Mathematics 1A

ITMTA1-B44

Integral Calculus

With

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Lecture 1
Week 7

5 Integrals

Introduction to Integral Calculus

Introduction

- Integration is a fundamental concept in calculus that allows us to find the area under curves, calculate accumulated quantities, solve differential equations, and explore various applications in physics, engineering, economics, and many other fields.

- In this lecture, we will explore the concept of integration, its applications, rules for solving problems involving functions, and techniques to calculate the area of regions bounded by graphs of functions.

Concept of Integration:

- Integration is the process of finding the antiderivative or integral of a function.
- It is the reverse operation of differentiation.
- The integral of a function $f(x)$ is denoted by:

$$\int \boldsymbol{f(x)\, dx}$$

Where:

- $\int$ represents the integration symbol;
- $f(x)$ is the integrand, and
- dx represents the differential of the independent variable x.
- Geometrically, integration represents the area under the curve of the function.

Applications of Integration:

- Integration has numerous applications across various disciplines. Some common applications include:
- **Area Calculation**: Integration helps calculate the area of regions bounded by curves or between two curves.
- **Physics**: Integration is used to calculate quantities such as displacement, velocity, acceleration, work, and energy.
- **Economics**: Integration is applied in economics to analyze consumer and producer surplus, revenue, and cost functions.
- **Engineering**: Integration is used to calculate the moments of inertia, fluid flow rates, and electrical circuits.
- **Probability**: Integration is employed in probability theory to calculate probabilities and cumulative distribution functions.

Indefinite Integral

Rules to Solve Problems of Functions:

- To solve integration problems, we need to follow certain rules. Some important rules include:

Power Rule:

- For any real number *n* (except -1), the integral of x^n with respect to x is given by:

$$\frac{(x^{(n+1)})}{(n+1)} + C$$

where C is the constant of integration.

Let us consider an example:

1. Solve $\int 5x^2\, dx$

<u>Solution</u>

$$\int 5x^2\, dx = \frac{(5x^{(2+1)})}{(2+1)} + C$$

Ans = $\mathbf{\frac{5x^3}{3} + C}$

Rules to Solve Problems of Functions:

Linearity:

- Integration is a linear operator, meaning that the integral of a sum or difference of functions is the sum or difference of their individual integrals.

Let us consider another example:

2. Solve $\int(\mathbf{5x^3 + x^2 - 10x})\boldsymbol{dx}$

<u>Solution</u>

This can be re-written as:

$$\int 5x^3 dx + \int x^2 dx - \int 10x\, dx$$

$$\int 5x^3 dx + \int x^2 dx - \int 10x\, dx = \frac{(5x^{(3+1)})}{(3+1)} + \frac{(x^{(2+1)})}{(2+1)} - \frac{(10x^{(1+1)})}{(1+1)} + C$$

$$= \frac{(5x^{(3+1)})}{(3+1)} + \frac{(x^{(2+1)})}{(2+1)} - \frac{(10x^{(1+1)})}{(1+1)} + C$$

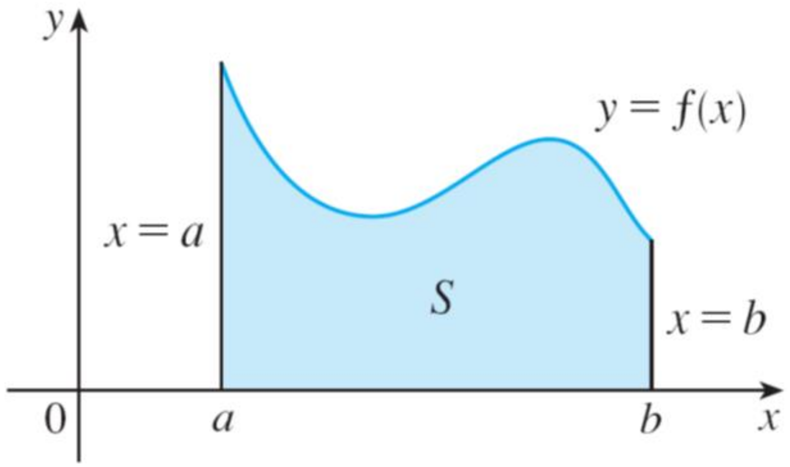
Ans = $\boldsymbol{\frac{5x^4}{4} + \frac{x^3}{3} - \frac{10x^2}{2} + C}$

The Area Problem

The Area Problem

Find the area of the region S that lies under the curve $y = f(x)$ from a to b.

