Mathematics 1A

ITMTA1-B44

Limits and Derivatives 2

With

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Lecture 7
Week 3

2 Limits and Derivatives

2.6 Limits at Infinity; Horizontal Asymptotes

Limits at Infinity; Horizontal Asymptotes

In this section we let *x* become arbitrarily large (positive or negative) and see what happens to *y*.

Let's begin by investigating the behavior of the function *f* defined by

$$f(x) = \frac{x^2 - 1}{x^2 + 1}$$

as *x* becomes large.

Limits at Infinity and Horizontal Asymptotes

The table gives values of this function correct to six decimal places, and the graph of *f* has been drawn by a computer in Figure 1.

x	*f*(*x*)
0	−1
± 1	0
± 2	0.600000
± 3	0.800000
± 4	0.882353
± 5	0.923077
± 10	0.980198
± 50	0.999200
± 100	0.999800
± 1000	0.999998

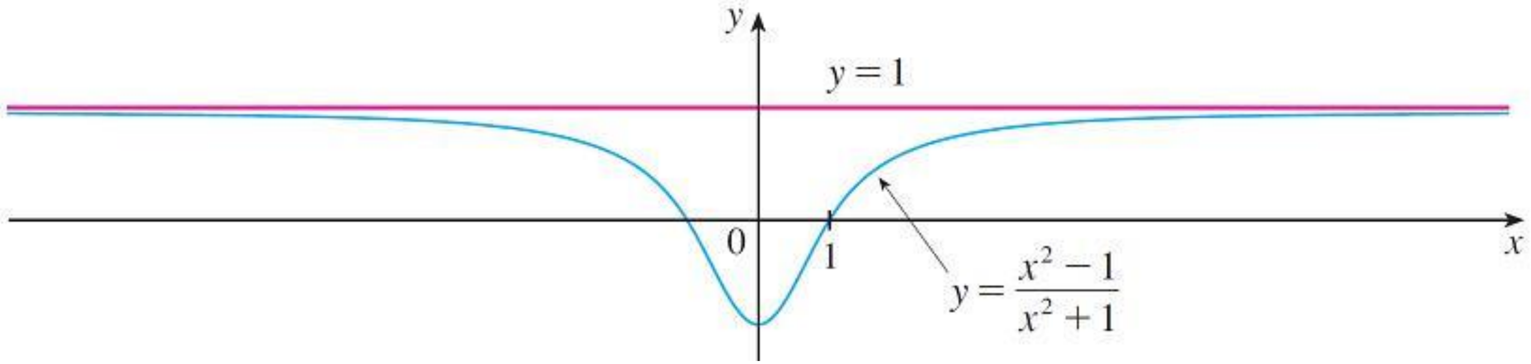


Figure 1

Limits at Infinity and Horizontal Asymptotes

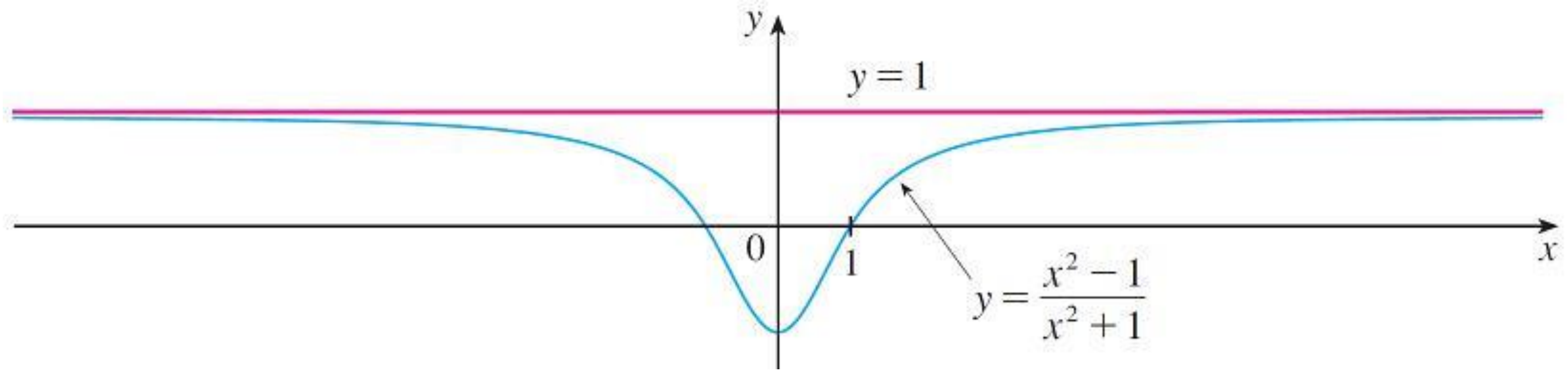


Figure 1

