A **ternary word** is a word over the alphabet $\{0, 1, 2\}$. Arrange the ternary words of the given length in increasing order of magnitude.

51. Length one. **52.** Length two.

Prove each.

***53.** The empty set is a subset of every set.
(*Hint*: Consider the implication $x \in \emptyset \to x \in A$.)

***54.** The empty set is unique.
(*Hint*: Assume there are two empty sets, $\emptyset_1$ and $\emptyset_2$. Then use Exercise 53.)

***55.** Let A, B, and C be arbitrary sets such that $A \subseteq B$ and $B \subseteq C$. Then $A \subseteq C$.
(**transitive property**)

***56.** If Σ is a nonempty alphabet, then Σ^* is infinite.
(*Hint*: Assume Σ^* is finite. Since $\Sigma \neq \emptyset$, it contains an element a. Let $x \in \Sigma^*$ with largest length. Now consider xa.)

2.2 Operations with Sets

Just as propositions can be combined in several ways to construct new propositions, sets can be combined in different ways to build new sets. You will find a close relationship between logic operations and set operations.

Union

The **union** of two sets A and B, denoted by $A \cup B$, is obtained by merging them; that is, $A \cup B = \{x|(x \in A) \vee (x \in B)\}$.

Notice the similarity between union and disjunction.

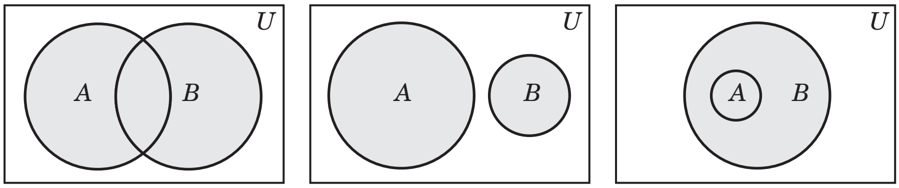
EXAMPLE 2.11 Let $A = \{\mathrm{a, b, c}\}$, $B = \{\mathrm{b, c, d, e}\}$, and $C = \{x, y\}$. Then $A \cup B = \{\mathrm{a, b, c, d, e}\} = B \cup A$ and $B \cup C = \{\mathrm{b, c, d, e, x, y}\} = C \cup B$. ■

The shaded areas in Figure 2.4 represent the set $A \cup B$ in three different cases.

Intersection

The **intersection** of two sets A and B, denoted by $A \cap B$, is the set of elements common to both A and B; that is, $A \cap B = \{x|(x \in A) \vee (x \in B)\}$.

Figure 2.4



$A \cup B$ | $A \cup B$, where A and B are disjoint | $A \cup B = B$

> Notice the relationship between intersection and conjunction.

EXAMPLE 2.12 Let $A = \{\text{Nov}, \text{Dec}, \text{Jan}, \text{Feb}\}$, $B = \{\text{Feb}, \text{Mar}, \text{Apr}, \text{May}\}$, and $C = \{\text{Sept}, \text{Oct}, \text{Nov}, \text{Dec}\}$. Then $A \cap B = \{\text{Feb}\} = B \cap A$ and $B \cap C = \emptyset = C \cap B$. (Notice that B and C are disjoint sets. More generally, two sets are disjoint if and only if their intersection is null.) ■

Figure 2.5

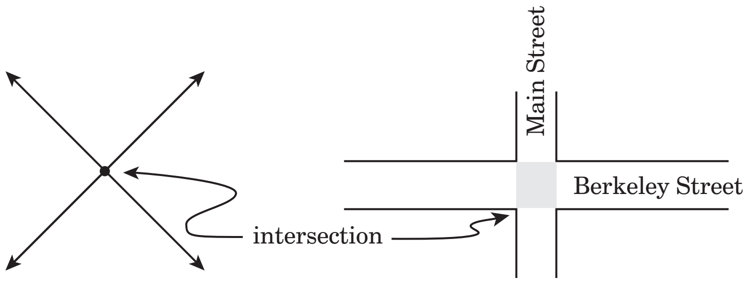
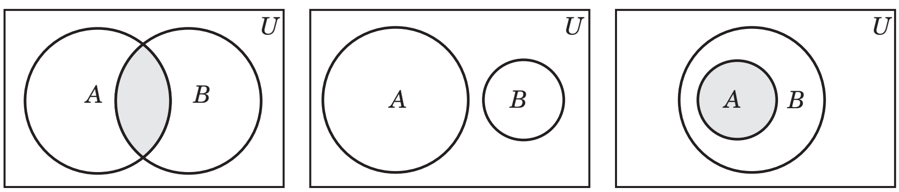


Figure 2.5 shows the intersection of two lines and that of two streets, and Figure 2.6 displays the set $A \cap B$ in three different cases.

Figure 2.6



$A \cap B$ | $A \cap B = \emptyset$ | $A \cap B = A$

EXAMPLE 2.13 Let $A = \{a, b, c, d, g\}$, $B = \{b, c, d, e, f\}$, and $C = \{b, c, e, g, h\}$. Find $A \cup (B \cap C)$ and $(A \cup B) \cap (A \cup C)$.

