Mathematics 1A

ITMTA1-B44

Application of Differentiation

With

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Lecture 1
Week 6

4 Applications of Differentiation

4.2 The Mean Value Theorem

Rolle’s Theorem

Rolle's Theorem

To arrive at the Mean Value Theorem we first need the following result.

Rolle's Theorem

Let f be a function that satisfies the following three hypotheses:

1. f is continuous on the closed interval $[a, b]$.

2. f is differentiable on the open interval (a, b).

3. $f(a) = f(b)$

Then there is a number c in (a, b) such that $f'(c) = 0$.

Before giving the proof let's take a look at the graphs of some typical functions that satisfy the three hypotheses.

Rolle’s Theorem

Figure 1 shows the graphs of four such functions.

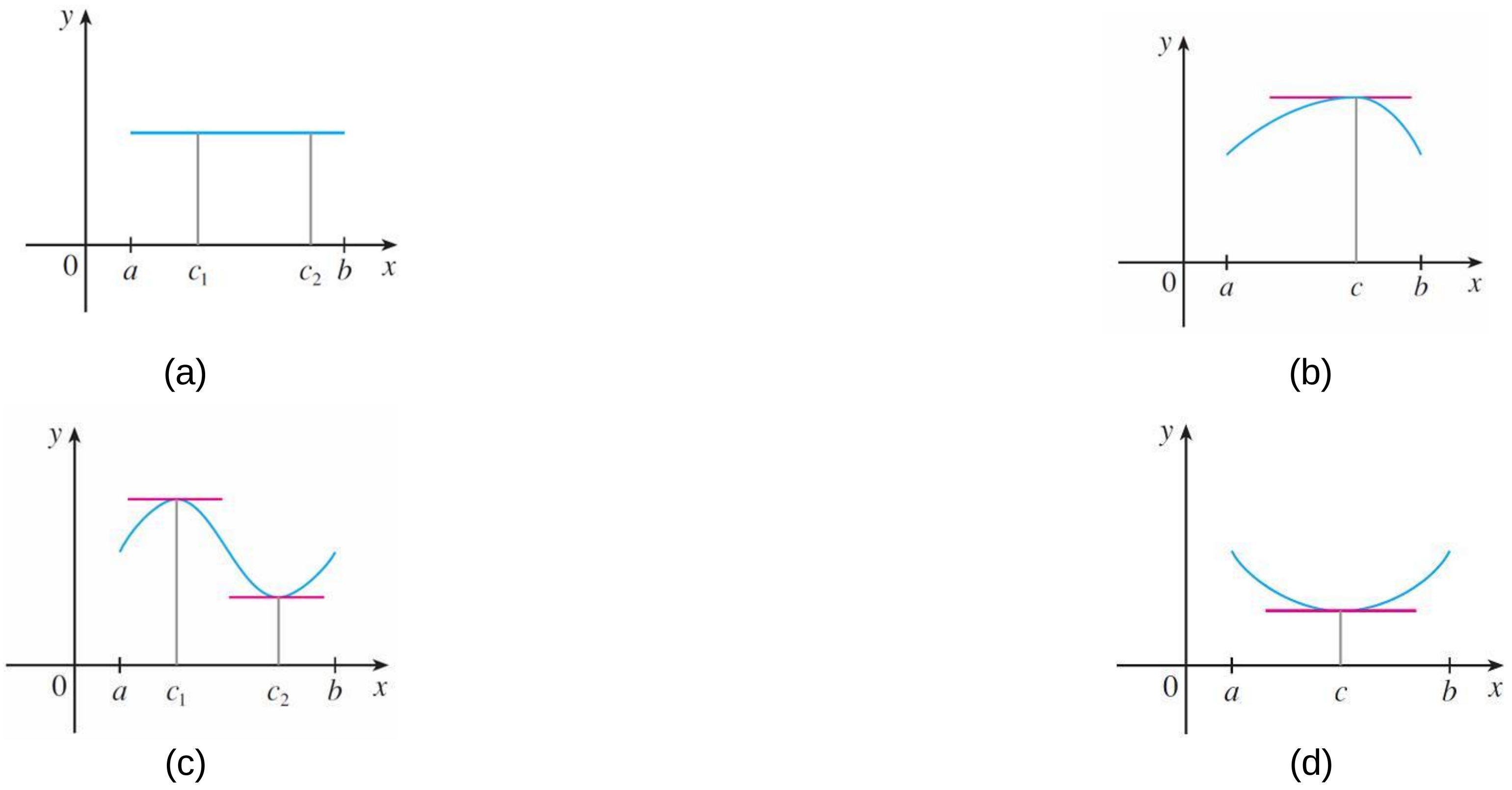

Figure 1

Rolle’s Theorem

In each case it appears that there is at least one point (c, $f(c)$) on the graph where the tangent is horizontal and therefore $f'(c) = 0$.

