Discrete Mathematics

Tutorial solution | Topic-9 | Relations

Question 1.

Let $\mathcal{S}$ be a set and $\mathcal{R}$ be a relation on $\mathcal{S}$. Explain what it means (you are expected to give mathematical definitions.) to say that $\mathcal{R}$ is:

1. reflexive;
2. symmetric;
3. anti-symmetric;
4. transitive;
5. an equivalence relation;
6. a partial order.

In each case give an example of a relation which has the given property and another relation which does not have it.

Solution:

1. $\mathcal{R}$ is reflexive if $\forall x \in S, x\S$

   Example: *Equality* is reflexive, but *is Less than(<)* is not reflexive

2. $\mathcal{R}$ is symmetric if $\forall x, y \in S, x\mathcal{R}y \Rightarrow y\mathcal{R}x$ Example: marriage is symmetric, but *is the parent of* is not symmetric.

3. $\mathcal{R}$ is antisymmetric if $\forall x, y \in S, x\mathcal{R}y \wedge y\mathcal{R}x \Rightarrow x = y$

   Example: Subset ($\subseteq$) is antisymmetric, but *is parent of* not.

4. $\mathcal{R}$ is transitive if $\forall x, y, z \in S, x\mathcal{R}y \wedge y\mathcal{R}z \Rightarrow x\mathcal{R}z$

   Example: been an ancestor is transitive, but been a parent is not

5. $\mathcal{R}$ is an equivalence relation if it is reflexive, symmetric and transitive

   Example: *is equal to* on the set of real number is an equivalence relation. *is less or equal* ($\leq$) is not an equivalence relation.

6. $\mathcal{R}$ is a partial order if it is reflexive, antisymmetric and transitive

   Example: The relation *divides* ($\mid$) on the set of positive integers (the natural numbers $\mathbb{N}$), certainly reflexive, antisymmetric, and transitive. But the same relation applied to the set of integers $\mathbb{Z}$, is not antisymmetric since $-2 \mid 2$ and $2 \mid -2$ but $2 \neq 2$.

Question 2.

Let $S = \{a, b, c\}$ and $A = \{(c, c), (a, b), (b, b), (b, c), (c, b)\}$.
Define a relation R on S by "x is related to y whenever$(x, y) \in A$".

1. Draw the relationship digraph.
2. The relation R is not reflexive. What pair (x, y) should be added to A to make R reflexive?
3. The relation R is not symmetric. What pair (x, y) should be added to A to make R symmetric?
4. The relation R is not anti-symmetric. What pair (x, y) should be removed to make R anti-symmetric?
5. The relation R is not transitive. What pair (x, y) should be added to A to make R transitive?

Solution:

1.
2. (a,a)
3. (b,a)
4. (b,c) or (c,b)
5. (a,c)

Question 3.

The following relations are defined on a set $S = \{a, b, c\}$.
R_1 is the relation given by $\{$(a,a),(a,b),(a,c),(b,a),(b,b),(c,a),(c,c)$\}$
R_2 is given by $\{$(a,a),(a,b),(b,a),(b,b),((c,c)$\}$
R_3 is given by $\{$(a,b),(a,c),(b,a),(b,c),(c,a),(c,b)$\}$
R_4 is given by $\{$(a,a),(a,b),(a,c),((b,b),(b,c),(c,c)$\}$
Complete the table below. If the relation is an equivalence relation give the equivalence classes. Also state whether any of the relations is a partial order, justifying your answer.

	reflexive	symmetric	antisymmetric	transitive	equivalence rel.	
$\mathcal{R}_1$						
$\mathcal{R}_2$						
$\mathcal{R}_3$						
$\mathcal{R}_4$						

Solution:

	reflexive	symmetric	antisymmetric	transitive	equivalence rel.	Partial order
$\mathcal{R}_1$	yes	yes	no	no	no	no
$\mathcal{R}_2$	yes	yes	no	yes	yes	no
$\mathcal{R}_3$	no	yes	no	no	no	no
$\mathcal{R}_4$	yes	no	yes	yes	no	yes

$\mathcal{R}_\in$ is an equivalence relation with equivalence classes $T = \{[a],[c]\}$

$\mathcal{R}_\triangle$ is the only relation which is a partial order.

