Trees problem sheet – Solutions

Question 1.

Define the following terms:

1. Tree
2. Spanning Tree
3. Degree of a vertex in a tree
4. Leaf node
5. Minimum Spanning Tree

Solution:

1. A **Tree** is a connected acyclic graph, meaning it has no cycles and is fully connected.
2. A **Spanning Tree** of a graph is a subgraph that includes all the vertices of the original graph and is a tree.
3. The **Degree of a vertex** in a tree is the number of edges connected to that vertex.
4. A **Leaf node** is a vertex in a tree that has a degree of 1 (except in the case of the root in a rooted tree).
5. A **Minimum Spanning Tree (MST)** is a spanning tree of a weighted graph that has the smallest possible sum of edge weights.

Question 2.

Let G be an unweighted undirected graph with five vertices labelled A, B, C, D and E. This graph is defined by it's adjacency matrix A_G given below. List at least four spanning trees of G.

$$\mathbf{A}_G = \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array} \begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

Solution:
Possible spanning trees include:

- Spanning Tree 1: $A - B - C - D - E$
- Spanning Tree 2: $A - C - D - E - B$
- Spanning Tree 3: $B - A - C - D - E$

- Spanning Tree 4: $B - D - C - A - E$
- Spanning Tree 5: $C - A - B - D - E$
- Etc.

Question 3.
Use Kruskal's algorithm to find the Minimum Spanning Tree (MST) of the following undirected weighted graph G with five vertices labelled A, B, C, D and E, represented by its Adjacency matrix, A_G given below. List the edges included in the MST and the total weight.

$$\mathbf{A}_G = \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array} \begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} 0 & 1 & 4 & 3 & 0 \\ 1 & 0 & 4 & 2 & 0 \\ 4 & 4 & 0 & 5 & 6 \\ 3 & 2 & 5 & 0 & 7 \\ 0 & 0 & 6 & 7 & 0 \end{bmatrix} \end{array}$$

Solution:

- List of edges sorted by weight: $(A, B, 1)$, $(B, D, 2)$, $(A, D, 3)$, $(A, C, 4)$, $(B, C, 4)$, $(C, D, 5)$, $(C, E, 6)$, $(D, E, 7)$.
- Start with edge $(A, B, 1)$.
- Add edge $(B, D, 2)$ (no cycle).
- Add edge $(A, D, 3)$ (no cycle).
- Skip edges $(A, C, 4)$ and $(B, C, 4)$ (would form a cycle).
- Add edge $(C, E, 6)$ (no cycle).
- Final MST edges: $(A, B, 1)$, $(B, D, 2)$, $(A, D, 3)$, $(C, E, 6)$.
- Total weight: $1 + 2 + 3 + 6 = 12$.

Question 4.
Given an undirected weighted graph G with five vertices labelled A, B, C, C and E, represented by its corresponding adjacency matrix, A_G, given below. Find the Minimum Spanning Tree (MST) using Prim's algorithm. Clearly show each step, including the selection of edges and the updating of the MST.

$$\mathbf{A}_G = \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array} \begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} 0 & 2 & 0 & 6 & 0 \\ 2 & 0 & 3 & 8 & 5 \\ 0 & 3 & 0 & 0 & 7 \\ 6 & 8 & 0 & 0 & 9 \\ 0 & 5 & 7 & 9 & 0 \end{bmatrix} \end{array}$$

Solution:

- Start with vertex A. Add edge $(A, B, 2)$ to the MST.
- Add edge $(B, C, 3)$ to the MST (smallest edge connected to MST).
- Add edge $(B, E, 5)$ to the MST (next smallest).
- Add edge $(A, D, 6)$ to the MST.
- Final MST edges: $(A, B, 2)$, $(B, C, 3)$, $(B, E, 5)$, $(A, D, 6)$.
- Total weight: $2 + 3 + 5 + 6 = 16$.

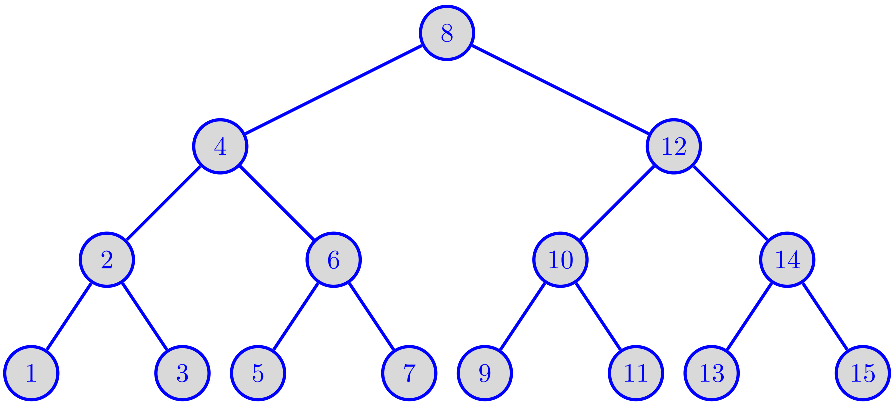
Question 5.
Define the following:

1. A binary tree.
2. A balanced binary tree.
3. A full binary tree.
4. A binary search tree.

Solution:

1. A binary tree is a type of tree data structure in which each node has at most two children, referred to as the left child and the right child.
2. A balanced binary tree is a binary tree where the height difference between the left and right subtrees of every node is at most 1.
3. A full binary tree (sometimes referred to as a proper binary tree) is a binary tree in which every node other than the leaves has exactly two children.
4. A binary search tree (BST) is a specific type of binary tree that maintains a particular property that makes searching, insertion, and deletion operations efficient. this property is all the values in the left subtree are less than the value of the node and all the values in the right subtree are greater than the value of the node.

Question 6.
Draw a binary search tree the store the records 1 to 15. What is height of this binary search tree?
Solution:



The height of a binary tree is defined as the length of the path from the root node to the deepest leaf node. The height of this tree is 3.