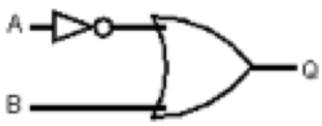
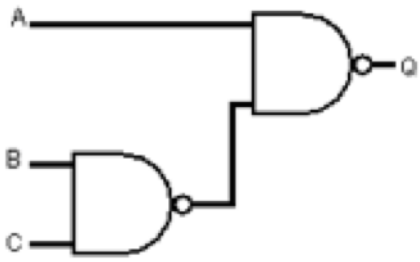
Discrete Mathematics

Tutorial sheet

Boolean Algebra

Question 1.
What is the output for each of the following logic circuits:

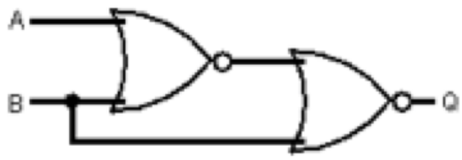
1. 

2. 

Solution:

1. $\overline{A} + B$
2. $\overline{A(\overline{BC})}$

Question 2.
Write down the truth table for the output Q of the following circuit.



Solution:

A	B	$A+B$	$\overline{A+B}$	$\overline{A+B}+B$	$Q=\overline{\overline{A+B}+B}$
0	0	0	1	1	0
0	1	1	0	1	0
1	0	1	0	0	1
1	1	1	0	1	0

Question 3.
Simplify each Boolean expression to one of the following expressions: $0, 1, A, B, AB, A+B, \overline{AB}, \overline{A}+\overline{B}, \overline{A}B$ and $A\overline{B}$

1. $\overline{\overline{A}+\overline{B}}$
2. $A(A+\overline{A})+B$

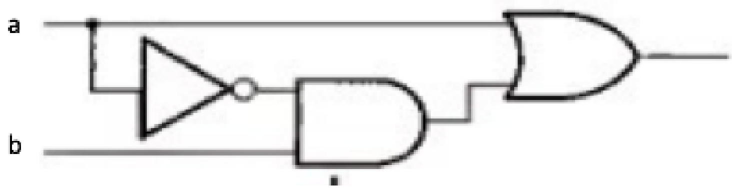
3. $(A+B)(\overline{A}+B)\overline{B}$

Solution:

1. $\overline{\overline{A}+\overline{B}} = A.B$ (De Morgan's law)
2. $A(A+\overline{A})+B$
$$\begin{aligned} A(A+\overline{A})+B &= A.A + A.\overline{A} + B && \text{Distributivity} \\ &= A + 0 + B && A.A = A \text{ and } A.\overline{A} = 0 \\ &= A + B && A + 0 = A \end{aligned}$$
3. $(A+B)(\overline{A}+B)\overline{B}$
$$\begin{aligned} (A+B)(\overline{A}+B)\overline{B}. &= (A+B)(\overline{A}.\overline{B} + B.\overline{B}) && \text{distributivity} \\ &= (A+B)(\overline{A}.\overline{B} + 0) && B.\overline{B} = 0 \\ &= (A+B)(\overline{A}.\overline{B}) && \\ &= (A.\overline{A}.\overline{B} + B.\overline{A}.\overline{B}) && \textit{distributivity} \\ &= (A.\overline{A}.\overline{B} + \overline{A}.B.\overline{B}) && \textit{commutativity} \\ &= (0.\overline{B} + \overline{A}.0) && A\overline{A} = B..\overline{B} = 0 \\ &= (0+0) && 0.\overline{B} = \overline{A}.0 = 0 \\ &= 0 && \end{aligned}$$

Question 4.

1. Use the laws of boolean algebra to simplify the boolean expression: $a + \overline{a}b = a + b$.
2. Use the truth table prove that $a + \overline{a}b = a + b$.
3. Use he results from 1 and 2 to find a simplified circuit for the following logic circuit:



Solution:

1.
$$\begin{aligned} a + \overline{a}b &= a.1 + \overline{a}b && \text{Identity law} \\ &= a.(1+b) + \overline{a}b && \text{Identity law} \\ & a.1 + ab + \overline{a}b && \text{Distributive law} \\ &= a.1 + b(a + \overline{a}) && \text{Distributive law} \\ &= a.1 + b.(1) && \text{Complement law} \\ &= a + b && \text{Identity law} \end{aligned}$$

2.

