Chapter 2

The Language of Sets

The essence of mathematics lies in its freedom.

—GEORG CANTOR

The concept of a set is so fundamental that it unifies mathematics and its cognates. It has revolutionized mathematical thinking, enabling us to express ourselves in clear and concise terms.

The foundation of set theory was laid by the eminent German mathematician Georg Cantor during the latter part of the 19th century. "Today, Cantor's set theory has penetrated into almost every branch of mathematics," as the mathematical historian Howard Eves writes in *An Introduction to the History of Mathematics*.

In this chapter we present the language of sets. We introduce the concept of a set, the various ways of describing a set and of constructing new sets from known sets, a variety of applications, and a brief introduction to fuzzy sets.

The following are some of the problems we shall pursue in this chapter:

- Find the number of positive integers $\leq N$ and divisible by a, b, or c.
- How many subsets does a finite set with n elements have?
- How would you define the set of legally paired parentheses?
- How many sequences of legally paired parentheses can be formed using n pairs of left and right parentheses?

2.1 The Concept of a Set

This section introduces the concept of a set, various methods of defining sets, and relationships between sets.

Georg Cantor *(1845–1918) was born in St. Petersburg, Russia, where his father was a successful merchant and broker. Cantor showed great interest in mathematics from early childhood. In 1856, the family moved to Germany. Six years later, he entered the University of Zurich, but in the following year he moved to the University of Halle to study mathematics, physics, and philosophy. There he was greatly influenced by the eminent mathematician Karl Weierstrass (1815–1897). Although his father wanted him to become an engineer, Cantor relentlessly pursued his interest in mathematics and received his doctorate of philosophy at 22 from the University of Berlin for his work in number theory.*

In 1869, Cantor began his professional career as an unsalaried lecturer at the University of Halle. Five years later, he published his revolutionary work on set theory. Cantor developed an arithmetic of transfinite numbers analogous to that of finite numbers, thus creating another area of mathematical study. He proved that the set of real numbers is uncountable and he also established the existence of infinitely many different transfinite cardinal numbers by ingenious methods. He also made significant contributions to indeterminate equations and trigonometric series. Deeply religious, Cantor was also interested in art, music, and philosophy.

Being unhappy with his low salary at the University, Cantor tried to secure a better-paid position at the University of Berlin, but was sabotaged by Leopold Kronecker (1823–1891), an eminent mathematician at the University, who severely criticized Cantor's views on sets.

Relentless attacks by contemporary mathematicians intensified the manic depression he suffered from. Cantor died in a mental hospital in Halle in 1918.

Cantor was "one of the greatest intellects of the nineteenth century," according to Bertrand Russell. He "was an imaginative genius whose work has inspired [every aspect of] mathematical thought," Hazel Perfect of the University of Sheffield wrote in 1994.

Set

A **set** is a collection of well-defined objects,* called **elements** (or **members**) of the set.

There should be no ambiguity in determining whether or not a given object belongs to the set. For example, the vowels of the English alphabet form a (well-defined) set, whereas beautiful cities in the United States do not form a set since its membership would be debatable.

Sets are denoted by capital letters and their elements by lowercase letters. If an object x is an element of a set A, we write $x \in A$; otherwise $x \notin A$. For example, let A be the set of New England states. Then $Connecticut \in A$, whereas $Michigan \notin A$.

There are two methods of defining sets.

Listing Method

A set can sometimes be described by listing its members within braces. For instance, the set B of New England states can be described as $B = \{\text{ME}, \text{VT}, \text{NH}, \text{MA}, \text{CT}, \text{RI}\}$.

*To be precise, this is a circular definition; *set* is an undefined term, like *point* and *line* in geometry.

The order in which the elements are enumerated is immaterial. Thus B can also be written as {VT, RI, MA, CT, NH, ME}. If an element is repeated, it is not counted more than once. For example, $\{x, x, y, x, y, z\} = \{x, y, z\}$.

A set with a large number of elements that follow a definite pattern is often described using ellipses (...) by listing a few elements at the beginning. For example, the set of letters of the alphabet can be written as {a, b, c, ..., z} and the set of odd positive integers as $\{1, 3, 5, \ldots\}$.

Set-Builder Notation

Another way of describing a set is by using the **set-builder notation**. Its general form is $\{x \mid \mathrm{P}(x)\}$, where $\mathrm{P}(x)$ is a predicate indicating the property (or properties) the object x has. You may read $\{x \mid \mathrm{P}(x)\}$ as *the set consisting of all objects x such that x has the property P(x)*. Here the vertical bar "|" means *such that*. (Again, the meaning of the vertical bar should be clear from the context.)

EXAMPLE 2.1 Let B be the set of all months of the year with exactly 30 days. Then

$$\begin{aligned} B &= \{x \mid x \text{ is a month of the year with exactly 30 days }\} \\ &= \{\text{September, April, June, November}\} \end{aligned}$$ ■

Next we present another of Russell's paradoxes introduced in 1901, which is quite similar to the barber paradox.

Russell's Paradox

Let $S = \{X \mid X \notin X\}$; that is, S consists of all sets that do not belong to themselves as elements. Does $S \in S$? If $S \in S$, then, by definition, $S \notin S$; on the other hand, if $S \notin S$, then, again by definition, $S \in S$. Thus, in either case, we have a contradiction. This paradox shows, not every predicate defines a set; that is, there is no set of all sets.

Next we present several relationships between sets.

Subset

If every element of A is also an element of B, A is a **subset** of B, denoted by $A \subseteq B$. In symbols, $(A \subseteq B) \leftrightarrow (\forall x)(x \in A \to x \in B)$. If $A \subseteq B$, we also say that B **contains** A and write $B \supseteq A$. If A is not a subset of B, we write $A \nsubseteq B$; thus $(A \nsubseteq B) \leftrightarrow (\exists x)(x \in A \wedge x \notin B)$.

EXAMPLE 2.2 Let A = set of states in the United States, B = set of New England states, and C = set of Canadian provinces. Then $B \subseteq A$, but $B \nsubseteq C$ and $A \nsubseteq C$. ■

To show that a set X is a subset of Y, select an arbitrary element x in X; then using the laws of logic and known facts, show that x is in Y also. We shall apply this technique in later sections. To show that $X \nsubseteq Y$, all you need is to find an element $x \in X$ which does not belong to Y.

Equal Sets

Two sets A and B are **equal**, denoted by $A = B$, if they contain the same elements. In other words, $A = B$ if $(A \subseteq B) \wedge (B \subseteq A)$. (We shall use this property to prove the equality of sets.) If $A \subseteq B$ and $A \neq B$, then A is a **proper subset** of B, denoted by $A \subset B$.

EXAMPLE 2.3 Consider the sets $A = \{x | x \text{ is a vowel of the alphabet}\}$, $B = \{\mathrm{a, e, i, o, u}\}$, $C = \{2, 3, 4\}$, and $D = \{x | x \text{ is a digit in the numeral } 23432.\}$ Then $A = B$, and $C = D$. ■

Does a set have to contain any element? Can there be a set with no elements? Suppose Fred went hunting in a nearby jungle and returned home with great tales, but no animals. The set of animals he caught is null. This leads us to the following definition.

Empty Set

The set containing no elements is the **empty** (or **null) set**; it is denoted by $\emptyset$ or $\{\}$.

EXAMPLE 2.4 The set of pink elephants is empty. So are the set of mountains on the earth that are 50,000 feet tall and the set of prime numbers between 23 and 28. ■

Many people mistakenly believe that $\{\emptyset\} = \emptyset$; this is *not* true, since $\{\emptyset\}$ contains an element $\emptyset$, whereas $\emptyset = \{\}$ contains no elements. Thus $\{\emptyset\} \neq \emptyset$.

Logically, it can be proved that $\emptyset$ is a subset of every set; that is, $\emptyset \subseteq A$ for every set A. Besides, although many people think that there are many empty sets, it can be proved that it is unique, meaning there is exactly one empty set. (See Exercises 53 and 54.)

Universal Set

It is always possible to choose a special set $U(\neq \emptyset)$ such that every set under discussion is a subset of U. Such a set is called a **universal set**, denoted by U. Thus $A \subseteq U$ for every set A.

EXAMPLE 2.5 Suppose we wish to discuss something about the sets $\{\mathrm{a}\}$, $\{\mathrm{b, c, d}\}$, and $\{\mathrm{b, d, e, f}\}$. Then $U = \{\mathrm{a, b, c, d, e, f}\}$ may be chosen as a valid universal set. (There are other valid choices also.) ■

EXAMPLE 2.6 (optional) Programming languages such as *Pascal* support the data type SET, although the implementations have a limit on the number of elements on the base-type of the set, that is, on the size of the universal set. For example, consider the Pascal declarations:

```
TYPE
   MONTHS = (JAN, FEB, MAR, APR, MAY, JUN, JUL, AUG, SEP, OCT, NOV, DEC);
   SETOFMONTHS = SET OF MONTHS;
VAR
   SPRING,SUMMER,FALL,WINTER:  SETOFMONTHS;
```

Here the universal set is

```
SETOFMONTHS = {JAN,FEB,MAR,APR,MAY,JUN,JUL,AUG,SEP,OCT,NOV,DEC}.
```

The above variable declarations define four set variables, namely, SPRING, SUMMER, FALL, and WINTER. The set values assigned to them must be subsets of SETOFMONTHS. For instance,

```
SPRING := [JAN,FEB,MAR];
```

is a legal Pascal assignment, although it is preposterous.

The set membership operator in Pascal is **IN** and can be used to determine if an element belongs to a set. For example, `FEB IN SPRING` is a legal boolean expression. Likewise, the set inclusion and containment operators are $<=$ and $>=$, respectively. ■

Disjoint Sets

Sets need not have common elements. Two such sets are **disjoint** sets.

For example, the sets {Ada, BASIC, FORTRAN} and {C++, Java} are disjoint; so are the sets $\{+, -, *, /\}$ and $\{\wedge, \vee, \rightarrow, \leftrightarrow\}$.

Venn Diagrams

Relationships between sets can be displayed using Venn diagrams, named after the English logician John Venn. In a Venn diagram, the universal set U is represented by the points inside a rectangle and sets by the points enclosed by simple closed curves inside the rectangle, as in Figure 2.1. Figure 2.2 shows $A \subseteq B$, whereas Figure 2.3 shows they are not disjoint.

Figure 2.1

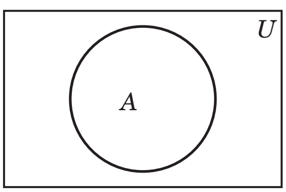


Figure 2.2

$A \subseteq B$.

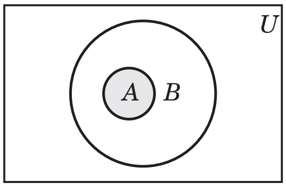
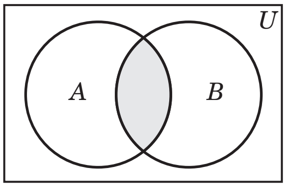


Figure 2.3

A and B may have common elements.



John Venn *(1834–1923) was born into a philanthropic family in Hull, England. After attending the high schools at Highgate and Islington, in 1853 he entered Gonville and Caius College, Cambridge, and graduated in mathematics three years later. He was elected a fellow of the College, a position he held until his death.*

In 1859 Venn was ordained in the Church of England, but after a brief period of church work, he returned to Cambridge as a lecturer on moral sciences. In 1883 he gave up his priesthood. The same year, he received a D.Sc. from Cambridge and was elected a fellow of the Royal Society of London.

Venn was greatly influenced by Boole's work in symbolic logic. Venn's masterpiece, Symbolic Logic *(1881), clarifies the inconsistencies and ambiguities in Boole's ideas and notations. He employed geometric diagrams to represent logical arguments, a technique originated by Leibniz and developed further by Euler. Venn added a rectangle to represent the universe of discourse.*

Venn published two additional books, The Logic of Chance *(1866) and* The Principles of Empirical Logic *(1889).*

Can the elements of a set be sets? Certainly. $\{\{a\}, \{b, c\}\}$, and $\{\emptyset, \{\emptyset\}, \{a, b\}\}$ are two such sets. In fact, the subsets of a set can be used to build a new set.

Power Set

The family of subsets of a set A is the **power set** of A, denoted by $P(A)$.

EXAMPLE 2.7 Find the power set $P(A)$ of the set $A = \{a, b\}$.

SOLUTION:
Since Ø is a subset of every set, Ø ∈ P(A). Also, {a} and {b} are subsets of A. Further, every set is a subset of itself, so A ∈ P(A). Thus, the various elements of P(A) are Ø, {a}, {b}, and A; that is, P(A) = {Ø, {a}, {b}, A}. ■

Sets can be classified as finite and infinite sets, as defined below.

Finite and Infinite Sets

A set with a definite number of elements is a **finite set**. A set that is not finite is **infinite**.

EXAMPLE 2.8 The sets {a,b,c} and the set of computers in the world are finite, but the set of integers and the set of points on a line are infinite. ■

> It may sometimes be difficult to know the exact number of elements in a finite set. But that does not affect its finiteness. For example, the set of residents in California at a given time is finite, although it is difficult to determine the actual count.

It is impossible to list all the elements of an infinite set. Consequently, the enumeration method with ellipsis or the set-builder notation is used to define infinite sets. In the former case, the ellipsis would come at the end of the list, for example, $\mathbb{N} = \{1, 2, 3, \ldots\}$.

