Mathematics 1A

ITMTA1-B44

Application of Differentiation

With

Amakan Elisha Agoni
Amakan.agoni@EDUVOS.com

Lecture 2
Week 6

4 Applications of Differentiation

4.3 What Derivatives Tell Us about the Shape of a Graph

What Does f' Say About f ?

What Does *f′* Say About *f* ?

To see how the derivative of *f* can tell us where a function is increasing or decreasing, look at Figure 1.

Between *A* and *B* and between *C* and *D*, the tangent lines have positive slope and so $f'(x) > 0$

Between *B* and *C* the tangent lines have negative slope and so $f'(x) < 0$. Thus it appears that f increases when $f'(x)$ is positive and decreases when $f'(x)$ is negative.

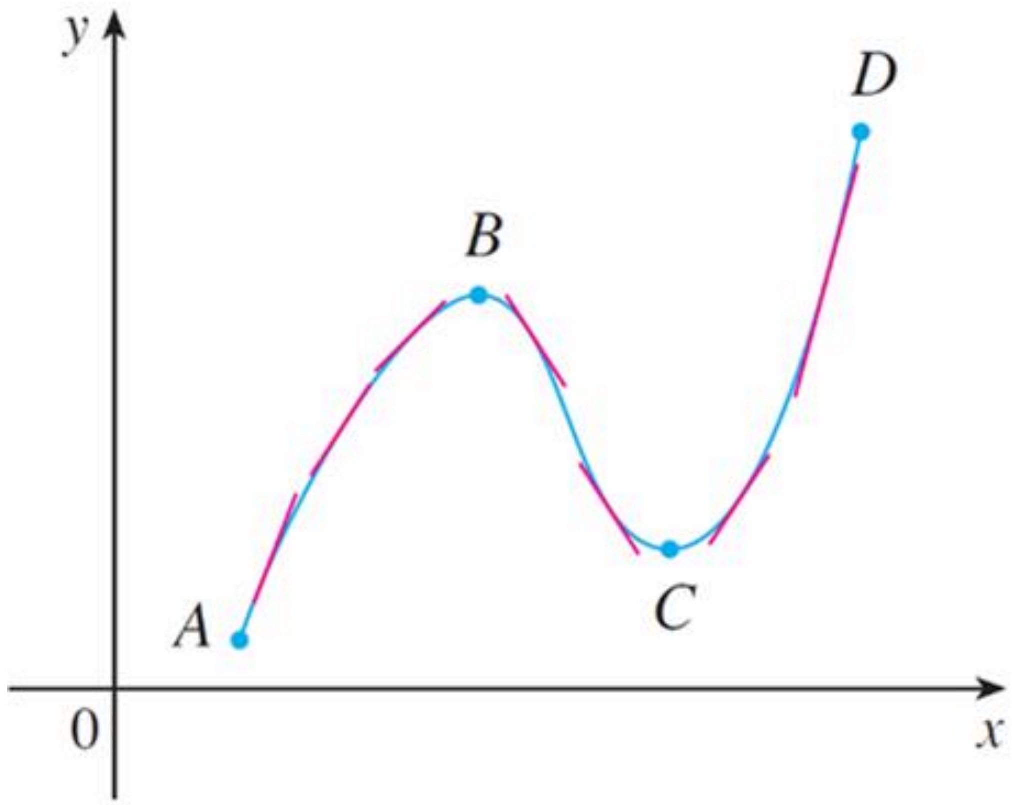


Figure 1

What Does f' Say About f ?

To prove that this is always the case, we use the Mean Value Theorem.

Increasing/Decreasing Test

(a) If $f'(x) > 0$ on an interval, then f is increasing on that interval.

(b) If $f'(x) < 0$ on an interval, then f is decreasing on that interval.

Example 1

Find where the function $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ is increasing and where it is decreasing.

Solution:

We start by differentiating *f*:

$$f'(x) = 12x^3 - 12x^2 - 24x = 12x(x-2)(x+1)$$

To use the test we have to know where $f'(x) > 0$ and where $f'(x) < 0$.

To solve these inequalities we first find where $f'(x) = 0$, namely at x = 0, 2, and −1.

Example 1 – Solution

These are the critical numbers of f, and they divide the domain into four intervals (see the number line in Figure 2).

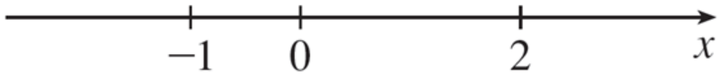


Figure 2

Within each interval, $f'(x)$ must be always positive or always negative.

