Mathematics 1A

ITMTA1-B44

Functions and Models

With

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Lecture 2
Week 2

1.4 Exponential Functions

Exponential Functions and Graphs

The function $f(x) = 2^x$ is called an *exponential function* because the variable, x, is the exponent. It should not be confused with the power function $g(x) = x^2$, in which the variable is the base.

In general, an **exponential function** is a function of the form

$$f(x) = b^x$$

where b is a positive constant.

Exponential Functions and Their Graphs

If $x = 0$, then $b^0 = 1$, and if $x = -n$, where n is a positive integer, then

$$b^{-n} = \frac{1}{b^n}$$

If x is a rational number, $x = \frac{p}{q}$, where p and q are integers and $q > 0$, then

$$b^x = b^{\frac{p}{q}} = \sqrt[q]{b^p} = \left(\sqrt[q]{b}\right)^p$$

But what is the meaning of b^x if x is an irrational number? For instance, what is meant by $2^{\sqrt{3}}$ or 5^{π} ?

Exponential Functions and Their Graphs

To help us answer this question we first look at the graph of the function

where x is rational. A representation of this graph is shown in Figure 1.

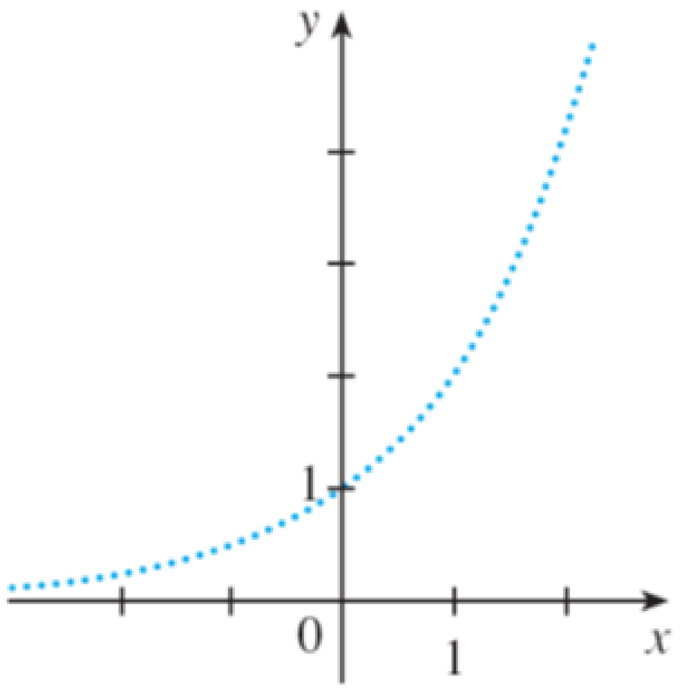
In particular, since the irrational number $\sqrt{3}$ satisfies

$$1.7 < \sqrt{3} < 1.8$$

we must have

$$2^{1.7} < 2^{\sqrt{3}} < 2^{1.8}$$

and we know what $2^{1.7}$ and $2^{1.8}$ mean because 1.7 and 1.8 are rational numbers.