Question 4.

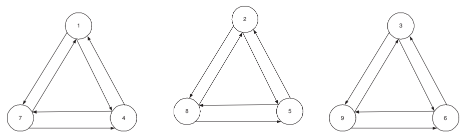
Let $\mathcal{S}$ =$\{$1,2,3,4,5,6,7,8,9$\}$ and let $\mathcal{P}$ be the partition on $\mathcal{S}$ given by

$$\{\{1,4,7\},\{2,5,8\},\{3,6,9\}\}.$$

Define $\mathcal{R}$ to be the equivalence relation associated to $\mathcal{P}$.

1. Give two conditions for $\mathcal{P}$ to be a partition.
2. Draw the relationship digraph.
3. Write down the equivalence class [5] as a set.

Solution:

1. $\mathcal{P}$ is a partition if
   - all the elements of S are in the partition.
   - each element of S appears in one partition only.
2. 
3. $[5] = \{2,5,8\}$

Question 5.

Let $S = \mathbb{Z} \times \mathbb{N}^+$ and Let $\mathbb{R}$ be relation on S defined as follows:

$$(a,b)\ \mathcal{R}\ (c,d) \text{ whenever } ad = bc$$

1. Show that $\mathcal{R}$ is an equivalence relation
2. Define the equivalence class generated by (a,b), for $a \in \mathbb{Z}$ and $b \in \mathbb{N}^+$

Solution:

1. $\mathcal{R}$ is reflexive as $\forall (a,b) \in S.\ (a,b)\ \mathcal{R}\ (a,b)$
   $\mathcal{R}$ is symmetric as $\forall (a,b),(c,d) \in S.\ (a,b)\ \mathcal{R}\ (c,d) \Rightarrow ad = bc \Rightarrow cb = da \Rightarrow (c,d)\ \mathcal{R}\ (a,b)$
   $\mathcal{R}$ is transitive as $\forall (a,b),(c,d),(e,f) \in S.\ (a,b)\ \mathcal{R}\ (c,d)$ and $(c,d)\ \mathcal{R}\ (e,f) \Rightarrow ad = bc$ and $cf = de \Rightarrow cbde = dacf \Rightarrow be = af \Rightarrow (a,b)\ \mathcal{R}\ (e,f)$
2. $[(a,b)] = \{(m,n) : (a,b) \in \mathbb{Z} \times \mathbb{N}^{+}$ and $\frac{a}{b} = \frac{m}{n}\}$

Question 6.
Let A and B be two sets where:
$A = \{France, Germany, Switzerland, England, Morocco\}$ and
$B = \{French, German, English, Arabic\}$. Let $\mathcal{R}$ be relation defined from A to B, given by $a\mathcal{R}b$ when b is a national language of a. The national language of of each of these countries is as follows: French for France, German for Germany, English for England, Arabic for Morocco, whereas, Switzerland has two national languages, French and German. Find the logical matrix for the relation $\mathcal{R}$.
Solution:

$$M_{\mathcal{R}} = \begin{array}{c} \\ France \\ Germany \\ England \\ Morocco \\ Switzerland \end{array} \begin{array}{c} \begin{array}{cccc} French & German & English & Arabic \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

Question 7.
For each of the following relations on the set of all people, state if i is an equivalence relation. Explain your answer.

1. $\mathcal{R}_1 = \{(x,y)|x$ and y are the same height $\}$.
2. $\mathcal{R}_2 = \{(x,y)|x$ and y have, at some time, lived in the same country$\}$.
3. $\mathcal{R}_3 = \{(x,y)|x$ and y have the same first name$\}$.
4. $\mathcal{R}_4 = \{(x,y)|x$ is taller than $y\}$.
5. $\mathcal{R}_5 = \{(x,y)|x$ and y have the same colour hair$\}$.