This means that S, illustrated in Figure 1, is bounded by the graph of a continuous function f [where $f(x) \geq 0$], the vertical lines $x = a$ and $x = b$, and the x-axis.



$$S = \{(x, y) | a \leq x \leq b, 0 \leq y \leq f(x)\}$$

Figure 1

The Area Problem

For a rectangle, the area is defined as the product of the length and the width. The area of a triangle is half the base times the height.

The area of a polygon is found by dividing it into triangles (as in Figure 2) and adding the areas of the triangles.

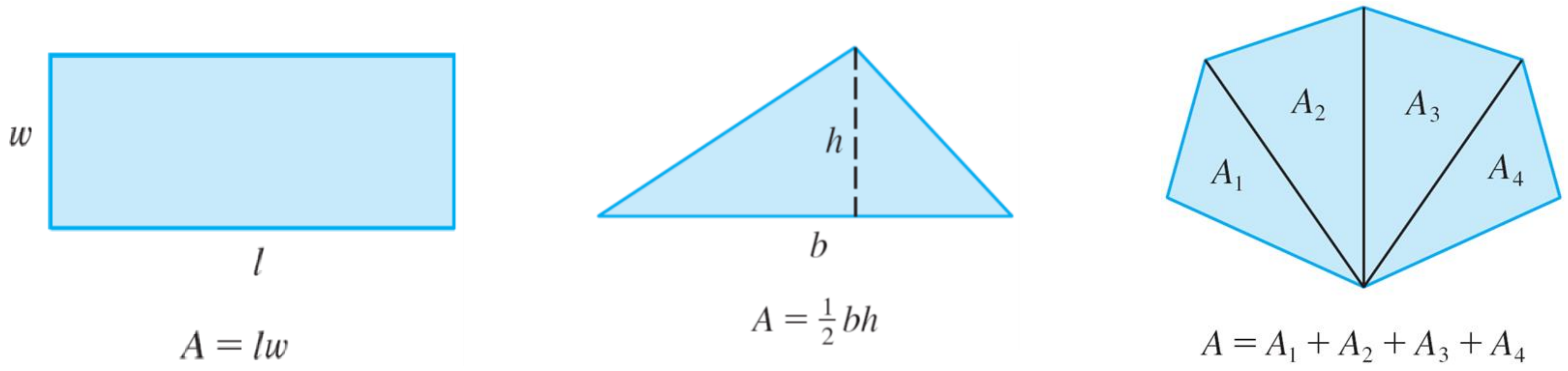


Figure 2

Example 1

Use rectangles to estimate the area under the parabola $y = x^2$ from 0 to 1 (the parabolic region *S* illustrated in Figure 3).