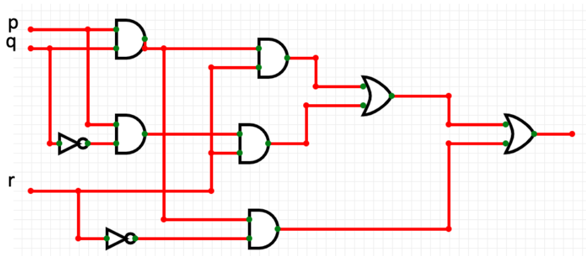
a	b	$\overline{a}$	$a+b$	$\overline{a}b$	$a+\overline{a}b$
0	0	1	0	0	0
0	1	1	1	1	1
1	0	0	1	0	1
1	1	0	1	0	1

The column for $(\overline{a+b})$ is the same as the column $\overline{a}\overline{b}$, hence, $(\overline{a+b}) = \overline{a}\overline{b}$.

3. 

Question 5.

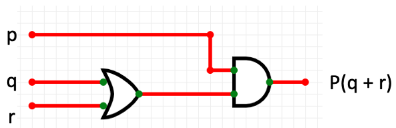
1. What is the output of the following logical circuit:



Solution:
$pqr + p\overline{q}r + pq\overline{r}$

2. Simplify the output form the circuit above and find a simpler circuit which has the same effect.

Solution:

1. $pqr + p\overline{q}r + pq\overline{r} = p(q + r)$. Hence, this circuit can be simplified to
2. 

Question 6.

Use the truth table prove De Morgan's laws: $\overline{ab} = \overline{a} + \overline{b}$ and $\overline{a+b} = \overline{a}.\overline{b}$

Solution:

a	b	$\overline{a}$	$\overline{b}$	$a+b$	$\overline{a+b}$	$\overline{a}.\overline{b}$	ab	$\overline{ab}$	$\overline{a}+\overline{b}$
0	0	1	1	0	1	1	0	1	1
0	1	1	0	1	0	0	0	1	1
1	0	0	1	1	0	0	0	1	1
1	1	0	0	1	0	0	1	0	0

The column for $(\overline{a+b})$ is the same as the column $\overline{a}\overline{b}$, hence, $(\overline{a+b}) = \overline{a}\overline{b}$.
The column for $(\overline{ab})$ is the same as the column $\overline{a}+\overline{b}$, hence, $(\overline{ab}) = \overline{a}+\overline{b}$.

Question 7.

Use the laws of boolean algebra to simplify the boolean expression:

$$\overline{ab}(\overline{a}+b)(\overline{b}+b)$$

Solution:

$$\begin{array}{lll}
\overline{ab}(\overline{a}+b)(\overline{b}+b) & = \overline{ab}(\overline{a}+b).1 & \text{Complement law} \\
 & = \overline{ab}(\overline{a}+b) & \text{idempotent law} \\
 & = \overline{ab}\overline{a} + \overline{ab}b & \text{Distributive law} \\
 & = (\overline{a}+\overline{b})\overline{a} + (\overline{a}+\overline{b})b & \text{De Morgan's law} \\
 & = \overline{a}\overline{a} + \overline{b}\overline{a} + \overline{a}b + \overline{b}b & \text{Distributive law} \\
 & = \overline{a} + \overline{a}(\overline{b}+b) + 0 & \text{Idempotent law , Complement laws and distributive} \\
 & = \overline{a} + \overline{a}.1 + 0 & \text{complement law} \\
 & = \overline{a} + \overline{a} & \text{identity law} \\
 & = \overline{a} & \text{idempotent law}
\end{array}$$

Question 8.