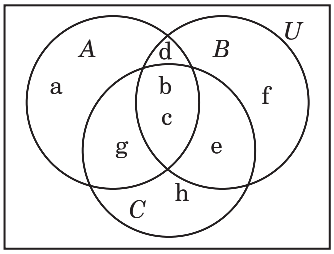
SOLUTION:

(1)
$$B \cap C = \{b, c, e\}$$
$$A \cup (B \cap C) = \{a, b, c, d, e, g\}$$

(2)
$$A \cup B = \{a, b, c, d, e, f, g\}$$
$$A \cup C = \{a, b, c, d, e, g, h\}$$
$$(A \cup B) \cap (A \cup C) = \{a, b, c, d, e, g\}$$
$$= A \cup (B \cap C)$$

See the Venn diagram in Figure 2.7.

Figure 2.7



■

A third way of combining two sets is by finding their difference, as defined below.

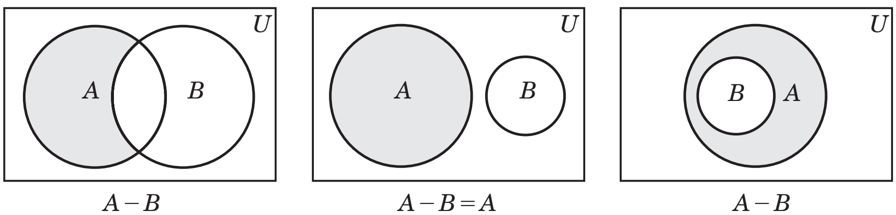
Difference

The **difference** of two sets A and B (or the **relative complement** of B in A), denoted by $A - B$ (notice the order), is the set of elements in A that are *not* in B. Thus $A - B = \{x \in A | x \notin B\}$.

EXAMPLE 2.14 Let $A = \{a, \ldots, z, 0, \ldots, 9\}$, and $B = \{0, \ldots, 9\}$. Then $A - B = \{a, \ldots, z\}$ and $B - A = \emptyset$.

The shaded areas in Figure 2.8 represent the set $A - B$ in three different cases.

Figure 2.8



■

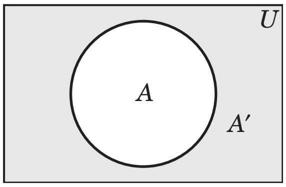
For any set $A \neq U$, although $A - U = \emptyset$, the difference $U - A \neq \emptyset$. This shows yet another way of obtaining a new set.

Complement

The difference $U - A$ is the (absolute) **complement** of A, denoted by A' (A prime). Thus $A' = U - A = \{x \in U \mid x \notin A\}$.

Figure 2.9 represents the complement of a set A. (Complementation corresponds to negation.)

Figure 2.9



EXAMPLE 2.15 Let $U = \{a, \ldots, z\}$. Find the complements of the sets $A = \{a, e, i, o, u\}$ and $B = \{a, c, d, e, \ldots, w\}$. Then $A' = U - A$ = set of all consonants in the alphabet, and $B' = U - B = \{b, x, y, z\}$. ■

EXAMPLE 2.16 Let $A = \{a, b, x, y, z\}$, $B = \{c, d, e, x, y, z\}$, and $U = \{a, b, c, d, e, w, x, y, z\}$. Find $(A \cup B)'$ and $A' \cap B'$.

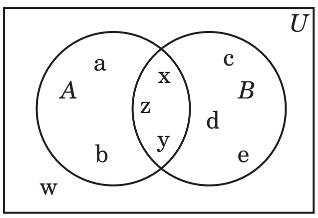
SOLUTION:

(1)
$$A \cup B = \{a, b, c, d, e, x, y, z\}$$
$$(A \cup B)' = \{w\}$$

(2)
$$A' = \{c, d, e, w\}$$
$$B' = \{a, b, w\}$$
$$A' \cap B' = \{w\} = (A \cup B)'$$

See Figure 2.10.

Figure 2.10



■

Since as a rule, $A - B \neq B - A$, by taking their union we can form a new set.

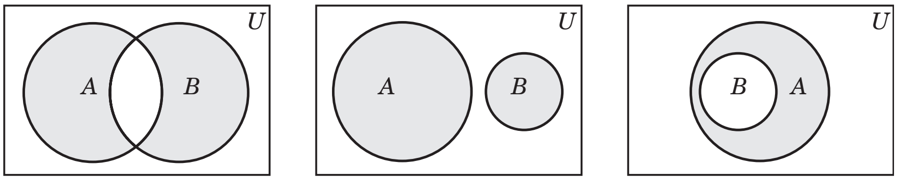
Symmetric Difference

The **symmetric difference** of A and B, denoted by $A \oplus B$, is defined by $A \oplus B = (A - B) \cup (B - A)$.

EXAMPLE 2.17 Let $A = \{\mathrm{a}, \ldots, \mathrm{z}, 0, \ldots, 9\}$ and $B = \{0, \ldots, 9, +, -, *, /\}$. Then $A - B = \{\mathrm{a}, \ldots, \mathrm{z}\}$ and $B - A = \{+, -, *, /\}$. So $A \oplus B = (A - B) \cup (B - A) = \{\mathrm{a}, \ldots, \mathrm{z}, +, -, *, /\}$. ■

The symmetric difference of A and B is pictorially displayed in Figure 2.11 in three different cases.