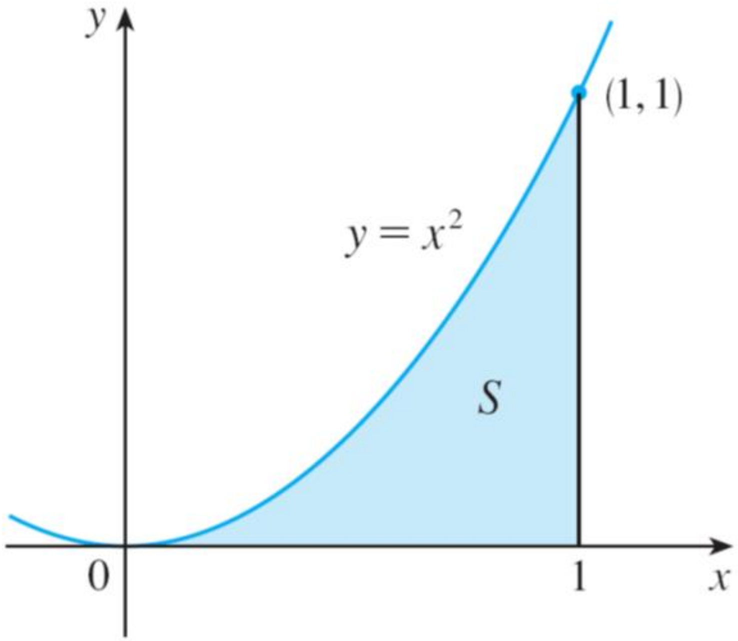


Figure 3

Example 1 – Solution

We first notice that the area of S must be somewhere between 0 and 1 because S is contained in a square with side length 1, but we can certainly do better than that.

Suppose we divide S into four strips S_1, S_2, S_3, and S_4 by drawing the vertical lines $x = \frac{1}{4}, x = \frac{1}{2}$, and $x = \frac{3}{4}$ as in Figure 4(a).

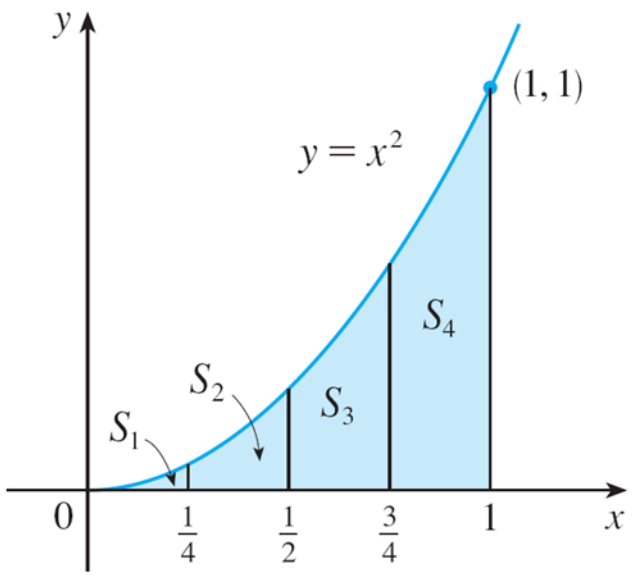


Figure 4(a)

Example 1 – Solution

We can approximate each strip by a rectangle that has the same base as the strip and whose height is the same as the right edge of the strip [see Figure 4(b)].

In other words, the heights of these rectangles are the values of the function $f(x) = x^2$ at the *right* endpoints of the subintervals $\left[0, \frac{1}{4}\right], \left[\frac{1}{4}, \frac{1}{2}\right], \left[\frac{1}{2}, \frac{3}{4}\right]$, and $\left[\frac{3}{4}, 1\right]$.

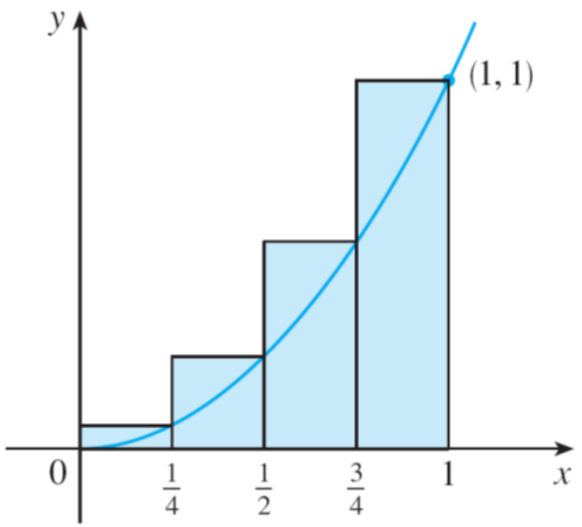


Figure 4(b)

Each rectangle has width $\frac{1}{4}$ and the heights are $\left(\frac{1}{4}\right)^2, \left(\frac{1}{2}\right)^2, \left(\frac{3}{4}\right)^2$, and 1^2.

