Mathematics 1A

ITMTA1-B44

Derivatives

With

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Lecture 1
Week 5

3 Differentiation Rules

3.4 The Chain Rule

The Chain Rule

Suppose you are asked to differentiate the function

$$F(\mathrm{x}) = \sqrt{x^2 + 1}$$

The differentiation formulas you learned in the previous lecture cannot be used to calculate $F'(\mathrm{x})$.

Observe that *F* is a composite function. In fact, if we let $y = f(u) = \sqrt{u}$ and let $u = g(x) = x^2 + 1,$ then we can write *y* = *F*(*x*) = *f*(*g*(*x*)), that is, $F = f \circ g$.

We know how to differentiate both *f* and *g*, so it would be useful to have a rule that tells us how to find the derivative of $F = f \circ g$ in terms of the derivatives of *f* and *g*.

The Chain Rule

It turns out that the derivative of the composite function $f \circ g$ is the product of the derivatives of f and g. This fact is one of the most important of the differentiation rules and is called the *Chain Rule.*

It seems plausible if we interpret derivatives as rates of change. Regard $\frac{du}{dx}$ as the rate of change of u with respect to x, $\frac{dy}{du}$ as the rate of change of y with respect to u, and $\frac{dy}{dx}$ as the rate of change of y with respect to x.

The Chain Rule

The Chain Rule If g is differentiable at x and f is differentiable at $g(x)$, then the composite function $F = f \circ g$ defined by $F(x) = f(g(x))$ is differentiable at x and F' is given by the product

$$F'(x) = f'\big(g(x)\big) \cdot g'(x)$$

In Leibniz notation, if $y = f(u)$ and $u = g(x)$ are both differentiable functions, then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

Formula 2 is easy to remember because if we think of $\frac{dy}{du}$ and $\frac{du}{dx}$ as quotients, then we could cancel du.

The Chain Rule

Let's make explicit the special case of the Chain Rule where the outer function *f* is a power function.

If $y = [g(x)]^n$, then we can write $y = f(u) = u^n$

where $u = g(x)$. By using the Chain Rule and then the Power Rule, we get

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = nu^{n-1}\frac{du}{dx} = n[g(x)]^{n-1}g'(x)$$

The Chain Rule

4 The Power Rule Combined with the Chain Rule If n is any real number and $u = g(x)$ is differentiable, then

$$\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}$$

Alternatively,

$$\frac{d}{dx}[g(x)]^n = n[g(x)]^{n-1} \cdot g'(x)$$

Example 1

Find $\mathrm{F}'(\mathrm{x})$ if $F(x) = \sqrt{x^2+1}$.

Solution 1:

(Using Formula 1): We have expressed *F* as $F(x) = (f \circ g) = f\big(g(x)\big)$where $f(u) = \sqrt{u}$ and $g(x) = x^2 + 1$.

Since

$$f'(u) = \frac{1}{2}u^{\frac{-1}{2}} = \frac{1}{2\sqrt{u}} \quad \text{and} \quad g'(x) = 2x$$

we have

$$F'(x) = f'\big(g(x)\big) \cdot g'(x)$$

$$= \frac{1}{2\sqrt{x^2+1}} \cdot 2x = \frac{x}{\sqrt{x^2+1}}$$

Example 1 – Solution 2

(Using Formula 2): If we let $u = x^2 + 1$ and $y = \sqrt{u}$, then

$$\begin{aligned} F'(x) &= \frac{dy}{du}\frac{du}{dx} \\ &= \frac{1}{2\sqrt{u}}(2x) \\ &= \frac{1}{2\sqrt{x^2+1}}(2x) \\ &= \frac{x}{\sqrt{x^2+1}} \end{aligned}$$

Example 3

Differentiate $y = (x^3 - 1)^{100}$.

