Discrete Mathematics

Tutorial sheet | Set Theory

Question 1.

Describe the following sets by the listing method:

1. $\{n : n \in \mathbb{Z} \text{ and } 5 \leq n < 8\}$
2. $\{3n : n \in \mathbb{Z} \text{ and } 5 \leq n < 8\}$
3. $\{2^n : n \in \mathbb{Z} \text{ and } 5 \leq n < 8\}$

Solution:

1. $\{n : n \in \mathbb{Z} \text{ and } 5 \leq n < 8\} = \{5, 6, 7\}$
2. $\{3n : n \in \mathbb{Z} \text{ and } 5 \leq n < 8\} = \{15, 18, 21\}$
3. $\{2^n : n \in \mathbb{Z} \text{ and } 5 \leq n < 8\} = \{2^5, 2^6, 2^7\} = \{32, 64, 128\}$

Question 2.

Let $\sum = \{x, y\}$ be an alphabet. List the element of the set L_1 and L_2

1. L_1 is the language consisting of all strings over $\sum$ of length **less or equal to 4** that are **palindromes**.
2. L_2 is the language consisting of all strings over $\sum$ of length **less or equal to 3** in which all the x's appear to the left of all the y's.

Solution:
ε represents the empty string.

1. $L_1 = \{\varepsilon, x, y, xx, yy, xxx, yyy, xyx, yxy, xxxx, yyyy, xyyx, yxxy\}$
2. $L_2 = \{\varepsilon, x, y, xx, xy, yy, xxx, xxy, xyy, yyy\}$

Question 3.

Describe the following sets by giving a suitable universal set and rules of inclusion:

1. $\{4, 8, 12, 16, 20\}$
2. $\{0, 2, -2, 4, -4, \cdots\}$

3. $\{2, 4, 8, 16, 32\}$

4. $\{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}\}$

Solution:

1. $\{4, 8, 12, 16, 20\} = \{4n : n \in \mathbb{Z} \text{ and } 1 \leq n < 6\}$
2. $\{0, 2, -2, 4, -4, \cdots\} = \{2n : n \in \mathbb{Z}\}$
3. $\{2, 4, 8, 16, 32\} = \{2^n : n \in \mathbb{Z} \text{ and } 1 \leq n < 6\}$
4. $\{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}\} = \{2^{-n} : n \in \mathbb{Z} \text{ and } 0 \leq n < 6\}$

Question 4.

Let $X = \{f, g, h, i, k\}$ and $Y = \{d, g, h, k\}$ be subsets of a universal set $U = \{d, e, f, g, h, i, j, k, l\}$. Find each of the following:

1. $\overline{X}$

2. $X \cap Y$

3. $X \cup \overline{Y}$

4. $X - Y$

5. $X \oplus Y$

6. $\overline{(X \cap Y)}$

Solution:

1. $\overline{X} = \{d, e, j, l\}$
2. $X \cap Y = \{g, h, k\}$
3. $X \cup \overline{Y} == \{f, g, h, i, j, k, e, l\}$
4. $X - Y = \{f, i\}$
5. $X \oplus Y = \{d, f, i\}$
6. $\overline{(X \cap Y)} = \{d, e, f, i, j, l\}$

Question 5.
Let $A = \{2, \frac{1}{2}, \sqrt{2}\}$ and $B = \{x \in \mathbb{Q} : x \notin \mathbb{Z}\}$ be two sets. List the following sets:

$$A \cap B, \quad A - B, \quad A \cap \mathbb{R}, \quad A \cap \mathbb{Z},$$

Solution:

$$A \cap B = \{\frac{1}{2}\}, \quad A - B = \{2, \sqrt{2}\}, \quad A \cap \mathbb{R} = \{2, \frac{1}{2}, \sqrt{2}\}, \quad \text{and} \quad A \cap \mathbb{Z} = \{2\}$$

Question 6.

