The Chain Rule - a More Formal Approach

Suggested Prerequesites: The definition of the derivative, The chain rule

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Leibniz's differential notation

$$\frac{dy}{dx}$$

leads us to consider treating derivatives as fractions, so that given a composite function $y(u(x))$, we guess that

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}.$$

This speculation turns out to be correct, but we would like a better justification that what is perhaps a happenstance of notation. Let's start with the definition of the derivative and try to arrive at this result:

Given: $y = f(u(x))$.

By simple algebra, we know that

$$\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta u}\frac{\Delta u}{\Delta x}.$$

Then:

$$\begin{aligned}\frac{dy}{dx} &= \lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}\\ &= \lim_{\Delta x\to 0}\left(\frac{\Delta y}{\Delta u}\frac{\Delta u}{\Delta x}\right)\\ &= \left[\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta u}\right]\left[\lim_{\Delta x\to 0}\frac{\Delta u}{\Delta x}\right].\end{aligned}$$

Differentiablility implies continuity; therefore

$$\Delta u \to 0 \quad \text{as} \quad \Delta x \to 0.$$

Then, we have

$$\begin{aligned}\frac{dy}{dx} &= \left[\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta u}\right]\left[\lim_{\Delta x\to 0}\frac{\Delta u}{\Delta x}\right]\\ &= \left[\lim_{\Delta u\to 0}\frac{\Delta y}{\Delta u}\right]\left[\lim_{\Delta x\to 0}\frac{\Delta u}{\Delta x}\right]\\ &= \frac{dy}{du}\frac{du}{dx},\end{aligned}$$

which is the Chain Rule.

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