Example 1 – Solution

If we let R_4 be the sum of the areas of these approximating rectangles, we get

$$\begin{aligned} R_4 &= \frac{1}{4}\cdot\left(\frac{1}{4}\right)^2 + \frac{1}{4}\cdot\left(\frac{1}{2}\right)^2 + \frac{1}{4}\cdot\left(\frac{3}{4}\right)^2 + \frac{1}{4}\cdot 1^2 \\ &= \frac{15}{32} \\ &= 0.46875 \end{aligned}$$

From Figure 4(b) we see that the area A of S is less than R_4, so

$$A < 0.46875$$

Example 1 – Solution

Instead of using the rectangles in Figure 4(b) we could use the smaller rectangles in Figure 5 whose heights are the values of *f* at the *left* endpoints of the subintervals. (The leftmost rectangle has collapsed because its height is 0.)

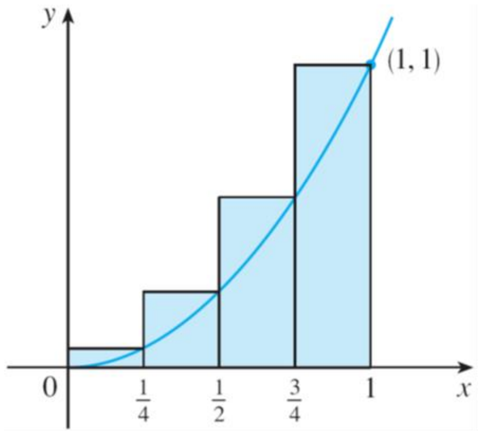


Figure 4(b)

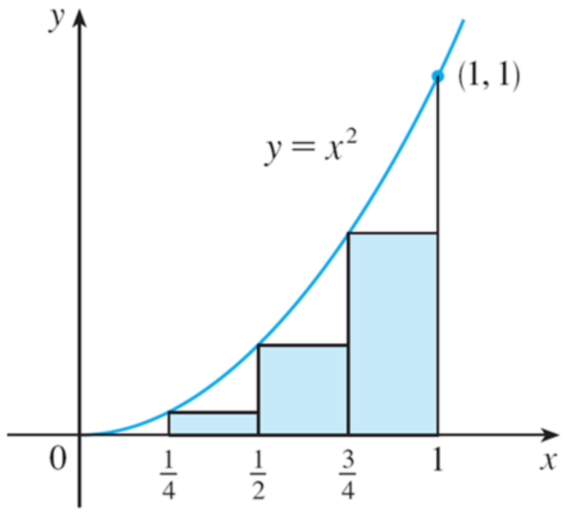


Figure 5

Example 1 – Solution

The sum of the areas of these approximating rectangles is

$$\begin{aligned} L_4 &= \frac{1}{4}\cdot 0^2 + \frac{1}{4}\cdot\left(\frac{1}{4}\right)^2 + \frac{1}{4}\cdot\left(\frac{1}{2}\right)^2 + \frac{1}{4}\cdot\left(\frac{3}{4}\right)^2 \\ &= \frac{7}{32} \\ &= 0.21875 \end{aligned}$$

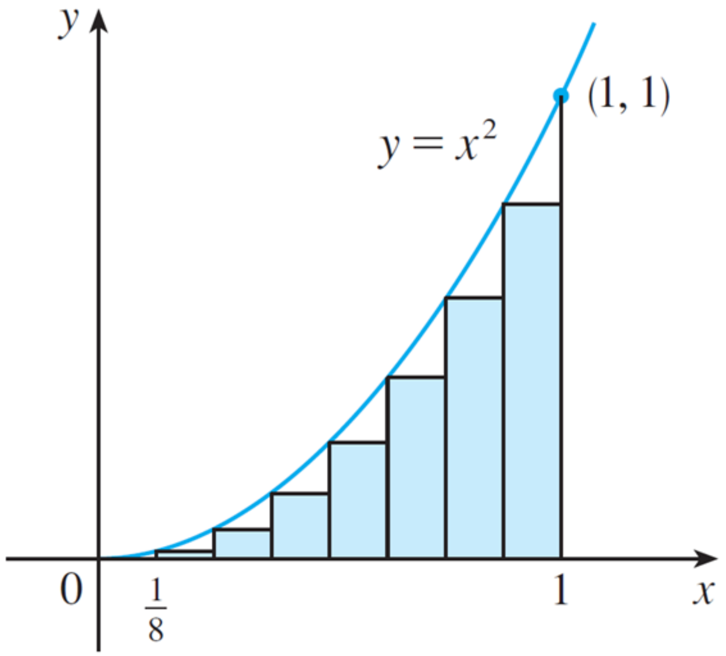
We see that the area of S is larger than L_4, so we have lower and upper estimates for A:

$$0.21875 < A < 0.46875$$

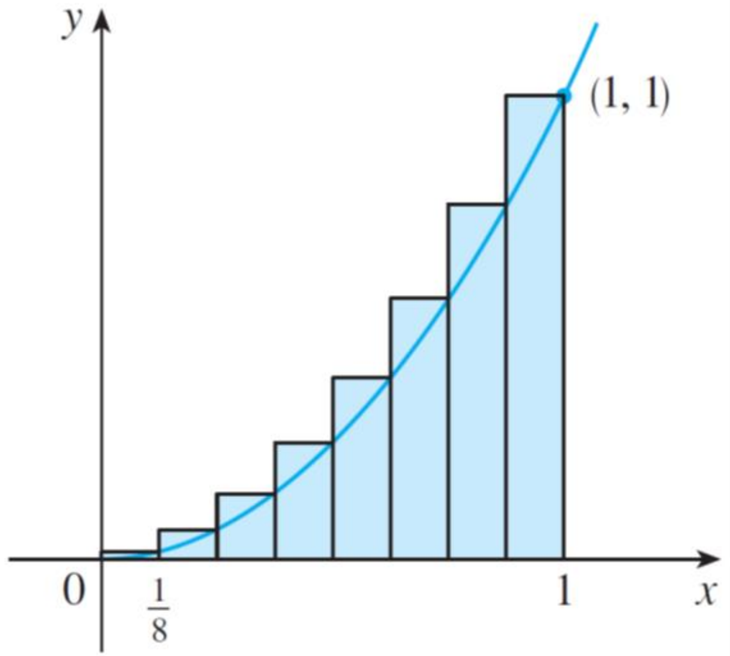
We can repeat this procedure with a larger number of strips.

Example 1 – Solution

Figure 6 shows what happens when we divide the region *S* into eight strips of equal width.



(a) Using left endpoints



(b) Using right endpoints

Approximating *S* with eight rectangles

Figure 6

Example 1 – Solution

By computing the sum of the areas of the smaller rectangles (L_8) and the sum of the areas of the larger rectangles (R_8), we obtain better lower and upper estimates for A:

$$0.2734375 < A < 0.3984375$$

So one possible answer to the question is to say that the true area of S lies somewhere between 0.2734375 and 0.3984375.

We could obtain better estimates by increasing the number of strips.

Example 1 – Solution

The table at the right shows the results of similar calculations (with a computer) using n rectangles whose heights are found with left endpoints (L_n) or right endpoints (R_n).