Let X and Y be two sets with $X = \{f, g, h, j, k\}$ and $Y = \{f, g\}$.

1. What is cardinality of X?
2. What is the total number of subsets of X?
3. Put the correct sign $\in \quad \notin \quad \subset \quad \subseteq$ between the following pairs:

$$f \quad X, \qquad Y \quad X, \qquad X \quad X, \qquad \emptyset \quad X, \qquad \text{and} \qquad h \quad Y$$

Solution:

1. The cardinality of $X = 5$?
2. The total number of subsets of $X = 2^5 = 32$?
3. $$f \in X, \quad Y \subset X, \quad X \subseteq X, \quad \emptyset \subset X, \quad \text{and} \quad h \notin Y$$

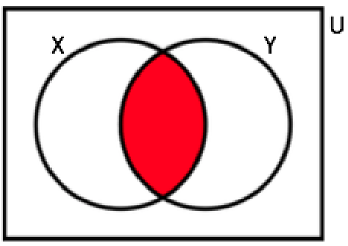
Question 7.

Let X and Y be two sets of the universal set U.

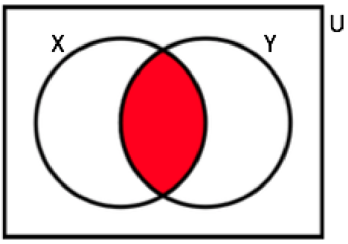
1. Use Venn diagram to show to show that $\overline{X \cap Y} = \overline{X} \cup \overline{Y}$.
2. Use membership tables to prove that $\overline{X \cap Y} = \overline{X} \cup \overline{Y}$.

Solution:

1. Venn diagram for $\overline{X \cap Y}$. the red area represents $X \cap Y$ where as the white area represents $\overline{X \cap Y}$.



The white area on the Venn diagram below represents $\overline{X} \cup \overline{Y}$.



Both areas representing $\overline{X \cap Y}$ and that representing $\overline{X} \cup \overline{Y}$ are the same, hence, $\overline{X \cap Y} = \overline{X} \cup \overline{Y}$

2.

X	Y	$\overline{X}$	$\overline{Y}$	$X \cap Y$	$\overline{X \cap Y}$	$\overline{X} \cup \overline{Y}$
0	0	1	1	0	1	1
0	1	1	0	0	1	1
1	0	0	1	0	1	1
1	1	0	0	1	0	0

The last two columns representing $\overline{X \cap Y}$ and $\overline{X} \cup \overline{Y}$ are the same, hence, $\overline{X \cap Y} = \overline{X} \cup \overline{Y}$

Question 8.

Let A and B and C be subsets of a universal set $\mathcal{U}$.

1. Draw a three binary digit labelled Venn diagram depicting A, B, C in such a way that they divide $\mathcal{U}$ into 8 disjoint regions.

2. The subset $X \subseteq \mathcal{U}$ is defined by the following membership table:

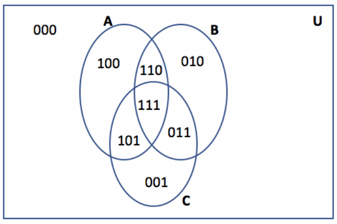
A	B	C	X
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	0
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	0

Identify the region X on your diagram. Describe the region you have identified in set notation as simply as you can.

3. let Y be the set represented by the region 000, 011, 101, 110, and 111. Describe the set Y using the set notation,

Solution:

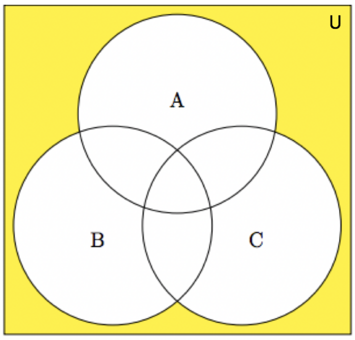
1. Venn diagram: the digit represents the set A, the second represents B and the third digit represnt C. for example the region (101) contains elements that are in A(1^{st} digit=1), not in B (2^{nd} digit =0) and in C (3^{rd} digit =1).