The following are some special infinite sets we will be using frequently:

$$\mathbf{Z} = \text{set of integers} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$$

$$\mathbb{N} = \mathbf{Z}^+ = \text{set of positive integers} = \{1, 2, 3, \ldots\}$$

$$\mathbf{Z}^- = \text{set of negative integers} = \{\ldots, -3, -2, -1\}$$

$$\mathbf{W} = \text{set of whole numbers} = \{0, 1, 2, 3, \ldots\}$$

$$\mathbf{Q} = \text{set of rational numbers} = \{p/q | p, q \in \mathbf{Z} \wedge q \neq 0\}$$

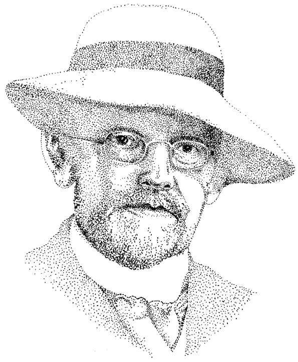
$$\mathbb{R} = \text{set of real numbers}$$

$$\mathbb{R}^+ = \text{set of positive real numbers} = \{x \in \mathbb{R} | x > 0\}$$

$$\mathbb{R}^- = \text{set of negative real numbers} = \{x \in \mathbb{R} | x < 0\}$$

A few additional subsets of $\mathbb{R}$, called **intervals**, will prove useful in our discussions. They are given below, where $a < b$:

closed interval	$[a, b] = \{x \in \mathbb{R} \mid a \le x \le b\}$
closed–open interval	$[a, b) = \{x \in \mathbb{R} \mid a \le x < b\}$
open–closed interval	$(a, b] = \{x \in \mathbb{R} \mid a < x \le b\}$
open interval	$(a, b) = \{x \in \mathbb{R} \mid a < x < b\}$



David Hilbert *(1862–1943) was born and educated in Königsberg, Germany (now in Russia). He made significant contributions to algebra, analysis, geometry, and mathematical physics. He described the importance of set theory in the development of mathematics: "No one shall expel us from the paradise which Cantor has created for us."*

A bracket at an endpoint indicates it is included in the set, whereas a parenthesis indicates it is not included.

The set $\{x \in \mathbb{R} | x \geq a\}$ is denoted by $[a, \infty)$ using the **infinity symbol** ∞. Likewise, the set $\{x \in \mathbb{R} | x \leq a\}$ is denoted by $(-\infty, a]$.

Next we present two interesting paradoxes related to infinite sets and proposed in the 1920s by the German mathematician David Hilbert.

The Hilbert Hotel Paradoxes

Imagine a grand hotel in a major city with an infinite number of rooms, all occupied. One morning a visitor arrives at the registration desk looking for a room. "I'm sorry, we are full," replies the manager, "but we can certainly accommodate you." How is this possible? Is she contradicting herself?

To give a room to the new guest, Hilbert suggested moving the guest in Room 1 to Room 2, the guest in Room 2 to Room 3, the one in Room 3 to Room 4, and so on; Room 1 is now vacant and can be given to the new guest. The clerk is happy that she can accommodate him by moving each guest one room down the hall.

The second paradox involves an infinite number of conventioneers arriving at the hotel, each looking for a room. The clerk realizes that the hotel can make a fortune if she can somehow accommodate them. She knows she can give each a room one at a time as above, but that will involve moving each guest constantly from one room to the next, resulting in total chaos and frustration.

So Hilbert proposed the following solution: move the guest in Room 1 to Room 2, the guest in Room 2 to Room 4, the one in Room 3 to Room 6, and so on. This puts the old guests in even-numbered rooms, so the new guests can be checked into the odd-numbered rooms.

Notice that in both cases the hotel could accommodate the guests only because it has infinitely many rooms.

A third paradox: Infinitely many hotels with infinitely many rooms are leveled by an earthquake. All guests survive and come to Hilbert Hotel. How can they be accommodated? See Example 3.23 for a solution.

We close this section by introducing a special set used in the study of formal languages.

Every word in the English language is an arrangement of the letters of the alphabet $\{A, B, \ldots, Z, a, b, \ldots, z\}$. The alphabet is finite and not every arrangement of the letters need make any sense. These ideas can be generalized as follows.

Alphabet

A finite set Σ of symbols is an **alphabet**. (Σ is the uppercase Greek letter *sigma*.) A **word** (or **string**) **over** Σ is a finite arrangement of symbols from Σ.

For instance, the only alphabet understood by a computer is the **binary alphabet** {0,1}; every word is a finite and unique arrangement of 0's and 1's. Every zip code is a word over the alphabet $\{0, \ldots, 9\}$.

Sets such as {a, b, c, ab, bc} are *not* considered alphabets since the string ab, for instance, can be obtained by juxtaposing, that is, placing next to each other, the symbols a and b.

Length of a Word

The **length** of a word w, denoted by $\|w\|$, is the number of symbols in it. A word of length zero is the **empty word** (or the **null word**), denoted by the lowercase Greek letter λ (**lambda**); It contains no symbols.

For example, $\|ab\| = 2$, $\|aabba\| = 5$, and $\|\lambda\| = 0$.

The set of words over an alphabet Σ is denoted by Σ^*. The empty word λ belongs to Σ^* for every alphabet Σ. In particular, if Σ denotes the English alphabet, then Σ^* consists of all words, both meaningful and meaningless. Consequently, the English language is a subset of Σ^*. More generally, we make the following definition.

Language

A **language** over an alphabet Σ is a subset of Σ^*.

The following two examples illustrate this definition.

EXAMPLE 2.9 The set of zip codes is a finite language over the alphabet $\Sigma = \{0, \ldots, 9\}$. ■

EXAMPLE 2.10 Let $\Sigma = \{a, b\}$. Then $\Sigma^* = \{\lambda, a, b, aa, ab, ba, bb, aaa, aab, aba, abb, baa, \ldots\}$, an infinite set. Notice that $\{aa, ab, ba, bb\}$ is a finite language over Σ, whereas $\{a, aa, aba, bab, aaaa, abba, \ldots\}$ is an infinite language. ■

Words can be combined to create new words, as defined below.

Concatenation

The **concatenation** of two words x and y over an alphabet, denoted by xy, is obtained by appending the word y at the end of x. Thus if $x = x_1 \ldots x_m$ and $y = y_1 \ldots y_n$, $xy = x_1 \ldots x_m y_1 \ldots y_n$.

For example, let Σ be the English alphabet, $x =$ CAN, and $y =$ ADA; then $xy =$ CANADA. Notice that concatenation is *not* a commutative operation; that is, $xy \neq yx$. It is, however, associative; that is, $x(yz) = (xy)z = xyz$.

Two interesting properties are satisfied by the concatenation operation:

- The concatenation of any word x with λ is itself; that is, $\lambda x = x = x\lambda$ for every $x \in \Sigma^*$.
- Let $x, y \in \Sigma^*$. Then $\|xy\| = \|x\| + \|y\|$. (See Section 5.1 for a proof.)

For example, let $\Sigma = \{\text{a,b}\}$, $x =$ aba, and $y =$ bbaab. Then $xy =$ ababbaab and $\|xy\| = 8 = 3 + 5 = \|x\| + \|y\|$.

A useful notation: As in algebra, the exponential notation can be employed to eliminate the repeating of symbols in a word. Let x be a symbol and n an integer ≥ 2; then x^n denotes the concatenation $xx \ldots x$ to $n - 1$ times. Using this compact notation, the words aaabb and ababab can be abbreviated as a^3b^2 and $(\text{ab})^3$, respectively. Notice, however, that $(\text{ab})^3 =$ ababab $\neq \text{a}^3\text{b}^3 =$ aaabbb.

Exercises 2.1

Rewrite each set using the listing method.

1. The set of months that begin with the letter A.

2. The set of letters of the word GOOGOL.

3. The set of months with exactly 31 days.

4. The set of solutions of the equation $x^2 - 5x + 6 = 0$.

Rewrite each set using the set-builder notation.

5. The set of integers between 0 and 5.

6. The set of January, February, May, and July.

7. The set of all members of the United Nations.

8. {Asia, Australia, Antarctica}

Determine if the given sets are equal.

9. $\{x, y, z\}, \{x, z, y\}$

10. $\{x|x^2 = 1\}, \{x|x^2 = x\}$

11. $\{x|x^2 = x\}, \{0, 1\}$

12. $\{x, \{y\}\}, \{\{x\}, y\}$

Mark each as true or false.

13. $\text{a} \in \{\text{alfa}\}$

14. $\text{b} \subseteq \{\text{a,b,c}\}$

15. $\{\text{x}\} \subseteq \{\text{x, y, z}\}$

16. $\{0\} = \emptyset$ **17.** $0 \in \emptyset$ **18.** $\{\emptyset\} = 0$

19. $\{\emptyset\} = \emptyset$ **20.** $\emptyset \subseteq \emptyset$ **21.** $\emptyset \in \{\emptyset\}$

22. $\{x|x \neq x\} = \emptyset$ **23.** $\{x, y\} = \{y, x\}$ **24.** $\{x\} \in \{\{x\}, y\}$

25. $\emptyset$ is a subset of every set. **26.** Every set is a subset of itself.

27. Every nonempty set has at least two subsets.

28. The set of people in the world is infinite.

29. The set of words in a dictionary is infinite.

Find the power set of each set.

30. $\emptyset$ **31.** $\{\mathrm{a}\}$ **32.** $\{\mathrm{a}, \mathrm{b}, \mathrm{c}\}$

33. Using Exercises 30–32, predict the number of subsets of a set with n elements.

In Exercises 34–37, n denotes a positive integer less than 10. Rewrite each set using the listing method.

34. $\{n|n$ is divisible by $2\}$ **35.** $\{n|n$ is divisible by $3\}$

36. $\{n|n$ is divisible by 2 and $3\}$ **37.** $\{n|n$ is divisible by 2 or $3\}$

Find the family of subsets of each set that do *not* contain consecutive integers.

38. $\{1, 2\}$ **39.** $\{1, 2, 3\}$

40. Let a_n denote the number of subsets of the set $S = \{1, 2, \ldots, n\}$ that do not contain consecutive integers, where $n \geq 1$. Find a_3 and a_4.

In Exercises 41–46, a language L over $\Sigma = \{a, b\}$ is given. Find five words in each language.

41. $L = \{x \in \Sigma^* | x$ begins with and ends in $b.\}$

42. $L = \{x \in \Sigma^* | x$ contains exactly one $b.\}$

43. $L = \{x \in \Sigma^* | x$ contains an even number of a's.$\}$

44. $L = \{x \in \Sigma^* | x$ contains an even number of a's followed by an odd number of b's.$\}$

Compute the length of each word over $\{\mathrm{a}, \mathrm{b}\}$.

45. aab **46.** aabbb

47. ab^4 **48.** $\mathrm{a}^3\mathrm{b}^2$

Arrange the binary words of the given length in increasing order of magnitude.

49. Length two. **50.** Length three.

A **ternary word** is a word over the alphabet $\{0, 1, 2\}$. Arrange the ternary words of the given length in increasing order of magnitude.

51. Length one. **52.** Length two.

Prove each.

***53.** The empty set is a subset of every set.
(*Hint*: Consider the implication $x \in \emptyset \rightarrow x \in A$.)

***54.** The empty set is unique.
(*Hint*: Assume there are two empty sets, $\emptyset_1$ and $\emptyset_2$. Then use Exercise 53.)

***55.** Let A, B, and C be arbitrary sets such that $A \subseteq B$ and $B \subseteq C$. Then $A \subseteq C$.
(**transitive property**)

***56.** If Σ is a nonempty alphabet, then Σ^* is infinite.
(*Hint*: Assume Σ^* is finite. Since $\Sigma \neq \emptyset$, it contains an element a. Let $x \in \Sigma^*$ with largest length. Now consider xa.)

2.2 Operations with Sets

Just as propositions can be combined in several ways to construct new propositions, sets can be combined in different ways to build new sets. You will find a close relationship between logic operations and set operations.

Union

The **union** of two sets A and B, denoted by $A \cup B$, is obtained by merging them; that is, $A \cup B = \{x|(x \in A) \vee (x \in B)\}$.

Notice the similarity between union and disjunction.

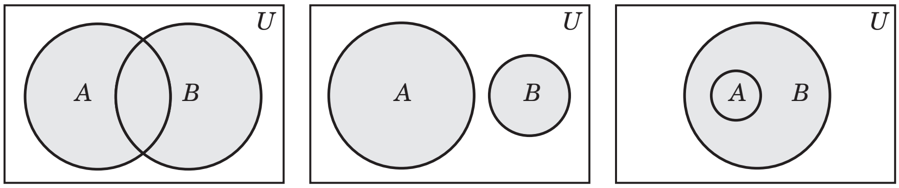
EXAMPLE 2.11 Let $A = \{\text{a, b, c}\}$, $B = \{\text{b, c, d, e}\}$, and $C = \{x, y\}$. Then $A \cup B = \{\text{a, b, c, d, e}\} = B \cup A$ and $B \cup C = \{\text{b, c, d, e, x, y}\} = C \cup B$. ■

The shaded areas in Figure 2.4 represent the set $A \cup B$ in three different cases.

Intersection

The **intersection** of two sets A and B, denoted by $A \cap B$, is the set of elements common to both A and B; that is, $A \cap B = \{x|(x \in A) \vee (x \in B)\}$.

Figure 2.4



$A \cup B$ | $A \cup B$, where A and B are disjoint | $A \cup B = B$

Notice the relationship between intersection and conjunction.

EXAMPLE 2.12 Let $A = \{\text{Nov, Dec, Jan, Feb}\}$, $B = \{\text{Feb, Mar, Apr, May}\}$, and $C = \{\text{Sept, Oct, Nov, Dec}\}$. Then $A \cap B = \{\text{Feb}\} = B \cap A$ and $B \cap C = \emptyset = C \cap B$. (Notice that B and C are disjoint sets. More generally, two sets are disjoint if and only if their intersection is null.) ■

Figure 2.5

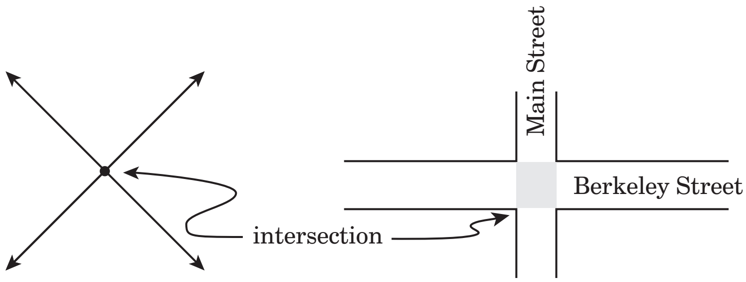
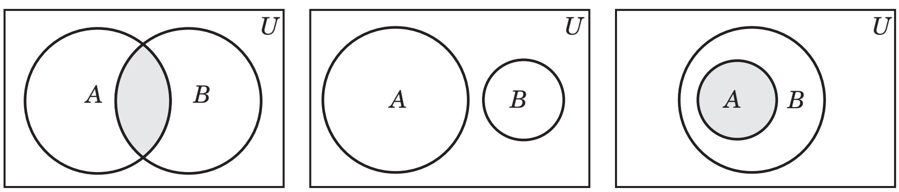


Figure 2.5 shows the intersection of two lines and that of two streets, and Figure 2.6 displays the set $A \cap B$ in three different cases.

