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Mathematics 1A

ITMTA1-B44

Limits and Derivatives 2

With

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Lecture 8
Week 3

2 Limits and Derivatives

2.6 Limits at Infinity; Horizontal Asymptotes

Evaluating Limits at Infinity

Evaluating Limits at Infinity

5 Theorem If $r > 0$ is a rational number, then

$$\lim_{x \to \infty} \frac{1}{x^r} = 0$$

If $r > 0$ is a rational number such that x^r is defined for all x, then

$$\lim_{x \to -\infty} \frac{1}{x^r} = 0$$

Example 3

Evaluate the following limit and indicate which properties of limits are used at each stage.

$$\lim_{x \to \infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 1}$$

Solution:

As *x* becomes large, both numerator and denominator become large, so it isn't obvious what happens to their ratio.

To evaluate the limit at infinity of any rational function, we first divide both the numerator and denominator by the highest power of *x* that occurs in the denominator.

Example 3 – Solution

In this case the highest power of *x* in the denominator is x^2, so we have

$$\lim_{x\to\infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 1} = \lim_{x\to\infty} \frac{\dfrac{3x^2 - x - 2}{x^2}}{\dfrac{5x^2 + 4x + 1}{x^2}}$$

$$= \lim_{x\to\infty} \frac{3 - \dfrac{1}{x} - \dfrac{2}{x^2}}{5 + \dfrac{4}{x} + \dfrac{1}{x^2}}$$

Example 3 – Solution

$$= \frac{\lim_{x\to\infty}\left(3 - \frac{1}{x} - \frac{2}{x^2}\right)}{\lim_{x\to\infty}\left(5 + \frac{4}{x} + \frac{1}{x^2}\right)}$$

(by Limit Law 5)

$$= \frac{\lim_{x\to\infty} 3 - \lim_{x\to\infty}\frac{1}{x} - 2\lim_{x\to\infty}\frac{1}{x^2}}{\lim_{x\to\infty} 5 + 4\lim_{x\to\infty}\frac{1}{x} + \lim_{x\to\infty}\frac{1}{x^2}}$$

(by 1, 2, and 3)

$$= \frac{3 - 0 - 0}{5 + 0 + 0}$$

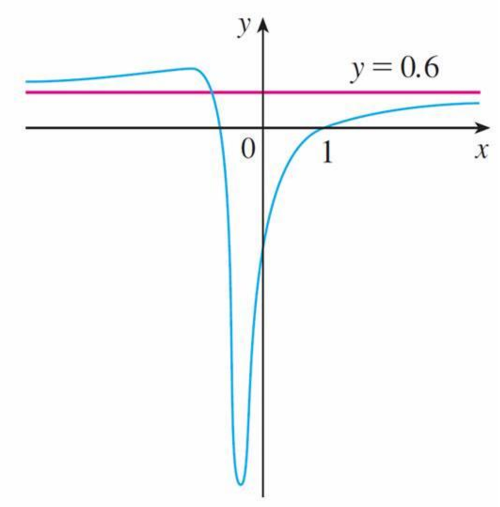
(by 8 and Theorem 5)

$$= \frac{3}{5}$$

Example 3 – Solution

A similar calculation shows that the limit as $x \to -\infty$ is y also approaches $\frac{3}{5}$.

$$y = \frac{3}{5} = 0.6.$$



$$y = \frac{3x^2 - x - 2}{5x^2 + 4x + 1}$$

Figure 7

Example 4

Find the horizontal asymptotes of the graph of the function

$$f(x) = \frac{\sqrt{2x^2+1}}{3x-5}$$

Solution:

Dividing both numerator and denominator by *x* (which is the highest power of *x* in the denominator) and using the properties of limits, we have

$$\lim_{x\to\infty} \frac{\sqrt{2x^2+1}}{3x-5} = \lim_{x\to\infty} \frac{\frac{\sqrt{2x^2+1}}{x}}{\frac{3x-5}{x}}$$

