Discrete Mathematics

Tutorial sheet | Introduction to Proofs

Question 1.
Prove that the sum of any two even integers is even. In an other way show that:

$\forall n, m \in \mathbb{Z}$, if n and m are even numbers then $n+m$ is also an even number.

Solution:
Let $n, m \in \mathbb{Z}$ and assume that n and m are even, we need to show that $n+m$ is also even.
n and m are two even integers, it follows by definition of even numbers that there exists two integers i and j such that $n = 2i$ and $m = 2j$.
Thus $n+m = 2i+2k = 2(i+j)$. Hence, there exists an integer $k = i+j$ such $n+m = 2k$. it follows by definition of even numbers that that $n+m$

Question 2.
Use direct proof to show that: $\forall n, m \in \mathbb{Z}$, if n is an even number and m is an odd number then $3n + 2m$ is also an even number.
Solution:
Let $n, m \in \mathbb{Z}$ and assume that n is even and m is odd, we need to show that $3n + 2m$ is also even.
Assume that n is even and m is odd, this implies that there exists two integers $i, j \in \mathbb{Z}$ such that: $n = 2i$ and $m = 2j + 1$.
Thus $3n + 2m = 3 \times 2i + 2 \times (2j + 1) = 6i + 4j + 2 = 2(3i + 2j + 1)$. Hence, there exists an integer $k = 3i + 2j + 1$ such $3n + 2m = 2k$. it follows, by definition of even numbers, that that $3n + 2m$ is an even number.

Question 3.
Prove that the sum of any two odd integers is even . In an other way show that:

$\forall n, m \in \mathbb{Z}$, if n and m are odd numbers then $n + m$ is an even number.

Solution:
Let $n, m \in \mathbb{Z}$ and assume that n and m are odd, we need to show that $n+m$ is also even.
n and m are two odd integers, it follows by definition of odd numbers that there exists two integers i and j such that $n = 2i + 1$ and $m = 2j + 1$.
Thus $n + m = 2i + 2j + 2 = 2(i + j + 1)$. Hence, there exists an integer $k = i + j + 1$ such $n + m = 2k$. it follows ,by definition of even numbers, that that $n + m$.

Question 4.

Show that for any odd number integer n, n^2 is also odd. in another way show that:

$\forall n \in \mathbb{Z}$, if n is odd then n^2 is also odd.

Solution:
Let $n \in \mathbb{Z}$ and assume that n is an odd number. we need to show that n^2 is also odd, which means we need to show that there an integer k such that $n^2 = 2k + 1$.
By definition of odd numbers, n is odd means that there exists an integer i such that $n = 2i + 1$s. it follows that $n^2 = (2i + 1)^2 = 4i^2 + 4i + 1 = 2(2i^2 + 2i) + 1$. $2i^2 + 2i$ is an integer as it is the sum of products of integers, Therefore, there exists k= $2i^2 + 2i$ such that $n^2 = 2k + 1$. it follows by definition of odd numbers that n^2 is an odd number.

Question 5.

Show that: $\forall x \in \mathbb{R}\ \forall m \in \mathbb{Z}, \lfloor x + m \rfloor = \lfloor x \rfloor + m$.

Solution:
Proof:. Let x be any real number and n be any integer. we must show that $\lfloor x + m \rfloor = \lfloor x \rfloor + m$.
Let n be an integer with $\lfloor x \rfloor = n$. By definition of the the floor function, it follows that:

$$n \leq x < n + 1$$

by adding the value m to all parts of this inequality we obtain:

$$(n + m) \leq (x + m) < (n + m) + 1$$

Hence, $\lfloor x + m \rfloor = n + m$
Finally, by substituting n by $\lfloor x \rfloor$, we get $\lfloor x + m \rfloor = \lfloor x \rfloor + m$. This ends the proof.

Question 6.