Figure 2.6



$A \cap B$ | $A \cap B = \emptyset$ | $A \cap B = A$

EXAMPLE 2.13 Let $A = \{a, b, c, d, g\}$, $B = \{b, c, d, e, f\}$, and $C = \{b, c, e, g, h\}$. Find $A \cup (B \cap C)$ and $(A \cup B) \cap (A \cup C)$.

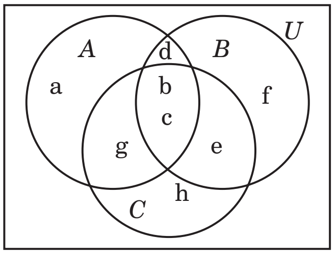
SOLUTION:

(1)
$$B \cap C = \{b, c, e\}$$
$$A \cup (B \cap C) = \{a, b, c, d, e, g\}$$

(2)
$$A \cup B = \{a, b, c, d, e, f, g\}$$
$$A \cup C = \{a, b, c, d, e, g, h\}$$
$$(A \cup B) \cap (A \cup C) = \{a, b, c, d, e, g\}$$
$$= A \cup (B \cap C)$$

See the Venn diagram in Figure 2.7.

Figure 2.7



■

A third way of combining two sets is by finding their difference, as defined below.

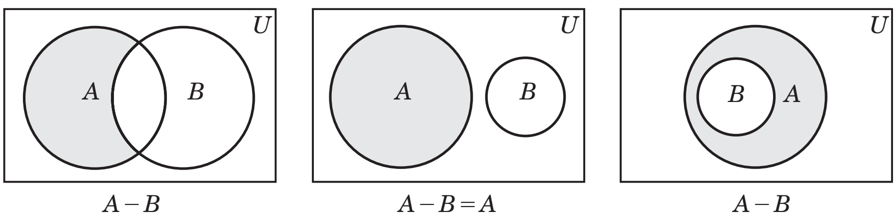
Difference

The **difference** of two sets A and B (or the **relative complement** of B in A), denoted by $A - B$ (notice the order), is the set of elements in A that are *not* in B. Thus $A - B = \{x \in A | x \notin B\}$.

EXAMPLE 2.14 Let $A = \{a, \ldots, z, 0, \ldots, 9\}$, and $B = \{0, \ldots, 9\}$. Then $A - B = \{a, \ldots, z\}$ and $B - A = \emptyset$.

The shaded areas in Figure 2.8 represent the set $A - B$ in three different cases.

Figure 2.8



■

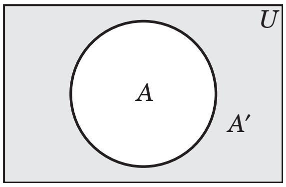
For any set $A \neq U$, although $A - U = \emptyset$, the difference $U - A \neq \emptyset$. This shows yet another way of obtaining a new set.

Complement

The difference $U - A$ is the (absolute) **complement** of A, denoted by A' (A prime). Thus $A' = U - A = \{x \in U \mid x \notin A\}$.

Figure 2.9 represents the complement of a set A. (Complementation corresponds to negation.)

Figure 2.9



EXAMPLE 2.15 Let $U = \{\text{a}, \ldots, \text{z}\}$. Find the complements of the sets $A = \{\text{a}, \text{e}, \text{i}, \text{o}, \text{u}\}$ and $B = \{\text{a}, \text{c}, \text{d}, \text{e}, \ldots, \text{w}\}$. Then $A' = U - A =$ set of all consonants in the alphabet, and $B' = U - B = \{\text{b}, \text{x}, \text{y}, \text{z}\}$. ■

EXAMPLE 2.16 Let $A = \{\text{a}, \text{b}, \text{x}, \text{y}, \text{z}\}$, $B = \{\text{c}, \text{d}, \text{e}, \text{x}, \text{y}, \text{z}\}$, and $U = \{\text{a}, \text{b}, \text{c}, \text{d}, \text{e}, \text{w}, \text{x}, \text{y}, \text{z}\}$. Find $(A \cup B)'$ and $A' \cap B'$.

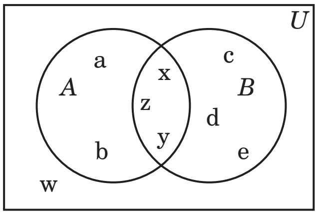
SOLUTION:

(1)
$$A \cup B = \{\text{a}, \text{b}, \text{c}, \text{d}, \text{e}, \text{x}, \text{y}, \text{z}\}$$
$$(A \cup B)' = \{\text{w}\}$$

(2)
$$A' = \{\text{c}, \text{d}, \text{e}, \text{w}\}$$
$$B' = \{\text{a}, \text{b}, \text{w}\}$$
$$A' \cap B' = \{\text{w}\} = (A \cup B)'$$

See Figure 2.10.

Figure 2.10



■

Since as a rule, $A - B \neq B - A$, by taking their union we can form a new set.

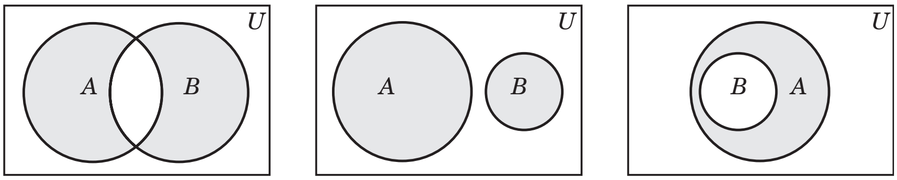
Symmetric Difference

The **symmetric difference** of A and B, denoted by $A \oplus B$, is defined by $A \oplus B = (A - B) \cup (B - A)$.

EXAMPLE 2.17 Let $A = \{a, \dots, z, 0, \dots, 9\}$ and $B = \{0, \dots, 9, +, -, *, /\}$. Then $A - B = \{a, \dots, z\}$ and $B - A = \{+, -, *, /\}$. So $A \oplus B = (A - B) \cup (B - A) = \{a, \dots, z, +, -, *, /\}$. ■

The symmetric difference of A and B is pictorially displayed in Figure 2.11 in three different cases.

Figure 2.11



$A \oplus B = A \cup B - A \cap B$ $\qquad$ $A \oplus B = A \cup B$ $\qquad$ $A \oplus B = A - B$

Set and Logic Operations

Set operations and logic operations are closely related, as Table 2.1 shows.

Table 2.1

Set operation	Logic operation
$A \cup B$	$p \vee q$
$A \cap B$	$p \vee q$
A'	$\sim p$
$A \oplus B$	p XOR q

The important properties satisfied by the set operations are listed in Table 2.2. (Notice the similarity between these properties and the laws of logic in Section 1.2.) We shall prove one of them. Use its proof as a model to prove the others as routine exercises.

We shall prove law 16. It uses De Morgan's law in symbolic logic, and the fact that $X = Y$ if and only if $X \subseteq Y$ and $Y \subseteq X$.

PROOF:

In order to prove that $(A \cup B)' = A' \cap B'$, we must prove two parts: $(A \cup B)' \subseteq A' \cap B'$ and $A' \cap B' \subseteq (A \cup B)'$.

- To prove that $(A \cup B)' \subseteq (A' \cap B')$:

  Let x be an arbitrary element of $(A \cup B)'$. Then $x \notin (A \cup B)$. Therefore, by De Morgan's law, $x \notin A$ and $x \notin B$; that is, $x \in A'$ and $x \in B'$. So $x \in A' \cap B'$. Thus every element of $(A \cup B)'$ is also an element of $A' \cap B'$; that is, $(A \cup B)' \subseteq A' \cap B'$.

Table 2.2

Laws of Sets	
Let A, B, and C be any three sets and U the universal set. Then:	
Idempotent laws	
1. $A \cup A = A$	2. $A \cap A = A$
Identity laws	
3. $A \cup \emptyset = A$	4. $A \cap U = A$
Inverse laws	
5. $A \cup A' = U$	6. $A \cap A' = \emptyset$
Domination laws	
7. $A \cup U = U$	8. $A \cap \emptyset = \emptyset$
Commutative laws	
9. $A \cup B = B \cup A$	10. $A \cap B = B \cap A$
Double complementation law	
11. $(A')' = A$	
Associative laws	
12. $A \cup (B \cup C) = (A \cup B) \cup C$	13. $A \cap (B \cap C) = (A \cap B) \cap C$
Distributive laws	
14. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	15. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
De Morgan's laws	
16. $(A \cup B)' = A' \cap B'$	17. $(A \cap B)' = A' \cup B'$
Absorption laws	
18. $A \cup (A \cap B) = A$	19. $A \cap (A \cup B) = A$
(Note: The following laws have no names.)	
20. If $A \subseteq B$, then $A \cap B = A$.	21. If $A \subseteq B$, then $A \cup B = B$.
22. If $A \subseteq B$, then $B' \subseteq A'$.	23. $A - B = A \cap B'$
24. $A \oplus B = A \cup B - A \cap B$	

- To prove that $A' \cap B' \subseteq (A \cup B)'$:

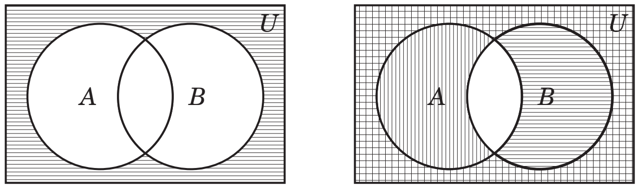
  Let x be any element of $A' \cap B'$. Then $x \in A'$ and $x \in B'$. Therefore, $x \notin A$ and $x \notin B$. So, by De Morgan's law, $x \notin (A \cup B)$. Consequently, $x \in (A \cup B)'$. Thus, since x is arbitrary, $A' \cap B' \subseteq (A \cup B)'$.

Thus, $(A \cup B)' = A' \cap B'$. See the Venn diagrams in Figure 2.12 also.

Note: Law 23 is a very useful result and will be used in the next section.

A few words of explanation: The commutative laws imply that the order in which the union (or intersection) of two sets is taken is irrelevant. The associative laws imply that when the union (or intersection) of three or more

Figure 2.12



$(A \cup B)' =$ shaded area $\quad A' \cap B' =$ cross-shaded area ■

sets is taken, the way the sets are grouped is immaterial; in other words, such expressions without parentheses are perfectly legal. For instance, $A \cup B \cup C = A \cup (B \cup C) = (A \cup B) \cup C$ is certainly valid. The two De Morgan's laws in propositional logic play a central role in deriving the corresponding laws in sets.

Again, as in propositional logic, parentheses are essential to indicate the groupings in the distributive laws. For example, if you do not parenthesize the expression $A \cap (B \cup C)$ in law 15, then the LHS becomes $A \cap B \cup C = (A \cap B) \cup C = (A \cup C) \cap (B \cup C) \neq (A \cap B) \cup (A \cap C)$.

Notice the similarity between the set laws and the laws of logic. For example, properties 1 through 19 and 22 have their counterparts in logic. Every corresponding law of logic can be obtained by replacing sets A, B, and C with propositions p, q, and r, respectively, the set operators $\cap$, $\cup$, and $'$ with the logic operators $\wedge$, $\vee$, and $\sim$ respectively, and equality $(=)$ with logical equivalence $(\equiv)$.

Using this procedure, the absorption law $A \cup (A \cap B) = A$, for instance, can be translated as $p \vee (p \wedge q) \equiv p$, which is the corresponding absorption law in logic.

Just as truth tables were used in Chapter 1 to establish the logical equivalence of compound statements, they can be applied to verify set laws as well. The next example illustrates this method.

EXAMPLE 2.18 Using a truth table, prove that $(A \cup B)' = A' \cap B'$.

SOLUTION:

Let x be an arbitrary element. Then x may or may not be in A. Likewise, x may or may not belong to B. Enter this information, as in logic, in the first two columns of the table, which are headed by $x \in A$ and $x \in B$.

The table needs five more columns, headed by $x \in (A \cup B)$, $x \in (A \cup B)'$, $x \in A'$, $x \in B'$, and $x \in (A' \cap B')$ (see Table 2.3). Again, as in logic, use the entries in the first two columns to fill in the remaining columns, as in the table.

Table 2.3

$x \in A$	$x \in B$	$x \in (A \cup B)$	$x \in (A \cup B)'$	$x \in A'$	$x \in B'$	$x \in (A' \cap B')$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

Note: The shaded columns are identical

Since the columns headed by $x \in (A \cup B)'$ and $x \in (A' \cap B')$ are identical, it follows that $(A \cup B)' = A' \cap B'$. ■

Using truth tables to prove set laws is purely mechanical and elementary. It does not provide any insight into the development of a mathematical proof. Such a proof does not build on previously known set laws, so we shall not resort to such proofs in subsequent discussions.

Just as the laws of logic can be used to simplify logic expressions and derive new laws, set laws can be applied to simplify set expressions and derive new laws. In order to be successful in this art, you must know the laws well and be able to apply them as needed. So, practice, practice, practice.

EXAMPLE 2.19 Using set laws, verify that $(X - Y) - Z = X - (Y \cup Z)$.

