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Mathematics 1A
ITMTA1-B44
Derivatives

With

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Lecture 1
Week 4

2 Limits and Derivatives

2.8 The Derivative as a Function

The Derivative Function

The Derivative Function

We have considered the derivative of a function *f* at a fixed number *x*:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Other Notations

Other Notations

If we use the traditional notation $y = f(x)$ to indicate that the independent variable is x and the dependent variable is y, then some common alternative notations for the derivative are as follows:

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}f(x) = Df(x) = D_x f(x)$$

The symbols D and $\frac{d}{dx}$ are called **differentiation operators** because they indicate the operation of **differentiation**, which is the process of calculating a derivative.

Other Notations

The symbol $\frac{dy}{dx}$, which was introduced by Leibniz, should not be regarded as a ratio (for the time being); it is simply a synonym for $f'(x)$.

Nonetheless, it is a very useful and suggestive notation, especially when used in conjunction with increment notation.

We can rewrite the definition of derivative in Leibniz notation in the form

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

Example 5

Where is the function $f(x) = |x|$ differentiable?

Solution:

If $x > 0$, then $|x| = x$ and we can choose h small enough that $x + h > 0$ and hence $|x + h| = x + h$. Therefore, for $x > 0$, we have

$$f'(x) = \lim_{h \to 0} \frac{|x + h| - |x|}{h} = \lim_{h \to 0} \frac{(x + h) - x}{h}$$
$$= \lim_{h \to 0} \frac{h}{h} = \lim_{h \to 0} 1 = 1$$

and so f is differentiable for any $x > 0$.

Example 5 – Solution

Similarly, for $x < 0$ we have $|x| = -x$ and h can be chosen small enough that $x + h < 0$ and so $|x + h| = -(x + h)$.

Therefore, for $x < 0$,

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{|x+h| - |x|}{h} \\ &= \lim_{h \to 0} \frac{-(x+h) - (-x)}{h} \\ &= \lim_{h \to 0} \frac{-h}{h} = \lim_{h \to 0} (-1) \end{aligned}$$

and so f is differentiable for any $x < 0$.

Higher Derivatives

Higher Derivatives

If *f* is a differentiable function, then its derivative f' is also a function, so f' may have a derivative of its own, denoted by $(f')' = f''$.

This new function f'' is called the **second derivative** of *f* because it is the derivative of the derivative of *f* .

Using Leibniz notation, we write the second derivative of *y* = *f*(*x*) as

$$\underbrace{\frac{d}{dx}}_{\text{derivative of}}\underbrace{\left(\frac{dy}{dx}\right)}_{\text{first derivative}} = \underbrace{\frac{d^2y}{dx^2}}_{\text{second derivative}}$$

Example 6

If $f(x) = x^3 - x,$ find and interpret $f''(x)$.

Solution:

The first derivative is $f'(x) = 3x^2 - 1.$

So the second derivative is

$$f''(x) = (f')'(x)$$

$$= \lim_{h \to 0} \frac{f'(x+h) - f'(x)}{h}$$

$$= \lim_{h \to 0} \frac{[3(x+h)^2 - 1] - [3x^2 - 1]}{h}$$

Example 6 – Solution

$$= \lim_{h \to 0} \frac{3x^2 + 6xh + 3h^2 - 1 - 3x^2 + 1}{h}$$

$$= \lim_{h \to 0} (6x + 3h)$$

$$f''(x) = 6x$$

Example 6 – Solution

The graphs of f, f', and f'' are shown in Figure 10.