We can determine which is the case for each interval from the signs of the three factors of $f'(x)$, namely, $12x$, $x - 2$, and $x + 1$, as shown in the chart.

Example 1 – Solution

A plus sign indicates that the given expression is positive, and a minus sign indicates that it is negative. The last column of the chart gives the conclusion based on the test.

For instance, $f'(x) < 0$ for $0 < x < 2$, so f is decreasing on $(0, 2)$. (It would also be true to say that f is decreasing on the closed interval $[0, 2]$.)

Interval	$12x$	$x - 2$	$x + 1$	$f'(x)$	f
$x < -1$	$-$	$-$	$-$	$-$	decreasing on $(-\infty, -1)$
$-1 < x < 0$	$-$	$-$	$+$	$+$	increasing on $(-1, 0)$
$0 < x < 2$	$+$	$-$	$+$	$-$	decreasing on $(0, 2)$
$x > 2$	$+$	$+$	$+$	$+$	increasing on $(2, \infty)$

Example 1 – Solution

The graph of *f* shown in Figure 3 confirms the information in the chart.

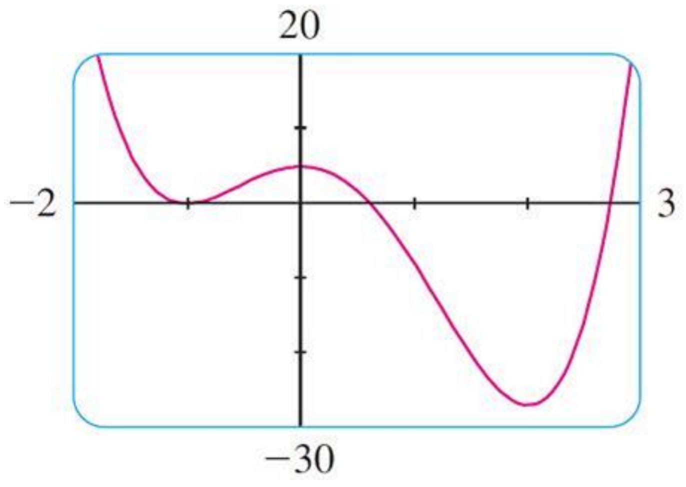


Figure 3

The First Derivative Test

The First Derivative Test

You can see from Figure 3 that $f(0) = 5$ is a local maximum value of f because f increases on $(-1, 0)$ and decreases on $(0, 2)$.

Or, in terms of derivatives,

$f'(x) > 0$ for $-1 < x < 0$ and $f'(x) < 0$ for $0 < x < 2$.

$f(x) = 3x^4 - 4x^3 - 12x^2 + 5$

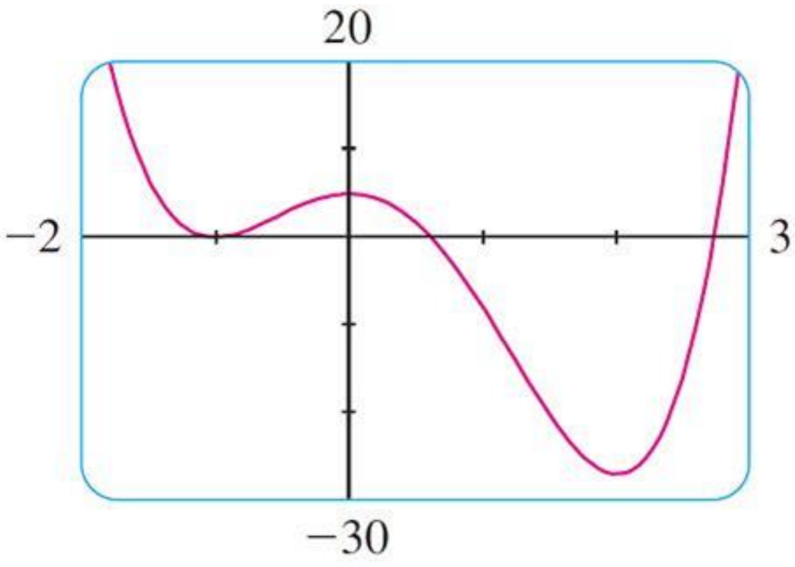


Figure 3

The First Derivative Test

In other words, the sign of $f'(x)$ changes from positive to negative at 0. This observation is the basis of the following test.

The First Derivative Test Suppose that *c* is a critical number of a continuous function *f*.

(a) If f' changes from positive to negative at *c*, then *f* has a local maximum at *c*.