Figure 2.11



$A \oplus B = A \cup B - A \cap B$ $A \oplus B = A \cup B$ $A \oplus B = A - B$

Set and Logic Operations

> Set operations and logic operations are closely related, as Table 2.1 shows.

Table 2.1

Set operation	Logic operation
$A \cup B$	$p \vee q$
$A \cap B$	$p \vee q$
A'	$\sim p$
$A \oplus B$	p XOR q

The important properties satisfied by the set operations are listed in Table 2.2. (Notice the similarity between these properties and the laws of logic in Section 1.2.) We shall prove one of them. Use its proof as a model to prove the others as routine exercises.

We shall prove law 16. It uses De Morgan's law in symbolic logic, and the fact that $X = Y$ if and only if $X \subseteq Y$ and $Y \subseteq X$.

PROOF:

In order to prove that $(A \cup B)' = A' \cap B'$, we must prove two parts: $(A \cup B)' \subseteq A' \cap B'$ and $A' \cap B' \subseteq (A \cup B)'$.

- To prove that $(A \cup B)' \subseteq (A' \cap B')$:

  Let x be an arbitrary element of $(A \cup B)'$. Then $x \notin (A \cup B)$. Therefore, by De Morgan's law, $x \notin A$ and $x \notin B$; that is, $x \in A'$ and $x \in B'$. So $x \in A' \cap B'$. Thus every element of $(A \cup B)'$ is also an element of $A' \cap B'$; that is, $(A \cup B)' \subseteq A' \cap B'$.

Table 2.2

Laws of Sets

Let A, B, and C be any three sets and U the universal set. Then:

Idempotent laws

1. $A \cup A = A$	2. $A \cap A = A$

Identity laws

3. $A \cup \emptyset = A$	4. $A \cap U = A$

Inverse laws

5. $A \cup A' = U$	6. $A \cap A' = \emptyset$

Domination laws

7. $A \cup U = U$	8. $A \cap \emptyset = \emptyset$

Commutative laws

9. $A \cup B = B \cup A$	10. $A \cap B = B \cap A$

Double complementation law

11. $(A')' = A$

Associative laws

12. $A \cup (B \cup C) = (A \cup B) \cup C$	13. $A \cap (B \cap C) = (A \cap B) \cap C$

Distributive laws

14. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	15. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

De Morgan's laws

16. $(A \cup B)' = A' \cap B'$	17. $(A \cap B)' = A' \cup B'$

Absorption laws

18. $A \cup (A \cap B) = A$	19. $A \cap (A \cup B) = A$

(Note: The following laws have no names.)

20. If $A \subseteq B$, then $A \cap B = A$.	21. If $A \subseteq B$, then $A \cup B = B$.
22. If $A \subseteq B$, then $B' \subseteq A'$.	23. $A - B = A \cap B'$
24. $A \oplus B = A \cup B - A \cap B$	

- To prove that $A' \cap B' \subseteq (A \cup B)'$:

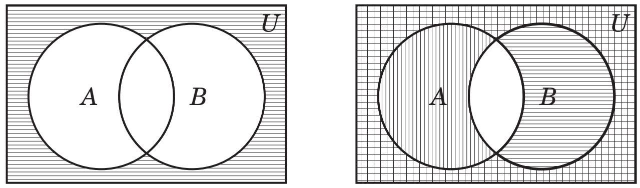
  Let x be any element of $A' \cap B'$. Then $x \in A'$ and $x \in B'$. Therefore, $x \notin A$ and $x \notin B$. So, by De Morgan's law, $x \notin (A \cup B)$. Consequently, $x \in (A \cup B)'$. Thus, since x is arbitrary, $A' \cap B' \subseteq (A \cup B)'$.

Thus, $(A \cup B)' = A' \cap B'$. See the Venn diagrams in Figure 2.12 also.

Note: Law 23 is a very useful result and will be used in the next section.

A few words of explanation: The commutative laws imply that the order in which the union (or intersection) of two sets is taken is irrelevant. The associative laws imply that when the union (or intersection) of three or more

Figure 2.12



$(A \cup B)' =$ shaded area $A' \cap B' =$ cross-shaded area

sets is taken, the way the sets are grouped is immaterial; in other words, such expressions without parentheses are perfectly legal. For instance, $A \cup B \cup C = A \cup (B \cup C) = (A \cup B) \cup C$ is certainly valid. The two De Morgan's laws in propositional logic play a central role in deriving the corresponding laws in sets.

Again, as in propositional logic, parentheses are essential to indicate the groupings in the distributive laws. For example, if you do not parenthesize the expression $A \cap (B \cup C)$ in law 15, then the LHS becomes $A \cap B \cup C = (A \cap B) \cup C = (A \cup C) \cap (B \cup C) \neq (A \cap B) \cup (A \cap C)$.

