3 - 0.09029t^2 + 23.61t - 3.083)$$

$$= 0.003906t^2 - 0.18058t + 23.61$$

Examples

We now apply the Closed Interval Method to the continuous function a on the interval $0 \leqslant t \leqslant 126$. Its derivative is

$$a'(t) = 0.007812t - 0.18058$$

The only critical number occurs when $a'(t) = 0$:

$$t_1 = \frac{0.18058}{0.007812} \approx 23.12$$

Evaluating $a(t)$ at the critical number and at the endpoints, we have

$$a(0) = 23.61 \qquad a(t_1) \approx 21.52 \qquad a(126) \approx 62.87$$

So the maximum acceleration is about 62.87 ft/s^2 and the minimum acceleration is about 21.52 ft/s^2.

Exercise

Chapter 4

4.1 Exercises p280