You can see that as *x* grows larger and larger, the values of *f* (*x*) get closer and closer to 1. In fact, it seems that we can make the values of *f* (*x*) as close as we like to 1 by taking *x* sufficiently large.

This situation is expressed symbolically by writing

$$\lim_{x \to \infty} \frac{x^2 - 1}{x^2 + 1} = 1$$

Limits at Infinity and Horizontal Asymptotes

In general, we use the notation

$$\lim_{x \to \infty} f(x) = L$$

to indicate that the values of $f(x)$ approach L as x becomes larger and larger.

1 Intuitive Definition of a Limit at Infinity

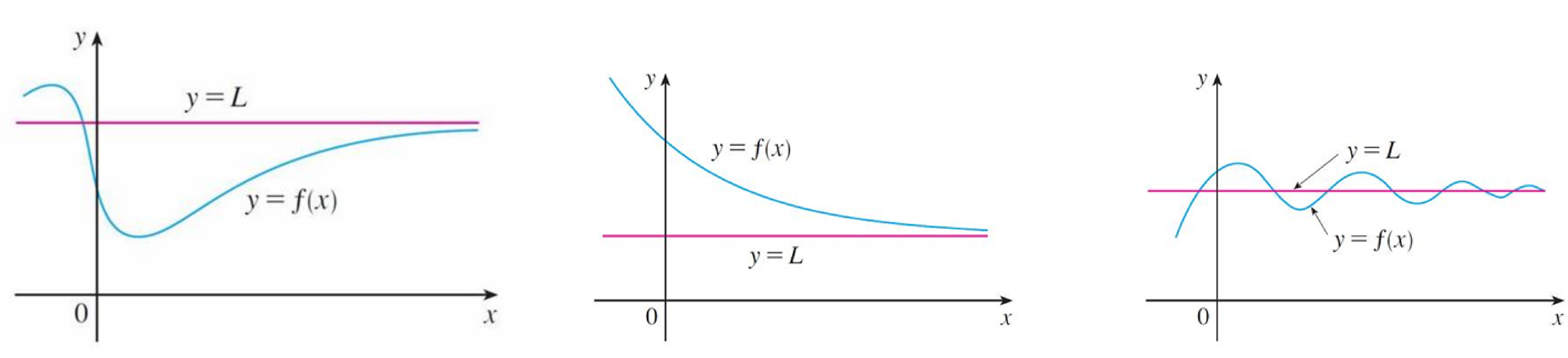
Let f be a function defined on some interval (a, ∞). Then

$$\lim_{x \to \infty} f(x) = L$$

means that the values of $f(x)$ can be made arbitrarily close to L by requiring x to be sufficiently large.

Limits at Infinity and Horizontal Asymptotes

Geometric illustrations of Definition 1 are shown in Figure 2.



Examples illustrating $\lim_{x \to \infty} f(x) = L$

Figure 2

Example 2

Find $\lim\limits_{x\to\infty} \frac{1}{x}$ and $\lim\limits_{x\to-\infty} \frac{1}{x}$.