Use proof by contraposition show that for any integer n, if n^2 is even then n is even

Solution:
Proof (by contraposition):
The contraposition is for every integer n if n is odd then n^2 is also odd (not even). Suppose n is any odd integer, we need to show that n^2 is odd.
By definition of odd numbers, n is odd means that there exists an integer i such that $n = 2i + 1$s. it follows that $n^2 = (2i + 1)^2 = 4i^2 + 4i + 1 = 2(2i^2 + 2i) + 1$. $2i^2 + 2i$ is an integer as it is the sum of products of integers, Therefore, there exists k= $2i^2 + 2i$ such that $n^2 = 2k + 1$. Hence n^2 is an odd number. Therefore, for all integer n , if n^2 is also even. n is even

Question 7.

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Use proof by contraposition show that for any integer n, if $5 \nmid n^2$ then $5 \nmid n$

Solution:
Proof (by contraposition):
The contrapositive is: for every integer n, if $5 \mid n$ then $5 \mid n^2$. Suppose n is any integer such that $5 \mid n$. We must show that $5 \mid n^2$. by definition of divisibility, $n = 5k$. By substitution, n=$(5k)$=$5(5k^2)$. hence, there exists an integer $i = 5k^2$ such that $n^2 = 5i$. Hence, $5 \mid n^2$. Therefore for any integer n, if $5 \nmid n^2$ then $5 \nmid n$

Question 8.

Use proof by contradiction to show that for any integer n, if n^2 is even then n is even

Solution:

Proof (by contradiction):

Assume there exists an integer n such that n^2 is even and n is odd. n is odd hence n can be written as $n = 2k + 1$ for some integer k. By substitution it follows that $n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$. $2k^2 + 2k$ is an integer because the products and sums of integers are integers. So there is an integer $i = 2k^2 + 2$ with $n^2 = 2i + 1$, and thus by definition n^2 is odd. This is a contradiction of the hypothesis as n^2 is even. This ends the proof.

Question 9.

Use proof by contradiction to show that for any integer n, $3n + 2$ is not divisible by 3.

Solution:

Proof (by contradiction):

Assume there is exists an integer m such that $3m + 2$ is divisible by 3. Hence, there exists an integer k such that $3m + 2 = 3k$ and thus $m = k - \frac{2}{3}$. Therefore m is not integer and this contradicts our initial hypothesis which says that m is an integers. Hence, $\forall n\mathbb{Z}$, $3n + 2$ is not divisible by 3. This ends the proof.

Question 10.

Use proof by contradiction to show that for any integer n, $7n + 4$ is not divisible by 7.

Solution:

Proof (by contradiction):

Assume there is exists an integer m such that $7m + 4$ is divisible by 7 Hence, there exists an integer k such that $7m + 4 = 7k$ and thus $m = k - \frac{4}{7}$. Therefore m is not integer and this contradicts our initial hypothesis which says that m is an integers. Hence, $\forall n\mathbb{Z}$, $7n + 4$ is not divisible by 7. This ends the proof.

Question 11.

Write the following series in $\sum$ notation:

1. $1 + 3 + 5 \cdots (2n - 1)$
2. $1 + 2 + 4 + 8 + 16 + \cdots + 1024$

Solution:

1. $1 + 3 + 5 \cdots (2n - 1) = \sum_{k=1}^{n} (2k - 1)$
2. $1 + 4 + 8 + 16 \cdots 1024 = \sum_{k=0}^{10} 2^k$

Question 12.
Given the following formulae

$$\sum_{k=1}^{n} 1 = n \qquad \sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

Evaluate the following

1. $\sum\limits_{k=1}^{10}(4k-2)$
2. $\sum\limits_{k=41}^{100} k$
3. $3+6+9+12+\cdots+300$

Solution:

1. $\sum\limits_{k=1}^{10}(4k-2) = \sum\limits_{k=1}^{10} 4k - \sum\limits_{k=1}^{10} -2 = 4\sum\limits_{k=1}^{10} k - 2\sum\limits_{k=1}^{10} 1 = 4\frac{10*11}{2} - 2*10 = 200$
2. $\sum\limits_{k=41}^{100} k = \sum\limits_{k=1}^{100} k - \sum\limits_{k=1}^{40} k = \frac{100*101}{2} - \frac{40*41}{2} = 5050 - 820 = 4230$
3. $3+6+9+12+\cdots+300 = \sum\limits_{k=1}^{100} 3k = 3\sum\limits_{k=1}^{100} k = 3\frac{100*101}{2} = 35050 = 15150$