Solution:

Taking $u = g(x) = x^3 - 1$ and n = 100 in (4), we have

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx}(x^3 - 1)^{100} \\
&= 100(x^3 - 1)^{99} \frac{d}{dx}(x^3 - 1) \\
&= 100(x^3 - 1)^{99} \cdot 3x^2 \\
&= 300x^2(x^3 - 1)^{99}
\end{aligned}
$$

The Chain Rule

When using Formula 2 we should bear in mind that $\frac{dy}{dx}$ refers to the derivative of y when y is considered as a function of x (the derivative of y with respect to x), whereas $\frac{dy}{du}$ refers to the derivative of y when considered as a function of u (the derivative of y with respect to u).

Example 4

- If $f(x) = (2x^3 - 5)^{10}$, find $\mathrm{f}'(\mathrm{x})$.

$f'(x) = g'\,[h(x)].h'(x)$

From the example above:

$$f'(x) = 10(2x^3 - 5)^9.(6x^2)$$

$$f'(x) = 10(6x^2)(2x^3 - 5)^9$$

$$f'(x) = 60x^2(2x^3 - 5)^9$$

Example 5 and 6

5. $f(x) = (x^2 + 5)^3$
$f'(x) = g'[h(x)].h'(x)$
$f'(x) = 3(x^2 + 5)^2.(2x)$
$f'(x) = 6x(x^2 + 5)^2$

6. $f(x) = \sqrt{6x^2 - 1}$
$f(x) = (6x^2 - 1)^{1/2}$
$$f'(x) = \frac{1}{2}(6x^2 - 1)^{-1/2}.12x$$
$$f'(x) = \frac{6x}{\sqrt{6x^2 - 1}}$$

Examples 7 and 8

7. $f(x) = (x^2 + 5x - 1)^7$

$f'(x) = \quad 7(x^2 + 5x - 1)^6(2x + 5)$

$f'(x) = 7(2x + 5)(x^2 + 5x - 1)^6$

8. $f(x) = 3x^2(\sqrt{7x^4 - 1}\,)$

$f'(x) = 3x^2[\frac{1}{2}(7x^4 - 1)^{-1/2})(28x^3)] + 6x\,(\sqrt{7x^4 - 1}\,)$

$f'(x) = \dfrac{42x^5}{\sqrt{7x^4 - 1}} + 6x\,(\sqrt{7x^4 - 1}\,)$

Activities on Chain Rule

Use chain use to solve for $f'(x)\ in$ the following functions, simplify completely

1. $f(x) = (x^2+5)^{10}$

$f'(x) = 10(x^2+5)^9(2x)$

$f'(x) = 20x(x^2+5)^9$

2. $f(x) = \sqrt{3x^2-7}$

$f(x) = (3x^2-7)^{1/2}$

$f'(x) = \frac{1}{2}(3x^2-7)^{-1/2}(6x)$

$f'(x) = \frac{3x}{\sqrt{3x^2-7}}$

3. $f(x) = (5x^2-4x)^{15}$

$f'(x) = 15(5x^2-4x)^{14}(10x-4)$

4. $f(x) = 2\,(\sqrt{5x^3-7}\,)$

$f(x) = 2\,(5x^3-7\,)^{1/2}$

$f'(x) = 2\,[\frac{1}{2}(5x^3-7\,)^{-1/2}(15x^2)]$

$$f'(x) = \frac{15x^2}{\sqrt{5x^3-7}}$$

Derivative of Trigonometric functions

Derivative of Trigonometric functions

Given a trigonometric function, we can find the derivative, f', using certain rules of differentiation:

Given $y = \sin x$,

$$\frac{dy}{dx} = \cos x$$

$y = cos\, x$,

$$\frac{dy}{dx} = -\sin x$$

Given $y = tan\, x$,

$$\frac{dy}{dx} = sec^2\, x$$

Also note that,

$$sec^2\, x = (\frac{1}{\cos x})^2$$

Derivative of Trigonometric functions

Here are simplified forms of inverse trigonometric functions:

1. $\frac{1}{\sin x} = cosec\ x$

2. $\frac{1}{\cos x} = sec\ x$

3. $\frac{1}{\tan x} = \cot x$

Also note that $(\sin x)^2$ simply implies $sin^2 x$ and not $\sin x^2$

Examples on Trigonometric functions

Find the first derivative of the following trigonometric functions:

1. $f(x) = -3\ cos\ x$
2. $f(x) = \ e^x + 2\ sin\ x$
3. $f(x) = \sqrt{\tan x}$

Derivatives of Trigonometric functions

Original Rule	Generalized Rule (Chain Rule)
$\frac{d}{dx}\sin x = \cos x$	$\frac{d}{dx}\sin \mathbf{u} = \cos \mathbf{u}\frac{du}{dx}$
$\frac{d}{dx}\cos x = -\sin x$	$\frac{d}{dx}\cos \mathbf{u} = -\sin \mathbf{u}\frac{du}{dx}$
$\frac{d}{dx}\tan x = \sec^2 x$	$\frac{d}{dx}\tan \mathbf{u} = \sec^2 \mathbf{u}\frac{du}{dx}$
$\frac{d}{dx}\cot x = -\mathrm{cosec}^2 x$	$\frac{d}{dx}\cot \mathbf{u} = -\mathrm{cosec}^2 \mathbf{u}\frac{du}{dx}$
$\frac{d}{dx}\sec x = \sec x \tan x$	$\frac{d}{dx}\sec \mathbf{u} = \sec \mathbf{u} \tan \mathbf{u}\frac{du}{dx}$
$\frac{d}{dx}\mathrm{cosec}\, x = -\mathrm{cosec}\, x \cot$	$\frac{d}{dx}\mathrm{cosec}\, \mathbf{u} = -\mathrm{cosec}\, \mathbf{u} \cot \mathbf{u}\frac{du}{dx}$

Class work

If $f(x) = \tan x$, show that $f'(x) = sec^2 x$.

Hint: $\tan x = \frac{\sin x}{\cos x}$

Also, $(\frac{1}{\cos x})^2 = sec^2\, x$

The Chain Rule

In general, if $y = \sin u$, where u is a differentiable function of x, then, by the Chain Rule,

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \cos u \frac{du}{dx}$$

Thus

$$\frac{d}{dx}(\sin u) = \cos\ u \frac{du}{dx}$$

In a similar fashion, all of the formulas for differentiating trigonometric functions can be combined with the Chain Rule.

The Chain Rule

The reason for the name “Chain Rule” becomes clear when we make a longer chain by adding another link.

Suppose that $y = f(u)$, $u = g(x)$, and $x = h(t)$, where f, g, and h are differentiable functions.

Then, to compute the derivative of y with respect to t, we use the Chain Rule twice:

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = \frac{dy}{du}\frac{du}{dx}\frac{dx}{dt}$$

The Chain Rule - Example

Suppose you are given $\mathrm{y} = f(t) = \sin\sqrt{t^3 - 3}$

We can let *y* = *f*(*u*), *u* = *g*(*x*), and *x* = *h*(*t*), where *f*, *g*, and *h* are differentiable functions.

Then, to compute the derivative of *y* with respect to *t*, we use the Chain Rule twice:

Therefore, $\mathrm{y} = f(u) = \sin u$, and $\mathrm{u} = g(x) = \sqrt{x}$, and $x = t^3 - 3$

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}\frac{dx}{dt} = \cos u \cdot \frac{1}{2} x^{-1/2} \cdot 3t^2$$

$$\frac{dy}{dx} = \cos\sqrt{t^3 - 3} \cdot \frac{1}{2\sqrt{t^3-3}} \cdot 3t^2$$

$$\frac{dy}{dx} = 3t^2 \frac{\cos\sqrt{t^3 - 3}}{2\sqrt{t^3 - 3}}$$

Examples on Trigonometric functions

Find the first derivative of the following trigonometric functions:

1. $f(t) = \sqrt{\sin\ (t^3)}$
2. $f(t) = e^3 \sin 3t - 2\tan t$

2. $f(t) = \sqrt{t^3 \tan(t^2 + 1)}$

Exercises

Find the first derivative of the following functions :

1. $f(x) = 2\sin x$
2. $f(x) = 2\,cos\,4x - e^{-x}$
3. $f(x) = 3^x \tan(4x - 5)$
4. $f(x) = \sqrt{e^{2x}\sin 7x^3}$
5. $f(x) = 2e^{-5x}\sin 3x$