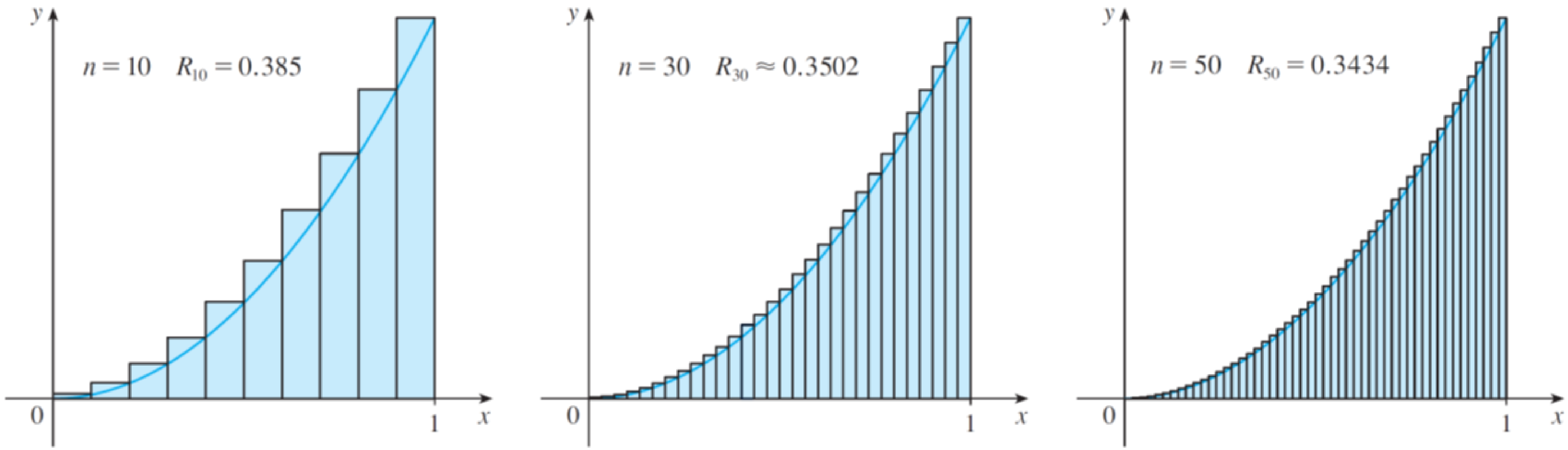
n	L_n	R_n
10	0.2850000	0.3850000
20	0.3087500	0.3587500
30	0.3168519	0.3501852
50	0.3234000	0.3434000
100	0.3283500	0.3383500
1000	0.3328335	0.3338335

In particular, we see by using 50 strips that the area lies between 0.3234 and 0.3434. With 1000 strips we narrow it down even more: A lies between 0.3328335 and 0.3338335.

A good estimate is obtained by averaging these numbers: $A \approx 0.3333335$.

The Area Problem

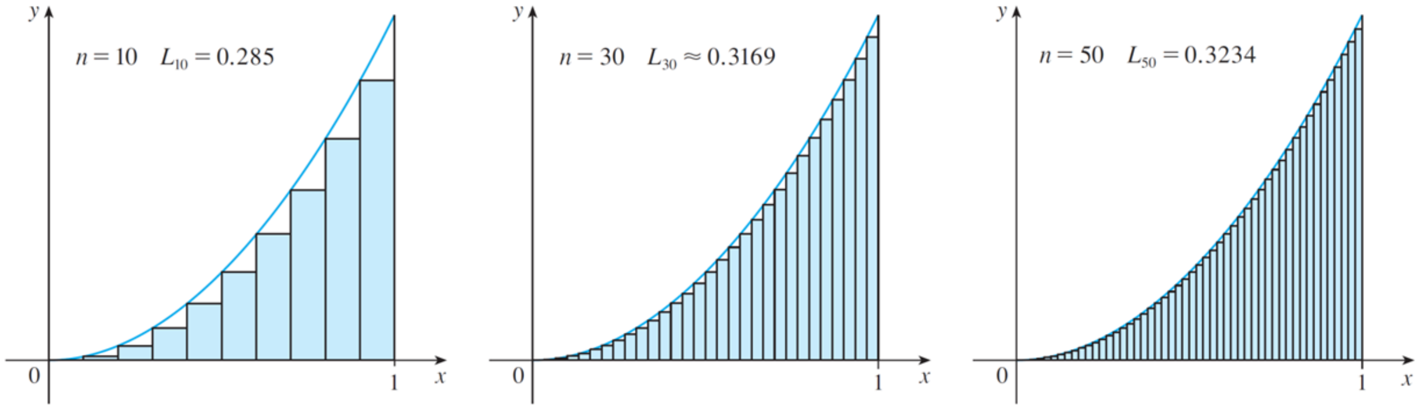
From Figures 8 and 9 it appears that as *n* increases, both L_n and R_n become better and better approximations to the area of *S*.