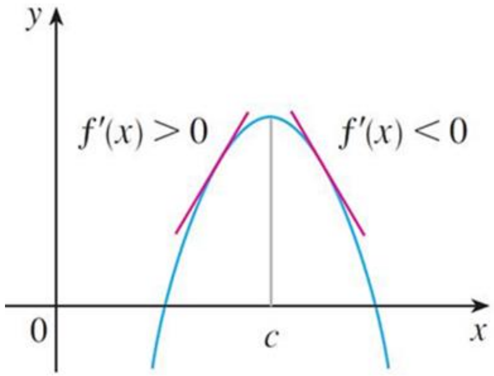
(b) If f' changes from negative to positive at *c*, then *f* has a local minimum at *c*.

(c) If f' is positive to the left and right of *c*, or negative to the left and right of *c*, then *f* has no local maximum or minimum at *c*.

The First Derivative Test

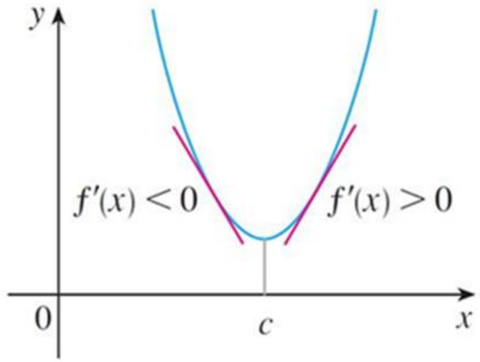
The First Derivative Test is a consequence of the I/D Test. In part (a), for instance, since the sign of $f'(x)$ changes from positive to negative at *c*, *f* is increasing to the left of *c* and decreasing to the right of *c*. It follows that *f* has a local maximum at *c*.

It is easy to remember the First Derivative Test by visualizing diagrams such as those in Figure 4.



Local maximum at *c*

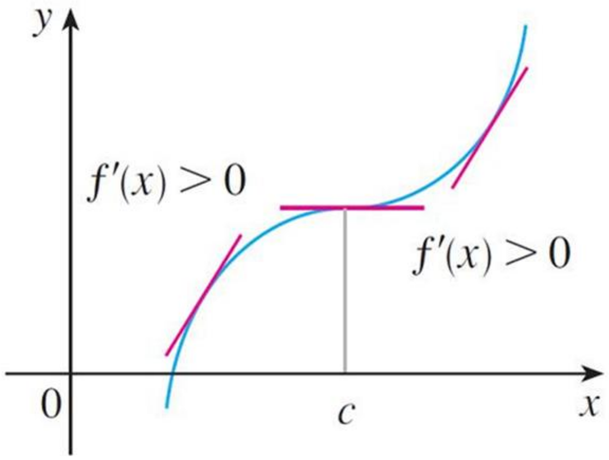
Figure 4(a)



Local minimum at *c*

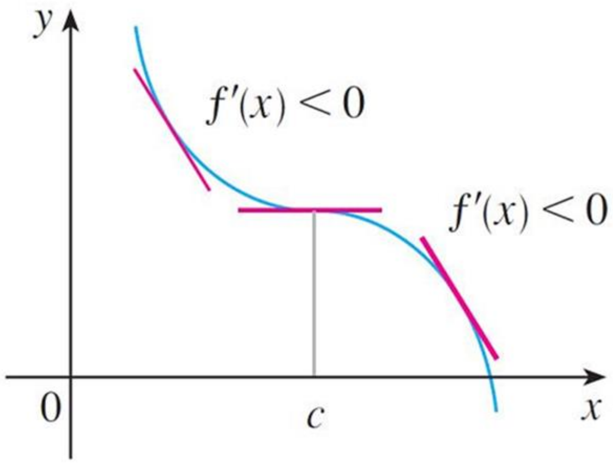
Figure 4(b)

The First Derivative Test



No maximum or minimum at *c*

Figure 3(c)



No maximum or minimum at *c*

Figure 3(d)

Example 3

Find the local maximum and minimum values of the function

$$g(x) = x + 2\sin x \quad 0 \leq x \leq 2\pi$$

Solution:

We start by finding the critical numbers. The derivative is:

$$g'(x) = 1 + 2\cos x$$

so $g'(x) = 0$ when $\cos x = -\frac{1}{2}$. The solutions of this equation are $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$.

Note: in the first quadrant, $\cos x = \frac{\pi}{3}$

In the second quadrant, the angle x is equal to $\pi - \frac{\pi}{3}$, which is $\frac{2\pi}{3}$.

In the third quadrant, the angle x is equal to $\pi + \frac{\pi}{3}$, which is $\frac{4\pi}{3}$.