$f(x) = x^3 - x$

$f'(x) = 3x^2 - 1$

$f''(x) = 6x,$

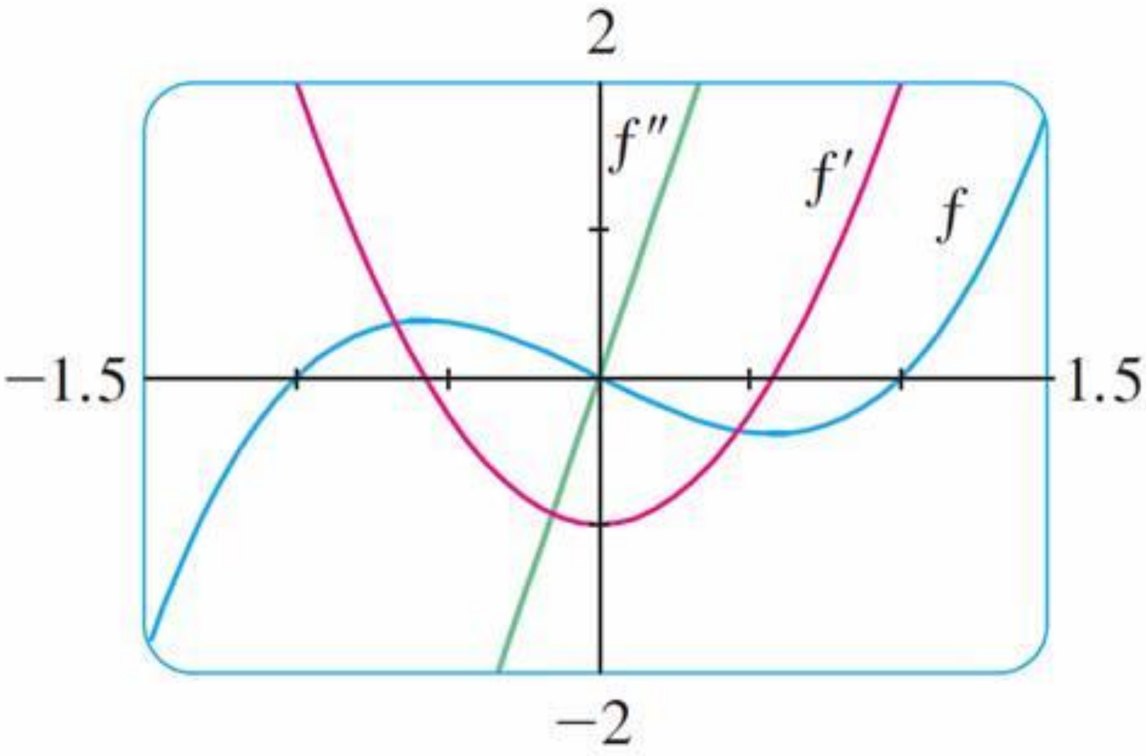

Figure 10

Example 6 – Solution

We can interpret $f''(x)$ as the slope of the curve $y = f'(x)$ at the point $(x, f'(x))$. In other words, it is the rate of change of the slope of the original curve $y = f(x)$.

Notice from Figure 10 that $f''(x)$ is negative when $y = f'(x)$ has negative slope and positive when $y = f'(x)$ has positive slope. So the graphs serve as a check on our calculations.

Higher Derivatives

In general, we can interpret a second derivative as a rate of change of a rate of change. The most familiar example of this is *acceleration,* which we define as follows.

If $s = s(t)$ is the position function of an object that moves in a straight line, we know that its first derivative represents the velocity $v(t)$ of the object as a function of time:

$$v(t) = s'(t) = \frac{ds}{dt}$$

Higher Derivatives

The instantaneous rate of change of velocity with respect to time is called the **acceleration** $a(t)$ of the object. Thus the acceleration function is the derivative of the velocity function and is therefore the second derivative of the position function:

$$a(t) = v'(t) = s''(t)$$

or, in Leibniz notation,

$$a = \frac{dv}{dt} = \frac{d^2s}{dt^2}$$

Higher Derivatives

The **third derivative** f''' is the derivative of the second derivative:

$$f''' = (f'')'. \mathrm{So} f'''(x)$$

can be interpreted as the slope of the curve $y = f''(x)$ or as the rate of change of $f''(x)$.

If $y = f(x)$, then alternative notations for the third derivative are

$$y''' = f'''(x) = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3}$$

Higher Derivatives

We can also interpret the third derivative physically in the case where the function is the position function $s = s(t)$ of an object that moves along a straight line.

Because $s''' = (s'')' = a'$, the third derivative of the position function is the derivative of the acceleration function and is called the **jerk**:

$$j = \frac{da}{dt} = \frac{d^3 s}{dt^3}$$

Higher Derivatives

Thus the jerk j is the rate of change of acceleration.

It is aptly named because a large jerk means a sudden change in acceleration, which causes an abrupt movement.

Higher Derivatives

The differentiation process can be continued. The fourth derivative f''''' is usually denoted by $f^{(4)}$.

In general, the *n*th derivative of *f* is denoted by $f^{(n)}$ and is obtained from *f* by differentiating *n* times.

If $y = f(x)$, we write

$$y^{(n)} = f^{(n)}(x) = \frac{d^n y}{dx^n}$$