# Discrete Mathematics

Tutorial sheet | Topic-9 | Relations

**Question 1.**

Let $\mathcal{S}$ be a set and $\mathcal{R}$ be a relation on $\mathcal{S}$. Explain what it means (you are expected to give mathematical definitions.) to say that $\mathcal{R}$ is:

1. reflexive;
2. symmetric;
3. anti-symmetric;
4. transitive;
5. an equivalence relation;
6. a partial order.

In each case give an example of a relation which has the given property and another relation which does not have it.

**Question 2.**

Let $S = \{a, b, c\}$ and $A = \{(c, c), (a, b), (b, b), (b, c), (c, b)\}$.
Define a relation $R$ on $S$ by "$x$ is related to $y$ whenever$(x, y) \in A$".

1. Draw the relationship digraph.
2. The relation $R$ is not reflexive. What pair $(x, y)$ should be added to $A$ to make $R$ reflexive?
3. The relation $R$ is not symmetric. What pair $(x, y)$ should be added to $A$ to make $R$ symmetric?
4. The relation $R$ is not anti-symmetric. What pair $(x, y)$ should be removed to make $R$ anti-symmetric?
5. The relation $R$ is not transitive. What pair $(x, y)$ should be added to $A$ to make $R$ transitive?

**Question 3.**

The following relations are defined on a set $S = \{a, b, c\}$.
$R_1$ is the relation given by {(a,a),(a,b),(a,c),(b,a),(b,b),(c,a),(c,c)}
$R_2$ is given by {(a,a),(a,b),(b,a),(b,b),((c,c)}
$R_3$ is given by {(a,b),(a,c),(b,a),(b,c),(c,a),(c,b)}
$R_4$ is given by {(a,a),(a,b),(a,c),((b,b),(b,c),(c,c)}
Complete the table below. If the relation is an equivalence relation give the equivalence classes. Also state whether any of the relations is a partial order, justifying your answer.

| | reflexive | symmetric | antisymmetric | transitive | equivalence rel. | |
|---|---|---|---|---|---|---|
| $\mathcal{R}_1$ | | | | | | |
| $\mathcal{R}_2$ | | | | | | |
| $\mathcal{R}_3$ | | | | | | |
| $\mathcal{R}_4$ | | | | | | |

**Question 4.**
Let $\mathcal{S}$ ={1,2,3,4,5,6,7,8,9} and let $\mathcal{P}$ be the partition on $\mathcal{S}$ given by

$$\{\{1, 4, 7\}, \{2, 5, 8\}, \{3, 6, 9\}\}.$$

Define $\mathcal{R}$ to be the equivalence relation associated to $\mathcal{P}$.

1. Give two conditions for $\mathcal{P}$ to be a partition.
2. Draw the relationship digraph.
3. Write down the equivalence class [5] as a set.

**Question 5.**

Let $S = \mathbb{Z} \times \mathbb{N}^+$ and Let $\mathbb{R}$ be relation on $S$ defined as follows:

$$(a, b)\ \mathcal{R}\ (c, d) \text{ whenever } ad = bc$$

1. Show that $\mathcal{R}$ is an equivalence relation
2. Define the equivalence class generated by (a,b), for $a \in \mathbb{Z}$ and $b \in \mathbb{N}^+$

**Question 6.**
Let $A$ and $B$ be two sets where:
$A = \{France, Germany, Switzerland, England, Morocco\}$ and
$B = \{French, German, English, Arabic\}$. Let $\mathcal{R}$ be relation defined from $A$ to $B$, given by $a\mathcal{R}b$ when $b$ is a national language of $a$. The national language

of of each of these countries is as follows: French for France, German for Germany, English for England, Arabic for Morocco, whereas, Switzerland has two national languages, French and German. Find the logical matrix for the relation $\mathcal{R}$ .

**Question 7.**

For each of the following relations on the set of all people, state if i is an equivalence relation. Explain your answer.

1. $\mathcal{R}_1 = \{(x, y)|x$ and $y$ are the same height $\}$.
2. $\mathcal{R}_2 = \{(x, y)|x$ and $y$ have, at some time, lived in the same country$\}$.
3. $\mathcal{R}_3 = \{(x, y)|x$ and $y$ have the same first name$\}$.
4. $\mathcal{R}_4 = \{(x, y)|x$ is taller than $y\}$.
5. $\mathcal{R}_5 = \{(x, y)|x$ and $y$ have the same colour hair$\}$.

**Question 8.**

Let $\mathcal{S}$ = {{1},{1,2},{1,2,3},{1,2,3,4},{1,2,3,4,5}}. Define a relation $\mathcal{R}$ between the elements of $\mathcal{S}$ by

$$X \text{is related to } Y \text{ whenever } X \subseteq Y.$$

1. Draw the relationship digraph.
2. Determine whether or not $\mathcal{R}$ is reflexive, symmetric, antisymmetric or transitive. Give a brief justification for each of your answers.
3. State, with reasons, whether or not $\mathcal{R}$ is an equivalence relation, whether or not it is a partial order and whether or not it is a total order.

**Question 9.**

Let $\mathcal{S}$ ={a,b,c,d} and let $A \subseteq \mathcal{S} \times \mathcal{S}$ be given by

$$\{(a, a), (a, c).(b, b), (b, d), (c, a), (c, c), (d, b), (d, d)\}.$$

A relation $\mathcal{R}$ on $\mathcal{S}$ is defined by

$$x \text{ is related to } y \text{ whenever } (x, y) \in A.$$

1. Draw the relationship digraph.

2. Determine whether or not $\mathcal{R}$ is reflexive, symmetric, antisymmetric or transitive, giving a brief justification for your answer.