Solution: Observe that when *x* is large, $\frac{1}{x}$ is small. For instance,

$$\frac{1}{100} = 0.01 \quad \frac{1}{10{,}000} = 0.0001 \quad \frac{1}{1{,}000{,}000} = 0.000001$$

In fact, by taking *x* large enough, we can make $1/x$ as close to 0 as we please.

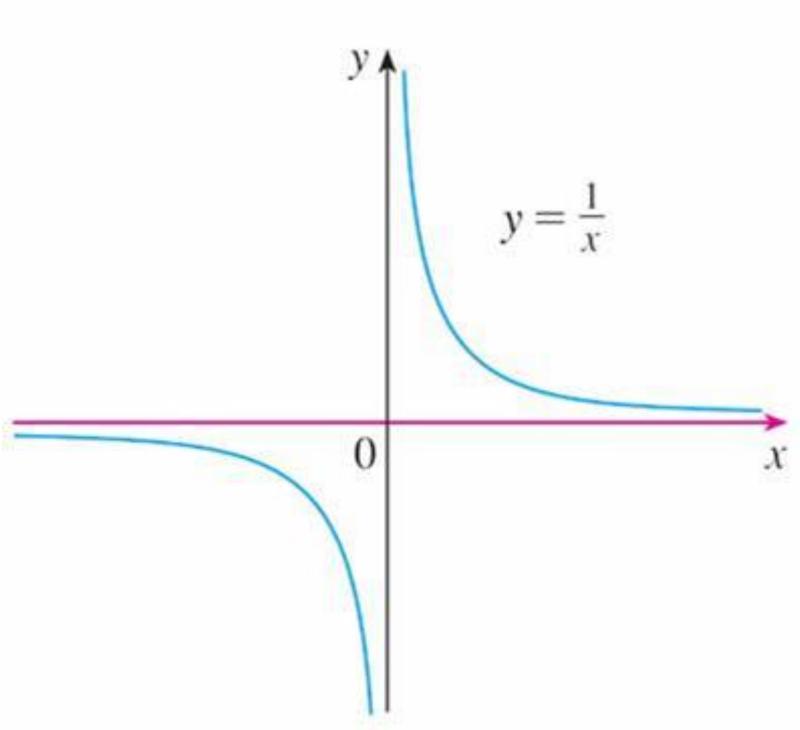
we have $\lim\limits_{x\to\infty} \frac{1}{x} = 0$

Similar reasoning shows that when *x* is large negative,

$1/x$ is small negative, so we also have $\lim\limits_{x\to-\infty} \frac{1}{x} = 0$

Example 2 – Solution

It follows that the line $y = 0$ (the x-axis) is a horizontal asymptote of the curve $y = 1/x$. (This is a hyperbola; see Figure 6.)



$$\lim_{x \to \infty} \frac{1}{x} = 0, \quad \lim_{x \to -\infty} \frac{1}{x} = 0$$

Figure 6

Exercises

19–32 Prove the statement using the ε, δ definition of a limit.

19. $\lim_{x \to 1} \frac{2 + 4x}{3} = 2$

20. $\lim_{x \to 10} \left(3 - \frac{4}{5}x\right) = -5$

21. $\lim_{x \to 2} \frac{x^2 + x - 6}{x - 2} = 5$

22. $\lim_{x \to -1.5} \frac{9 - 4x^2}{3 + 2x} = 6$

23. $\lim_{x \to a} x = a$

24. $\lim_{x \to a} c = c$

25. $\lim_{x \to 0} x^2 = 0$

26. $\lim_{x \to 0} x^3 = 0$

27. $\lim_{x \to 0} |x| = 0$

28. $\lim_{x \to -6^+} \sqrt[8]{6 + x} = 0$

29. $\lim_{x \to 2} (x^2 - 4x + 5) = 1$

30. $\lim_{x \to 2} (x^2 + 2x - 7) = 1$