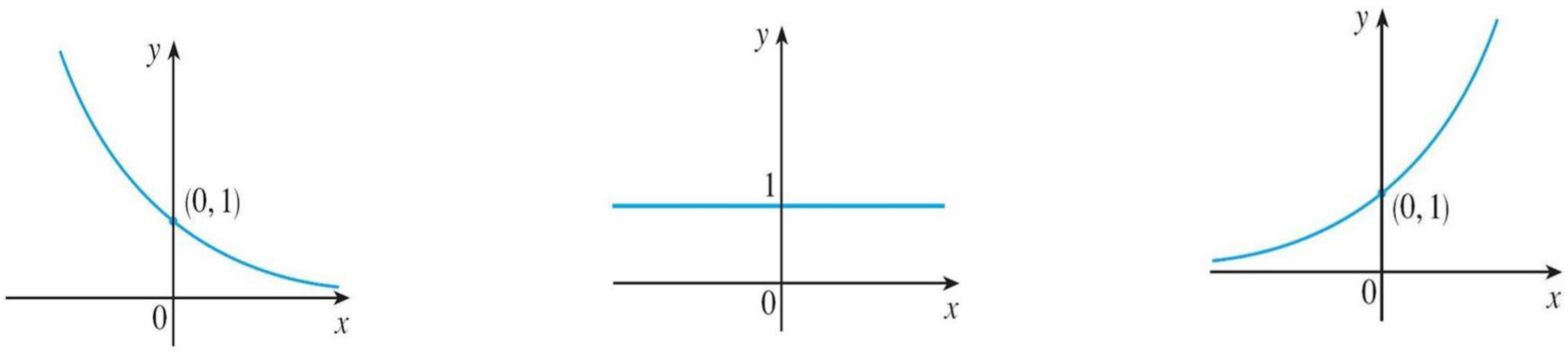


Representation of $y = 2^x, x$ rational

Figure 1

Exponential Functions and Their Graphs

There are three cases of exponential function are illustrated in Figure 4.



(a) $y = b^x,\ 0 < b < 1$ **(b)** $y = 1^x$ **(c)** $y = b^x,\ b > 1$

Figure 2

Exponential Functions and Their Graphs

Observe that if $b \neq 1$, then the exponential function $y = b^x$ has domain $\mathbb{R}$ and range $(0, \infty)$.

Notice also that, since $\left(\frac{1}{b}\right)^x = \frac{1}{b^x} = b^{-x}$, the graph of $y = \left(\frac{1}{b}\right)^x$

is just the reflection of the graph of $y = b^x$ about the y-axis.

Exponential Functions and Their Graphs (12 of 12)

One reason for the importance of the exponential function lies in the following properties.

If x and y are rational numbers, then these laws are well known from elementary algebra. It can be proved that they remain true for arbitrary real numbers x and y.

Laws of Exponents If a and b are positive numbers and x and y are any real numbers, then

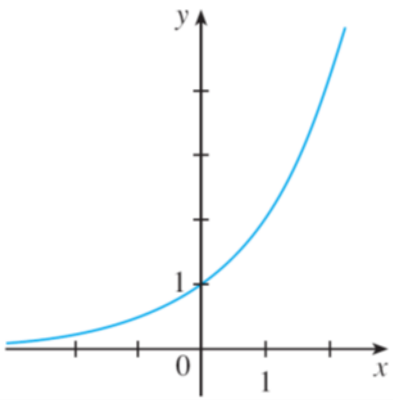
1. $b^{x+y} = b^x b^y$
2. $b^{x-y} = \dfrac{b^x}{b^y}$
3. $\left(b^x\right)^y = b^{xy}$
4. $(ab)^x = a^x b^x$

Example 1

Sketch the graph of the function $y = 3 - 2^x$ and determine its domain and range.

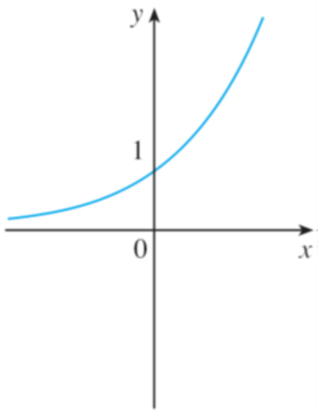
Solution:

First we reflect the graph of $y = 2^x$ [shown in Figures 2 and 5(a)] about the x-axis to get the graph of $y = -2^x$ in Figure 5(b).

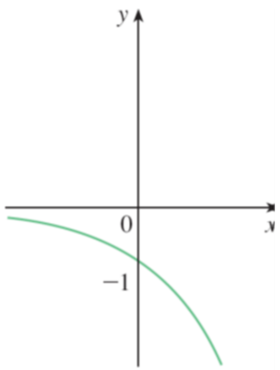


$y = 2^x$, x real

Figure 2



(a) $y = 2^x$

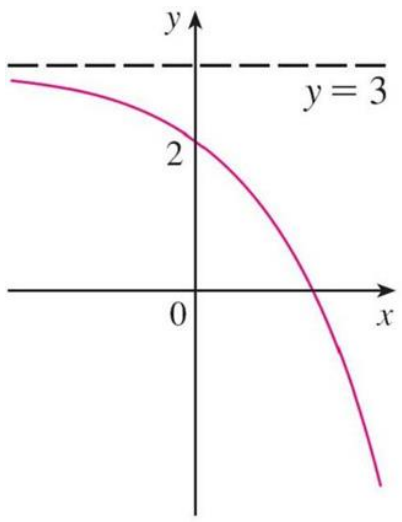


(b) $y = -2^x$

Figure 5

Example 1 – Solution

Then we shift the graph of $y = -2^x$ upward 3 units to obtain the graph of $y = 3 - 2^x$ in Figure 5(c).