Thus Rolle’s Theorem is plausible.

The Mean Value Theorem

The Mean Value Theorem

Our main use of Rolle's Theorem is in proving the following important theorem, which was first stated by another French mathematician, Joseph-Louis Lagrange.

The Mean Value Theorem

Let *f* be a function that satisfies the following hypotheses:

1. *f* is continuous on the closed interval [*a*, *b*].

2. *f* is differentiable on the open interval (*a*, *b*).

Then there is a number *c* in (*a*, *b*) such that

$$1 \quad f'(c) = \frac{f(b) - f(a)}{b - a}$$

or, equivalently,

$$2 \quad f(b) - f(a) = f'(c)(b - a)$$

The Mean Value Theorem

Before proving this theorem, we can see that it is reasonable by interpreting it geometrically. Figures 3 and 4 show the points $A(a, f(a))$ and $B(b, f(b))$ on the graphs of two differentiable functions.

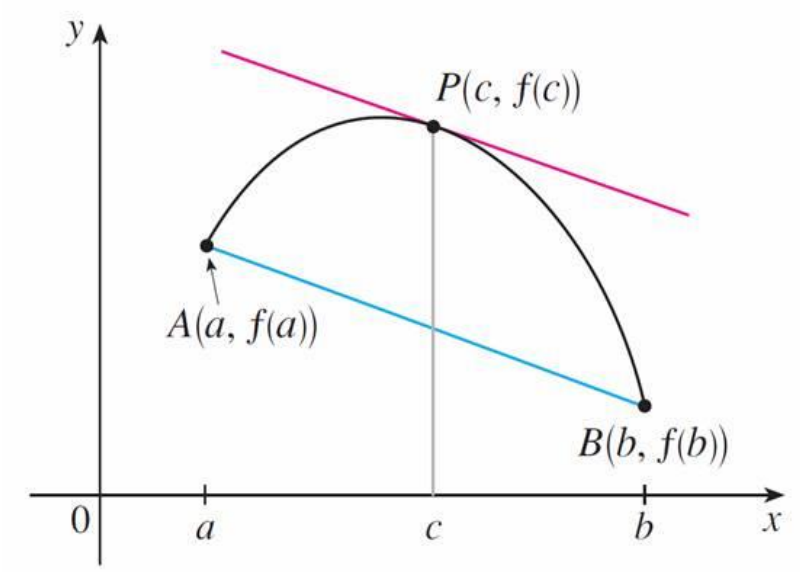

Figure 3

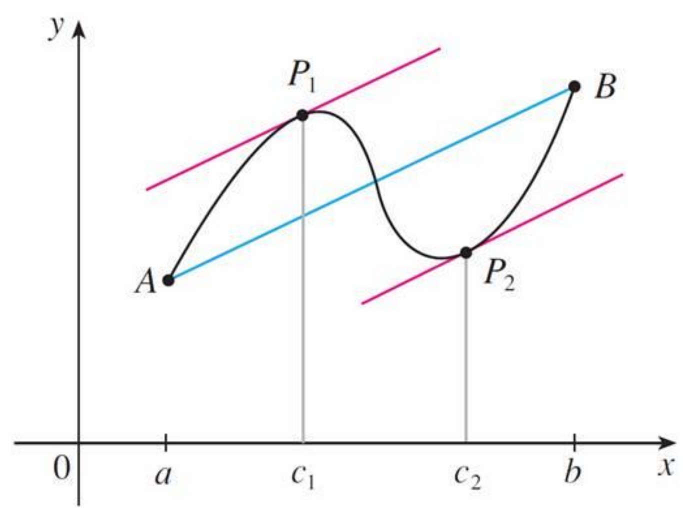

Figure 4

The Mean Value Theorem

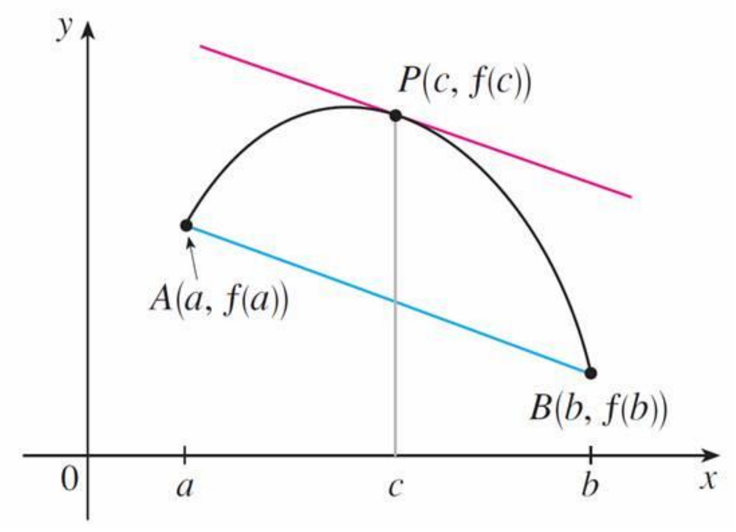

Figure 3

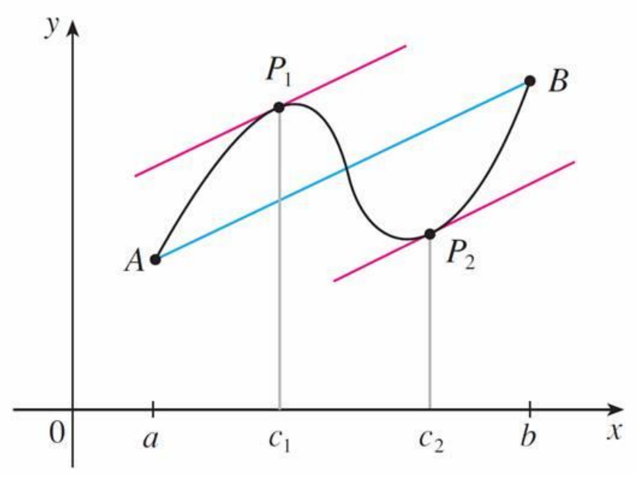

Figure 4

The slope of the secant line *AB* is

$$3 \quad m_{AB} = \frac{f(b) - f(a)}{b - a}$$

which is the same expression as on the right side of Equation 1.

The Mean Value Theorem

