Discrete Mathematics

Tutorial sheet | Propositional Logic

Question 1.
Which of the following statements are propositions:

1. $2 + 2 = 4$
2. $2 + 2 = 5$
3. $x^2 + 2 = 11$
4. $x + y > 0$
5. This coffee is strong

Solution:

1. $2 + 2 = 4$ is a proposition as it is always true.
2. $2 + 2 = 5$ is a proposition as it is always false
3. $x^2 + 2 = 11$ is **NOT** a proposition as its value depends on the value of x, it is true for x = 3 and x =-3, whereas for other values it is false.
4. $x + y > 0$ is **NOT** a proposition as its value depends on the values of x and y, when x =-1 and y= 2 it is true, whereas when x = -1 and y = -1 it is false.
5. This coffee is strong. is **NOT** a proposition as it is matter of opinion, the coffee may be strong for somebody and may not for somebody else.

Question 2.
Let s and i be the following propositions:
s: "stocks are increasing"
i: "interest rates are steady"
Write each of the following sentences symbolically:

1. Stocks are increasing but interest rates are steady
2. Neither are stocks increasing nor are interest rates steady

Solution:

1. Stocks are increasing but interest rates are steady. $= s \wedge i$
2. Neither are stocks increasing nor are interest rates steady $= \neg s \wedge \neg i = \neg(s \vee i)$

Question 3.
Let h, s and r be the following three propositions:
h:"It is hot"
s:"It is sunny"
r:"It is raining"
Write each of the following sentences symbolically:

1. It is not hot but it is sunny
2. It is neither hot nor sunny
3. It is either hot and sunny or it is raining
4. It is sunny or it is raining but not both

Solution:

1. It is not hot but it is sunny $= \neg h \wedge s$
2. It is neither hot nor sunny $= \neg h \wedge \neg s = \neg(h \vee s)$
3. It is either hot and sunny or it is raining $= (h \wedge s) \vee r$
4. It is sunny or it is raining but not both. $= s \oplus r$

Question 4.
Let l denote one of the letters in the word "$software$". The following propositions relate to l

$$p : \text{"}l \text{ is a vowel"}; \quad q : \text{"}l \text{ comes after the letter } k \text{ in the alphabet"}.$$

Use the *listing method* to specify the truth sets corresponding to each of the following statements:

$$\neg q; \qquad p \wedge \neg q; \qquad \neg p \vee q.$$

Solution:

$q = \{o, s, t, r, w\}$	$\neg q = \{a, e, f\}$
$p = \{a, e, o\}$	$p \wedge \neg q = \{a, e\}$
$\neg p = \{s, f, t, w, r\}$	$\neg p \vee q = \{o, f, t, w, r, s\}$

Question 5.
Let p and q be two propositions. Construct a truth table to show the truth value of each of the following logical statements:

$$p \vee q, \qquad \neg p \vee \neg q, \qquad p \wedge q, \qquad \neg(p \wedge q)$$

What can we say about the following two statements: $\neg p \vee \neg q$ and $\neg(p \wedge q)$?
Solution:

p	q	$\neg p$	$\neg q$	$p \vee q$	$p \wedge q$	$\neg(p \wedge q)$	$\neg p \vee \neg q$
0	0	1	1	0	0	1	1
0	1	1	0	1	0	1	1
1	0	0	1	1	0	1	1
1	1	0	0	1	1	0	0

The columns for $\neg(p \wedge q)$ and $\neg p \vee \neg q$ are the same. Hence, $\neg(p \wedge q)$ and $\neg p \vee \neg q$ are equivalent statements (De Morgan's Law)
The same way we can show the second De Morgan's statement

$$\neg(p \vee q) = \neg p \wedge \neg q$$

Question 6.

Let h, s and r be the following three propositions:
h:"It is hot"
s:"It is sunny"
r:"It is raining"
Write each of the following sentences symbolically:

1. It is sunny or it is raining but not both
2. It is hot only if it is sunny
3. It is hot only if it is sunny and not raining.

Solution:

1. It is sunny or it is raining but not both $= s \oplus r$
2. It is hot only if it is sunny $= h \to s$
3. It is hot only if it is sunny and not raining. $= h \to (s \wedge \neg r)$

Question 7.