Use the laws of boolean algebra to simplify the boolean expression:

$$\overline{a}(a+b) + (b+aa)(a+\overline{b})$$

Solution:

$$\begin{array}{lll}
\overline{a}(a+b) + (b+aa)(a+\overline{b}) & = \overline{a}(a+b) + (b+a)(a+\overline{b}) & \text{idempotent law} \\
 & = \overline{a}(a+b) + (a+b)(a+\overline{b}) & \text{Commutative law} \\
 & = (a+b).(\overline{a} + (a+\overline{b})) & \text{Distributive law} \\
 & = (a+b).((\overline{a}+a) + \overline{b}) & \text{Associative law} \\
 & = (a+b).(1+\overline{b}) & \text{Complement law} \\
 & = (a+b).1 & \text{Annulment law} \\
 & = (a+b). & \text{Identity law}
\end{array}$$

Question 9.

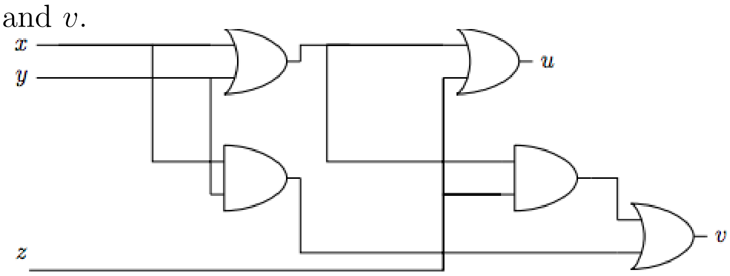
Prove that in a boolean algebra $a^2 = a$. You are required to explain your answer by making a reference to a boolean algebra axioms (laws).

Solution:

$$\begin{aligned} a &= a.1 && (a.1 = a) \\ &= a.(a+\overline{a}) && (\overline{a}+a=1) \\ &= a.a + a.\overline{a} && (\text{distributiviy of . over } +) \\ &= a^2 + 0 && (a.\overline{a} = 0) \\ &= a^2 && a+0=a \end{aligned}$$

Question 10.

The following diagram shows a circuit with three inputs and two outputs, u and v.



1. List the logic gates used in this circuit.

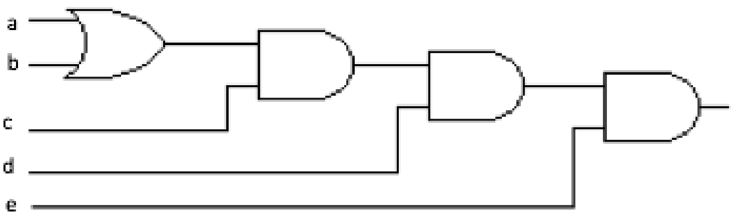
2. Describe each output u and v as a Boolean expression in terms of x, y and z.

Solution:

1. 2 and-gates and 3 or-gates
2. $u = x + y + z$ where as $v = ((x + y).z) + (x.y) = x.y + x.z + y.z$

Question 11.

Derive the Boolean expression for the following logic circuit shown below



Solution:
(a+b).c.d.e

Question 12.

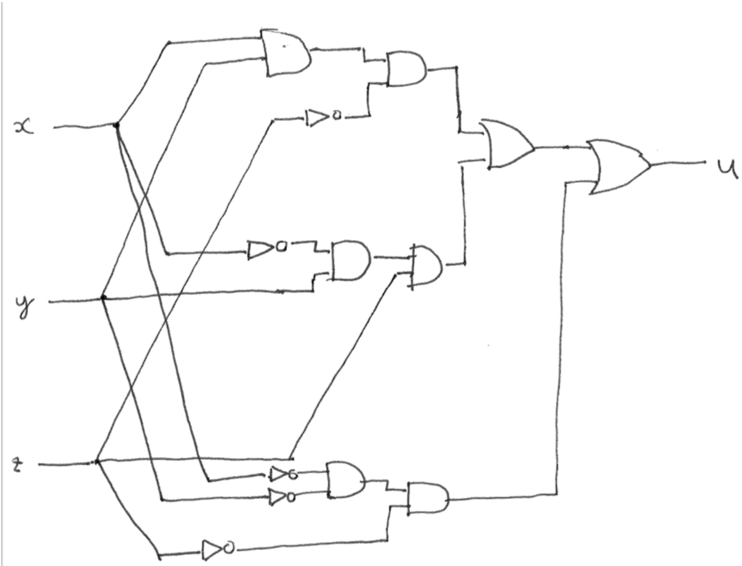
1. Write down a boolean expression for the following input/output behaviour.