(c) $y = 3 - 2^x$

Figure 5

The domain is $\mathbb{R}$ and the range is $(-\infty, 3)$.

Applications of Exponential Functions

Applications of Exponential Functions (1 of 3)

- The exponential function occurs very frequently in mathematical models of nature and society.
- Here we indicate briefly how it arises in the description of increasing population or decreasing viral loads.
- First we consider a population of bacteria in a homogeneous nutrient medium.
- Suppose that by sampling the population at certain intervals it is determined that the population doubles every hour.

Applications of Exponential Functions (2 of 3)

If the number of bacteria at time t is $p(t)$, where t is measured in hours, and the initial population is $p(0) = 1000$, then we have

$$p(1) = 2p(0) = 2 \times 1000$$

$$p(2) = 2p(1) = 2^2 \times 1000$$

$$p(3) = 2p(2) = 2^3 \times 1000$$

It seems from this pattern that, in general,

$$p(t) = 2^t \times 1000 = (1000)2^t$$

Applications of Exponential Functions (3 of 3)

This population function is a constant multiple of the exponential function $y = 2^t$, so it exhibits the rapid growth.

Under ideal conditions (unlimited space and nutrition and absence of disease) this exponential growth is typical of what actually occurs in nature.

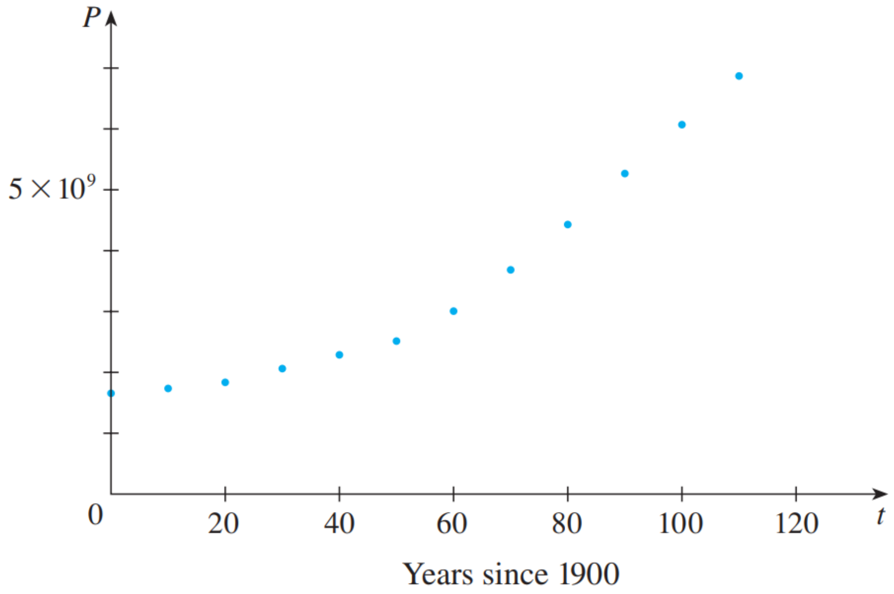
Example 3 (1 of 3)

Table 1 shows data for the population of the world in the 20th century and Figure 8 shows the corresponding scatter plot.

t (years since 1900)	Population *P* (millions)
0	1650
10	1750
20	1860
30	2070
40	2300
50	2560
60	3040
70	3710
80	4450
90	5280
100	6080
110	6870