Let p, q be propositions. Construct a truth table to show the truth value of each of the following statements:

$$p \to q, \qquad \neg p \vee q, \qquad \neg q \to \neg p.$$

What can we say the above three logical statements?
Solution:

p	q	$\neg p$	$\neg q$	$p \to q$	$\neg p \vee q$	$\neg q \to \neg p$
0	0	1	1	1	1	1
0	1	1	0	1	1	1
1	0	0	1	0	0	0
1	1	0	0	1	1	1

The columns of the three statements are the same and hence,

$$p \to q = \neg p \vee q = \neg q \to \neg p$$

$\neg q \to \neg p$ is the contrapositive of $p \to q$.

Question 8.

Let p and q be the following propositions concerning a positive integer n.

$$p: \text{"}n \text{ is divisible by 5"}; \quad q: \text{"}n \text{ is even"}.$$

1. Express in words the following statements.

$$(i)\ p \vee \neg q; \quad (ii)\ p \wedge q.$$

2. List the elements of the truth sets corresponding to each of the statements in (1).

3. Express each of the following conditional statements symbolically.

$$(i)\ \textit{if } n \textit{ is odd then } n \textit{ is divisible by } 5.$$
$$(ii)\ n \textit{ is even or } n \textit{ is divisible by } 5 \textit{ but not both.}$$

Solution:

1. $p \vee \neg q$ "n is divisible by 5 or an odd number"
   $p \wedge q$ "n is an even number divisible by 5"
2. $p \vee \neg q = \{1, 3, 5, 7, 9, 10, 11, 13, \cdots\}$
   $p \wedge q = \{10, 20, 30, 40, \cdots\}$
3. (i) $\neg q \to p$ (ii) $p \oplus q$

Question 9.

Let p and q be two propositions. Show that $p \vee \neg(p \wedge q)$ is a tautology.

Solution:

$p \vee \neg(p \wedge q) = p \vee \neg p \vee \neg q = T \vee q = T$

You can also show this using a truth table.

Question 10.

Copy and complete the following table by giving the truth value of each of the statements $p,\ q,\ p \to q,\ q \to p$ and $p \leftrightarrow q$.

p	q	$p \to q$	$q \to p$	$p \leftrightarrow q$
0	0			
0	1			
1	0			
1	1			

Solution:

p	q	$p \to q$	$q \to p$	$p \leftrightarrow q$
0	0	1	1	1
0	1	1	0	0
1	0	0	1	0
1	1	1	1	1

Question 11.

Write the inverse, the converse and the contrapositive of the following statement.

If it is November 5th then we have fireworks.

Solution:

Inverse: if it is not November the 5^{th} then we don't have fireworks

Converse: if we have fireworks then it is November the 5^{th}

Contrapositive: If we don't have fireworks then it is not November the 5th.

Question 12.

Let p denote the following statement about integers n:

If n is divisible by 15, then it is divisible by 3 or divisible by 5.

Write the inverse, the converse and the contrapositive of p.

Solution:

Inverse: If n is not divisible by 15, then it isn't divisible by either 3 or 5.

Converse: If n is divisible by 3 or divisible by 5, then it is divisible by 15.

Contrapositive: If n is not divisible by 3 and not divisible by 5, then it is not divisible by 15.

Question 13.

Let p and q be two propositions. Show, by constructing the truth table or otherwise, that the following statements are equivalent:

$$p \to q \text{ and } \neg(\neg(p \wedge q) \wedge p)$$

Solution:

$\neg(\neg(p \wedge q) \wedge p)$	$=$	$\neg((\neg p \vee \neg q) \wedge p)$	De Morgan's law
	$=$	$\neg((\neg p \wedge p) \vee (\neg q \wedge p))$	Distributive
	$=$	$\neg(F \vee (\neg q \wedge p))$	
	$=$	$\neg(\neg q \wedge p)$	
	$=$	$(q \vee \neg p)$	De Morgans law
	$=$	$p \to q$	

This can also be shown by using a truth table.

Question 14.

Let p and q be two propositions, show that $(p \wedge \neg q) \vee (p \wedge q) = p$.

Solution:

$(p \wedge \neg q) \vee (p \wedge q)$	$=$	$p \wedge (\neg q \vee q)$	distributive law
	$=$	$p \wedge T$	negation law
	$=$	p	identity law

This can also be shown by using a truth table.

Question 15.

Let p, q and r be three propositions, show that $p \to (q \to r)$ and $(p \wedge q) \to r$ are two equivalent statements.

Solution:

$$\begin{array}{rcll} p \to (q \to r) & = & \neg p \vee (\neg q \vee r) & \\ & = & (\neg p \vee \neg q) \vee r & \text{associative law} \\ & = & \neg(p \wedge q) \vee r & \text{De Morgan's law} \\ & = & (p \wedge q) \to r & \end{array}$$

This can also be shown by using a truth table.

End of questions