Unit 2: The Derivative

Much of the work we are going to do in this course consists of taking one or more functions, and producing a new function. The first of these methods is **differentiation**: we take a function, call it $f(x)$ for convenience, and produce a new function which we call the **derivative** of $f(x)$.

We have several notations for this process. (In fact, we probably have too many notations. Such is mathematics.) If we again denote our function by $f(x)$, its derivative can be denoted by

$$\frac{d}{dx}f(x) = \frac{df}{dx} = D_x f(x) = f'(x).$$

If we express our function in the x-y plane as $y=f(x)$, then we gain the notations

$$f'(x) = \frac{d}{dx}y(x) = \frac{dy}{dx}.$$

There is, of course, a reason we want the derivative of a function. The derivative is the *slope* of the original function at any given point. This will be explained in the reading along with pretty pictures of tangent lines. The method used to define the tangent is to approach the tangent line through a sequence of secants; again, refer to pictures in the various texts. Following the secant lines, we arrive at our definition of the derivative:

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}.$$

(Sometimes Δx, ``Delta -*x*", is used instead of *h*. The triangle symbol is the Greek letter "Delta".)
This definition raises several interesting questions: what is this *limit*? (This problem was brushed off for a few centuries. Its eventual answer forms the subject material of analysis.) How do we know it exists? (Sometimes it doesn't. See discussion of $|x|$ in the text.). We follow a long tradition of beginning calculus courses in firmly brushing these questions under the rug. In most of the cases of interest, we are dealing with a ``smooth" function where we may take the derivative with impunity.

There is another interpretation of the derivative that you should also understand: the derivative indicates the *rate of change* of the original function. This interpretation is especially important when the function is expressed in terms of time. For example we may have the height of a rocket expressed as $h = f(t)$, giving the height for every instant. The upward component of the velocity of the rocket, *i.e.*, the rate of change of the height, is given by

$$v(t) = h'(t) = f'(t) = \frac{df}{dt}.$$

Become accustomed to this idea now; you will see it again and again.

Now that we've developed a *feel* for the derivative, we need method for calculating the derivative without applying the definition over and over again. Using the definiton, the following *rules* of differentiation can be derived. You should make sure to go over the derivation of these rules:

1. **The constant rule**: if c is a constant, then

$$\frac{d}{dx} c = 0.$$

2. **The sum and difference rules**:

$$\frac{d}{dx}\left[f(x)+g(x)\right]=\frac{d}{dx}f(x)+\frac{d}{dx}g(x) \qquad \text{and}$$
$$\frac{d}{dx}\left[f(x)-g(x)\right]=\frac{d}{dx}f(x)-\frac{d}{dx}g(x).$$

3. **The product rule**:

$$\frac{d}{dx}\left[f(x)\,g(x)\right]=f(x)\,\frac{d}{dx}g(x)+g(x)\,\frac{d}{dx}f(x).$$

4. **The quotient rule**:

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]=\frac{f'(x)\,g(x)-f(x)\,g'(x)}{\left(g(x)\right)^2}.$$

5. **The power rule** (first version):

$$\frac{d}{dx}x^n=n\,x^{n-1}.$$

Note that this is only the first version of the power rule; we will generalize this rule in the next unit, when we have the chain rule. The proofs of the various rules follow from the definition of the derivative and some algebraic manipulation. You should be able to follow the proofs, but you needn't worry about reproducing them. The power rule is an interesting case in that it can be proven using induction for positive integral powers.

(The following discussion will be extended when the World Wide Math Integer Arithmetic page is completed.)
To over-generalize madly, induction arguments can come up whenever the natural numbers are involved in a formula. Both texts have further examples of

induction proofs in their respective appendices. While induction proofs aren't vital to your understanding of calculus, they are a piece of mathematical culture you should learn. The general form is to prove a base case, either $n = 0$ or $n = 1$ or wherever you want to start: for the case of the **power rule**,

$$\frac{d}{dx}x^n = n\,x^{n-1}.$$

Next is the *inductive* step: assume that the proposition is true for all values less than n, then prove it for $n+1$:

$$n = 0, \qquad \frac{d}{dx}x^0 = \frac{d}{dx}1 = 0 = 0 \cdot x^{-1}.$$

$$\frac{d}{dx}x^n = n\,x^{n-1}$$

$$\begin{aligned}\frac{d}{dx}x^{n+1} = \frac{d}{dx}\left[x \cdot x^n\right] &= x^n + x\frac{d}{dx}x^n \\ &= x^n + x \cdot n \cdot x^{n-1} \\ &= (n+1)\,x^n.\end{aligned}$$

Note that the **product rule** was used in the above. Mastering these rules for differentiation is important; work enough practice problems so that you can differentiate *blindfolded!!*

Now is a convenient time as any to mention that you can take derivatives multiples times. The derivative of a function is a perfectly good function in its own right: it can be graphed, and one can ask for its slope at any point. Relative to the function we started with, this is a *second order* derivative. We, of course, have lots of fun notation for this notion, and for the obvious next steps:

$$
\begin{aligned}
y = f(x) &= x^4 + x^3 + x^2 + x + 1 \\
\frac{d}{dx}\, y(x) = f'(x) &= 4x^3 + 3x^2 + 2x + 1 \\
\frac{d^2}{dx^2}\, y(x) = f''(x) &= 12x^2 + 6x + 2 \\
\frac{d^3}{dx^3}\, y(x) = f'''(x) = f^{\text{iii}} &= 24x + 6 \\
\frac{d^4}{dx^4}\, y(x) = f''''(x) = f^{\text{iv}}(x) &= 24 \\
\frac{d^5}{dx^5}\, y(x) = f^{\text{v}}(x) &= 0.
\end{aligned}
$$

Objectives:

After completing this unit you should be able to:

1. Understand the derivative as expressing the slope of a curve at each point on the curve.
2. Understand the derivative as expressing a rate of change.
3. Be able to use

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

to set up derivatives, and to calculate derivatives in some cases.

4. Derive the rules of differentiation.
5. Calculate the derivative of any rational function.
6. Calculate higer order derivatives.

Suggested Procedure:

1. Read *Simmons*, first edition chapter 2 and sections 3.1, 3.2, and 3.5, or
2. Read *Simmons*, second edition chapter 2 and sections 3.1, 3.2, and 3.6.
3. Read the following World Web Math Pages:
   - The definition of differentiation
   - The notation of differentitation
   - Derivatives of polynomial functions
   - The Product Rule
   - The Quotient Rule
4. Do some problems in *Simmons*
   - 2.2 : # 1, 3, 5, 7, 8
   - 2.3 : # 1, 4, 12
   - 2.4 : # 1, 4, 8, 10, 11
   - 3.1 : # 1, 9 - 12
   - 3.2 : # 1, 2, 5, 7, 8, 9
5. Take the **Practice Unit Test**, Xdvi or PDF
6. Ask your instructor for a unit test.

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