Since $f'(c)$ is the slope of the tangent line at the point (c, $f(c)$), the Mean Value Theorem, in the form given by Equation 1, says that there is at least one point P (c, $f(c)$) on the graph where the slope of the tangent line is the same as the slope of the secant line AB.

In other words, there is a point P where the tangent line is parallel to the secant line AB.

Example 3

To illustrate the Mean Value Theorem with a specific function, let's consider

$$f(x) = x^3 - x, \qquad \text{over the interval } [0, 2]$$

$$a = 0, b = 2.$$

Since *f* is a polynomial, it is continuous and differentiable for all *x*, so it is certainly continuous on [0, 2] and differentiable on (0, 2).

Therefore, by the Mean Value Theorem, there is a number *c* in (0, 2) such that

$$f(2) - f(0) = f'(c)(2 - 0)$$

Example 3 (2 of 3)

Now $f(2) = 6$,

$f(0) = 0$, and

$f'(x) = 3x^2 - 1,$ Where $x = c$, and $f'(c) = 3c^2 - 1,$ so this equation becomes

$$f(b) - f(a) = f'(c)(b - a)$$
$$6 = \left(3c^2 - 1\right)2$$

$$6 = 6c^2 - 2 \qquad c^2 = \frac{6+2}{6} = \frac{8}{6}$$

which gives $c^2 = \frac{4}{3}$, that is, $c = \pm\frac{2}{\sqrt{3}}$ $or\ c = -1.155\ or + 1.155.$

But *c* must lie in (0, 2), so $c = \frac{2}{\sqrt{3}} = 1.155.$

Example 3

Figure 6 illustrates this calculation:

the tangent line at this value of *c* is parallel to the secant line *OB*.