Solution:

1. $\mathcal{R}_1$. is reflexive as for all x, x is has the same height as x. $\mathcal{R}_1$. is symmetric as for all x, y, if x has the same height as y, then y has the same height as x. $\mathcal{R}_1$. is transitive as for all x, y, z, if x has the same height as y **AND** y has the same height as z, then x has the same height as z. Thus, $\mathcal{R}_1$ an equivalence relation.

2. $\mathcal{R}_2$. is reflexive as for all $x, x\mathcal{R}_2x$. $\mathcal{R}_2$. is symmetric as for all x, y,if x and y have at some time lived in the same country, then y and x have at some time lived in the same country. $\mathcal{R}_2$ is not transitive as for all x, y, z, if x and y have at some time lived in the same country, **AND** y and z have at some time lived in the same country doesn't necessarily mean that x and z have at some time lived in the same country. $\mathcal{R}_2$ is then not an equivalence relation.
3. $\mathcal{R}_3$. is reflexive as for all $x, x\mathcal{R}_3x$. $\mathcal{R}_3$ is symmetric as for all x, y,if $x\mathcal{R}_3y$ then x and y have the same first name, hence, $y\mathcal{R}_3x$. $\mathcal{R}_3$ is transitive as for x, y, z if x has the same first name as y and y has the same first name as z then x and z have the same first name as well. $\mathcal{R}_3$ is then an equivalence relation.
4. $\mathcal{R}_4$ is not reflexive as for all x, x is not taller than itself. $\mathcal{R}_4$ is not symmetric either as if x is taller than y then y is not taller than x. Hence, $\mathcal{R}_4$ is not an equivalence relation.
5. $\mathcal{R}_5$ is reflexive as for all $x, x\mathcal{R}_5x$. $\mathcal{R}_x$ is symmetric as for all x, y,if $x\mathcal{R}_5y$ then x and y have the same colour hair , hence, $y\mathcal{R}_5x$. $\mathcal{R}_5$ is transitive as for x, y, z if x has the same colour hair as y and y has the same colour hair as z then x and z have the same colour hair as well. $\mathcal{R}_5$ is then an equivalence relation.

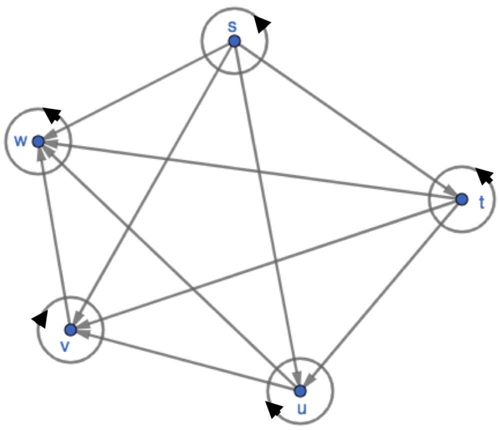
Question 8.

Let $\mathcal{S}$ = {{1},{1,2},{1,2,3},{1,2,3,4},{1,2,3,4,5}}. Define a relation $\mathcal{R}$ between the elements of $\mathcal{S}$ by

$$X \text{is related to } Y \text{ whenever } X \subseteq Y.$$

1. Draw the relationship digraph.
2. Determine whether or not $\mathcal{R}$ is reflexive, symmetric, antisymmetric or transitive. Give a brief justification for each of your answers.
3. State, with reasons, whether or not $\mathcal{R}$ is an equivalence relation, whether or not it is a partial order and whether or not it is a total order.

Solution:

1. 

$$s = \{1\}, \quad t = \{1,2\}, \quad u = \{1,2,3\}, \quad v = \{1,2,3,4\}, \quad w = \{1,2,3,4,5\}$$

2. $\mathcal{R}$ is reflexive, $\forall x \in S,\ x\mathcal{R}x$.
   $\mathcal{R}$ is not symmetric, $\{1\} \subseteq \{1,2\}$ but $\{1,2\} \not\subseteq \{1\}$.
   $\mathcal{R}$ is antisymmetric, $\forall X, Y \in S,\ X \subseteq Y$ and $Y \subseteq X \Rightarrow X = Y$.
   $\mathcal{R}$ is transitive , $\forall X, Y, Z \in S,\ X \subseteq Y$ and $Y \subseteq Z \Rightarrow X \subseteq Z$.
3. $\mathcal{R}$ is not an equivalence relation since $\mathcal{R}$ is not symmetric.
   $\mathcal{R}$ is reflexive, antisymmetric and transitive then it is a partial order.
   $\mathcal{R}$ is total order as it is a partial order and every two element of S are comparable with respect to the relation $\mathcal{R}$.