x	y	z	u
0	0	0	1
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	0
1	1	0	1
1	1	1	0

2. Construct the corresponding circuit of the above expression using not-gates, and-gates and or-gates only.

Solution:

1. In order to answer this questions we need to check all combination that makes the output 1 the and-gates then use or-gates to link all the possible true outputs. by doing this we will get:
   $u = (x.y.\overline{z}) + (\overline{x}.y.z) + (\overline{x}.\overline{y}.\overline{z})$.

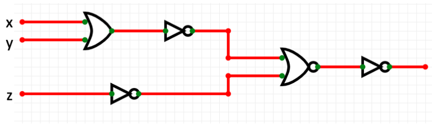
2. 

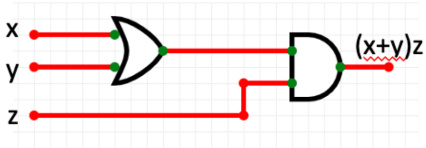
Question 13.

Given the following boolean expression $\overline{\overline{(x+y)} + \overline{z}}$.

1. Construct its corresponding circuit.
2. Use DeMorgan's laws to find a simpler form for this expression
3. Construct the circuit the simplified expression.

Solution:

1. 

2. $(x + y)z$
3. 

Question 14.

Simplifying the following boolean expression using Karnaugh Map

$$\overline{a}\overline{b}\overline{c} + \overline{a}b\overline{c} + ab\overline{x}$$

Solution:

c \ ab	00	01	11	10
0	1	1	1	
1		1		

We can now group the 1's as follows:

c \ ab	00	01	11	10
0	1	1	1	
1		1		

Now for each grouping, we are not going to consider any variable that is changing.
For the first grouping in, yellow, consists of two 1's which correspond to a = 0, b =0, c = 0 and a= 0, b = 1, c = 0. In another way, the value in this first grouping is independent from the value of b. so when $a = 0$ and $c = 0$ the output is 1. hence, this can be reduced to just $\overline{ac}$
For the green grouping which consists of two 1's corresponding to a = 0, b =1, c = 0 and a= 0, b = 1, c = 1. In another way, the value in this first grouping is independent from the value of c. so when $a = 0$ and $b = 1$ the output is 1. hence this can be reduced to $\overline{a}b$
Finally, for he second grouping in orange consists of two 1's which correspond to a = 0, b =1, c = 0 and a= 1, b = 1, c = 0. In another way, the value in this first grouping is independent from the value of a. so when $b = 1$ and $c = 0$ the output is 1. Hence this can be reduced to just $b\overline{c}$.
The final reduced expression is then $\overline{ac} + \overline{a}b + b\overline{c}$

Question 15.
Given the following boolean function

$$f(a,b,c,d) = \overline{a}\overline{b}cd + \overline{a}bcd + abcd + a\overline{b}cd + ab\overline{c}\overline{d} + ab\overline{c}d + abc\overline{c}$$

1. Fill in the missing value in the following Karnaugh map of $f(a,b,c)$:

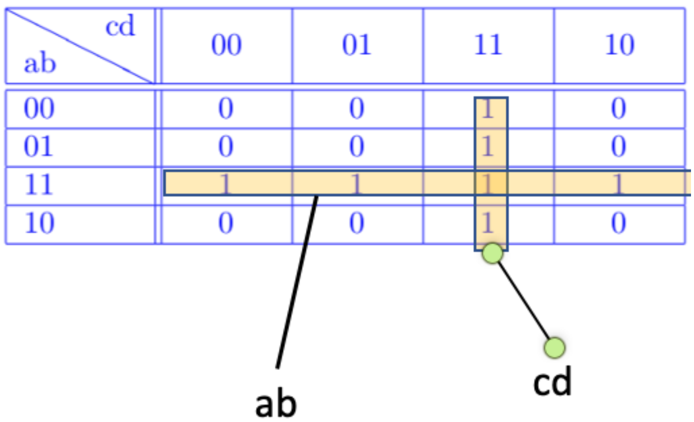
cd \ ab	00	01	11	10
00				
01				
11				
10				