Notice the similarity between the set laws and the laws of logic. For example, properties 1 through 19 and 22 have their counterparts in logic. Every corresponding law of logic can be obtained by replacing sets A, B, and C with propositions p, q, and r, respectively, the set operators $\cap$, $\cup$, and $'$ with the logic operators $\wedge$, $\vee$, and $\sim$ respectively, and equality $(=)$ with logical equivalence $(\equiv)$.

Using this procedure, the absorption law $A \cup (A \cap B) = A$, for instance, can be translated as $p \vee (p \wedge q) \equiv p$, which is the corresponding absorption law in logic.

Just as truth tables were used in Chapter 1 to establish the logical equivalence of compound statements, they can be applied to verify set laws as well. The next example illustrates this method.

EXAMPLE 2.18 Using a truth table, prove that $(A \cup B)' = A' \cap B'$.

SOLUTION:

Let x be an arbitrary element. Then x may or may not be in A. Likewise, x may or may not belong to B. Enter this information, as in logic, in the first two columns of the table, which are headed by $x \in A$ and $x \in B$.

The table needs five more columns, headed by $x \in (A \cup B)$, $x \in (A \cup B)'$, $x \in A'$, $x \in B'$, and $x \in (A' \cap B')$ (see Table 2.3). Again, as in logic, use the entries in the first two columns to fill in the remaining columns, as in the table.

Table 2.3

$x \in A$	$x \in B$	$x \in (A \cup B)$	$x \in (A \cup B)'$	$x \in A'$	$x \in B'$	$x \in (A' \cap B')$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

Note: The shaded columns are identical

Since the columns headed by $x \in (A \cup B)'$ and $x \in (A' \cap B')$ are identical, it follows that $(A \cup B)' = A' \cap B'$. ■

Using truth tables to prove set laws is purely mechanical and elementary. It does not provide any insight into the development of a mathematical proof. Such a proof does not build on previously known set laws, so we shall not resort to such proofs in subsequent discussions.

Just as the laws of logic can be used to simplify logic expressions and derive new laws, set laws can be applied to simplify set expressions and derive new laws. In order to be successful in this art, you must know the laws well and be able to apply them as needed. So, practice, practice, practice.

EXAMPLE 2.19 Using set laws, verify that $(X - Y) - Z = X - (Y \cup Z)$.

PROOF:

$$\begin{aligned}
(X - Y) - Z &= (X - Y) \cap Z' && A - B = A \cap B' \\
&= (X \cap Y') \cap Z' && A - B = A \cap B' \\
&= X \cap (Y' \cap Z') && \text{associative law 13} \\
&= X \cap (Y \cup Z)' && \text{De Morgan's law 16} \\
&= X - (Y \cup Z) && A - B = A \cap B'
\end{aligned}$$ ■

EXAMPLE 2.20 Simplify the set expression $(A \cap B') \cup (A' \cap B) \cup (A' \cap B')$.

SOLUTION:

(You may supply the justification for each step.)

$$\begin{aligned}
(A \cap B') \cup (A' \cap B) \cup (A' \cap B') &= (A \cap B') \cup [(A' \cap B) \cup (A' \cap B')] \\
&= (A \cap B') \cup [A' \cap (B \cup B')] \\
&= (A \cap B') \cup (A' \cap U) \\
&= (A \cap B') \cup A'
\end{aligned}$$

$$
\begin{aligned}
&= A' \cup (A \cap B') \\
&= (A' \cup A) \cap (A' \cup B') \\
&= U \cap (A' \cup B') \\
&= A' \cup B' \qquad \blacksquare
\end{aligned}
$$

Often subscripts are used to name sets, so we now turn our attention to such sets.

Indexed Sets

Let I, called the **index set**, be the set of subscripts i used to name the sets A_i. Then the union of the sets A_i as i varies over I is denoted by $\bigcup_{i \in I} A_i$. Similarly, $\bigcap_{i \in I} A_i$ denotes the intersection of the sets A_i as i runs over I. In particular, let $I = \{1, 2, \ldots, n\}$. Then $\bigcup_{i \in I} A_i = A_1 \cup A_2 \cup \cdots \cup A_n$, which is often written as $\bigcup_{i=1}^{n} A_i$ or simply $\bigcup_{1}^{n} A_i$. Likewise, $\bigcap_{i \in I} A_i = \bigcap_{i=I}^{n} A_i = \bigcap_{1}^{n} A_i = A_1 \cap A_2 \cap \cdots \cap A_n$. If $I = \mathbb{N}$, the expression $\bigcup_{i \in \mathbb{N}} A_i$ is written as $\bigcup_{i \in 1}^{\infty} A_i = \bigcup_{1}^{\infty} A_i$, using the infinity symbol ∞; similarly, $\bigcap_{i \in \mathbb{N}} A_i = \bigcap_{i=1}^{\infty} A_i = \bigcap_{1}^{\infty} A_i$.

Before we proceed to define a new binary operation on sets $\bigcap_{i=1}^{\infty}$, we define an ordered set.