World Population

Table 1



Scatter plot for world population growth

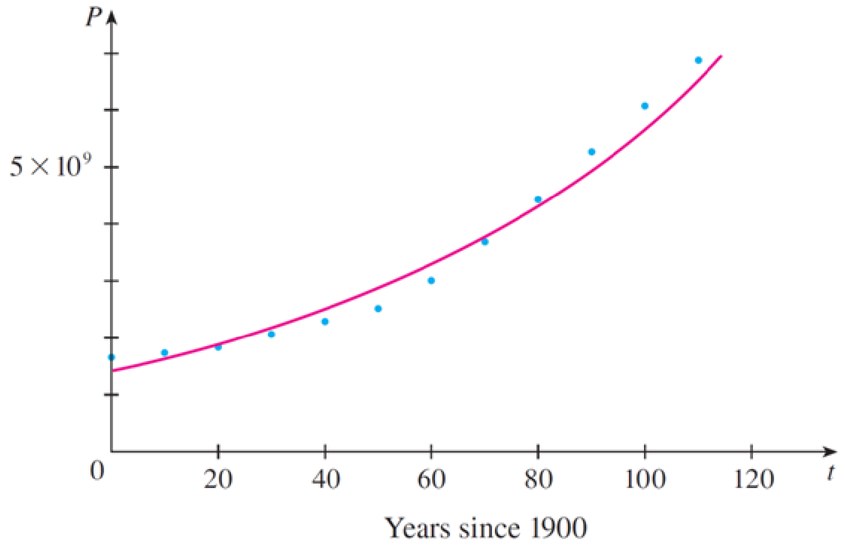
Figure 8

Example 3 (2 of 3)

The pattern of the data points in Figure 8 suggests exponential growth, so we use a graphing calculator (or computer) with exponential regression capability to apply the method of least squares and obtain the exponential model

$$P(t) = \left(1.43653 \times 10^9\right) \cdot (1.01395)^t$$

where t = 0 corresponds to 1900. Figure 9 shows the graph of this exponential function together with the original data points.



Exponential model for world population growth

Figure 9

Example 3 (3 of 3)

We see that the exponential curve fits the data reasonably well.

The period of relatively slow population growth is explained by the two world wars and the Great Depression of the 1930s.

The Number *e*

The Number *e* (1 of 6)

Of all possible bases for an exponential function, there is one that is most convenient for the purposes of calculus.

The choice of a base *b* is influenced by the way the graph of $y = b^x$ crosses the *y*-axis.

The Number e (2 of 6)

Figures 12 and 13 show the tangent lines to the graphs of $y = 2^x$ and $y = 3^x$ at the point (0, 1).

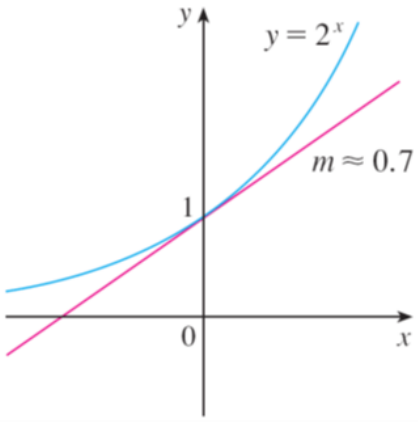


Figure 12

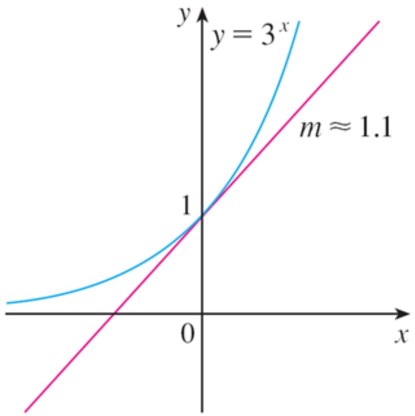


Figure 13

(For present purposes, you can think of the tangent line to an exponential graph at a point as the line that touches the graph only at that point.) If we measure the slopes of these tangent lines at (0, 1), we find that $m \approx 0.7$ for $y = 2^x$ and $m \approx 1.1$ for $y = 3^x$.

The Number *e* (3 of 6)

It turns out, that some of the formulas of calculus will be greatly simplified if we choose the base b so that the slope of the tangent line to $y = b^x$ at (0, 1) is *exactly* 1. (See Figure 14.)

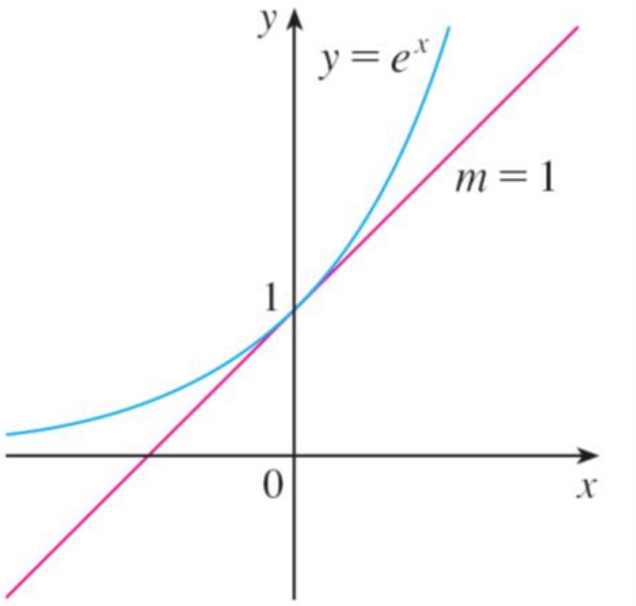


Figure 14

The Number *e* (4 of 6)