Question 9.

Let $\mathcal{S}$ ={a,b,c,d} and let $A \subseteq \mathcal{S} \times \mathcal{S}$ be given by

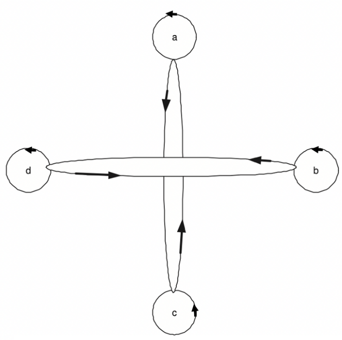
$$\{(a,a),(a,c).(b,b),(b,d),(c,a),(c,c),(d,b),(d,d)\}.$$

A relation $\mathcal{R}$ on $\mathcal{S}$ is defined by

$$x \text{ is related to } y \text{ whenever } (x,y) \in A.$$

1. Draw the relationship digraph.
2. Determine whether or not $\mathcal{R}$ is reflexive, symmetric, antisymmetric or transitive, giving a brief justification for your answer.
3. State, with reasons, whether or not $\mathcal{R}$ is and equivalence relation, whether or not it is a partial order and whether or not it is a total order.

Solution:

1. 

2. $\mathcal{R}$ is reflexive $for x \in S,\ x\mathcal{R}x$
   $\mathcal{R}$ is symmetric, $\forall x, y \in S, x\mathcal{R}y \Rightarrow y\mathcal{R}x$.
   $\mathcal{R}$ is not antisymmetric, $a\mathcal{R}b$ but $b \not\mathcal{R} a$.
   $\mathcal{R}$ is transitive if $\forall x, y, z \in S, x\mathcal{R}y \wedge y\mathcal{R}z \Rightarrow x\mathcal{R}z$

3. $\mathcal{R}$ is an equivalence relation. $\mathcal{R}$ is not a partial order as $\mathcal{R}$ is not antisymmetric. $\mathcal{R}$ is not a total order as it is not a partial order

Question 10.
Let $\mathcal{R}$ be a relation from a set A to a set B. The inverse of $\mathcal{R}$, denoted $\mathcal{R}^{-1}$, is the relation from B to A defined by $\mathcal{R}^{-1} = \{(y, x) : (x, y) \in \mathcal{R}\}$.
Given a relation $\mathcal{R}$ from $A = \{2, 3, 4\}$ to $B = \{3, 4, 5, 6, 7\}$ defined by $(x, y) \in \mathcal{R}$ if x divides y.

1. List the elements of of $\mathcal{R}$ and write down the matrix, $M_{\mathcal{R}}$, of $\mathcal{R}$.

2. List the elements of of $\mathcal{R}^{-1}$ and write down the matrix, $M_{\mathcal{R}^{-1}}$, of $\mathcal{R}$.

Solution:

1. $\mathcal{R} = \{(2, 4), (2, 6), (3, 3), (3, 6), (4, 4)\}$

$$M_{\mathcal{R}} = \begin{array}{c} \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

2. $\mathcal{R}^{-1} = \{(4, 2), (6, 2), (3, 3), (6, 3), (4, 4)\}$. The relation $\mathcal{R}^{-1}$ can be defined from B to A by $(x, y) \in \mathcal{R}^{-1}$ if x is divisible by y

$$M_{\mathcal{R}^{-1}} = \begin{array}{c} \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array} \begin{array}{c} \begin{array}{ccc} 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

Question 11.

Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be the relations on a set $S = \{1, 2, 3, 4\}$ given by:
$\mathcal{R}_1 = \{(1, 1), (1, 2), (3, 4), (4, 2), (2.4)\}$ $\mathcal{R}_2 = \{(1, 1), (3, 2), (4, 4), (2, 2), (4, 2)\}$.