Ordered Set

Recall that the set $\{a_1, a_2, \ldots, a_n\}$ is an unordered collection of elements. Suppose we assign a position to each element. The resulting set is an **ordered set** with n elements or an ***n*-tuple**, denoted by $(a_1, a_2, \ldots, a_n)$. (Notice the use of parentheses versus braces.) The set (a_1, a_2) is an **ordered pair.**

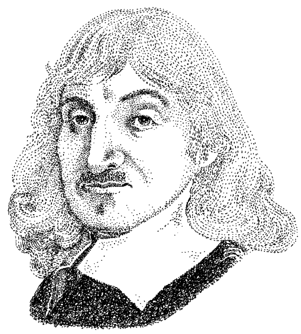
Two n-tuples are **equal** if and only if their corresponding elements are equal. That is, $(a_1, a_2, \ldots, a_n) = (b_1, b_2, \ldots, b_n)$ if and only if $a_i = b_i$ for every i.

EXAMPLE 2.21 Every numeral and word can be considered an n-tuple. For instance,

$$345 = (\underset{\text{hundreds}}{3},\ \underset{\text{tens}}{4},\ \underset{\text{ones}}{5})$$

$$
\begin{aligned}
\text{computer} &= (\text{c}, \text{o}, \text{m}, \text{p}, \text{u}, \text{t}, \text{e}, \text{r}) \\
1001011 &= (1, 0, 0, 1, 0, 1, 1) && \leftarrow \text{ASCII}^{*} \text{ code for letter K} \\
11010010 &= (1, 1, 0, 1, 0, 0, 1, 0) && \leftarrow \text{EBCDIC}^{**} \text{ code for letter K} \qquad \blacksquare
\end{aligned}
$$

*American Standard Code for Information Interchange.
**Extended Binary Coded Decimal Interchange Code.



René Descartes *(1596–1650) was born near Tours, France. At eight, he entered the Jesuit school at La Fleche, where because of poor health he developed the habit of lying in bed thinking until late in the morning; he considered those times the most productive. He left the school in 1612 and moved to Paris, where he studied mathematics for a brief period.*

After a short military career and travel through Europe for about 5 years, he returned to Paris and studied mathematics and philosophy. He then moved to Holland, where he lived for 20 years writing several books. In 1637 he wrote Discours*, which contains his contributions to analytic geometry.*

In 1649 Descartes moved to Sweden at the invitation of Queen Christina. There he contracted pneumonia and died.

We are now ready to define the next and final operation on sets.

Cartesian Product

The **cartesian product** of two sets A and B, denoted by $A \times B$, is the set of all ordered pairs (a, b) with $a \in A$ and $b \in B$. Thus $A \times B = \{(a, b) | a \in A \wedge b \in B\}$. $A \times A$ is denoted by A^2.

It is named after the French philosopher and mathematician René Descartes.

EXAMPLE 2.22 Let $A = \{\mathrm{a}, \mathrm{b}\}$ and $B = \{\mathrm{x}, \mathrm{y}, \mathrm{z}\}$. Then

$$A \times B = \{(\mathrm{a}, \mathrm{x}), (\mathrm{a}, \mathrm{y}), (\mathrm{a}, \mathrm{z}), (\mathrm{b}, \mathrm{x}), (\mathrm{b}, \mathrm{y}), (\mathrm{b}, \mathrm{z})\}$$

$$B \times A = \{(\mathrm{x}, \mathrm{a}), (\mathrm{x}, \mathrm{b}), (\mathrm{y}, \mathrm{a}), (\mathrm{y}, \mathrm{b}), (\mathrm{z}, \mathrm{a}), (\mathrm{z}, \mathrm{b})\}$$

$$A^2 = A \times A = \{(\mathrm{a}, \mathrm{a}), (\mathrm{a}, \mathrm{b}), (\mathrm{b}, \mathrm{a}), (\mathrm{b}, \mathrm{b})\}$$

(Notice that $A \times B \neq B \times A$.) ■

The various elements of $A \times B$ in Example 2.22 can be displayed in a rectangular fashion, as in Figure 2.13, and pictorially, using dots as in Figure 2.14. The circled dot in row a and column y, for instance, represents the element (a, y). The pictorial representation in Figure 2.14 is the **graph** of $A \times B$.

Figure 2.13

Elements of A				
	a	(a, x)	(a, y)	(a, z)
	b	(b, x)	(b, y)	(b, z)
		x	y	z

Elements of B

Figure 2.14

Pictorial representation of $A \times B$.

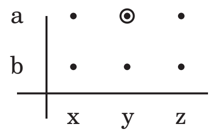
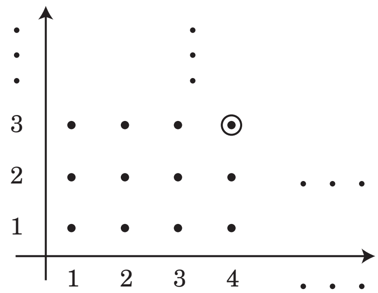


Figure 2.15 shows the graph of the infinite set $\mathbb{N}^2 = \mathbb{N} \times \mathbb{N}$. The circled dot in column 4 and row 3, for instance, represents the element (4,3). The horizontal and vertical dots indicate that the pattern is to be continued indefinitely in both directions.