Right endpoints produce upper sums because $f(x) = x^2$ is increasing.

Figure 8

The Area Problem



Left endpoints produce lower sums because $f(x) = x^2$ is increasing.

Figure 9

Therefore we *define* the area *A* to be the limit of the sums of the areas of the approximating rectangles, that is,

$$A = \lim_{n \to \infty} R_n = \lim_{n \to \infty} L_n = \frac{1}{3}$$

The Area Problem - The Definite Integral

If is a function defined for $a \leq x \leq b$, we divide the interval $[a, b]$ into n subintervals of equal width

$$\Delta x = \frac{(b-a)}{n}.$$

We let $x_0 (= a), x_1, x_2, \ldots, x_n (= b)$ be the endpoints of these subintervals and we let $x_1^*, x_2^*, \ldots, x_n^*$ be any **sample points** in these subintervals, so x_i^* lies in the *i*th subinterval $[x_{i-1}, x_i]$. Then the **definite integral of *f* from *a* to *b*** is

$$\int_a^b f(x)dx = \sum_{i=1}^{n} f(x_i^*)\Delta x$$

The Area Problem

We start by subdividing S into n strips S_1, S_2, …, S_n of equal width as in Figure 10.

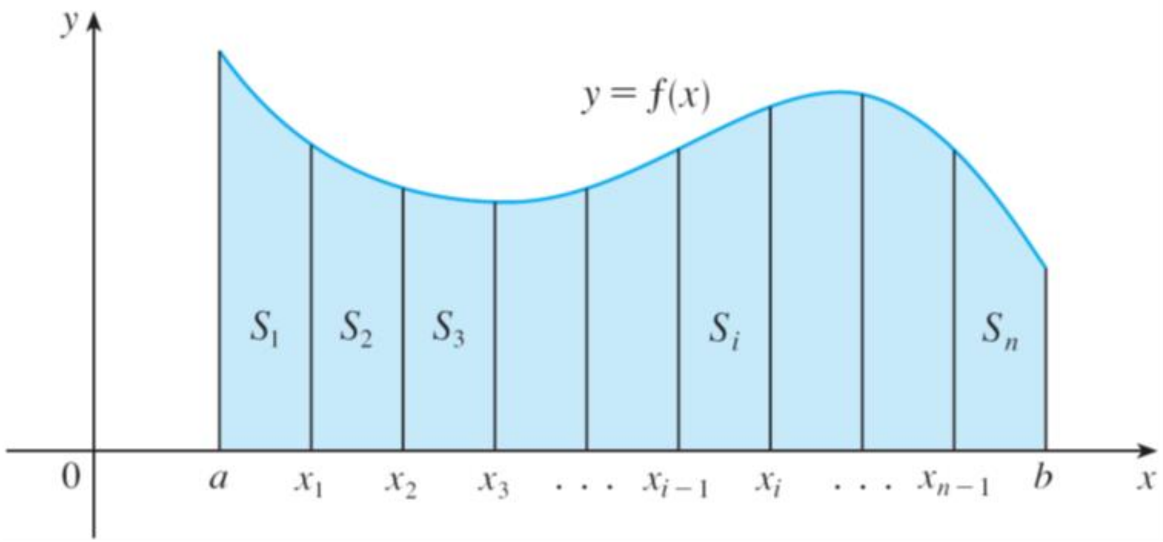


Figure 10

The width of the interval [a, b] is $b - a$, so the width of each of the n strips is

$$\Delta x = \frac{b-a}{n}$$

The Area Problem

These strips divide the interval [*a*, *b*] into *n* subintervals

$$[x_0, x_1], \qquad [x_1, x_2], \qquad [x_2, x_3], \qquad \ldots, \qquad [x_{n-1}, x_n]$$

where $x_0 = a$ and $x_n = b$.