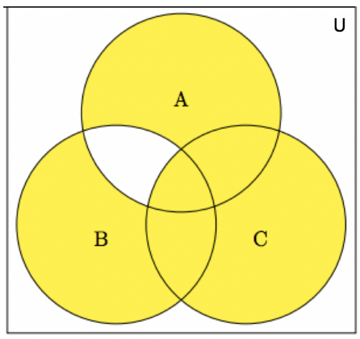


2. X is represented the region 001, 100, 101 and 110. $X = (A \cup C) - (B \cap C)$
3. $Y = \overline{A \oplus B \oplus C}$

Question 9.

Given three sets A, B and C, subsets of the universal set U. For each of each of the following Venn diagram write, in terms of A, B and C, the set representing the area coloured in yellow:

1. 

2. 

Solution:

1. $\bar{(}A) \cap \overline{B} \cap \overline{C}$

2. $(A \oplus B) \cup C$

Question 10.

Let $A = \{t, u, v, w\}$ and let S_1 be the set of all subsets of A that do not contain w and S_2 be the set of all subsets of A that contain w.

1. Find S_1 and S_2.
2. Are S_1 and S_2 disjoint?
3. Find $S_1 \cup S_2$.
4. What is the relation between $S_1 \cup S_2$ and $\mathcal{P}(A)$?

Solution:

1. $S_1 = \{\emptyset, \{t\}, \{u\}, \{v\}, \{t, u\}, \{t, v\}, \{u, v\}, \{t, u, v\}\}$
   $S_2 = \{\{w\}, \{t, w\}, \{u, w\}, \{v, w\}, \{t, u, w\}, \{t, v, w\}, \{u, v, w\}, \{t, u, v, w\}\}$.
2. $S_1 \cap S_2 = \emptyset$ hence, S_1 and S_2 are disjoint.
3. $S_1 \cup S_2 = \{\emptyset, \{t\}, \{u\}, \{v\}, \{t, u\}, \{t, v\}, \{u, v\}, \{t, u, v\},$
   $\{w\}, \{t, w\}, \{u, w\}, \{v, w\}, \{t, u, w\}, \{t, v, w\}, \{u, v, w\}, \{t, u, v, w\}\}$.
4. $S_1 \cup S_2 = \mathcal{P}(A)$.

Question 11.

Let $A = \{1, 2\}$ and let $B = \{2, 3\}$. Find each of the following:

1. $\mathcal{P}(A \cap B)$
2. $\mathcal{P}(A \cup B)$
3. $\mathcal{P}(A \times B)$

Solution:

1. $A \cap B = \{2\}$, hence, $\mathcal{P}(A \cap B) = \{\emptyset, \{2\}\}$
2. $A \cup B = \{1, 2, 3\}$, hence,$\mathcal{P}(A \cup B) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}\{1, 3\}, \{2, 3\}, \{1, 2, 3\}, \}$
3. $A \times B = \{(1, 2), (1, 3), (2, 2), (2, 3)\}$
   $\mathcal{P}(A \times B) = \{\emptyset, \{(1, 2)\}, \{(1, 3)\}, \{(2, 2)\}, \{(2, 3)\}, ,$
   $\{(1, 2), (1, 3)\}, \{(1, 2), (2, 2)\}, \{(1, 2), (2, 3)\}, \{(1, 3), (2, 2)\}, \{(1, 3), (2, 3)\}, \{(2, 2), (2, 3)\},$
   $\{(1, 2), (1, 3), (2, 2)\}, \{(1, 2), (1, 3), (2, 3)\}, \{(1, 2), (2, 2), (2, 3)\}, \{(1, 3), (2, 2), (2, 3)\},$
   $\{(1, 2), (1, 3), (2, 2), (2, 3)\}\}$

Question 12.