PROOF:

$$
\begin{aligned}
(X - Y) - Z &= (X - Y) \cap Z' && A - B = A \cap B' \\
&= (X \cap Y') \cap Z' && A - B = A \cap B' \\
&= X \cap (Y' \cap Z') && \text{associative law 13} \\
&= X \cap (Y \cup Z)' && \text{De Morgan's law 16} \\
&= X - (Y \cup Z) && A - B = A \cap B'
\end{aligned}
$$

■

EXAMPLE 2.20 Simplify the set expression $(A \cap B') \cup (A' \cap B) \cup (A' \cap B')$.

SOLUTION:

(You may supply the justification for each step.)

$$
\begin{aligned}
(A \cap B') \cup (A' \cap B) \cup (A' \cap B') &= (A \cap B') \cup [(A' \cap B) \cup (A' \cap B')] \\
&= (A \cap B') \cup [A' \cap (B \cup B')] \\
&= (A \cap B') \cup (A' \cap U) \\
&= (A \cap B') \cup A'
\end{aligned}
$$

$$\begin{aligned} &= A' \cup (A \cap B') \\ &= (A' \cup A) \cap (A' \cup B') \\ &= U \cap (A' \cup B') \\ &= A' \cup B' \end{aligned}$$ ■

Often subscripts are used to name sets, so we now turn our attention to such sets.

Indexed Sets

Let I, called the **index set**, be the set of subscripts i used to name the sets A_i. Then the union of the sets A_i as i varies over I is denoted by $\bigcup_{i \in I} A_i$. Similarly, $\bigcap_{i \in I} A_i$ denotes the intersection of the sets A_i as i runs over I. In particular, let $I = \{1, 2, \ldots, n\}$. Then $\bigcup_{i \in I} A_i = A_1 \cup A_2 \cup \cdots \cup A_n$, which is often written as $\bigcup_{i=1}^{n} A_i$ or simply $\bigcup_{1}^{n} A_i$. Likewise, $\bigcap_{i \in I} A_i = \bigcap_{i=I}^{n} A_i = \bigcap_{1}^{n} A_i = A_1 \cap A_2 \cap \cdots \cap A_n$. If $I = \mathbb{N}$, the expression $\bigcup_{i \in \mathbb{N}} A_i$ is written as $\bigcup_{i \in 1}^{\infty} A_i = \bigcup_{1}^{\infty} A_i$, using the infinity symbol ∞; similarly, $\bigcap_{i \in \mathbb{N}} A_i = \bigcap_{i=1}^{\infty} A_i = \bigcap_{1}^{\infty} A_i$.

Before we proceed to define a new binary operation on sets $\bigcap_{i=1}^{\infty}$, we define an ordered set.

Ordered Set

Recall that the set $\{a_1, a_2, \ldots, a_n\}$ is an unordered collection of elements. Suppose we assign a position to each element. The resulting set is an **ordered set** with n elements or an ***n*-tuple**, denoted by $(a_1, a_2, \ldots, a_n)$. (Notice the use of parentheses versus braces.) The set (a_1, a_2) is an **ordered pair.**

Two n-tuples are **equal** if and only if their corresponding elements are equal. That is, $(a_1, a_2, \ldots, a_n) = (b_1, b_2, \ldots, b_n)$ if and only if $a_i = b_i$ for every i.

EXAMPLE 2.21 Every numeral and word can be considered an n-tuple. For instance,

$$345 = (3,\ 4,\ 5)$$

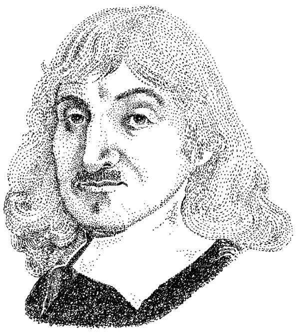
ones (5)
tens (4)
hundreds (3)

$$\text{computer} = (\text{c, o, m, p, u, t, e, r})$$

$1001011 = (1, 0, 0, 1, 0, 1, 1)$ ← ASCII* code for letter K

$11010010 = (1, 1, 0, 1, 0, 0, 1, 0)$ ← EBCDIC** code for letter K ■

*American Standard Code for Information Interchange.
**Extended Binary Coded Decimal Interchange Code.



René Descartes *(1596–1650) was born near Tours, France. At eight, he entered the Jesuit school at La Fleche, where because of poor health he developed the habit of lying in bed thinking until late in the morning; he considered those times the most productive. He left the school in 1612 and moved to Paris, where he studied mathematics for a brief period.*

After a short military career and travel through Europe for about 5 years, he returned to Paris and studied mathematics and philosophy. He then moved to Holland, where he lived for 20 years writing several books. In 1637 he wrote Discours*, which contains his contributions to analytic geometry.*

In 1649 Descartes moved to Sweden at the invitation of Queen Christina. There he contracted pneumonia and died.

We are now ready to define the next and final operation on sets.

Cartesian Product

The **cartesian product** of two sets A and B, denoted by $A \times B$, is the set of all ordered pairs (a, b) with $a \in A$ and $b \in B$. Thus $A \times B = \{(a, b) | a \in A \wedge b \in B\}$. $A \times A$ is denoted by A^2.

It is named after the French philosopher and mathematician René Descartes.

EXAMPLE 2.22 Let $A = \{\mathrm{a}, \mathrm{b}\}$ and $B = \{\mathrm{x}, \mathrm{y}, \mathrm{z}\}$. Then

$$A \times B = \{(\mathrm{a}, \mathrm{x}), (\mathrm{a}, \mathrm{y}), (\mathrm{a}, \mathrm{z}), (\mathrm{b}, \mathrm{x}), (\mathrm{b}, \mathrm{y}), (\mathrm{b}, \mathrm{z})\}$$

$$B \times A = \{(\mathrm{x}, \mathrm{a}), (\mathrm{x}, \mathrm{b}), (\mathrm{y}, \mathrm{a}), (\mathrm{y}, \mathrm{b}), (\mathrm{z}, \mathrm{a}), (\mathrm{z}, \mathrm{b})\}$$

$$A^2 = A \times A = \{(\mathrm{a}, \mathrm{a}), (\mathrm{a}, \mathrm{b}), (\mathrm{b}, \mathrm{a}), (\mathrm{b}, \mathrm{b})\}$$

(Notice that $A \times B \neq B \times A$.) ■

The various elements of $A \times B$ in Example 2.22 can be displayed in a rectangular fashion, as in Figure 2.13, and pictorially, using dots as in Figure 2.14. The circled dot in row a and column y, for instance, represents the element (a, y). The pictorial representation in Figure 2.14 is the **graph** of $A \times B$.

Figure 2.13

Elements of A				
	a	(a, x)	(a, y)	(a, z)
	b	(b, x)	(b, y)	(b, z)
		x	y	z

Elements of B

Figure 2.14

Pictorial representation of $A \times B$.

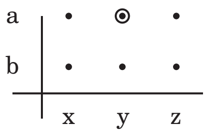
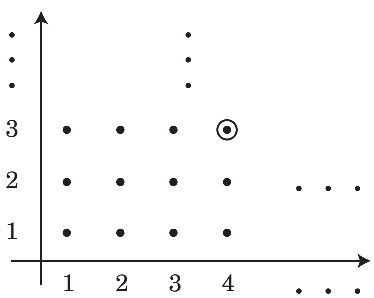


Figure 2.15 shows the graph of the infinite set $\mathbb{N}^2 = \mathbb{N} \times \mathbb{N}$. The circled dot in column 4 and row 3, for instance, represents the element (4,3). The horizontal and vertical dots indicate that the pattern is to be continued indefinitely in both directions.

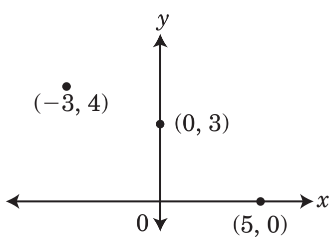
Figure 2.15



More generally, $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ consists of all possible ordered pairs (x, y) of real numbers. It is represented by the familiar ***xy*-plane** or the **cartesian plane** used for graphing (see Figure 2.16).

Figure 2.16

The cartesian plane $\mathbb{R}^2$.



The following example presents an application of cartesian product.

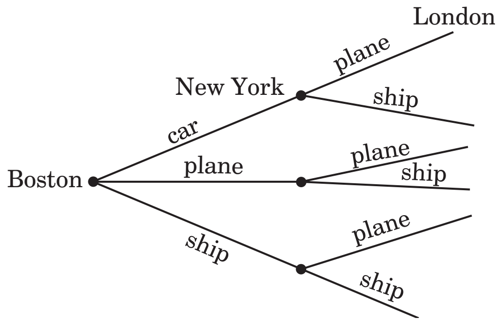
EXAMPLE 2.23

Linda would like to make a trip from Boston to New York and then to London. She can travel by car, plane, or ship from Boston to New York, and by plane or ship from New York to London. Find the set of various modes of transportation for the entire trip.

SOLUTION:

Let A be the set of means of transportation from Boston to New York and B the set from New York to London. Clearly $A = \{\text{car, plane, ship}\}$ and $B = \{\text{plane, ship}\}$. So the set of possible modes of transportation is given by

Figure 2.17



$A \times B = \{$(car, plane), (car, ship), (plane, plane), (plane, ship), (ship, plane), (ship, ship)$\}$. See Figure 2.17. ■

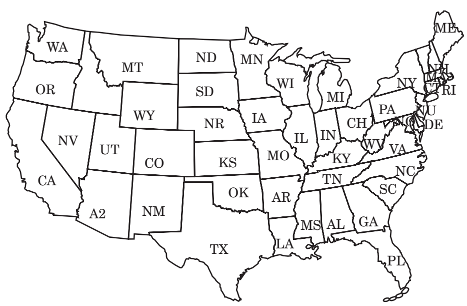
The definition of the product of two sets can be extended to n sets. The **cartesian product of n sets** $A_1, A_2, \ldots, A_n$ consists of all possible n-tuples $(a_1, a_2, \ldots, a_n)$, where $a_i \in A_i$ for every i; it is denoted by $A_1 \times A_2 \times \cdots \times A_n$. If all A_i's are equal to A, the product set is denoted by A^n.

EXAMPLE 2.24 Let $A = \{\mathrm{x}\}$, $B = \{\mathrm{y}, \mathrm{z}\}$, and $C = \{1, 2, 3\}$. Then

$$\begin{aligned} A \times B \times C &= \{(a, b, c) | a \in A, b \in B, \text{ and } c \in C\} \\ &= \{(\mathrm{x}, \mathrm{y}, 1), (\mathrm{x}, \mathrm{y}, 2), (\mathrm{x}, \mathrm{y}, 3), (\mathrm{x}, \mathrm{z}, 1), (\mathrm{x}, \mathrm{z}, 2), (\mathrm{x}, \mathrm{z}, 3)\} \end{aligned}$$

Finally, take a look at the map of the continental United States in Figure 2.18. It provides a geographical illustration of partitioning, a concept that can be extended to sets in an obvious way.

Figure 2.18

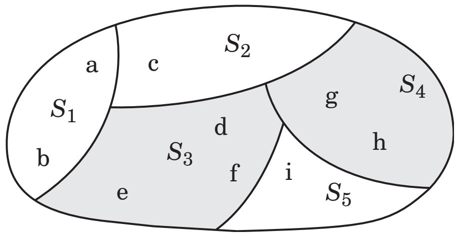


■

Partition

Consider the set $S = \{\mathrm{a, b, c, d, e, f, g, h, i}\}$ and the subsets $S_1 = \{\mathrm{a, b}\}$, $S_2 = \{\mathrm{c}\}$, $S_3 = \{\mathrm{d, e, f}\}$, $S_4 = \{\mathrm{g, h}\}$, and $S_5 = \{\mathrm{i}\}$. Notice that these subsets have three interesting properties: (1) They are nonempty; (2) they are pairwise disjoint; that is, no two subsets have any common elements; (3) their union is S. (See Figure 2.19.) The set $P = \{S_1, S_2, S_3, S_4, S_5\}$ is called a **partition** of S.

Figure 2.19



More generally, let I be an index set and P a family of subsets S_i of a nonempty set S, where $i \in I$. Then P is a **partition** of S if:

- Each set S_i is nonempty.
- The subsets are pairwise disjoint; that is, $S_i \cap S_j = \emptyset$ if $i \neq j$.
- The union of the subsets S_i is S; that is, $\bigcup_{i \in I} S_i = S$.

(Each subset S_i is a **block** of the partition.) Thus a partition of S is a collection of nonempty, pairwise disjoint subsets of S whose union is S.

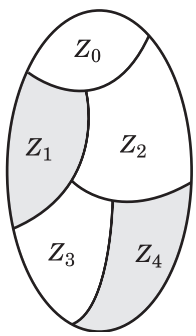
EXAMPLE 2.25 Let Z_r denote the set of integers which, when divided by 5, leave r as the remainder. Then $0 \le r < 5$ (see Section 4.1):

$$\mathbf{Z}_0 = \{\ldots, -5, 0, 5, \ldots\}$$
$$\mathbf{Z}_1 = \{\ldots, -4, 1, 6, \ldots\}$$
$$\mathbf{Z}_2 = \{\ldots, -3, 2, 7, \ldots\}$$
$$\mathbf{Z}_3 = \{\ldots, -2, 3, 8, \ldots\}$$
$$\mathbf{Z}_4 = \{\ldots, -1, 4, 9, \ldots\}$$

$P = \{\mathbf{Z}_0, \mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3, \mathbf{Z}_4\}$ is a partition of the set of integers. See Figure 2.20. (This example is discussed in more detail in Section 7.4.) ■

Figure 2.20

Set of integers **Z**.



The sports pages of newspapers provide fine examples of partitions, as the next example illustrates.