In fact, there *is* such a number and it is denoted by the letter *e*. (This notation was chosen by the Swiss mathematician Leonhard Euler in 1727, probably because it is the first letter of the word *exponential*.)

The Number e (5 of 6)

In view of Figures 12 and 13, it comes as no surprise that the number e lies between 2 and 3 and the graph of $y = e^x$ lies between the graphs of $y = 2^x$ and $y = 3^x$. (See Figure 15.)

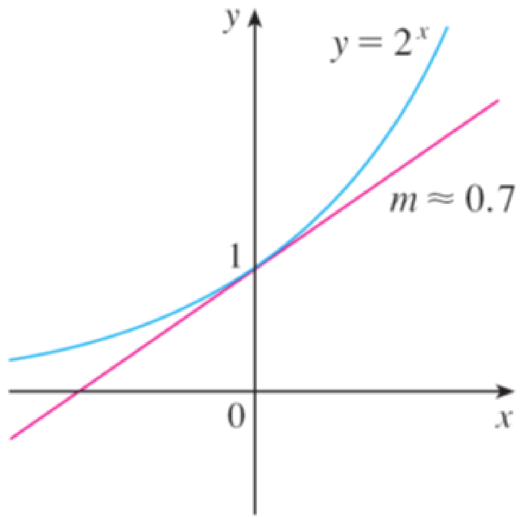


Figure 12

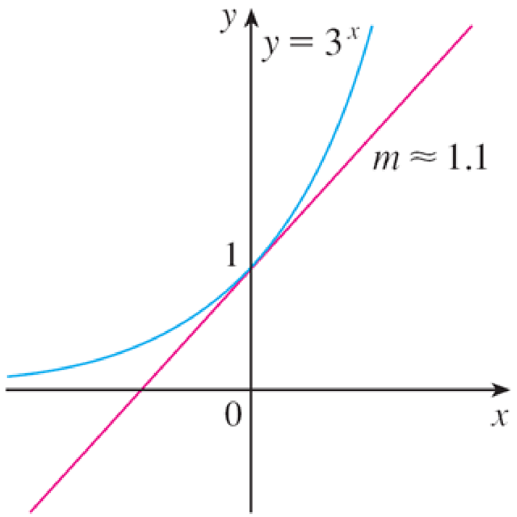
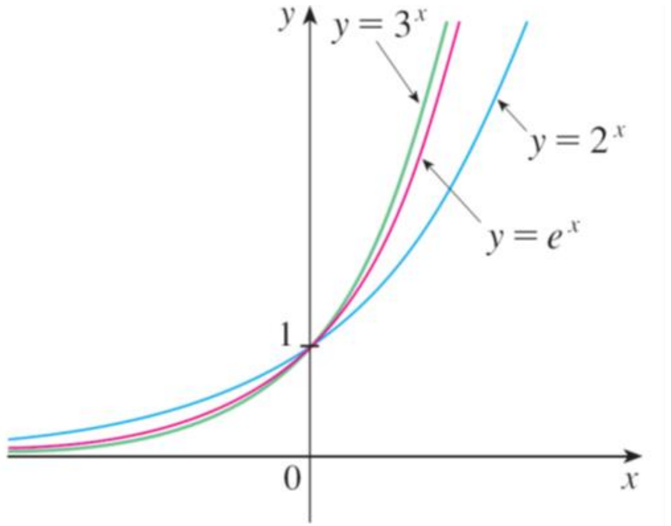


Figure 13



The graph of $y = e^x$ lies between the graphs of $y = 2^x$ and $y = 3^x$.

Figure 15

The Number *e* (6 of 6)

We will see that the value of *e*, correct to five decimal places, is

$$e \approx 2.71828$$

We call the function $f(x) = e^x$ the **natural exponential function**.