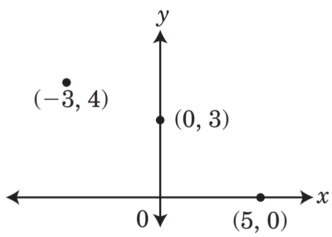
Figure 2.15



More generally, $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ consists of all possible ordered pairs (x, y) of real numbers. It is represented by the familiar ***xy*-plane** or the **cartesian plane** used for graphing (see Figure 2.16).

Figure 2.16

The cartesian plane $\mathbb{R}^2$.



The following example presents an application of cartesian product.

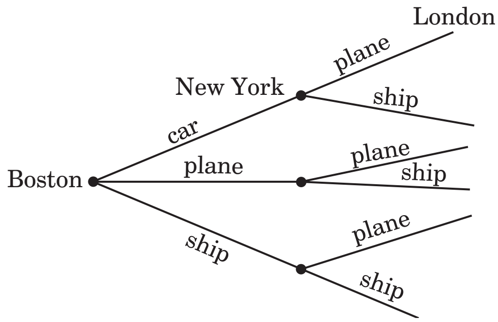
EXAMPLE 2.23

Linda would like to make a trip from Boston to New York and then to London. She can travel by car, plane, or ship from Boston to New York, and by plane or ship from New York to London. Find the set of various modes of transportation for the entire trip.

SOLUTION:

Let A be the set of means of transportation from Boston to New York and B the set from New York to London. Clearly $A = \{\text{car, plane, ship}\}$ and $B = \{\text{plane, ship}\}$. So the set of possible modes of transportation is given by

Figure 2.17



$A \times B = \{$(car, plane), (car, ship), (plane, plane), (plane, ship), (ship, plane), (ship, ship)$\}$. See Figure 2.17. ■

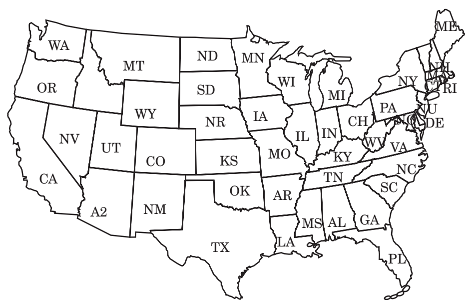
The definition of the product of two sets can be extended to n sets. The **cartesian product of *n* sets** $A_1, A_2, \ldots, A_n$ consists of all possible n-tuples $(a_1, a_2, \ldots, a_n)$, where $a_i \in A_i$ for every i; it is denoted by $A_1 \times A_2 \times \cdots \times A_n$. If all A_i's are equal to A, the product set is denoted by A^n.

EXAMPLE 2.24 Let $A = \{\mathrm{x}\}$, $B = \{\mathrm{y}, \mathrm{z}\}$, and $C = \{1, 2, 3\}$. Then

$$\begin{aligned} A \times B \times C &= \{(a, b, c) | a \in A, b \in B, \text{ and } c \in C\} \\ &= \{(\mathrm{x}, \mathrm{y}, 1), (\mathrm{x}, \mathrm{y}, 2), (\mathrm{x}, \mathrm{y}, 3), (\mathrm{x}, \mathrm{z}, 1), (\mathrm{x}, \mathrm{z}, 2), (\mathrm{x}, \mathrm{z}, 3)\} \end{aligned}$$

Finally, take a look at the map of the continental United States in Figure 2.18. It provides a geographical illustration of partitioning, a concept that can be extended to sets in an obvious way.

Figure 2.18

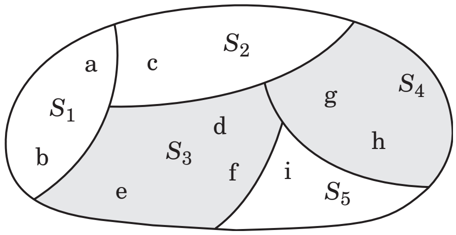


■

Partition

Consider the set $S = \{\mathrm{a, b, c, d, e, f, g, h, i}\}$ and the subsets $S_1 = \{\mathrm{a, b}\}$, $S_2 = \{\mathrm{c}\}$, $S_3 = \{\mathrm{d, e, f}\}$, $S_4 = \{\mathrm{g, h}\}$, and $S_5 = \{\mathrm{i}\}$. Notice that these subsets have three interesting properties: (1) They are nonempty; (2) they are pairwise disjoint; that is, no two subsets have any common elements; (3) their union is S. (See Figure 2.19.) The set $P = \{S_1, S_2, S_3, S_4, S_5\}$ is called a **partition** of S.

Figure 2.19



More generally, let I be an index set and P a family of subsets S_i of a nonempty set S, where $i \in I$. Then P is a **partition** of S if:

- Each set S_i is nonempty.
- The subsets are pairwise disjoint; that is, $S_i \cap S_j = \emptyset$ if $i \neq j$.
- The union of the subsets S_i is S; that is, $\bigcup_{i \in I} S_i = S$.

(Each subset S_i is a **block** of the partition.) Thus a partition of S is a collection of nonempty, pairwise disjoint subsets of S whose union is S.