EXAMPLE 2.26 In 2003, the set of teams S in the National Football League was divided into two conferences, American and National, and each conference into four divisions — East, South, North, and West. Let E_1, S_1, N_1, and W_1 denote the set of teams in East, South, North, and West Divisions in the American Conference, respectively, and E_2, S_2, N_2, and W_2 the corresponding sets in the National Conference. Then:

$$E_1 = \{\text{Buffalo, Miami, New England, NY Jets}\}$$
$$S_1 = \{\text{Indianapolis, Tennessee, Houston, Jacksonville}\}$$
$$N_1 = \{\text{Baltimore, Cincinnati, Cleveland, Pittsburgh}\}$$
$$W_1 = \{\text{Denver, Kansas City, Oakland, San Diego}\}$$
$$E_2 = \{\text{Washington, Philadelphia, Dallas, NY Giants}\}$$
$$S_2 = \{\text{Atlanta, Tampa Bay, Carolina, New Orleans}\}$$
$$N_2 = \{\text{Chicago, Detroit, Minnesota, Green Bay}\}$$
$$W_2 = \{\text{Arizona, Seattle, St. Louis, San Francisco}\}$$

Clearly, $P = \{E_1, S_1, N_1, W_1, E_2, S_2, N_2, W_2\}$ is a partition of S.

We close this section with a brief introduction to fuzzy sets.

Fuzzy Sets (optional)

Fuzzy sets, a generalization of ordinary sets, were introduced in 1965 by Lotfi A. Zadeh of the University of California at Berkeley. They have applications to human cognition, communications, decision analysis, psychology, medicine, law, information retrieval, and, of course, artificial intelligence. Like fuzzy logic, they model the fuzziness in the natural language — for example, in terms like *young, healthy, wealthy*, and *beautiful*.

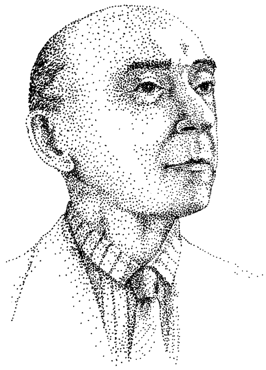
In fuzzy set theory, every element x in the universal set U has a certain degree of membership $d_U(x)$, where $0 \leq d_U(x) \leq 1; d_U(x)$ indicates the degree of fuzziness. Accordingly, a fuzzy set S is denoted by listing its elements along with their degrees of membership; an element with zero degree of membership is not listed.

For example, let U be the fuzzy set of wealthy people and $S = \{\text{Tom } 0.4, \text{Dick } 0.7, \text{Harry } 0.6\}$. Then Harry belongs to S with degree of membership 0.6; $d_S(\text{Harry}) = 0.6$ measures Harry's degree of wealthiness.

The concept of an ordinary subset can be extended to fuzzy sets also.

Fuzzy Subset

Let A and B be fuzzy sets. Then A is a **fuzzy subset** of B if $A \subseteq B$ and $d_A(x) \leq d_B(x)$ for every element x in A.



Lotfi A. Zadeh *(1921–) was born in Baku, Azerbaijan. An alumnus of the University of Tehran (1942) and the Massachusetts Institute of Technology (1946), he received his Ph.D. from Columbia University in 1949 for his dissertation on frequency analysis of time-varying networks. He began his professional career in the Department of Electrical Engineering at Columbia. In 1959, he joined the Department of Electrical Engineering and Computer Science at the University of California, Berkeley, serving as its chair during the years 1963–1968. Currently, he is a professor at Berkeley and Director of Berkeley Initiative in Soft Computing.*

Zadeh's earlier "work was centered on systems analysis, decision analysis, and information systems. Since then his current research has shifted to the theory of fuzzy sets and its applications to artificial intelligence (AI). His research interest now is focused on fuzzy logic, soft computing, computing with words, and the newly developed computational theory of perceptions and precisiated natural language," according to the University of California Web site.

A truly gifted mind and an expert on AI, Zadeh has authored about 200 journal articles on a wide variety of subjects relating to the conception, design, and analysis of information/intelligent systems. He serves on the editorial boards of more than 50 journals and on the advisory boards of a number of institutions related to AI.

Zadeh is a recipient of numerous awards and medals, including the IEEE Education Medal, IEEE Richard W. Hamming Medal, IEEE Medal of Honor, the ASME Rufus Oldenburger Medal, B. Bolzano Medal of the Czech Academy of Sciences, Kampe de Feriet Medal, AACC Richard E. Bellman Central Heritage Award, the Grigore Moisil Prize, Honda Prize, Okawa Prize, AIM Information Science Award, IEEE-SMC J. P. Wohl Career Achievement Award, SOFT Scientific Contribution Memorial Award of the Japan Society for Fuzzy Theory, IEEE Millennium Medal, and the ACM 2000 Allen Newell Award. He has received honorary doctorates from many universities from around the world.

For example, let $S = \{\text{Betsey } 0.6, \text{Mat } 0.5\}$ and $T = \{\text{Betsey } 0.8, \text{Jonathan } 0.3, \text{Mary } 0.5, \text{Mat } 0.7\}$ by fuzzy sets of smart people. Then S is a fuzzy subset of T.

Operations on ordinary sets can be extended to fuzzy sets as well.

Operations on Fuzzy Sets

Let A and B be any fuzzy set. The **union** of A and B is $A \cup B$, where $d_{A\cup B}(x) = \max\{d_A(x), d_B(x)\}$; their **intersection** is $A \cap B$, where $d_{A\cap B}(x) = \min\{d_A(x), d_B(x)\}$; and the **complement** of A is A', where $d_{A'}(x) = 1 - d_A(x)$; in A' only the degrees of membership change.

Using the sets S and T above,

$$S \cup T = \{\text{Betsey } 0.8, \text{Jonathan } 0.3, \text{Mary } 0.5, \text{Mat } 0.7\}$$
$$S \cap T = \{\text{Betsey } 0.6, \text{Mat } 0.5\}$$
$$S' = \{\text{Betsey } 0.4, \text{Mat } 0.5\}$$

Additional opportunities to practice the various operations are given in the exercises.

Exercises 2.2

Let $A = \{a, e, f, g, i\}$, $B = \{b, d, e, g, h\}$, $C = \{d, e, f, h, i\}$, and $U = \{a, b, \ldots, k\}$. Find each set.

1. C' **2.** $B \cap C'$ **3.** $C \cap A'$ **4.** $(A \cup B)'$

5. $(B \cap C)'$ **6.** $(A \cup C')'$ **7.** $(B \cap C')'$ **8.** $A \oplus B$

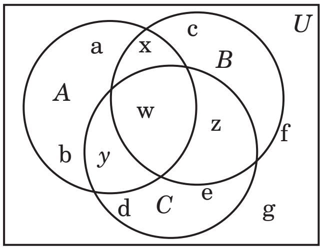
9. $(A - B) - C$ **10.** $A - (B - C)$ **11.** $(A \cup B) - C$ **12.** $(A \cap B) - C$

Using the Venn diagram in Figure 2.21 find each set.

13. $(A \cup B) \cap C$ **14.** $A \cap (B \cup C)$ **15.** $A - (B - C)$

16. $(A \oplus B) \cup C$ **17.** $A \cap (B \oplus C)$ **18.** $A - (B \oplus C)$

Figure 2.21



Let $A = \{b, c\}$, $B = \{x\}$, and $C = \{x, z\}$. Find each set.

19. $A \times B$ **20.** $B \times A$ **21.** $A \times \emptyset$ **22.** $A \times B \times \emptyset$

23. $A \times (B \cup C)$ **24.** $A \times (B \cap C)$ **25.** $A \times B \times C$ **26.** $A \times C \times B$

Mark each as true or false, where A, B, and C are arbitrary sets and U the universal set.

27. $A - \emptyset = A$ **28.** $\emptyset - A = -A$ **29.** $\emptyset - \emptyset = 0$

30. $A - A = 0$ **31.** $A - B = B - A$ **32.** $A - A' = \emptyset$

33. $(A')' = A$ **34.** $(A \cap B)' = A' \cap B'$ **35.** $(A \cup B)' = A' \cup B'$

36. $A \subseteq A \cup B$ **37.** $A \subseteq A \cap B$ **38.** $B \cap (A - B) = \emptyset$

Give a counterexample to disprove each proposition.

39. $(A - B) - C = A - (B - C)$ **40.** $A \cup (B - C) = (A \cup B) - (A \cup C)$

41. $A \cup (B \oplus C) = (A \cup B) \oplus (A \cup C)$ **42.** $A \oplus (B \cap C) = (A \oplus B) \cap (A \oplus C)$

Determine if each is a partition of the set $\{a, \ldots, z, 0, \ldots, 9\}$.

43. $\{\{a, \ldots, z\}, \{0, \ldots, 9\}, \emptyset\}$

44. $\{\{a, \ldots, j\}, \{i, \ldots, t\}, \{u, \ldots, z\}, \{0, \ldots, 9\}\}$

45. $\{\{\mathrm{a}, \ldots, \mathrm{l}\}, \{\mathrm{n}, \ldots, \mathrm{t}\}, \{\mathrm{u}, \ldots, \mathrm{z}\}, \{0, \ldots, 9\}\}$

46. $\{\{\mathrm{a}, \ldots, \mathrm{u}\}, \{\mathrm{v}, \ldots, \mathrm{z}\}, \{0, 3\}, \{1, 2, 4, \ldots, 9\}$

Prove each, where A, B, and C are any sets.

47. $(A')' = A$

48. $A \cup (A \cap B) = A$

49. $A \cap (A \cup B) = A$

50. $(A \cap B)' = A' \cup B'$

51. $A \oplus A = \emptyset$

52. $A \oplus U = A'$

53. $A \oplus B = B \oplus A$

54. $A - B = A \cap B'$

55. $(A \cup B \cup C)' = A' \cap B' \cap C'$

56. $(A \cap B \cap C)' = A' \cup B' \cup C'$

Simplify each set expression.

57. $A \cap (A - B)$

58. $(A - A') \cup (B - A)$

59. $(A - B') - (B - A')$

60. $(A \cup B) \cup (A \cap B')'$

61. $(A \cup B) - (A \cap B)'$

62. $(A \cup B)' \cap (A \cap B')$

63. $(A \cap B)' \cup (A \cup B')$

64. $(A \cup B')' \cap (A' \cap B)$

65. $(A' \cup B')' \cup (A' \cap B)$

***66.** State De Morgan's laws for sets A_i, $i \in I$. (I is an index set.)

***67.** State the distributive laws using the sets A and B_i, $i \in I$.

∘ The **sum** of two fuzzy sets A and B is the fuzzy set $A \oplus B$, where $d_{A \oplus B}(x) = 1 \wedge [d_A(x) + d_B(x)]$; their **difference** is the fuzzy set $A - B$, where $d_{A-B}(x) = 0 \vee [d_A(x) - d_B(x)]$; and their **cartesian product** is the fuzzy set $A \times B$, where $d_{A \times B}(x, y) = d_A(x) \wedge d_B(x)$. Use the fuzzy sets $A = \{$Angelo 0.4, Bart 0.7, Cathy 0.6$\}$ and $B = \{$Dan 0.3, Elsie 0.8, Frank 0.4$\}$ to find each fuzzy set.

68. $A \cup B$

69. $A \cap B$

70. A'

71. $A \cup B'$

72. $A \cap B'$

73. $A \cap A'$

74. $A \oplus B$

75. $A - B$

76. $B - A$

77. $A \times B$

78. $B \times A$

79. $A \times A$

∘ Let A and B be any fuzzy sets. Prove each.

***80.** $(A \cup B)' = A' \cap B'$

***81.** $(A \cap B)' = A' \cup B'$

*2.3 Computer Operations with Sets (optional)

Sets and the various set operations can be implemented in a computer in an elegant manner.

Computer Representation

Although the elements of a set have no inherent order, when the set is represented in a computer, an order is imposed upon them to permit implementation. The universal set U with n elements is represented as an array with n cells, each containing a 1:

	$n-1$				2	1	0
U	1	1	...	1	1	1	1

The elements are represented by the **b**inary dig**it**s (or **bit**s) 0 and 1 in the right-to-left fashion.

Subsets of U are represented by assigning appropriate bits to the various cells. A bit 1 in a cell indicates the corresponding element belongs to the set, whereas a 0 would indicate the element does *not* belong to the set.

EXAMPLE 2.27 Using $U = \{\mathrm{a}, \mathrm{b}, \ldots, \mathrm{h}\}$, represent the sets $A = \{\mathrm{a}, \mathrm{b}, \mathrm{g}\}$ and $B = \{\mathrm{c}, \mathrm{e}, \mathrm{h}\}$ as 8-bit strings.

SOLUTION:
Remember, the elements are represented in the right-to-left order. Thus:

	h	g	f	e	d	c	b	a
U	1	1	1	1	1	1	1	1
A	0	1	0	0	0	0	1	1
B	1	0	0	1	0	1	0	0

■

Next we discuss how the various subsets of a finite set can be found methodically.

Table 2.4

Subset	Bit String
Ø	000
{x}	001
{y}	010
{x, y}	011
{z}	100
{x, z}	101
{y, z}	110
{x, y, z}	111

Interestingly enough, there is a close relationship between sets and bit strings. Table 2.4, for instance, lists the various subsets of the set $\{\mathrm{x}, \mathrm{y}, \mathrm{z}\}$. Notice that the table contains all possible three-bit strings and their decimal values increase from 0 to 7. (See Section 4.3 for a discussion of nondecimal bases.)

Next we present a systematic procedure to find the bit string of the subset that "follows" a given subset with bit string $b_2b_1b_0$. Such a recipe for solving a problem in a finite number of steps is called an **algorithm**.*

Next-Subset Algorithm

Take a good look at each string in Table 2.4. Can you find a rule to obtain each, except 000, from the preceding string? It is fairly simple: From right to left, locate the first 0. Change it to 1 and the 1's to its right to 0's.

For example, suppose you would like to find the subset following $\{x, y\}$ with bit string $b_2b_1b_0 = 011$. From right to left, the first 0 is b_2. Change it to 1, and b_1 and b_0 to 0's. The resulting string is 100 and the corresponding subset is $\{z\}$.