1. Find the matrix representation $\mathcal{R}_1$ and that of $\mathcal{R}_2$.

2. Find the matrix of the intersection of both matrices in (1).

3. Find the matrix of the union both matrices in (1).

4. list the element of $\mathcal{R}_1 \cap \mathcal{R}_2$.

5. list the element of $\mathcal{R}_1 \cup \mathcal{R}_2$.

Solution:

1.

$$M_{\mathcal{R}_1} = \begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

$$M_{\mathcal{R}_2} = \begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \end{array}$$

2.

$$M_{\mathcal{R}_1 \cap \mathcal{R}_2} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \wedge \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} = \begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

3.

$$M_{\mathcal{R}_1 \cup \mathcal{R}_2} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \vee \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} = \begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \end{array}$$

4. $\mathcal{R}_1 \cap \mathcal{R}_2 = \{(1,1), (4,2)\}$.

5. $\mathcal{R}_1 \cup \mathcal{R}_2 = \{(1,1), (1,2), (3,4), (4,2), (2.4), (3,2), (4,4), (2,2)\}$.

Question 12.
Let $\mathcal{R}$ be a relation on set A.

1. How can we quickly determine whether a relation R is reflexive by examining the matrix of $\mathcal{R}$?

2. How can we quickly determine whether a relation R is symmetric by examining the matrix of $\mathcal{R}$?

3. How can we quickly determine whether a relation R is anti- symmetric by examining the matrix of $\mathcal{R}$?

Solution:
Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be two relations

1. A relation is reflexive if and only if its matrix has 1Õs on the main diagonal.
2. A relation $\mathcal{R}$ is symmetric if and only if its matrix $M_{\mathcal{R}}$ satisfies the following: For all i and j, the $(ij)^{th}$ entry of $M_{\mathcal{R}}$ is equal to the j $(ji)^{th}$ entry of $M_{\mathcal{R}}$.
3. A relation $\mathcal{R}$ is anti-symmetric if and only if its matrix $M_{\mathcal{R}}$ satisfies the following: For all i and j with $i \neq j$, if the $(ij)^{th}$ entry of $M_{\mathcal{R}}$ is non-zeror then the $(ji)^{th}$ entry of $M_{\mathcal{R}}$ is equal to zero.

Question 13.

For each of following relations on a set $A = \{a, b, c\}$ defined by their corresponding Matrices, say wether is is reflexive, symmetric or anti-symmetric.

1. $M_{\mathcal{R}_1} = \begin{array}{c c} & \begin{array}{ccc} a & b & c \end{array} \\ \begin{array}{c} a \\ b \\ c \end{array} & \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \end{array}$

2. $M_{\mathcal{R}_2} = \begin{array}{c c} & \begin{array}{ccc} a & b & c \end{array} \\ \begin{array}{c} a \\ b \\ c \end{array} & \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \end{array}$

3. $M_{\mathcal{R}_3} = \begin{array}{c c} & \begin{array}{ccc} a & b & c \end{array} \\ \begin{array}{c} a \\ b \\ c \end{array} & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$

4. $M_{\mathcal{R}_4} = \begin{array}{c c} & \begin{array}{ccc} a & b & c \end{array} \\ \begin{array}{c} a \\ b \\ c \end{array} & \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \end{array}$

Solution:

1. $M_{\mathcal{R}_1} = \begin{array}{c c} & \begin{array}{ccc} a & b & c \end{array} \\ \begin{array}{c} a \\ b \\ c \end{array} & \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \end{array}$

   $\mathcal{R}_1$ is not reflexive because the leading diagonal is not all 1's as $m_{33} = 0$ which means that c is not related to itself.

   $\mathcal{R}_1$ is not symmetric as $m_{12} = 1$ whereas $m_{21} = 0$. This matrix is not symmetric.

   $\mathcal{R}_1$ is anti-symmetric as if $m_{ij} = 1$ then $m_{ji} = 0$ for all $i \neq j$.