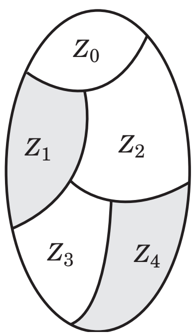
EXAMPLE 2.25 Let Z_r denote the set of integers which, when divided by 5, leave r as the remainder. Then $0 \leq r < 5$ (see Section 4.1):

$$\mathbf{Z}_0 = \{\ldots, -5, 0, 5, \ldots\}$$
$$\mathbf{Z}_1 = \{\ldots, -4, 1, 6, \ldots\}$$
$$\mathbf{Z}_2 = \{\ldots, -3, 2, 7, \ldots\}$$
$$\mathbf{Z}_3 = \{\ldots, -2, 3, 8, \ldots\}$$
$$\mathbf{Z}_4 = \{\ldots, -1, 4, 9, \ldots\}$$

$P = \{\mathbf{Z}_0, \mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3, \mathbf{Z}_4\}$ is a partition of the set of integers. See Figure 2.20. (This example is discussed in more detail in Section 7.4.) ■

Figure 2.20

Set of integers **Z**.



The sports pages of newspapers provide fine examples of partitions, as the next example illustrates.

EXAMPLE 2.26 In 2003, the set of teams S in the National Football League was divided into two conferences, American and National, and each conference into four divisions — East, South, North, and West. Let E_1, S_1, N_1, and W_1 denote the set of teams in East, South, North, and West Divisions in the American Conference, respectively, and E_2, S_2, N_2, and W_2 the corresponding sets in the National Conference. Then:

$$E_1 = \{\text{Buffalo, Miami, New England, NY Jets}\}$$
$$S_1 = \{\text{Indianapolis, Tennessee, Houston, Jacksonville}\}$$
$$N_1 = \{\text{Baltimore, Cincinnati, Cleveland, Pittsburgh}\}$$
$$W_1 = \{\text{Denver, Kansas City, Oakland, San Diego}\}$$
$$E_2 = \{\text{Washington, Philadelphia, Dallas, NY Giants}\}$$
$$S_2 = \{\text{Atlanta, Tampa Bay, Carolina, New Orleans}\}$$
$$N_2 = \{\text{Chicago, Detroit, Minnesota, Green Bay}\}$$
$$W_2 = \{\text{Arizona, Seattle, St. Louis, San Francisco}\}$$

Clearly, $P = \{E_1, S_1, N_1, W_1, E_2, S_2, N_2, W_2\}$ is a partition of S.

We close this section with a brief introduction to fuzzy sets.

Fuzzy Sets (optional)

Fuzzy sets, a generalization of ordinary sets, were introduced in 1965 by Lotfi A. Zadeh of the University of California at Berkeley. They have applications to human cognition, communications, decision analysis, psychology, medicine, law, information retrieval, and, of course, artificial intelligence. Like fuzzy logic, they model the fuzziness in the natural language — for example, in terms like *young, healthy, wealthy*, and *beautiful*.

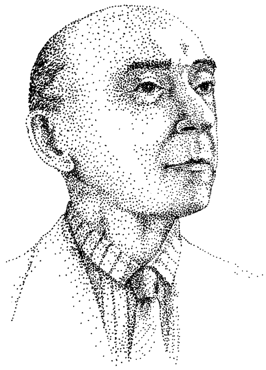
In fuzzy set theory, every element x in the universal set U has a certain degree of membership $d_U(x)$, where $0 \le d_U(x) \le 1$; $d_U(x)$ indicates the degree of fuzziness. Accordingly, a fuzzy set S is denoted by listing its elements along with their degrees of membership; an element with zero degree of membership is not listed.

For example, let U be the fuzzy set of wealthy people and $S = \{\text{Tom } 0.4, \text{Dick } 0.7, \text{Harry } 0.6\}$. Then Harry belongs to S with degree of membership 0.6; $d_S(\text{Harry}) = 0.6$ measures Harry's degree of wealthiness.

The concept of an ordinary subset can be extended to fuzzy sets also.

Fuzzy Subset

Let A and B be fuzzy sets. Then A is a **fuzzy subset** of B if $A \subseteq B$ and $d_A(x) \le d_B(x)$ for every element x in A.



Lotfi A. Zadeh *(1921–) was born in Baku, Azerbaijan. An alumnus of the University of Tehran (1942) and the Massachusetts Institute of Technology (1946), he received his Ph.D. from Columbia University in 1949 for his dissertation on frequency analysis of time-varying networks. He began his professional career in the Department of Electrical Engineering at Columbia. In 1959, he joined the Department of Electrical Engineering and Computer Science at the University of California, Berkeley, serving as its chair during the years 1963–1968. Currently, he is a professor at Berkeley and Director of Berkeley Initiative in Soft Computing.*

Zadeh's earlier "work was centered on systems analysis, decision analysis, and information systems. Since then his current research has shifted to the theory of fuzzy sets and its applications to artificial intelligence (AI). His research interest now is focused on fuzzy logic, soft computing, computing with words, and the newly developed computational theory of perceptions and precisiated natural language," according to the University of California Web site.