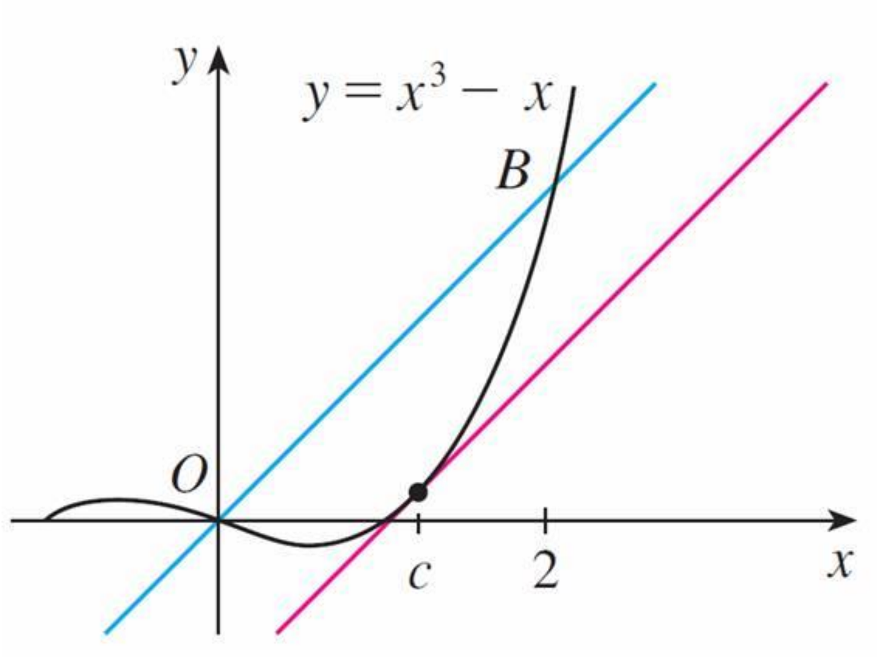

Figure 6

Example 5

Suppose that $f(0) = -3$ and $f'(x) \leq 5$ for all values of x. How large can $f(2)$ possibly be?

Solution:

We are given that f is differentiable (and therefore continuous) everywhere.

In particular, we can apply the Mean Value Theorem on the interval [0, 2]. There exists a number c such that

$$f(2) - f(0) = f'(c)(2 - 0)$$

Example 5 – Solution

So

$$f(2) = f(0) + 2f'(c)$$

$$f(2) = -3 + 2f'(c)$$

We are given that $f'(x) \leq 5$ for all *x*, so in particular we know that $f'(c) \leq 5.$ Multiplying both sides of this inequality by 2, we have $2f'(c) \leq 10, so$

$$f(2) = -3 + 2f'(c)$$

We can substitute $2f'(c) = 10$

$f(2) = -3 + 10$

The largest possible value for *f*(2) is 7.

Exercise

Question 1

Determine whether the Mean Value Theorem applies to the following function, $f(x) = \frac{x}{(x+2)}$ on the interval $[-1, 2]$. If so, determine the point(s) that are guaranteed to exist by the Mean Value Theorem.

Draw a sketch of the function and the line that passes through $(a, f(a))$ and $(b, f(b))$. Mark the points P at which the slope of the function equals the slope of the secant line. Sketch the tangent line at P.

Question 2

Verify that the function, $f(x) = x^3 - 2x^2 - 4x + 2$, satisfies the three hypotheses of Rolle's Theorem on the interval, [−2, 2]. Then find all numbers c that satisfy the conclusion of Rolle's Theorem.

Exercise

Solution 1:

Derivate: $f'(x) = \frac{2}{(x+2)^2}$ $\leftrightarrow$ $f'(\mathrm{c}) = \frac{2}{(\mathrm{c}+2)^2}$

Interval: $[-1, 2]$. $f(-1) = -1$ and $f(2) = 0.5$

Guaranteed points: $c = -4 \; and \; 0$, Discard -4 $\quad Answer \; is \; \boldsymbol{c = 0}$

Draw a sketch of the function and the line that passes through $(a, f(a))$ and $(b, f(b))$. Mark the points P at which the slope of the function equals the slope of the secant line. Sketch the tangent line at P.

Exercise

Solution 2:

$$f(x) = x^3 - 2x^2 - 4x + 2$$

f is a polynomial, so it is continuous, and differentiable on $\mathbb{R}$, and hence, continuous on $[-2,2]$ and differentiable on $(-2, 2)$.

Since $f(-2) = -6$ and $f(2) = -6$

f satisfies all the hypotheses of Rolle's Theorem.
$f'(c) = 3c^2 - 4c - 4 = 0$

$f'(c) = (3c + 2)(c - 2) = 0$

$c = -\frac{2}{3}$ or 2

$c = -\frac{2}{3}$ is in the open interval $(-2, 2)$ but 2 in not, so only $-\frac{2}{3}$ satisfies the conclusion of Rolle's Theorem.