The endpoints of the subintervals are

$$\begin{aligned}
x_0 &= a, \\
x_1 &= a + (1)\Delta x, \\
x_2 &= a + (2)\Delta x, \\
x_3 &= a + (3)\Delta x, \\
&\vdots \\
x_{n-1} &= a + (n-1)\Delta x, \\
x_n &= a + (n)\Delta x,
\end{aligned}$$

The Area Problem

Therefore we define the area *A* of the region *S* in the following way.

Definition

Given: $\int_a^b f(x)dx = \sum_{i=1}^{n} f(x_i^*)\Delta x$ Over an interval $[a, b]$ $or\ a \leq x \leq b$

The **area** *A* of the region *S* that lies under the graph of the continuous function *f* is the limit of the sum of the areas of approximating rectangles:

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} [f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x]$$

$$A = R_n = \Delta x[f(x_1) + f(x_2) + \cdots + f(x_n)]$$

The Area Problem

It can be proved that the limit in Definition 2 always exists, since we are assuming that *f* is continuous. It can also be shown that we get the same value if we use left endpoints:

$$A = \lim_{n\to\infty} L_n = \lim_{n\to\infty} [f(x_0)\Delta x + f(x_1)\Delta x + \cdots + f(x_{n-1})\Delta x]$$

$$A = L_n = \Delta x[f(x_0) + f(x_2) + \cdots + f(x_{n-1})]$$

The Definite Integral

In the notation $\int_a^b f(x)dx,\ \ f(x)$ is called the **integrand** and *a* and *b* are called the **limits of integration**; *a* is the **lower limit** and *b* is the **upper limit**.

The *dx* simply indicates that the independent variable is *x*.

Example 1

Evaluate the Riemann right sum for $f(x) = x^3 - 6x, \quad 0 \le x \le 3,$ with n = 6 subintervals and taking the sample endpoints to be right endpoints.

Solution:

With n = 6 subintervals, the interval width is $\Delta x = \frac{3-0}{6} = \frac{1}{2}$

$$A = R_n = \Delta x[f(x_1) + f(x_2) + \cdots + f(x_n)]$$

$$A = R_n = \Delta x[f(x_1) + f(x_2) + f(x_3) + f(x_4) + f(x_5) + f(x_6)]$$

So, the right endpoints are given by: $x_i = a + i\Delta x$

$x_1 = 0.5 \quad x_2 = 1.0 \quad x_3 = 1.5 \quad x_4 = 2.0 \quad x_5 = 2.5 \quad x_6 = 3.0$

Example 1 – Solution

So the Riemann Right sum is

$$
\begin{aligned}
A = R_n = R_6 &= \sum_{i=1}^{6} f(x_i)\Delta x \\
&= f(0.5)\Delta x + f(1.0)\Delta x + f(1.5)\Delta x + f(2.0)\Delta x + f(2.5)\Delta x + f(3.0)\Delta x \\
&= \frac{1}{2}(-2.875 - 5 - 5.625 - 4 + 0.625 + 9) \\
&= -3.9375
\end{aligned}
$$

Exercise

Evaluate the Riemann left sum for the same function for 6 intervals

$A = L_n =$?

Exercise

Find the Riemann Left and Right sum if $\int_1^2 2x^3\,dx,\ \ over\ 4\ intervals$.

$\Delta x = 0.25$

So, the right endpoints are given by: $x_i = a + i\Delta x$

x_0=1 x_1=1.25 x_2=1.50 x_3=1.75 x_4=2.0

Riemann Left sum

$A = L_n = \Delta x[f(x_0) + f(x_1) + f(x_2) + f(x_3)]$

$A = L_n = 0.25[f(1) + f(1.25) + f(1.50) + f(1.75)]$

$A = L_n = 5.84375$

Riemann Right sum

$A = R_n = \Delta x[f(x_1) + f(x_2) + f(x_3) + f(x_4)]$

$A = R_n = 0.25[f(1.25) + f(1.50) + f(1.75) + f(2)]$

$A = R_n = 9.34375$