Question 13.
Given the following arithmetic sequence:

$$a_n : 2, 5, 8, 11, 14, \cdots$$

1. Find the common difference d
2. Calculate the next term;
3. Write down the n^{th} term in terms of n.
4. Let $S_n = \sum\limits_{k=1}^{n} a_n$ be the sum of the first n^{th} terms of this sequence. Write down S_n in terms of n and a_1.
5. Workout the value of S_{100}

Solution:

1. 3

2. 17
3. $a_n = a_1 + (n-1)d = 2 + (n-1)3 = 3n - 1.$
4. Let $S_n = \sum_{k=1}^{n} a_n = \frac{n(2a_1+(n-1)d)}{2} = \frac{n(2*2+(n-1)3)}{2} = \frac{3n^2+n}{2}$
5. Workout the value of $S_{100} = \frac{3*100^2+100}{2} = \frac{30000+100}{2} = 15050$

Question 14.

Let the sequence u_n be defined by the recurrence relation

$$u_{n+1} = u_n + 2n, \quad \text{for } n = 1, 2, 3, \ldots \text{ and let } u_1 = 1.$$

Use mathematical induction to show that the nth term, where $n \geq 0$, is given by

$$u_n = n^2 - n + 1.$$

Solution:

- **Base step (Base case)** for n=1, $u_1 = 1 = 1^2 - 1 + 1$, true
- **Induction hypothesis:**
  Assume for $n = k, u_k = k^2 - k + 1$.
- **Induction step:**
  Show that for n=k+1, $u_{k+1} = (k+1)^2 - (k+1) + 1 = k^2 + k + 1$.

  $$\begin{aligned} u_{k+1} &= u_k + 2k && \text{(By definition)} \\ &= k^2 - k + 1 + 2k && \text{(Induction hypothesis)} \\ &= k^2 + k + 1 && \text{True} \end{aligned}$$

  Therefore, $u_n = n^2 - n + 1, \quad \forall\, n \geq 1$.

Question 15.

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Let S_n be a series defined as follows:

$$S_n = 1^2 + 2^2 + 3^3 + \cdots + n^2 = \sum_{k=1}^{n} k^2$$

Use mathematical induction to prove that each positive integer n, $S_n = \frac{n(n+1)(2n+1)}{6}$.
Solution:

- **Base step (Base case)** for n=1, $S_1 = 1 = 1^2 = \frac{1(1+1)(2\times1+1)}{6} = \frac{1\times(2)\times(3)}{6} = \frac{6}{6} = 1$, This shows that he formula is true for n = 1.

- **Induction hypothesis:**

  We now assume for $n = k, S_k = \frac{k(k+1)(2k+1)}{6}$.

- **Inductive step:**

  We now need to show that for n=k+1, $S_{k+1} = \frac{(k+1)(k+1+1)(2(k+1)+1)}{6} = \frac{(k+1)(k+2)(2k+3)}{6} = \frac{(k+1)(2k^+3k+6)}{6}$.

  $$\begin{aligned} S_{k+1} &= 1^2 + 2^2 + 3^3 + \cdots + k^2 + (k+1)^2 && \text{by definition} \\ &= S_k + (k+1)^2 \\ &= \tfrac{k(k+1)(2k+1)}{6} + (k+1)^2 && \text{induction hypothesis} \\ &= \tfrac{k(k+1)(2k+1)}{6} + \tfrac{6(k+1)^2}{6} \\ &= \tfrac{k(k+1)(2k+1)+6(k+1)^2}{6} \\ &= \tfrac{(k+1)[k(2k+1)+6(k+1)]}{6} \\ &= \tfrac{(k+1)[2k^+7k+6]}{6} && \text{True} \end{aligned}$$

  Therefore,

  $$S_n = \frac{n(n+1)(2n+1)}{6}, \ \forall n \geq 1.$$

Question 16.

Let $S_n = \sum_{i=1}^{i=n}(2i-1) = n^2$ for all $n \in \mathbb{Z}^+$.