This rule can be generalized and translated into an algorithm. See Algorithm 2.1. Use it to find the subsets following $\{z\}$ and $\{y, z\}$.

```
Algorithm next-subset (b_{n-1}b_{n-2}...b_0)
(* This algorithm finds the bit string of the subset that
   follows a given subset of an n-element set S. *)
Begin (* next-subset *)
  find the first 0 from the right
  change it to 1
  replace the bits to its right with 0's
End (* next-subset *)
```

Algorithm 2.1

The next-subset algorithm can be employed to find all subsets of a finite set S. Algorithm 2.2 shows the steps involved. Use it to find the subsets of $\{x, y, z\}$.

```
Algorithm subsets (S)
(* Using the next-subset algorithm, this algorithm finds the bit
   representations of all subsets of an n-element set S. *)
Begin (* subsets *)
  b_{n-1}b_{n-2}...b_0 ← 00...0  (* initialize string *)
  done ← false (* boolean flag *)
  while not done do
  begin (* while *)
    find the subset following b_{n-1}b_{n-2}...b_0.
```

*The word *algorithm* is derived from the last name of the ninth-century Arabian astronomer and mathematician Abu-Abdullah Muhammed ibn-Musa al-Khowarizmi (Muhammed, the father of Abdullah and the son of Moses of Khwarizm). He was a teacher in the mathematical school in Baghdad, Iraq. His last name indicates he or his family originally came from Khwarizm (now called Khiva) in Uzbekistan.

His books on algebra and Indian numerals had a significant influence in Europe in the 12th century through their Latin translations. The term *algebra* is derived from the title of his algebra book *Kitab al-jabr w'al-muqabalah*.

```
      if every bit b_i = 1 then (* terminate the loop *)
        done ← true
    endwhile
  End (* subsets *)
```

Algorithm 2.2

Next we show how the set operations can be implemented in a computer.

Computer Operations

The representation of sets as n-bit strings allows us to use logic operations to perform set operations. They are implemented through the bit operations — AND, OR, XOR, COMP — defined by Table 2.5, where COMP indicates *one's complement*: $\text{comp}(1) = 0$ and $\text{comp}(0) = 1$.

Table 2.5

logic operators

bit	AND		OR		XOR		COMP
y \ x	0	1	0	1	0	1	
0	0	0	0	1	0	1	1
1	0	1	1	1	1	0	0

The various set operations are accomplished by performing the corresponding logic operations, as shown in Table 2.6. Notice that the logic operation corresponding to $A - B$ makes sense since $A - B = A \cap B'$, by law 23 in Table 2.2.

Table 2.6

Set operations	Logic operations
$A \cap B$	A AND B
$A \cup B$	A OR B
A'	COMP(A)
$A \oplus B$	A XOR B
$A - B$	A AND (COMP(B))

EXAMPLE 2.28 Let $U = \{a, b, \ldots, h\}$, $A = \{a, b, c, e, g\}$, and $B = \{b, e, g, h\}$. Using bit representations, find the sets $A \cap B$, $A \cup B$, $A \oplus B$, B', and $A - B$ as 8-bit words.

SOLUTION:

$$A = 0\ 1\ 0\ 1\ 0\ 1\ 1\ 1$$

$$B = 1\ 1\ 0\ 1\ 0\ 0\ 1\ 0$$

Using Tables 2.5 and 2.6, we have:

(1) $A \cap B = 0\ 1\ 0\ 1\ 0\ 0\ 1\ 0$ (2) $A \cup B = 1\ 1\ 0\ 1\ 0\ 1\ 1\ 1$
(3) $A \oplus B = 1\ 0\ 0\ 0\ 0\ 1\ 0\ 1$ (4) $B' = 0\ 0\ 1\ 0\ 1\ 1\ 0\ 1$
(5)

$$\begin{aligned} A &= 0\ 1\ 0\ 1\ 0\ 1\ 1\ 1 \\ B' &= 0\ 0\ 1\ 0\ 1\ 1\ 0\ 1 \\ \text{So}\quad A - B &= A \text{ AND } (\text{COMP}(B)) \\ &= 0\ 0\ 0\ 0\ 0\ 1\ 0\ 1 \end{aligned}$$

Using the bit representations, you may verify that $A \cap B = \{\text{b}, \text{e}, \text{g}\}$, $A \cup B = \{\text{a}, \text{b}, \text{c}, \text{e}, \text{g}, \text{h}\}$, $A \oplus B = \{\text{a}, \text{c}, \text{h}\}$, $B' = \{\text{a}, \text{c}, \text{d}, \text{f}\}$, and $A - B = \{\text{a}, \text{c}\}$. ■

Exercises 2.3

Using the universal set $U = \{\text{a}, \ldots, \text{h}\}$, represent each set as an 8-bit word.

1. $\{\text{a}, \text{c}, \text{e}, \text{g}\}$ **2.** $\{\text{b}, \text{d}, \text{f}\}$ **3.** $\{\text{a}, \text{e}, \text{f}, \text{g}, \text{h}\}$ **4.** $\emptyset$

Use Algorithm 2.1 to find the subset of the set $\{s_0, s_1, s_2, s_3\}$ that follows the given subset.

5. $\{s_3\}$ **6.** $\{s_0, s_3\}$ **7.** $\{s_2, s_3\}$ **8.** $\{s_0, s_2, s_3\}$

Using Algorithm 2.2, find the subsets of each set.

9. $\{s_0, s_1\}$ **10.** $\{s_0, s_1, s_2, s_3\}$

Using the sets $A = \{\text{a}, \text{b}, \text{e}, \text{h}\}$, $B = \{\text{b}, \text{c}, \text{e}, \text{f}, \text{h}\}$, $C = \{\text{c}, \text{d}, \text{f}, \text{g}\}$, and $U = \{\text{a}, \ldots, \text{h}\}$, find the binary representation of each set.

11. $A \cap B$ **12.** $A \cup B$ **13.** B' **14.** $A - B$

15. $C - B$ **16.** $A \oplus B$ **17.** $B \oplus C$ **18.** $C \oplus A$

19. $A \cap C'$ **20.** $A \cup B'$ **21.** $A \cap (B \cap C)$ **22.** $A \cup (B \cap C)$

23. $A - (B \oplus C)$ **24.** $(A \oplus B) - C$ **25.** $A \oplus (B \oplus C)$ **26.** $(A \oplus B) \oplus C$

2.4 The Cardinality of a Set

This section presents four formulas involving finite sets, which we shall use frequently. Recall that every finite set has a fixed number of elements, so we make the following definition.

Cardinality

The **cardinality** of A, denoted by $|A|$, is the number of elements in it.*

*It should be clear from the context whether the symbol "| |" refers to absolute value or cardinality.

For example, $|\emptyset| = 0$, $|\{\emptyset\}| = 1$, and $|\{a, b, c\}| = 3$.

Let A and B be any two finite sets. How is $|A \cup B|$ related to $|A|$ and $|B|$? First, let's study an example.

EXAMPLE 2.29 Let $A = \{a, b, c\}$ and $B = \{b, c, d, e, f\}$. Clearly, $|A| = 3$, $|B| = 5$, $|A \cup B| = 6$, and $|A \cap B| = 2$, so $|A \cup B| = |A| + |B| - |A \cap B|$. ■

More generally, we have the following result:

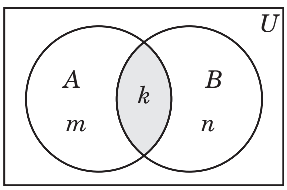
THEOREM 2.1 (**Inclusion–Exclusion Principle**) Let A and B be two finite sets. Then $|A \cup B| = |A| + |B| - |A \cap B|$.

PROOF:

Suppose $|A \cap B| = k$. Since $A \cap B \subseteq A$ and $A \cap B \subseteq B$, we can assume that $|A| = k + m$ and $|B| = k + n$ for some nonnegative integers m and n (see Figure 2.22). Then:

$$\begin{aligned} |A \cup B| &= m + k + n \\ &= (m + k) + (n + k) - k \\ &= |A| + |B| - |A \cap B| \end{aligned}$$

Figure 2.22



This completes the proof. ■

In addition, if A and B are disjoint sets, then $|A \cap B| = |\emptyset| = 0$, so $|A \cup B| = |A| + |B|$. Thus we have the following result.

COROLLARY 2.1
(**Addition Principle**) Let A and B be finite disjoint sets. Then $|A \cup B| = |A| + |B|$.

The next example demonstrates the inclusion–exclusion principle.

EXAMPLE 2.30 Find the number of positive integers ≤ 300 and divisible by 2 or 3.

SOLUTION:

Let $A = \{x \in \mathbb{N} | x \leq 300$ and is divisible by $2\}$ and $B = \{x \in \mathbb{N} | x \leq 300$ and is divisible by $3\}$. Then $A \cap B$ consists of positive integers ≤ 300 that are divisible by 2 and 3, that is, divisible by 6. Thus $A = \{2, 4, \ldots, 300\}$,

$B = \{3, 6, \ldots, 300\}$, and $A \cap B = \{6, 12, \ldots, 300\}$. Clearly, $|A| = 150$, $|B| = 100$, and $|A \cap B| = 50$, so by Theorem 2.1,

$$\begin{aligned}|A \cup B| &= |A| + |B| - |A \cap B| \\ &= 150 + 100 - 50 = 200\end{aligned}$$

Thus there are 200 positive integers ≤ 300 and divisible by 2 or 3. (See Examples 3.11, and 3.12 in Section 3.2.)

Theorem 2.1 can be extended to any finite number of finite sets. For instance, the next example derives the formula for three finite sets.

EXAMPLE 2.31 Let A, B, and C be three finite sets. Prove that

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C|.$$

PROOF:

$$\begin{aligned}|A \cup B \cup C| &= |A \cup (B \cup C)| \\ &= |A| + |B \cup C| - |A \cap (B \cup C)| && \text{by Theorem 2.1} \\ &= |A| + |B \cup C| - |(A \cap B) \cup (A \cap C)| && \text{by the distributive law} \\ &= |A| + (|B| + |C| - |B \cap C|) - [|A \cap B| + |A \cap C| \\ &\quad - |(A \cap B) \cap (A \cap C)|] \\ &= |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C|,\end{aligned}$$

since $A \cap C = C \cap A$ and $(A \cap B) \cap (A \cap C) = A \cap B \cap C$. ■

The next example shows how useful sets are in data analysis.

EXAMPLE 2.32 A survey among 100 students shows that of the three ice cream flavors vanilla, chocolate, and strawberry, 50 students like vanilla, 43 like chocolate, 28 like strawberry, 13 like vanilla and chocolate, 11 like chocolate and strawberry, 12 like strawberry and vanilla, and 5 like all of them. Find the number of students surveyed who like each of the following flavors.

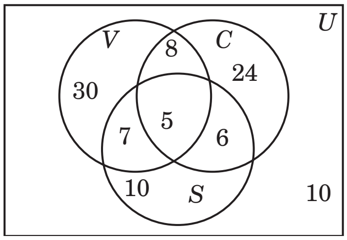
(1) Chocolate but not strawberry.
(2) Chocolate and strawberry, but not vanilla.
(3) Vanilla or chocolate, but not strawberry.

SOLUTION:

Let V, C, and S symbolize the set of students who like vanilla, chocolate, and strawberry flavors, respectively. Draw three intersecting circles to represent them in the most general case, as in Figure 2.23.

Our first goal is to distribute the 100 students surveyed into the various regions. Since five students like all flavors, $|V \cap C \cap S| = 5$. Twelve students like both strawberry and vanilla, so $|S \cap V| = 12$. But five of them like chocolate also. Therefore, $|(S \cap V) - C| = 7$. Similarly, $|(V \cap C) - S| = 8$ and $|(C \cap S) - V| = 6$.

Figure 2.23



Of the 28 students who like strawberry, we have already accounted for $7+5+6=18$. So the remaining 10 students belong to the set $S-(V\cup C)$. Similarly, $|V-(C\cup S)|=30$ and $|C-(S\cup V)|=24$.

Thus far, we have accounted for 90 of the 100 students. The remaining 10 students lie outside the region $V\cup S\cup C$, as in Figure 2.23. The required answers can now be directly read from this Venn diagram:

(1) $|C-S| = 24+8 = 32$. So 32 students like chocolate but not strawberry.

(2) $|(C\cap S)-V| = 6$. Therefore, 6 students like both chocolate and strawberry, but not vanilla.

(3) $30+8+24 = 62$ students like vanilla or chocolate, but not strawberry. They are represented by the region $(V\cup C)-S$. ■

Finally, suppose a set contains n elements. How many subsets does it have? Before we answer this partially, let us study the next example, which uses the addition principle.

EXAMPLE 2.33 Let s_3 denote the number of subsets of the set $S=\{\mathrm{a,b,c}\}$. Let $S^*=S-\{\mathrm{b}\}$. We shall use the subsets of S^* in a clever way to find s_3 and all subsets of S. Let A denote the subsets of S^*. Then $A=\{\emptyset,\{\mathrm{a}\},\{\mathrm{c}\},\{\mathrm{a,c}\}\}$. Clearly every element of A is also a subset of S.

Now add b to every element in A. Let B denote the resulting set: $B=\{\{\mathrm{b}\},\{\mathrm{b,a}\},\{\mathrm{b,c}\},\{\mathrm{b,a,c}\}\}$. Every subset of S either contains b or does not contain b; so, by the addition principle, $s_3=|A|+|B|=4+4=8$. ■

More generally, we have the following result.

THEOREM 2.2 Let s_n denote the number of subsets of a set S with n elements. Then $s_n=2s_{n-1}$, where $n\geq 1$.

PROOF:

Let $x\in S$. Let $S^*=S-\{x\}$. Then S^* contains $n-1$ elements and hence has s_{n-1} subsets by definition. Each of them is also a subset of S. Now insert x in each of them. The resulting s_{n-1} sets are also subsets of S. Since every subset of S either contains x or does not contain x, the addition principle indicates a total of $s_{n-1}+s_{n-1}=2s_{n-1}$ subsets of S. (Notice that $s_0=1$. Why?) ■

Consequently, if you know the number of subsets of a set with $n - 1$ elements, this theorem can be employed to compute the number of subsets of a set with n elements. For instance, by Example 2.33, a set with three elements has eight subsets; therefore, a set with four elements has $2 \cdot 8 = 16$ subsets.