2. $M_{\mathcal{R}_2} = \begin{array}{c c} & \begin{array}{ccc} a & b & c \end{array} \\ \begin{array}{c} a \\ b \\ c \end{array} & \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \end{array}$

   $\mathcal{R}_2$ is reflexive as the leading diagonal is all 1's.

   $\mathcal{R}_2$ is not symmetric as $m_{12} = 1$ whereas $m_{21} = 0$. This matrix is not symmetric.

   $\mathcal{R}_2$ is anti-symmetric as if $m_{ij} = 1$ then $m_{ji} = 0$ for all $i \neq j$.

3. $M_{\mathcal{R}_3} = \begin{array}{c c} & \begin{array}{ccc} a & b & c \end{array} \\ \begin{array}{c} a \\ b \\ c \end{array} & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$

   $\mathcal{R}_3$ is reflexive as the leading diagonal is all 1's.

   $\mathcal{R}_3$ is symmetric as this matrix is symmetric.

   $\mathcal{R}_3$ is anti-symmetric as if $m_{ij} = 1$ then $m_{ji} = 0$ for all $i \neq j$.

4. $M_{\mathcal{R}_4} = \begin{array}{c c} & \begin{array}{ccc} a & b & c \end{array} \\ \begin{array}{c} a \\ b \\ c \end{array} & \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \end{array}$

$\mathcal{R}_4$ is reflexive as the leading diagonal is all 1's.
$\mathcal{R}_4$ is symmetric as this matrix is symmetric.
$\mathcal{R}_4$ is not anti-symmetric as for example $m_{i2} = m_{21} = 1$ or $m_{23} = m_{32} = 1$

Question 14.

Let $A = \{1, 2, ..., 10\}$ and let $\mathcal{R}$ be a relation on A defined by $x\mathcal{R}y$ if 3 divides $x - y$.

1. Show that the relation $\mathcal{R}$ is an equivalence relation on A.

2. List all the equivalence classes of $\mathcal{R}$.

Solution:

1. To show that $\mathcal{R}$ is an equivalence relation on a set A, we need to show that it is reflexive, symmetric and transitive.

   Reflexive: for all x in A 3 divided $x - x = 0$, hence $x\mathcal{R}x$. Thus $\mathcal{R}$ is a reflexive relation.

   Symmetric: for all $x, y \in A$ if 3 divides $x - y$ then 3 divides $y - x$. Hence, $\mathcal{R}$ is a symmetric relation.

   Transitive: for all $x, y, z \in A$ if 3 divides $x - y$ and 3 divides $y - z$ then 3 divides $x - y + y - z = x - z$. Hence, for all $x, y, z \in A$ if $x\mathcal{R}y$ and $y\mathcal{R}z$ then $x\mathcal{R}z$. Thus, $\mathcal{R}$ is a transitive relation.

2. $[1] = \{x \in A : 3 \text{ divides } x - 1\} = \{1, 4, 7, 10\}$.

   $[2] = \{x \in A : 3 \text{ divides } x - 2\} = \{2, 5, 8\}$.

   $[3] = \{x \in A : 3 \text{ divides } x - 3\} = \{3, 6, 9\}$.

Question 15.
Let $A = \{1, 2, ..., 10\}$ and let $\mathcal{R}$ be a relation on A defined by $x\mathcal{R}y$ if $x \mod 2 = y \mod 2$.

1. Show that the relation $\mathcal{R}$ is an equivalence relation on A.

2. List all the equivalence classes of $\mathcal{R}$.

3. is $\mathcal{R}$ a partial or a total order?

Solution:

1. To show that $\mathcal{R}$ is an equivalence relation on a set A, we need to show that it is reflexive, symmetric and transitive.

   Reflexive: for all x in A $x \mod 2 = x \mod 2$, hence $x\mathcal{R}x$. Thus $\mathcal{R}$ is a reflexive relation.

   Symmetric: for all $x, y \in A$ if $x \mod 2 = y \mod 2$ then $y'mod2 = x \mod 2$ as the equal is commutative. Hence, $\mathcal{R}$ is a symmetric relation.