Example 4 – Solution

$$= \lim_{x\to\infty} \frac{\sqrt{\dfrac{2x^2+1}{x^2}}}{\dfrac{3x-5}{x}} \qquad \left(\text{since } \sqrt{x^2} = x \text{ for } x > 0\right)$$

$$= \frac{\lim\limits_{x\to\infty} \sqrt{2 + \dfrac{1}{x^2}}}{\lim\limits_{x\to\infty} \left(3 - \dfrac{5}{x}\right)}$$

$$= \frac{\sqrt{\lim\limits_{x\to\infty} 2 + \lim\limits_{x\to\infty} \dfrac{1}{x^2}}}{\lim\limits_{x\to\infty} 3 - 5 \lim\limits_{x\to\infty} \dfrac{1}{x}} = \frac{\sqrt{2+0}}{3 - 5 \cdot 0} = \frac{\sqrt{2}}{3}$$

Example 4 – Solution

Therefore the line $y = \sqrt{2}/3$ is a horizontal asymptote of the graph of *f*.

In computing the limit as $x \to -\infty$, we must remember that for $x < 0$, we have $\sqrt{x^2} = |x| = -x$.

So when we divide the numerator by x, for $x < 0$ we get

$$\frac{\sqrt{2x^2+1}}{x} = \frac{\sqrt{2x^2+1}}{-\sqrt{x^2}}$$

$$= -\sqrt{\frac{2x^2+1}{x^2}}$$

$$= -\sqrt{2+\frac{1}{x^2}}$$

Example 4 – Solution

Therefore

$$\begin{aligned}
\lim_{x\to-\infty}\frac{\sqrt{2x^2+1}}{3x-5} &= \lim_{x\to-\infty}\frac{-\sqrt{2+\frac{1}{x^2}}}{3-\frac{5}{x}} \\
&= \frac{-\sqrt{2+\lim\limits_{x\to-\infty}\frac{1}{x^2}}}{3-5\lim\limits_{x\to-\infty}\frac{1}{x}} \\
&= -\frac{\sqrt{2}}{3}
\end{aligned}$$

Example 4 – Solution

Thus the line $y = -\sqrt{2}/3$ is also a horizontal asymptote. See Figure 8.

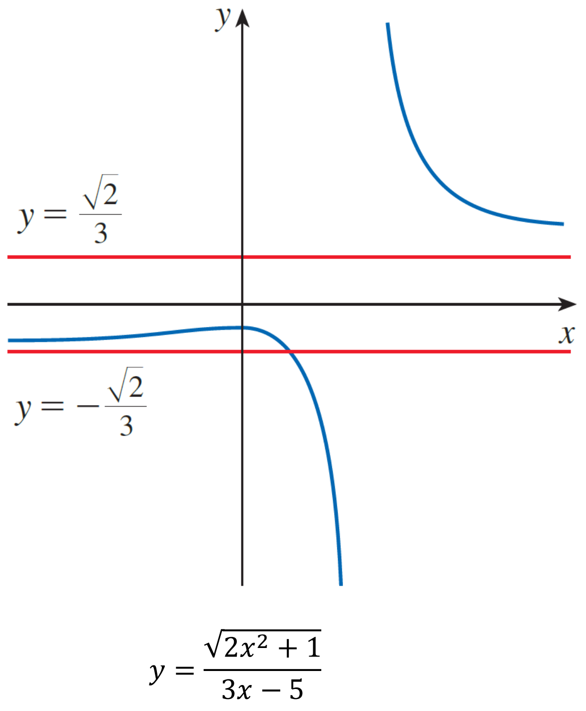


Figure 8

Example 4 – Solution

Compute $\lim_{x\to\infty}\left(\sqrt{x^2+1}-x\right)$.