3. State, with reasons, whether or not $\mathcal{R}$ is and equivalence relation, whether or not it is a partial order and whether or not it is a total order.

**Question 10.**
Let $\mathcal{R}$ be a relation from a set $A$ to a set $B$. The inverse of $\mathcal{R}$, denoted $\mathcal{R}^{-1}$, is the relation from $B$ to $A$ defined by $\mathcal{R}^{-1} = \{(y, x) : (x, y) \in \mathcal{R}\}$.
Given a relation $\mathcal{R}$ from $A = \{2, 3, 4\}$ to $B = \{3, 4, 5, 6, 7\}$ defined by $(x, y) \in \mathcal{R}$ if $x$ divides $y$.

1. List the elements of of $\mathcal{R}$ and write down the matrix, $M_{\mathcal{R}}$, of $\mathcal{R}$.

2. List the elements of of $\mathcal{R}^{-1}$ and write down the matrix, $M_{\mathcal{R}^{-1}}$, of $\mathcal{R}$.

**Question 11.**

Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be the relations on a set $S = \{1, 2, 3, 4\}$ given by:
$\mathcal{R}_1 = \{(1, 1), (1, 2), (3, 4), (4, 2), (2.4)\}$ $\mathcal{R}_2 = \{(1, 1), (3, 2), (4, 4), (2, 2), (4, 2)\}$.

1. Find the matrix representation $\mathcal{R}_1$ and that of $\mathcal{R}_2$.

2. Find the matrix of the intersection of both matrices in (1).

3. Find the matrix of the union both matrices in (1).

4. list the element of $\mathcal{R}_1 \cap \mathcal{R}_2$.

5. list the element of $\mathcal{R}_1 \cup \mathcal{R}_2$.

**Question 12.**
Let $\mathcal{R}$ be a relation on set $A$.

1. How can we quickly determine whether a relation R is reflexive by examining the matrix of $\mathcal{R}$?

2. How can we quickly determine whether a relation R is symmetric by examining the matrix of $\mathcal{R}$?

3. How can we quickly determine whether a relation R is anti- symmetric by examining the matrix of $\mathcal{R}$?

**Question 13.**
For each of following relations on a set $A = \{a, b, c\}$ defined by their corresponding Matrices, say wether is is reflexive, symmetric or anti-symmetric.

1. $M_{\mathcal{R}_1} = \begin{array}{c} \\ a \\ b \\ c \end{array} \begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \end{array}$

2. $M_{\mathcal{R}_2} = \begin{array}{c} \\ a \\ b \\ c \end{array} \begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \end{array}$

3. $M_{\mathcal{R}_3} = \begin{array}{c} \\ a \\ b \\ c \end{array} \begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$

4. $M_{\mathcal{R}_4} = \begin{array}{c} \\ a \\ b \\ c \end{array} \begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \end{array}$

**Question 14.**
Let $A = \{1, 2, ..., 10\}$ and let $\mathcal{R}$ be a relation on $A$ defined by $x\mathcal{R}y$ if 3 divides $x - y$.

1. Show that the relation $\mathcal{R}$ is an equivalence relation on $A$.
2. List all the equivalence classes of $\mathcal{R}$.

**Question 15.**
Let $A = \{1, 2, ..., 10\}$ and let $\mathcal{R}$ be a relation on $A$ defined by $x\mathcal{R}y$ if $x \mod 2 = y \mod 2$.

1. Show that the relation $\mathcal{R}$ is an equivalence relation on $A$.
2. List all the equivalence classes of $\mathcal{R}$.
3. is $\mathcal{R}$ a partial or a total order?

**Question 16.**
Let $A = \{1, 2, ..., 10\}$ and let $\mathcal{R}$ be a relation on $A$ defined by $x\mathcal{R}y$ if $x + y \mod 2 = 0$.

1. Show that the relation $\mathcal{R}$ is an equivalence relation on $A$.

2. List all the equivalence classes of $\mathcal{R}$.

3. is $\mathcal{R}$ a partial or a total order?

**Question 17.**
Let $\mathcal{R}$ be are relation on the set $A = \{1, 2, 3, 4, 5\}$ defined by the rule $x\mathcal{R}y$ if $x = y - 1$. Is this relation reflexive, symmetric, antisymmetric, transitive, equivalence, and/or a partial order?
**Question 18.**
Let $A = \{1, 2, ..., 10\}$ and let $\mathcal{R}$ be a relation on $A \times A$ defined by $(a, b)\mathcal{R}(c, d)$ if $a + d = b + c$. Show that $\mathcal{R}$ is an equivalence relation on $A \times A$.
**Question 19.**
Let $X = \{1, 2, 3, 4\}, Y = \{3, 4\}$, and $C = \{1, 3\}$ and let $\mathcal{R}$ be a relation on $\mathcal{P}(X)$, the set of all subsets of $X$, defined as

$$\forall A, B \in \mathcal{P}(X), \quad A\mathcal{R}B \text{ if } A \cup Y = B \cup Y$$

1. Show that $\mathcal{R}$ is an equivalence relation.

2. List the elements of $[C]$, the equivalence class containing $C$.