A truly gifted mind and an expert on AI, Zadeh has authored about 200 journal articles on a wide variety of subjects relating to the conception, design, and analysis of information/intelligent systems. He serves on the editorial boards of more than 50 journals and on the advisory boards of a number of institutions related to AI.

Zadeh is a recipient of numerous awards and medals, including the IEEE Education Medal, IEEE Richard W. Hamming Medal, IEEE Medal of Honor, the ASME Rufus Oldenburger Medal, B. Bolzano Medal of the Czech Academy of Sciences, Kampe de Feriet Medal, AACC Richard E. Bellman Central Heritage Award, the Grigore Moisil Prize, Honda Prize, Okawa Prize, AIM Information Science Award, IEEE-SMC J. P. Wohl Career Achievement Award, SOFT Scientific Contribution Memorial Award of the Japan Society for Fuzzy Theory, IEEE Millennium Medal, and the ACM 2000 Allen Newell Award. He has received honorary doctorates from many universities from around the world.

For example, let $S = \{\text{Betsey } 0.6, \text{Mat } 0.5\}$ and $T = \{\text{Betsey } 0.8, \text{Jonathan } 0.3, \text{Mary } 0.5, \text{Mat } 0.7\}$ by fuzzy sets of smart people. Then S is a fuzzy subset of T.

Operations on ordinary sets can be extended to fuzzy sets as well.

Operations on Fuzzy Sets

Let A and B be any fuzzy set. The **union** of A and B is $A \cup B$, where $d_{A\cup B}(x) = \max\{d_A(x), d_B(x)\}$; their **intersection** is $A \cap B$, where $d_{A\cap B}(x) = \min\{d_A(x), d_B(x)\}$; and the **complement** of A is A', where $d_{A'}(x) = 1 - d_A(x)$; in A' only the degrees of membership change.

Using the sets S and T above,

$$S \cup T = \{\text{Betsey } 0.8, \text{Jonathan } 0.3, \text{Mary } 0.5, \text{Mat } 0.7\}$$
$$S \cap T = \{\text{Betsey } 0.6, \text{Mat } 0.5\}$$
$$S' = \{\text{Betsey } 0.4, \text{Mat } 0.5\}$$

Additional opportunities to practice the various operations are given in the exercises.

Exercises 2.2

Let $A = \{a, e, f, g, i\}$, $B = \{b, d, e, g, h\}$, $C = \{d, e, f, h, i\}$, and $U = \{a, b, \ldots, k\}$. Find each set.

1. C' **2.** $B \cap C'$ **3.** $C \cap A'$ **4.** $(A \cup B)'$

5. $(B \cap C)'$ **6.** $(A \cup C')'$ **7.** $(B \cap C')'$ **8.** $A \oplus B$

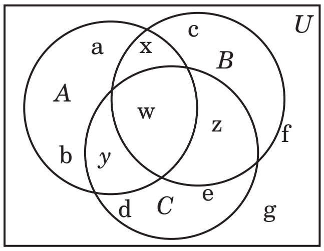
9. $(A - B) - C$ **10.** $A - (B - C)$ **11.** $(A \cup B) - C$ **12.** $(A \cap B) - C$

Using the Venn diagram in Figure 2.21 find each set.

13. $(A \cup B) \cap C$ **14.** $A \cap (B \cup C)$ **15.** $A - (B - C)$

16. $(A \oplus B) \cup C$ **17.** $A \cap (B \oplus C)$ **18.** $A - (B \oplus C)$

Figure 2.21



Let $A = \{b, c\}$, $B = \{x\}$, and $C = \{x, z\}$. Find each set.

19. $A \times B$ **20.** $B \times A$ **21.** $A \times \emptyset$ **22.** $A \times B \times \emptyset$

23. $A \times (B \cup C)$ **24.** $A \times (B \cap C)$ **25.** $A \times B \times C$ **26.** $A \times C \times B$

Mark each as true or false, where A, B, and C are arbitrary sets and U the universal set.

27. $A - \emptyset = A$ **28.** $\emptyset - A = -A$ **29.** $\emptyset - \emptyset = 0$

30. $A - A = 0$ **31.** $A - B = B - A$ **32.** $A - A' = \emptyset$

33. $(A')' = A$ **34.** $(A \cap B)' = A' \cap B'$ **35.** $(A \cup B)' = A' \cup B'$

36. $A \subseteq A \cup B$ **37.** $A \subseteq A \cap B$ **38.** $B \cap (A - B) = \emptyset$

Give a counterexample to disprove each proposition.

39. $(A - B) - C = A - (B - C)$ **40.** $A \cup (B - C) = (A \cup B) - (A \cup C)$

41. $A \cup (B \oplus C) = (A \cup B) \oplus (A \cup C)$ **42.** $A \oplus (B \cap C) = (A \oplus B) \cap (A \oplus C)$

Determine if each is a partition of the set $\{a, \ldots, z, 0, \ldots, 9\}$.

43. $\{\{a, \ldots, z\}, \{0, \ldots, 9\}, \emptyset\}$

44. $\{\{a, \ldots, j\}, \{i, \ldots, t\}, \{u, \ldots, z\}, \{0, \ldots, 9\}\}$