1. Find S_1 and S_2.
2. Prove by induction that $S_n = n^2$ for all $n \in \mathbb{Z}^+$.

Solution:

1. $S_1 = 2 * 1 - 1 = 1$ [1 mark] and $S_2 = 2 * 1 - 1 + 2 * 2 - 1 = 1 + 3 = 4$ [1 mark].
2. Prove by induction that $S_n = n^2$ for all $n \in \mathbb{Z}^+$.

   base case:. $S_1 = 1 = 1^2$, this true. [1 mark]
   Induction hypothesis: Assume that for $n = k, S_k = k^2$.
   Induction step: we need to show that $S_{k+1} = (k+1)^2$.

   $$\begin{aligned} S_{k+1} &= S_k + 2(k+1) - 1 && \text{by definition} \\ &= k^2 + 2k + 1 && \text{induction hypothesis} \\ &= (k+1)^2. && [1 mark] \end{aligned}$$

   Hence, $S_n = n^2$ for all $n \in \mathbb{Z}^+$.

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Question 17.

Use mathematical induction to show that for all integer $n \geq 3$, $2n + 1 < 2^n$
Solution:

- **Base step (Base case)** We need show that the formula is true for n=3: $2 \times 3 + 1 = 3 < 2^3 = 8$, thus the formula is true for $n = 1$

- **Induction hypothesis:**

  We now assume that for $n = k, 2k + 1 < 2^k$ is true.

- **Inductive step:**

  We now need to shat that this formula is also true for $n = k + 1$. Hence, we need to show that $2(k + 1) + 1 < 2^{k+1}$

  $$\begin{aligned} 2(k+1)+1 &= 2k+1+2 && \\ &< 2^k + 2 && \text{induction hypothesis} \\ &< 2^k + 2^k && \text{as } 2 < 2^k \text{ for } k \geq 3 \\ &< 2 \times 2^k && \\ &< 2^{k+1} && .\text{True} \end{aligned}$$

  Therefore, for all integer $n \geq 3$, $2n + 1 < 2^n$

Question 18.

Given the following sequence defined by

$$u_{n+2} = 4u_{n+1} - 3u_n$$

and initial terms $u_1 = 4$ and $u_2 = 10$.

1. Calculate u_3

2. Use strong mathematical induction to prove that

$$u_n = 3^n + 1, \quad \forall\, n \geq 1.$$

<u>Solution:</u>

1. $u_3 = 4u_2 - 3u_1 = 40 - 12 = 28$, and

2. - **Basis step (Base case)**

     We need to show that the formula we want to prove is true for n = 1 and n= 2. $u_1 = 4 = 3^1 + 1$ true, and $u_2 = 10 = 3^2 + 1$. thus, it's true for $n = 1$ and $n = 2$.

   - **induction hypothesis:**

     We assume that for $u_n = 3^n + 1$. any any $n \leq k$

   - **Induction step:**

     Show that for n=k+1, $u_{k+1} = 3^{k+1} + 1$.

     $$\begin{aligned} u_{k+1} &= 4u_k - 3u_{k-1} && \text{(by definition)} \\ &= 4 \times (3^k + 1) - 3 \times (3^{k-1} + 1) && \text{(induction hypothesis).} \\ & 3 \times 3^k + 1 = 3^{k+1} + 1 && \text{true.} \end{aligned}$$

     therefore,

     $$u_n = 3^n + 1, \quad \forall\, n \geq 1.$$

Question 19.

Use strong mathematical induction to prove that if n is an integer greater than 1, then it is either a prime or can be written as the product of primes.
Solution:

- **Basis step (Base case):** : for $n = 2$, Since 2 is a prime number, he property holds for $n = 2$
- **induction hypothesis:** We assume that for $n = 2, 3, \cdots, k$ n is either prime or product of primes.
- **Induction step:**. we want to prove the same thing about k + 1, which means we need to show that $k+1$ is either a prime or can be written as the product of primes.

  we have two cases: k + 1 is either (i) prime or (ii) composite.

  (i) if $k + 1$ then the property holds.

  (ii) if (k+1) is composite then $k + 1$ can be written as pq wehere $2 \leq p, q \leq k$,

  by the induction hypothesis p, q are either primies or product of primes

  Thus, $k + 1$ can also be written as product of primes.

  Therefore, for all integer $n > 1$, n is a prime or can be written as a product or primes.