Example 5

Graph the function $y = \frac{1}{2}e^{-x} - 1$ and state the domain and range.

Solution:

We start with the graph of $y = e^x$ from Figures 14 and 16(a) and reflect about the y-axis to get the graph of $y = e^{-x}$ in Figure 16(b). (Notice that the tangent line to the graph at the y-intercept has slope of −1).

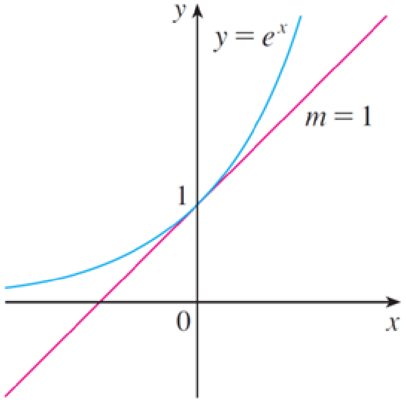
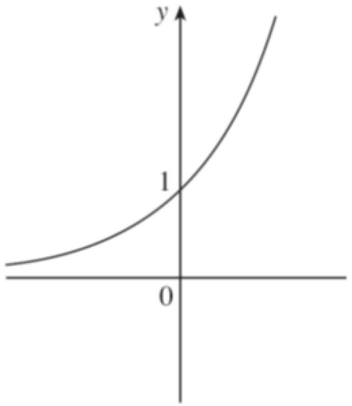
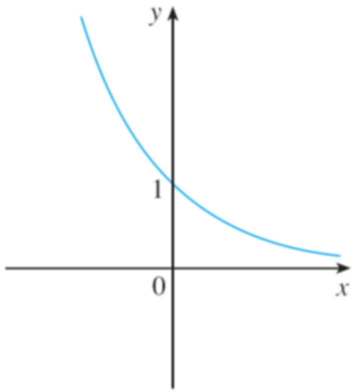


Figure 14



(a) $y = e^x$



(b) $y = e^{-x}$

Figure 16

Example 5 – Solution

Then we compress the graph vertically by a factor of 2 to obtain the graph of $y = \frac{1}{2}e^{-x}$ in Figure 16(c). Finally, we shift the graph downward one unit to get the desired graph in Figure 16(d).

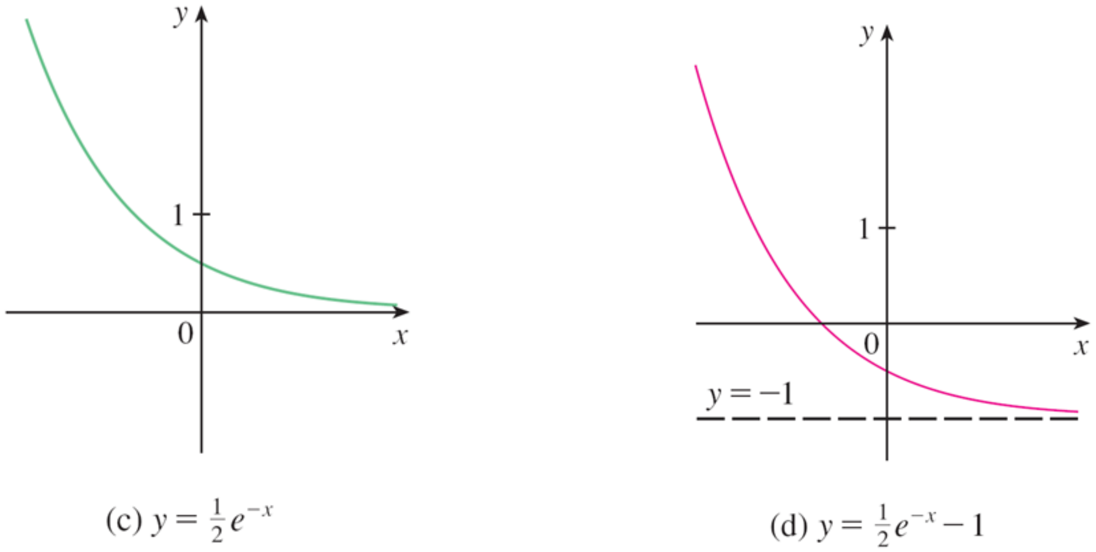


Figure 16

The domain is $\mathbb{R}$ and the range is $(-1, \infty)$.