2. Use K-map in (1) to find the minimum sum of products of $f(a,b,c)$.

Solution:

1.

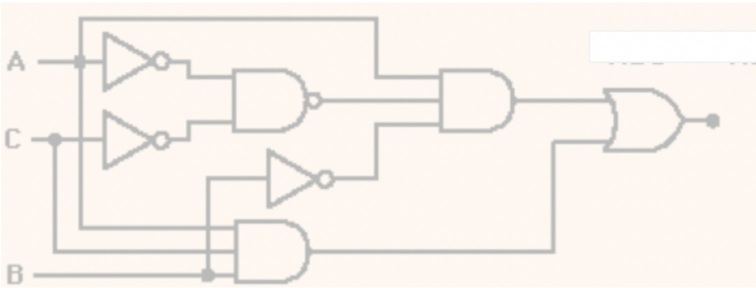
cd \ ab	00	01	11	10
00	0	0	1	0
01	0	0	1	0
11	1	1	1	1
10	0	0	1	0



2.

Hence, $f(a,b,c) = ab + cd$.

Question 16.
Given the following circuit:



1. Find the output of this circuit.
2. Use the laws of algebra to give a simpler expression for this output.
3. Use the result in 2 to draw a simpler circuit equivalent circuit.

Solution:

1. $ABC + A\overline{B}(\overline{\overline{AC}})$

2. $$\begin{array}{lll} ABC + A\overline{B}(\overline{\overline{AC}}) & = ABC + A\overline{B}(\overline{\overline{A}} + \overline{\overline{C}}) & \text{De Morgan's law} \\ & = ABC + A\overline{B}\overline{\overline{A}} + A\overline{B}\overline{\overline{C}} & \text{Distributive law} \\ & = ABC + A\overline{B}A + A\overline{B}C & \text{Complement law} \\ & = ACB + AA\overline{B} + AC\overline{B} & \text{Associative law} \\ & = ACB + A\overline{B} + AC\overline{B} & \text{Idempotent law} \\ & = AC(B + \overline{B}) + A\overline{B} & \text{Idempotent law} \\ & = AC.(1) + A\overline{B} & \text{Complement law} \\ & = AC + A\overline{B} & \text{Identity law} \\ & = A(C + \overline{B}) & \text{Identity law} \end{array}$$

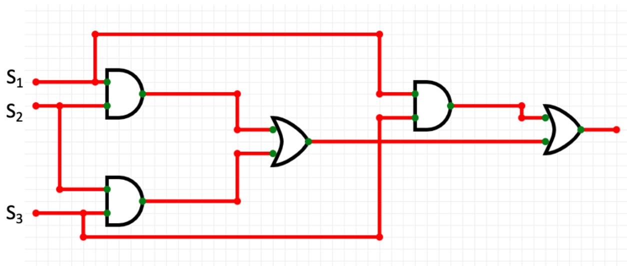
3. $ABC + A\overline{B}(\overline{\overline{AC}}) = A(C + \overline{B})$, hence the simplified circuit is :

Question 17.

A set of three sensors in a factory detects whether the pollution level it is outputting from an incinerator exceeds the safety limit. In which case the incinerator is shut down. An alarm A goes off if at least two the three sensors s_1, s_2 and s_3 detect a pollution level above the limit. Draw a logic circuit for the system showing the inputs s_1, s_2 and s_3 and the output A.

Solution:

The output of this circuit should be $A = s_1.s_2 + s_1.s_3 + s_2.s_3$. The logical circuit for the output A is



End of questions