   Transitive: for all $x, y, z \in A$ if $x \mod 2 = y \mod 2$ and $y \mod 2 = z \mod 2$ then $x \mod 2 = z \mod 2$. Hence, for all $x, y, z \in A$ if $x\mathcal{R}y$ and $y\mathcal{R}z$ then $x\mathcal{R}z$. Thus, $\mathcal{R}$ is a transitive relation.

2. $[1] = \{x \in A : x \mod 2 = 1 \mod 2 = 1\} = \{1, 3, 5, 7, 9\}$.

   $[2] = \{x \in A : x \mod 2 = 2mod2 = 0\} = \{2, 4, 6, 8, 10\}$.

3. partial order: $\mathcal{R}$ is not a partial order as it is not antisymmetric: $2\mathcal{R}4$ and $4\mathcal{R}2$ but $2 \neq 4$. Hence, $\mathcal{R}$ is not a total order.

Question 16.
Let $A = \{1, 2, ..., 10\}$ and let $\mathcal{R}$ be a relation on A defined by $x\mathcal{R}y$ if $x + y \mod 2 = 0$.

1. Show that the relation $\mathcal{R}$ is an equivalence relation on A.

2. List all the equivalence classes of $\mathcal{R}$.

3. is $\mathcal{R}$ a partial or a total order?

Solution:

1. To show that $\mathcal{R}$ is an equivalence relation on a set A, we need to show that it is reflexive, symmetric and transitive.

   Reflexive: for all x in A $x + x = 2x \mod 2 = 0$, hence $x\mathcal{R}x$. Thus $\mathcal{R}$ is a reflexive relation.

   Symmetric: for all $x, y \in A$ if $x + y \mod 2 = 0$ then $y + x \mod 2 = 0$ as the addition is commutative. Hence, $\mathcal{R}$ is a symmetric relation.

   Transitive: for all $x, y, z \in A$ if $x+y = 2m$ and $y+z = 2b$ then $x+y+y+z = 2(n+m)$, which leads to $x+z = 2(n+m-y)$, hence $x+z \mod 2 = 0$ Hence, for all $x, y, z \in A$ if $x\mathcal{R}y$ and $y\mathcal{R}z$ then $x\mathcal{R}z$. Thus, $\mathcal{R}$ is a transitive relation.

2. $[1] = \{x \in A : 1 + x \mod 2 = 0\} = \{1, 3, 5, 7, 9\}$.
   $[2] = \{x \in A : 2 + x \mod 2 = 0\} = \{2, 4, 6, 8, 10\}$.

3. partial order: $\mathcal{R}$ is not a partial order as it is not antisymmetric: $2\mathcal{R}4$ and $4\mathcal{R}2$ but $2 \neq 4$. Hence, $\mathcal{R}$ is not a total order.

Question 17.

Let $\mathcal{R}$ be are relation on the set $A = \{1, 2, 3, 4, 5\}$ defined by the rule $x\mathcal{R}y$ if $x = y - 1$. Is this relation reflexive, symmetric, antisymmetric, transitive, equivalence, and/or a partial order?

Solution:

Reflexive: $x \neq x - 1$ for all $x \in A$. Thus this relation is not reflexive.

Symmetric: $2\mathcal{R}3$ as $2 = 3 - 1$ however, $3 \not\mathcal{R}2$ as $3 \neq 2 - 1$. Thus $\mathcal{R}$ is not symmetric.

Antisymmetric: we can show this by contradiction: Assume there is exists $x, y \in A$ with $x \neq y$ and $x = y - 1$ and $y = x - 1$. This implies that $x = x - 2$, hence, $0 = -2$ (contradiction). Thus $\mathcal{R}$ is antisymmetric.

Transitive: it suffices to give a counter-example to show that this relation is not transitive. $2\mathcal{R}3(2 = 3 - 1)$ and $3\mathcal{R}4(3 = 4 - 1)$, however, $2 \not\mathcal{R}4$ as $2 \neq 4 - 1$. Hence, $mathcal R$ is not a transitive relation.

equivalence: for a relation to be an equivalence relation , it has be reflexive, symmetric and transitive. however, $\mathcal{R}$ is neither symmetric nor transitive. Hence, $\mathcal{R}$ is not an equivalence relation.

partial order: for a relation to be partial order, it has be reflexive, antisymmetric and transitive. however, $\mathcal{R}$ is neither reflexive nor transitive. Hence, $\mathcal{R}$ is not a partial order.