Example 3 – Solution

Because g is differentiable everywhere, the only critical numbers are $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$.

We split the domain into intervals according to the critical numbers. Within each interval, $g'(x)$ is either always positive or always negative and so we analyze g in the following chart.

Interval	$g'(x) = 1 + 2\cos x$	g
$0 < x < 2\pi/3$	$+$	increasing on $(0, 2\pi/3)$
$2\pi/3 < x < 4\pi/3$	$-$	decreasing on $(2\pi/3, 4\pi/3)$
$4\pi/3 < x < 2\pi$	$+$	increasing on $(4\pi/3, 2\pi)$

Example 3 – Solution

Because $g'(x)$ changes from positive to negative at $\frac{2\pi}{3}$, the First Derivative Test tells us that there is a local maximum at $\frac{2\pi}{3}$ and the local maximum value is

$$\begin{aligned} g\left(\frac{2\pi}{3}\right) &= \frac{2\pi}{3} + 2\sin\frac{2\pi}{3} \\ &= \frac{2\pi}{3} + 2\left(\frac{\sqrt{3}}{2}\right) \\ &= \frac{2\pi}{3} + \sqrt{3} \\ &\approx 3.83 \end{aligned}$$

Example 3 – Solution

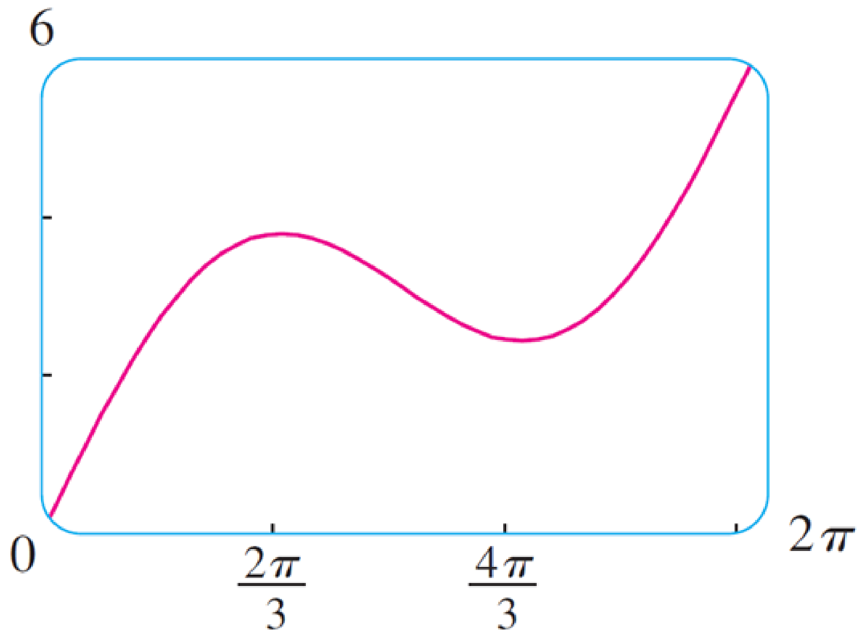
Likewise $g'(x)$, changes from negative to positive at $\frac{4\pi}{3}$ and so

$$\begin{aligned} g\left(\frac{4\pi}{3}\right) &= \frac{4\pi}{3} + 2\sin\frac{4\pi}{3} \\ &= \frac{4\pi}{3} + 2\left(-\frac{\sqrt{3}}{2}\right) \\ &= \frac{4\pi}{3} - \sqrt{3} \\ &\approx 2.46 \end{aligned}$$

is a local minimum value.

Example 3 – Solution

The graph of *g* in Figure 5 supports our conclusion.



$g(x) = x + 2 \sin x$

Figure 5

Exercises

Find the local maximum and minimum values of the following functions and indicate where the function is increasing/decreasing. Also determine if it satisfy the MVT and at which point.

1. $f(x) = x^3 - 6x^2 + 5, \quad [-3, 5]$
2. $f(x) = 3x^4 - 4x^3 - 12x^2 + 1, \quad [-2, 3]$
3. $f(x) = (x^2 - 1)^3, \quad [-1, 2]$
4. $f(x) = x + \dfrac{1}{x}, \quad [0.2, 4]$
5. $f(x) = \dfrac{x}{x^2 - x + 1}, \quad [0, 3]$
6. $f(t) = t\sqrt{4 - t^2}, \quad [-1, 2]$
7. $f(t) = \sqrt[3]{t}\,(8 - t), \quad [0, 8]$
8. $f(t) = 2\cos t + \sin 2t, \quad [0, \pi/2]$