SOLUTION Because both $\sqrt{x^2+1}$ and x are large when x is large, it's difficult to see what happens to their difference, so we use algebra to rewrite the function. We first multiply numerator and denominator by the conjugate radical:

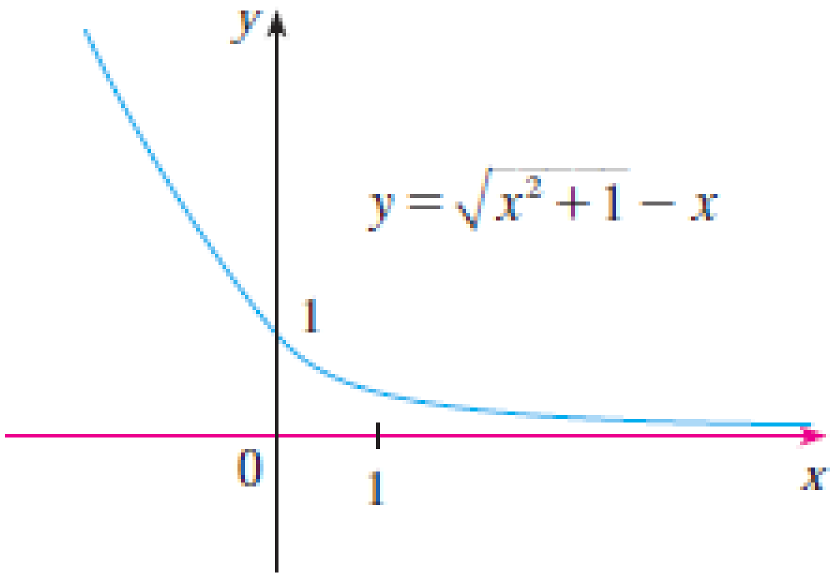
$$\lim_{x\to\infty}\left(\sqrt{x^2+1}-x\right)=\lim_{x\to\infty}\left(\sqrt{x^2+1}-x\right)\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}+x}$$

$$=\lim_{x\to\infty}\frac{(x^2+1)-x^2}{\sqrt{x^2+1}+x}=\lim_{x\to\infty}\frac{1}{\sqrt{x^2+1}+x}$$

Example 4 – Solution

Notice that the denominator of this last expression $\left(\sqrt{x^2+1}+x\right)$ becomes large as $x \to \infty$ (it's bigger than x). So

$$\lim_{x\to\infty}\left(\sqrt{x^2+1}-x\right)=\lim_{x\to\infty}\frac{1}{\sqrt{x^2+1}+x}=0$$



Exercises

1. Compute $\lim_{x \to \infty} x - \sqrt{x^2 + 1}$

    1. $\lim_{x \to \infty} x - \sqrt{x^2 + 1} = 0$

2. Compute $\lim_{x \to \infty} \frac{\sqrt{5x^2 - x}}{11x + 2}$

    2. $\lim_{x \to \infty} \frac{\sqrt{5x^2 - x}}{11x + 2} = \frac{\sqrt{5}}{11}$

3. Compute $\lim_{x \to -\infty} \frac{\sqrt{5x^2 - x}}{11x + 2}$

    3. $\lim_{x \to -\infty} \frac{\sqrt{5x^2 - x}}{11x + 2} = -\frac{\sqrt{5}}{11}$

4. Compute $\lim_{x \to \infty} \frac{5x^3 - x^2 - x}{x^3 + 2x - 2x + 1}$

    4. $\lim_{x \to \infty} \frac{5x^3 - x^2 - x}{x^3 + 2x - 2x + 1} = 5$

Example 5 – Solution

$$\boxed{\lim_{x \to -\infty} e^x = 0}$$

Example 6 – Solution

Evaluate $\lim_{x \to 0^-} e^{1/x}$.

SOLUTION If we let $t = 1/x$, we know that $t \to -\infty$ as $x \to 0^-$.

$$\lim_{x \to 0^-} e^{1/x} = \lim_{t \to -\infty} e^t = 0$$