Question 18.

Let $A = \{1, 2, ..., 10\}$ and let $\mathcal{R}$ be a relation on $A \times A$ defined by $(a, b)\mathcal{R}(c, d)$ if $a + d = b + c$. Show that $\mathcal{R}$ is an equivalence relation on $A \times A$.

Solution:

To show this relation is an equivalence relation, we need to show that is reflexive, symmetric, and transitive. Reflexive: for all $(a, b) \in A \times A$ we have $(a, b)\mathcal{R}(a, b)$ as a+b= a+b.

Symmetric: for all $(a, b), (c, d) \in A \times A$ if $(a, b)\mathcal{R}(c, d)$ then $a + d = b + c$. this implies $c+b = d+a$ as both $=$ and $+$ are commutative. hence, $(c, d)\mathcal{R}(a, b)$. Thus $\mathcal{R}$ is symmetric.

Transitive: for all $(a, b), (c, d), (e, f) \in A \times A$ we need to show that if $(a, b)\mathcal{R}(c, d)$ and $(c, d)\mathcal{R}(e, f)$ then $(a, b)\mathcal{R}(e, f)$.

$(a, b)\mathcal{R}(c, d)$ and $(c, d)\mathcal{R}(e, f)$ implies that:

(1) $a + d = b + c$ and (2) $c + f = d + e$, hence

(1) $a + d = b + c$ and (2) $c = d + e - f$. if we substitute c fro m(2) in (1), we get:

$a + d = b + d + e - f$ this implies that $a + e = b + e$, hence, $(a, b)\mathcal{R}(e, f)$. Thus, $\mathcal{R}$ is a transitive relation.

Therefore, $\mathcal{R}$ is an equivalence relation.

Question 19.

Let $X = \{1, 2, 3, 4\}, Y = \{3, 4\}$, and $C = \{1, 3\}$ and let $\mathcal{R}$ be a relation on $\mathcal{P}(X)$, the set of all subsets of X, defined as

$$\forall A, B \in \mathcal{P}(X), \quad A\mathcal{R}B \text{ if } A \cup Y = B \cup Y$$

1. Show that $\mathcal{R}$ is an equivalence relation.

2. List the elements of $[C]$, the equivalence class containing C.

Solution:

Writing down the $\mathcal{P}(X)$ is not necessary but we can do it as it has only 16 elements.
$\mathcal{P}(X) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\},$
$\{1,2,3\}, \{1,2,4\}, \{1,3,4\}, \{2,3,4\}, \{1,2,3,4\}\}$

1. We need to show that $\mathcal{R}$ is reflexive, symmetric and transitive.

   Reflexive: for all $A \in \mathcal{P}(X)$ we have $A \cup Y = A \cup Y$. Hence, $\mathcal{R}$ is reflexive.

   Symmetric: for all $A, B \in \mathcal{P}(X)$ if $A\mathcal{R}B$ then $A \cup Y = B \cup Y$. This implies that $B \cup Y = A \cup Y$ as equality is commutative. hence, $B\mathcal{R}A$. Thus $\mathcal{R}$ is symmetric.

   transitive: for all $A, B, C \in \mathcal{P}(X)$ we need to show if $A\mathcal{R}B$ and $B\mathcal{R}C$ then $A\mathcal{R}C$.

   $A\mathcal{R}B$ and $B\mathcal{R}C$ implies that $A \cup Y = B \cup Y$ and $B \cup Y = C \cup Y$. This implies that $A \cup Y = C \cup Y$, hence, $A\mathcal{R}C$. Thus $\mathcal{R}$ is an equivalence relation.

2. $Y = \{3,4\}$, and $C = \{1,3\}$

   $[C] = \{B : B \in \mathcal{P}(X) \text{ AND } B \cup Y = C \cup Y = \{1,3,4\}\}$

   $[C] = \{\{1\}, \{1,3\}, \{1,4\}, \{1,3,4\}\}$