The technique used in the proof of Theorem 2.2 can be applied to write an algorithm for finding the power set of a set S. See Algorithm 2.3. It uses the fact that if A is a subset of S and $s \in S$, then $A \cup \{s\}$ is also a subset of S.

```
Algorithm subsets(S)
(* This algorithm finds the power set of a set S with n elements
   s1, s2, ..., sn. Sj denotes the jth element in the power set. *)
   Begin (* subsets *)
     power set ← {Ø}    (* initialize power set *)
     numsubsets ← 1 (* initialize the number of subsets *)
     for i = 1 to n do (* si denotes the ith element in S *)
     begin (* for *)
       j ← 1            (* j-th element in P(S) *)
       temp ← numsubsets (* temp is a temporary variable*)
       while j ≤ temp do  (* construct a new subset *)
       begin (* while *)
         add Sj ∪ {si} to the power set
         j ← j + 1
         numsubsets ← numsubsets + 1
       endwhile
     endfor
   End (* subsets *)
```

Algorithm 2.3 ■

Although Theorem 2.2 does not give us an explicit formula for the number of subsets, it can be used to find the formula. The next theorem gives us the explicit formula, which we shall prove in Section 4.4 (see Example 4.18).

THEOREM 2.3 A set with n elements has 2^n subsets, where $n \geq 0$. ■

For example, a set with four elements has $2^4 = 16$ subsets!

Exercises 2.4

Find the cardinality of each set.

1. The set of letters of the English alphabet.
2. The set of letters of the word TWEEDLEDEE.
3. The set of months of the year with 31 days.
4. The set of identifiers in Java that begin with 3.

Let A and B be two sets such that $|A| = 2a - b, |B| = 2a$, $|A \cap B| = a - b$, and $|U| = 3a + 2b$. Find the cardinality of each set.

5. $A \cup B$ **6.** $A - B$ **7.** B' **8.** $A - A'$

9. Find $|A|$ if $|A| = |B|$, $|A \cup B| = 2a + 3b$, and $|A \cap B| = b$.

10. Find $|A \cap B|$ if $|A| = a + b = |B|$ and $|A \cup B| = 2a + 2b$.

11. Find $|A \cap B|$ if $|A| = 2a$, $|B| = a$, and $|A \cup B| = 2a + b$.

Let A and B be finite sets such that $A \subseteq B$, $|A| = b$, $|B| = a + b$. Find the cardinality of each set.

12. $A \cup B$ **13.** $A - B$ **14.** $B - A$ **15.** $A \cap B$

Let A and B be finite disjoint sets, where $|A| = a$, and $|B| = b$. Find the cardinality of each set.

16. $A \cup B$ **17.** $A - B$ **18.** $B - A$

19–21. Find the cardinality of each set in Exercises 16–18, where $A \subseteq B$, B is finite, $|A| = a$, and $|B| = b$.

22. A survey conducted recently among 300 adults in Omega City shows 160 like to have their houses painted green, and 140 like them blue. Seventy-five adults like both colors. How many do not like either color?

23. A survey was taken to determine the preference between two laundry detergents, Lex and Rex. It was found that 15 people liked Lex only, 10 liked both, 20 liked Rex only, and 5 liked neither of them. How many people were surveyed?

Find the number of positive integers ≤ 500 and divisible by:

24. Two or three. **25.** Two, three, or five.

26. Two or three, but not six. **27.** Neither two, three, nor five.

Find the number of positive integers ≤ 1776 and divisible by:

28. Two, three, or five. **29.** Two, three, or five, but not six.

30. Two, three, or five, but not 15. **31.** Two, three, or five, but not 30.

According to a survey among 160 college students, 95 students take a course in English, 72 take a course in French, 67 take a course in German, 35 take a course in English and in French, 37 take a course in French and in German, 40 take a course in German and in English, and 25 take a course in all three languages. Find the number of students in the survey who take a course in:

32. English, but not German. **33.** English, French, or German.

34. English or French, but not German.

35. English and French, but not German.

36. English, but neither French nor German.

37. Neither English, French, nor German.

A recent survey by the MAD corporation indicates that of the 700 families interviewed, 220 own a television set but no stereo, 200 own a stereo but no camera, 170 own a camera but no television set, 80 own a television set and a stereo but no camera, 80 own a stereo and a camera but no television set, 70 own a camera and a television set but no stereo, and 50 do not have any of these. Find the number of families with:

38. Exactly one of the items.

39. Exactly two of the items.

40. At least one of the items.

41. All of the items.

Using Algorithm 2.3, find the power set of each set. List the elements in the order obtained.

42. $\{a, b\}$

43. $\{a, b, c\}$

A finite set with a elements has b subsets. Find the number of subsets of a finite set with the given cardinality.

44. $a+1$

45. $a+2$

46. $a+5$

47. $2a$

Let A, B, and C be subsets of a finite set U. Derive a formula for each.

48. $|A' \cap B'|$

49. $|A' \cap B' \cap C'|$

***50.** State the inclusion–exclusion principle for four finite sets A_i, $1 \leq i \leq 4$. (The formula contains 15 terms.)

***51.** Prove the formula in Exercise 50.

****52.** State the inclusion–exclusion principle for n finite sets A_i, $1 \leq i \leq n$.

2.5 Recursively Defined Sets

A new way of defining sets is using recursion. (It is a powerful problem-solving technique discussed in detail in Chapter 5.)

Notice that the set of numbers $S = \{2, 2^2, 2^{2^2}, 2^{2^{2^2}}, \ldots\}$ has three interesting characteristics:

(1) $2 \in S$.

(2) If $x \in S$, then $2^x \in S$.

(3) Every element of S is obtained by a finite number of applications of properties 1 and 2 only.

Property 1 identifies explicitly the primitive element in S and hence ensures that it is nonempty. Property 2 establishes a systematic procedure to construct new elements from known elements. How do we know, for instance, that $2^{2^2} \in S$? By property 1, $2 \in S$; then, by property (2), $2^2 \in S$; now choose $x = 2^2$ and apply property 2 again; so $2^{2^2} \in S$. Property 3 guarantees that in no other way can the elements of S be constructed. Thus the various elements of S can be obtained systematically by applying the above properties.

These three characteristics can be generalized and may be employed to define a set S implicitly. Such a definition is a recursive definition.

Recursively Defined Set

A **recursive definition** of a set S consists of three clauses:

- The **basis clause** explicitly lists at least one primitive element in S, ensuring that S is nonempty.
- The **recursive clause** establishes a systematic recipe to generate new elements from known elements.
- The **terminal clause** guarantees that the first two clauses are the only ways the elements of S can be obtained.

The terminal clause is generally omitted for convenience.

EXAMPLE 2.34 Let S be the set defined recursively as follows.

(1) $2 \in S$. (2) If $x \in S$, then $x^2 \in S$.

Describe the set by the listing method.

SOLUTION:

- $2 \in S$, by the basis clause.
- Choose $x = 2$. Then by the recursive clause, $4 \in S$.
- Now choose $x = 4$ and apply the recursive clause again, so $16 \in$ S. Continuing like this, we get $S = \{2, 4, 16, 256, 65536, \ldots\}$. ■

The next three examples further elucidate the recursive definition.

EXAMPLE 2.35 Notice that the language $L = \{\text{a}, \text{aa}, \text{ba}, \text{aaa}, \text{aba}, \text{baa}, \text{bba}, \ldots\}$ consists of words over the alphabet $\Sigma = \{\text{a}, \text{b}\}$ that end in the letter a. It can be defined recursively as follows.

- $\text{a} \in L$.
- If $x \in L$, then $\text{a}x, \text{b}x \in L$.

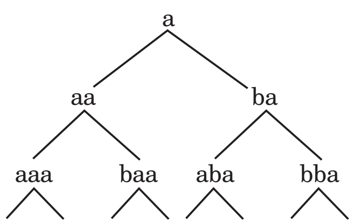
For instance, the word aba can be constructed as follows:

- $\text{a} \in L$. Choosing $x = \text{a}$, $\text{b}x = \text{ba} \in L$.

- Now choose $x = \text{ba}$. Then $\text{a}x = \text{aba} \in L$.

The tree diagram in Figure 2.24 illustrates systematically how to derive the words in L.

Figure 2.24

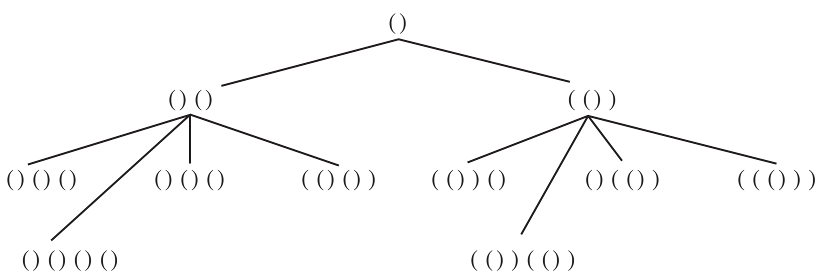


EXAMPLE 2.36 **(Legally Paired Parentheses)** An important problem in computer science is to determine whether or not a given expression is legally parenthesized. For example, (( )), ( ) ( ), and (( ) ( )) are validly paired sequences of parentheses, but) ( ), ( ) (, and) ( ) (are not. The set S of sequences of legally paired parentheses can be defined recursively as follows:

- $() \in S$.
- If $x, y \in S$, then xy and (x) belong to S.

The tree diagram in Figure 2.25 shows the various ways of constructing the elements in S.

Figure 2.25



A simplified recipe to determine if a sequence of parentheses is legally paired is given in Algorithm 2.4.

```
Algorithm Legally Paired Sequence
(* This algorithm determines if a nonempty sequence of parentheses is
   legally paired. Count keeps track of the number of parentheses. It is
   incremented by 1 if the current parenthesis is a left parenthesis.
   and decremented by 1 if it is a right parenthesis. *)
   Begin (* algorithm *)
     count ← 0                (* initialize *)
     read a symbol
     if symbol = left paren then
```

```
      while not the end of the sequence do
      begin (* while *)
        if symbol = left paren then
          count ← count + 1
        else  (* symbol = right parenthesis *)
          count ← count − 1
        read the next symbol
      endwhile
      if count = 0 then
        legal sequence
      else
        invalid sequence
  else
    invalid sequence (* begins with a right paren *)
  End (* algorithm *)
```

Algorithm 2.4

This example is studied further in Chapters 6 and 9. ■

EXAMPLE 2.37 A legal expression in propositional logic is called a **well-formed formula** (wff). For convenience, we restrict our discussion to the logical variables p, q, and r, and the operators $\wedge$, $\vee$, and $\sim$. Then the set of well-formed formulas can be defined recursively:

- The logic variables are wffs.
- If x and y are wffs, then so are (x), $\sim(x)$, $(x \wedge y)$, and $(x \vee y)$.

For instance, the expression $((p) \wedge ((\sim(q)) \vee (r)))$ is a wff, but $(q \wedge (\sim r))$ is not (why?). (Parentheses are often omitted when ambiguity is impossible.) ■

Exercises 2.5

In Exercises 1–6, a set S is defined recursively. Find four elements in each case.

1. i) $1 \in S$
ii) $x \in S \to 2x \in S$

2. i) $1 \in S$
ii) $x \in S \to 2^x \in S$

3. i) $e \in S$
ii) $x \in S \to e^x \in S$

4. i) $3 \in S$
ii) $x \in S \to \lg x \in S$[†]

5. i) $\lambda \in L$
ii) $x \in L \to x\text{bb} \in L$

6. i) $\lambda \in L$
ii) $x \in L \to \text{a}x\text{b} \in L$

[†] $\lg x$ means $\log_2 x$.

In Exercises 7–10, identify the set S that is defined recursively.

7. i) $1 \in S$
ii) $x, y \in S \to x + y \in S$

8. i) $1 \in S$
ii) $x, y \in S \to x \pm y \in S$

9. i) $2 \in S$
ii) $x, y \in S \to x \pm y \in S$

10. i) $\emptyset \in S$
ii) $x \in X, A \in S \to \{\mathrm{x}\} \cup A \in S$

Define each language L over the given alphabet recursively.

11. $\{0, 00, 10, 100, 110, 0000, 1010, \ldots\}$, $\Sigma = \{0, 1\}$.

12. $L = \{1, 11, 111, 1111, 11111, \ldots\}$, $\Sigma = \{0, 1\}$.

13. $L = \{x \in \Sigma^* | x = \mathrm{b}^n\mathrm{ab}^n, n \geq 0\}$, $\Sigma = \{\mathrm{a}, \mathrm{b}\}$.

14. The language L of all palindromes over $\Sigma = \{\mathrm{a}, \mathrm{b}\}$. (A **palindrome** is a word that reads the same both forwards and backwards. For instance, abba is a palindrome.)

***15.** $\{\mathrm{b}, \mathrm{bb}, \mathrm{bbb}, \mathrm{bbbb}, \ldots\}$, $\Sigma = \{\mathrm{a}, \mathrm{b}\}$.

***16.** $\{\mathrm{b}, \mathrm{aba}, \mathrm{aabaa}, \mathrm{aaabaaa}, \ldots\}$, $\Sigma = \{\mathrm{a}, \mathrm{b}\}$.

***17.** $\{\mathrm{a}, \mathrm{aaa}, \mathrm{aaaaa}, \mathrm{aaaaaaa}, \ldots\}$, $\Sigma = \{\mathrm{a}, \mathrm{b}\}$.

***18.** $\{1, 10, 11, 100, 101, \ldots\}$, $\Sigma = \{0, 1\}$.

Determine if each sequence of parentheses is legal.

19. (()()) **20.** (())(**21.** (()() **22.** (()())()

The nth **Catalan number** C_n, named after the Belgian mathematician, Eugene Charles Catalan (1814–1894), is defined by

$$\mathrm{C}_n = \frac{(2n)!}{n!(n+1)!}, \quad n \geq 0$$

where $n!$ (n **factorial**) is defined by $n! = n(n-1)\ldots 3 \cdot 2 \cdot 1$ and $0! = 1$. Catalan numbers have many interesting applications in computer science. For example, the number of well-formed sequences of n pairs of left and right parentheses is given by the nth Catalan number. Compute the number of legally paired sequences with the given pairs of left and right parentheses.