Given three sets A, B and C. Prove that the expression $\overline{\overline{(A \cup B) \cap C} \cup \overline{B}}$ is equivalent to $B \cap C$ by re-writing the expression using algebraic laws, state the name of each law used.

Solution:

$\overline{\overline{(A \cup B) \cap C} \cup \overline{B}}$	$= \overline{\overline{(A \cup B) \cap C}} \cap \overline{\overline{B}}$	De Morgan's law
	$= ((A \cup B) \cap C) \cap B$	double complement
	$= (A \cup B) \cap (C \cap B)$	associativity of the intersection
	$= (A \cup B) \cap (B \cap C)$	comutativity of the intersection
	$= ((A \cup B) \cap B) \cap C$	Associativity of the intersection
	$= B \cap C$	Absorption law

Question 13.

Given three sets A, B and C. Using set identities, prove that:

$$(A \cup B) - (C - A) = A \cup (B - C).$$

Solution:

$(A \cup B) - (C - A)$	$= (A \cup B) \cap \overline{(C - A)}$	set difference law
	$= (A \cup B) \cap \overline{(C \cap \overline{A})}$	set difference law
	$= (A \cup B) \cap \overline{(\overline{A} \cap C)}$	commutativity of the intersection
	$= (A \cup B) \cap (\overline{\overline{A}} \cup \overline{C})$	De Morgan's law
	$= (A \cup B) \cap (A \cup \overline{C})$	double complement law
	$= A \cup (B \cap \overline{C})$	distributivity of union over intersection
	$= A \cup (B - C)$	set difference law

Question 14.

Given two sets A and B. Simplify $\overline{(\overline{A} \cup \overline{B}) - A}$.

Solution:

$\overline{(\overline{A} \cup \overline{B}) - A}$	$= \overline{(\overline{A} \cup \overline{B}) \cap \overline{A}}$	set difference law
	$= \overline{(\overline{A} \cup \overline{B})} \cup \overline{\overline{A}}$	De Morgan's law
	$= (\overline{\overline{A}} \cap \overline{\overline{B}}) \cup \overline{\overline{A}}$	De Morgan's law
	$= (A \cap B) \cup A$	double complement law
	$= A \cup (A \cap B)$	commutativity of the union
	$= A$	absorption law

Question 15.

Show that for all sets A and B, $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$.

Solution:

Suppose A and B are any two sets, and suppose $X \in \mathcal{P}(A) \cup \mathcal{P}(B)$. We need to show that X is also an element of $\mathcal{P}(A \cup B)$.

$X \in \mathcal{P}(A) \cup \mathcal{P}(B)$ implies that $X'\mathcal{P}(A)$ or $X \in \mathcal{P}(B)$. Assume $X \in \mathcal{P}(A)$ then $X \subseteq A$, which implies that $X \subseteq (A \cup B)$ and thus $X \in \mathcal{P}(A \cup B)$. Similarly, if $X \in \mathcal{P}(B)$ then $X \subseteq B$, which implies that $X \subseteq (A \cup B)$ and thus $X \in \mathcal{P}(A \cup B)$. Therefore, $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$ for all sets A and B.

Question 16.

Let A, B and C be three sets. Prove that if $C \subseteq (B - A)$ then $A \cap C = \emptyset$

Solution:
Assume $C \subseteq (B - A)$ and let x be in C, hence x is also an element of $B - A$, hus x is an element of B but not an element of A. Therefore any element x of C is not an element of A. thus $A \cap C = \emptyset$.
This can be proved by contradiction (See later: topic 5) Let A, B and C be any three sets such that $C \subseteq (B - A)$ and suppose that $A \cap C \neq \emptyset$. Then, by definition of the intersection, there exists an element $x \in A$ and $x \in C$. Since, $x \in X$ and $C \subseteq (B - A)$ then $x \in B$ and $x \notin A$. so $x \in A$ and $x \notin A$ is a contradiction. Thus $A \cap C = \emptyset$

End of questions