Combinatorics problem sheet

Question 1

How many numbers are there between 99 and 1000, having at least one of their digits 7?

Model Answer:

Total number of 3 digit numbers having at least one of their digits as 7 = (Total numbers of three-digit numbers) – (Total number of 3 digit numbers in which 7 does not appear at all)
$= (9 \times 10 \times 10) - (8 \times 9 \times 9)$
$= 900 - 648$
$= 252$

Question 2

How many 5-digit telephone numbers can be constructed using the digits 0 to 9, if each number starts with 67 and no digit appears more than once?

Model Answer:

The first two digits of the telephone number are fixed as 67. Therefore, we only need to determine the remaining 3 digits.

- We have 8 digits left to choose from (0-9 excluding 6 and 7).
- For the third digit of the telephone number, we have 8 choices.
- For the fourth digit of the telephone number, we have 7 choices (since one digit has already been used).
- For the fifth digit of the telephone number, we have 6 choices left.

The total number of 5-digit telephone numbers is:

$$8 \times 7 \times 6 = 336$$

Thus, there are **336** possible 5-digit telephone numbers that can be constructed using the digits 0 to 9, starting with "67" and with no digit appearing more than once.

Question 3

How many different ways can the letters of the word COMBINATORICS be arranged?

Model Answer:

The word COMBINATORICS has 13 letters. The letters C, O, and I each repeat twice.
The formula to calculate the number of permutations of a multiset is:

$$\text{Number of permutations} = \frac{n!}{n_1! \times n_2! \times \cdots \times n_k!}$$

Where n is the total number of letters, and $n_1, n_2, \ldots$ are the frequencies of the repeating letters.

$$\text{Number of permutations} = \frac{13!}{2! \times 2! \times 2!} = \frac{13 \times 12 \times 11 \times \cdots \times 1}{2 \times 2 \times 2} = \frac{6227020800}{8} = 778377600$$

There are **778377600** different ways to arrange the letters of the word COMBINATORICS.

Question 4

In how many ways can you choose 4 candies from a jar of 6 different types of candies (let's say A, B, C, D, E, F) if each type of candy can be chosen multiple times?

Model Answer:

This is a problem of combinations with repetition. The formula to calculate combinations with repetition is:

$$\text{Number of combinations} = \binom{n+r-1}{r}$$

Where n is the number of types of candies, and r is the number of candies to choose.

$$\text{Number of combinations} = \binom{6+4-1}{4} = \binom{9}{4} = \frac{9 \times 8 \times 7 \times 6}{4 \times 3 \times 2 \times 1} = 126$$

So, there are **126** different ways to choose 4 candies from 6 different types with repetition allowed.

Question 5

A committee of 3 persons is to be constituted from a group of 6 men and 4 women. In how many ways can this be done? How many of these committees would consist of 1 man and 2 women?

Model Answer:

We need to determine the total number of ways to form a committee of 3 persons from a group of 6 men and 4 women.

The total number of people is $6 + 4 = 10$. We need to choose 3 people out of 10.

The number of ways to choose 3 people from 10 is given by the combination formula:

$$\binom{10}{3} = \frac{10 \times 9 \times 8}{3 \times 2 \times 1} = 120$$

So, there are **120** ways to form the committee.

Now, we need to determine how many of these committees consist of 1 man and 2 women.

- First, choose 1 man from the 6 men. The number of ways to do this is:

$$\binom{6}{1} = 6$$

- Then, choose 2 women from the 4 women. The number of ways to do this is:

$$\binom{4}{2} = \frac{4 \times 3}{2 \times 1} = 6$$

- The total number of committees with 1 man and 2 women is:

$$6 \times 6 = 36$$

So, there are **36** committees that consist of 1 man and 2 women.

Question 6

Determine the number of 5 card combinations out of a deck of 52 cards, if there is exactly one ace in each combination.

Model Answer:
Step 1: Choose the Ace. There are 4 aces in a deck of 52 cards. We need to choose 1 ace from these 4 aces. The number of ways to choose 1 ace from 4 is given by the combination formula:

$$\binom{4}{1} = 4$$

Step 2: Choose the Remaining 4 Cards. After choosing 1 ace, we need to choose 4 more cards from the remaining 48 cards (since we cannot choose another ace). The number of ways to choose 4 cards from the remaining 48 cards is:

$$\binom{48}{4} = \frac{48 \times 47 \times 46 \times 45}{4 \times 3 \times 2 \times 1} = 194580$$

Step 3: Multiply the Results. The total number of 5-card combinations with exactly one ace is:

$$4 \times 194580 = 778320$$

Thus, there are **778,320** 5-card combinations out of a deck of 52 cards where each combination contains exactly one ace.

Question 7

Expand the expression $(x + 2y)^4$ using the binomial theorem

Model Answer:
The binomial theorem states:

$$(a + b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

For $(x + 2y)^4$, $a = x$, $b = 2y$, and $n = 4$:

$$(x + 2y)^4 = \binom{4}{0} x^4 (2y)^0 + \binom{4}{1} x^3 (2y)^1 + \binom{4}{2} x^2 (2y)^2 + \binom{4}{3} x^1 (2y)^3 + \binom{4}{4} x^0 (2y)^4$$

Expanding each term:

$$= 1 \cdot x^4 + 4 \cdot x^3 \cdot 2y + 6 \cdot x^2 \cdot (2y)^2 + 4 \cdot x \cdot (2y)^3 + 1 \cdot (2y)^4$$

$$= x^4 + 8x^3 y + 24x^2 y^2 + 32xy^3 + 16y^4$$

So, $(x + 2y)^4$ expands to $x^4 + 8x^3y + 24x^2y^2 + 32xy^3 + 16y^4$.

Question 8

Suppose you have 10 pigeons and 9 pigeonholes. Prove that at least one pigeonhole must contain more than one pigeon.

Model Answer:

The **Pigeonhole Principle** states that if n items are put into m containers, with $n > m$, then at least one container must contain more than one item. Here, $n = 10$ pigeons and $m = 9$ pigeonholes. Since 10 pigeons are more than 9 pigeonholes, by the pigeonhole principle, **at least one pigeonhole must contain more than one pigeon**.

Question 9

In a group of 100 students, 60 study Math, 45 study Physics, and 25 study both. How many students study either Math or Physics?

Model Answer:

Use the **Inclusion-Exclusion Principle** to find the number of students studying either Math or Physics:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Where $|A|$ is the number of students studying Math, $|B|$ is the number of students studying Physics, and $|A \cap B|$ is the number of students studying both.

$$|A \cup B| = 60 + 45 - 25 = 80$$

So, **80** students study either Math or Physics.

Question 10

How many distinct seating arrangements can be made for 7 people around a circular table if 3 of them are identical twins and 4 of them are distinct?

Model Answer:

Since the arrangement is circular, fix one position and arrange the remaining $n-1$ people. The total number of people is 7, but 3 are identical twins, reducing the effective number of distinct positions to $7 - 1 = 6$. The formula for the distinct circular arrangements is:

$$\text{Number of arrangements} = \frac{6!}{3!}$$

Where 3! accounts for the identical twins:

$$\text{Number of arrangements} = \frac{720}{6} = 120$$

So, there are **120** distinct seating arrangements.