23. Three **24.** Four **25.** Five **26.** Six

27. List the well-formed sequences of parentheses with three pairs of left and right parentheses.

28. Redo Exercise 27 with four pairs of left and right parentheses.

Using Example 2.37, determine if each is a wff in propositional logic.

29. $(p \wedge ((\sim(q)) \vee r))$

30. $((\sim(p)) \vee ((q) \wedge (\sim r))$

31. $(((\sim p) \vee q) \wedge (\sim q) \vee (\sim p)))$ **32.** $((p \vee q) \wedge ((\sim(q)) \vee (\sim(r))))$

33. Determine if the following recursive definition yields the set S of legally paired parentheses. If not, find a validly paired sequence that cannot be generated by this definition.

i) $() \in S$. ii) If $x \in S$, then $()x$, (x), $x() \in S$.

34. Define the set of words S over an alphabet Σ recursively. Assume $\lambda \in S$.
(*Hint*: use concatenation.)

35. Let Σ be an alphabet. Define Σ^* recursively.
(*Hint*: use concatenation.)

***36.** Define the language L of all binary representations of nonnegative integers recursively.

Chapter Summary

This chapter presented the concept of a set, different ways of describing a set, relations between sets, operations with sets and their properties, and formal languages. How sets and set operations work in a typical computer were also discussed.

Set

- A **set** is a well-defined collection of objects (page 68).
- A set can be described using words, listing the elements, or by the set-builder notation (page 69).
- $A \subseteq B$ if and only if every element of A is also an element of B (page 69).
- $(A = B) \leftrightarrow (A \subseteq B) \wedge (B \subseteq A)$ (page 70).
- The **null set** Ø contains no elements (page 70).
- The **universal set** U contains all elements under discussion (page 70).
- A and B are **disjoint sets** if $A \cap B = \emptyset$ (page 71).
- The **power set** $P(A)$ of a set A is the family of all subsets of A (page 72).
- A set with a definite number of elements is **finite**; if a set is not finite, it is **infinite** (page 73).

Formal Language

- An **alphabet** Σ is a finite set of symbols; $\{0,1\}$ is the binary alphabet (page 75).
- A **word** over Σ is a finite arrangement of symbols from Σ. A word of length zero is the empty word λ (page 75).
- Σ^* consists of all possible words over Σ (page 75).
- A **formal language** over Σ is a subset of Σ^* (page 75).
- The **concatenation** of two words x and y is the word xy. (page 76).

Set Operations

- **Union** $A \cup B = x|(x \in A) \vee (x \in B)$ (page 78).
- **Intersection** $A \cap B = \{x|(x \in A) \wedge (x \in B)\}$ (page 78).
- **Difference** $A - B = \{x \in A|x \notin B\}$ (page 80).
- **Complement** $A' = U - A = \{x \in U|x \notin A\}$ (page 81).
- **Symmetric difference** $A \oplus B = (A - B) \cup (B - A)$ (page 82).
- **Cartesian product** $A \times B = \{(a,b)|(a \in A) \wedge (b \in B)\}$ (page 87).
- The fundamental properties of set operations are listed in Table 2.2 (page 83).

Partition

- A **partition** of a set S is a finite collection of nonempty, pairwise disjoint subsets of S whose union is S (page 90).

Computer Implementation

- Set operations are implemented in a computer using the bit operations in Table 2.5 and the logic operations in Table 2.6. (page 97).

Cardinality

- **Inclusion–exclusion principle** $|A \cup B| = |A| + |B| - |A \cap B|$ (page 99).
- **Addition principle** $|A \cup B| = |A| + |B|$, where $A \cap B = \emptyset$ (page 99).
- A set with n elements has 2^n subsets (page 102).

Recursion

- The **recursive** definition of a set consists of a basis clause, recursive clause, and a terminal clause (page 105).

Review Exercises

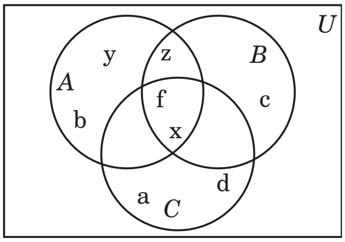
Using the Venn diagram in Figure 2.26, find each.

1. $A - (B \cap C)$ **2.** $(A \cup B) - C$ **3.** $A - (B - C)$

4. $(A - B) - C$ **5.** $A \oplus B$ **6.** $(A - B) \times (B - C)$

7. $A - (B \oplus C)$ **8.** $A \cup (B \oplus C')$

Figure 2.26



9. Find the sets A and B if $A \cap B' = \{\text{a}, \text{c}\}$, $B \cap A' = \{\text{b}, \text{e}, \text{g}\}$, $A \cap B = \{\text{d}, \text{f}\}$, and $A' \cap B' = \{\text{i}\}$.

Let A, B, and C be sets such that $A - (B \cup C) = \{\text{b,e}\}$, $B - (C \cup A) = \{\text{k}\}$, $C - (A \cup B) = \{\text{h}\}$, $A \cap B = \{\text{f}, \text{g}\}$, $B \cap C = \{\text{j}\}$, $C \cap A = \{\text{i}\}$, and $A \cap B \cap C = \emptyset$. Find each set.

10. $A - (B \cap C)$ **11.** $(A \oplus B) - C$ **12.** $A \oplus (B \oplus C)$

Find the power set of each set.

13. $\{\emptyset, \{\emptyset\}\}$ ***14.** $\{2, \{3\}, \{2, 3\}\}$

15. Let $A = \{n \in \mathbb{N} | n < 20$ and n is divisible by $2\}$, $B = \{n \in \mathbb{N} | n < 20$ and n is divisible by $3\}$, and $C = \{n \in \mathbb{N} | n < 20$ and n is divisible by $5\}$. Determine if they form a partition of the set $\{n \in \mathbb{N} | n < 20\}$.

○ Let $U = \{1, \ldots, 8\}$, $A = \{1, 3, 5, 7, 8\}$, and $B = \{2, 3, 6, 7\}$. Find the binary representation of each set.

16. $A - (A \cap B)$ **17.** $A - B'$ **18.** $A - (A \oplus B)$ **19.** $A \oplus (A \oplus B)$

A survey found that 45% of women like plain yogurt, 55% like flavored yogurt, and 23% like both. Compute the percentage of women who like each.

20. Plain yogurt, but not flavored.

21. Plain or flavored yogurt, but not both.

A survey was taken among the students on campus to find out whether they prefer vanilla or strawberry ice cream and whether they prefer chocolate or

Table 2.7

		Pudding			
		Chocolate	Tapioca	Neither	Total
Ice Cream	Vanilla	68	53	12	133
	Strawberry	59	48	9	116
	Neither	23	21	7	51
	Total	150	122	28	300

tapioca pudding. The results are summarized in Table 2.7. Find the number of students who:

22. Like strawberry ice cream and tapioca pudding.

23. Do not like pudding.

24. Like at least one of the ice cream flavors.

25. Like neither ice cream nor pudding.

Find the number of positive integers ≤ 4567 and divisible by:

26. Two, three, or five.

27. Two, five, or seven, but not 35.

Find four elements in each set S defined recursively.

28. i) $1 \in S$
ii) $x \in S \to 1 + x \in S$

29. i) $3 \in S$
ii) $x \in S \to \lg x \in S$

30. i) $\sqrt{2} \in S$
ii) $x \in S \to \sqrt{2 + x} \in S$

31. i) $1 \in S$
ii) $x \in S \to \sqrt{1 + 2x} \in S$

Define each set S recursively.

32. $\{2, 4, 16, 256, \ldots\}$

33. $\{1, 3, 7, 15, 31, \ldots\}$

34. $\{\mathrm{b}, \mathrm{ba}^2, \mathrm{ba}^4, \mathrm{ba}^6, \ldots\}$

35. $\{\lambda, ba, b^2a^2, b^3a^3, \ldots\}$

Find five words in each language L over the alphabet $\Sigma = \{\mathrm{a}, \mathrm{b}\}$.

36. $\{x \in \Sigma^* | x \text{ contains exactly one } a\}$

37. $\{x \in \Sigma^* | x \text{ contains an odd number of } a\text{'s}\}$

Define each language L over the given alphabet recursively.

38. $\{x \in \Sigma^* | x \text{ contains exactly one } a\}$, $\Sigma = \{\mathrm{a}, \mathrm{b}\}$.

39. $\{x \in \Sigma^* | x \text{ ends in ab}\}$, $\Sigma = \{a, b\}$.

40. $\{2, 3, 4, 5, 6, \ldots\}$, $\Sigma = \{2, 3\}$.

41. $\{1, 010, 00100, 0001000, 000010000, \ldots\}$, $\Sigma = \{0, 1\}$.

Determine if each is a well-formed formula.

42. $(p \wedge ((\sim(q)) \vee (r)))$

43. $(((p) \wedge (q)) \vee (\sim(q) \wedge (r)))$

Let A, B, and C be any sets. Prove each.

***44.** $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

***45.** $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

***46.** $A \cup (B - C) = (A \cup B) - (C - A)$

***47.** $A \cap (B - C) = (A \cap B) - (A \cap C)$

Simplify each set expression.

48. $(A' \cup B')' \cup (A' \cap B)$

***49.** $[A - (B \cup C)] \cap [(B \cap C) - A]$

∘ Consider the fuzzy sets, where $A = \{\text{Mike } 0.6, \text{ Andy } 0.3, \text{ Jeff } 0.7\}$ and $B = \{\text{Jean } 0.8, \text{ June } 0.5\}$. Find each fuzzy set.

50. $A \cup B'$ **51.** $A' \cap B$ **52.** $A \oplus B'$ **53.** $A \times B$

∘ Let A and B be any fuzzy sets. Prove each.

***54.** $(A \cup B)' = A' - B$

***55.** $(A - B)' = A' \cup B$

Supplementary Exercises

Prove each, where A, B, and C are arbitrary sets.

1. $A - (B \cup C) = (A - B) \cap (A - C)$

2. $[A \cap (A - B)] \cup (A' \cup B)' = A - B$

***3.** $A \cap (B \oplus C) = (A \cap B) \oplus (A \cap C)$

***4.** $A \oplus (B \oplus C) = (A \oplus B) \oplus C$

Simplify each set expression.

5. $(A \cap B) \cap (B \cap C) \cap (C \cap A)$

***6.** $[(A \cup B) \cap C] \cup [A \cap (B \cup C)]$

***7.** $(A \cup B') \cap (A' \cup B) \cap (A' \cup B')$

***8.** $[(A \cup B') \cup (A' \cup B)]' \cap (A' \cap B')$

Find the number of positive integers ≤ 1000 and *not* divisible by:

9. 2, 3, or 5.

***10.** 2, 3, 5, or 7.

11. Define recursively the language $\{0^n 1^n | n \geq 0\}$ over $\Sigma = \{0, 1\}$.

12. Define recursively a word w over a finite alphabet Σ.

Let $x = x_1 x_2 \ldots x_n \in \sum^*$. Then the string $x_n \ldots x_2 x_1$ is called the **reverse** of x, denoted by x^{R}. For example, the reverse of the binary word 01101 is 10110. Let $x, y \in \sum^n$. Prove each.

13. $(xy)^{\mathrm{R}} = y^{\mathrm{R}} x^{\mathrm{R}}$

14. The string x is palindromic if and only if $x^{\mathrm{R}} = x$.

15. The word xx^{R} is palindromic.

Computer Exercises

Write a program to do each task, where n denotes a positive integer ≤ 20.

1. Read in k subsets of the set $S = \{1, 2, \ldots, n\}$ and determine if the subsets form a partition of S.

2. Read in two sets A and B, where $U = \{1, 2, 3, \ldots, n\}$. Print the bit-representations of A and B. Use them to find the elements in $A \cup B$, $A \cap B$, A', $A - B$, $A \oplus B$, and $A \times B$, and their cardinalities.

3. Find all subsets of the set $\{1, 2, \ldots, n\}$.

4. Read in sequences of left and right parentheses, each containing at most 25 symbols. Determine if each word consists of legally paired parentheses.

5. Print the Catalan numbers C_0 through C_n.

Exploratory Writing Projects

Using library and Internet resources, write a team report on each of the following in your own words. Provide a well-documented bibliography.

1. Write an essay on the life and contributions of G. Cantor.

2. Explain the various occurrences of the ordered pair notation in everyday life.

3. Explain how the addition principle is used to define the addition of positive integers. Give concrete examples.

4. Explain how the concept of partitioning is used in everyday life. In sports. In computer science. Give concrete examples.

5. Study a number of mathematical paradoxes and explain them.

6. Discuss the various string operations and list the programming languages that support them.

7. Describe fuzzy sets and their applications, and L. A. Zadeh's contributions to them.

8. Write a biography of Abu-Abdullah Muhammed ibn-Musa al-Khowarizmi and the origin of the word *algorithm*.

9. Extend the concept of the cardinality of a finite set to infinite sets. Describe the arithmetic of transfinite cardinal numbers.

10. Discuss the halting problem.

Enrichment Readings

1. R. R. Christian, *Introduction to Logic and Sets*, Blaisdell, Waltham, MA, 1965.
2. M. Guillen, *Bridges to Infinity: The Human Side of Mathematics*, J. P. Tarcher, Inc., Los Angeles, 1983, pp. 41–60.
3. P. R. Halmos, *Naive Set Theory*, Van Nostrand, New York, 1960.
4. S. Sahni, *Concepts in Discrete Mathematics*, 2nd ed., Camelot, Fridley, MN, 1985, pp. 93–111.
5. P. C. Suppes, *Axiomatic Set Theory*, Van Nostrand, New York, 1960.
6. R. L. Wilder, *Evolution of Mathematical Concepts: An Elementary Study*, Wiley, New York, 1968.

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