

Combined Tutorial Exercises

Mathematics, Statistics, and Optimization

June 20, 2025

1. Algebra and Calculus

1. Inequalities.

(a) Solve $4x - 7 > 3x + 2$.

(b) Solve $\frac{x^2-4}{x-3} \leq 2$.

(c) Solve $|2x + 5| < 9$.

(d) Solve $\frac{1}{x} + \frac{2}{x-2} \geq 0$.

2. Distance and Midpoint.

(a) Distance and midpoint of $(1, 2)$ and $(5, 6)$.

(b) Distance and midpoint of $(-3, 4)$ and $(3, -2)$.

(c) Distance and midpoint of $(a, 0)$ and $(0, a)$, $a \neq 0$.

3. Limits and Continuity.

(a) $\lim_{x \rightarrow 2} \frac{x^2-4}{x-2}$.

(b) Determine continuity at $x = 1$ for

$$f(x) = \begin{cases} x^2 - 1, & x \neq 1, \\ 2, & x = 1. \end{cases}$$

4. Derivatives.

(a) $\frac{d}{dx}(x^3 e^{-x})$.

(b) Using first principles, $f'(x)$ for $f(x) = \sqrt{x+2}$.

5. Integrals.

(a) $\int_0^1 (3x^2 - 2x + 1) dx$.

(b) $\int \frac{1}{x^2 + 1} dx$.

2. Matrices and Linear Algebra

1. Solve by Gaussian elimination:

$$\begin{cases} 2x + 3y - z = 7, \\ -x + 4y + 2z = -1, \\ 3x - y + z = 4. \end{cases}$$

2. Determinants:

$$\det \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}, \quad \det \begin{bmatrix} 2 & 0 & 1 \\ -1 & 3 & 2 \\ 4 & 1 & 5 \end{bmatrix}.$$

3. Eigenpairs and diagonalization of

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

4. Inverse of

$$A = \begin{pmatrix} 4 & 7 \\ 2 & 3 \end{pmatrix}.$$

3. Probability and Statistics

1. Discrete: die twice, $P(\text{sum} = 8)$; $X \sim \text{Bern}(p)$, $E[X]$, $\text{Var}(X)$.
2. Continuous: $X \sim N(\mu, \sigma^2)$, $P(\mu - \sigma < X < \mu + \sigma)$; $X \sim \text{Exp}(\lambda)$, mean, median.
3. Testing: Type I/II errors; $n = 25$, $\bar{x} = 102$, $\sigma = 5$, test $H_0 : \mu = 100$ at $\alpha = 0.05$.
4. CI: 95% for mean, known variance; interpretation.

4. Regression and Classification

1. Simple regression on $(1, 2), (2, 4), (3, 5), (4, 7)$: $\beta_0, \beta_1, \hat{y}(2.5)$, SSR.
2. Multiple regression: normal eqns for $X \in \mathbb{R}^{n \times 3}$; dims of $X^T X, X^T y, \beta$.
3. Logistic: logit and log-likelihood; gradient-ascent update.

5. Optimization and Lagrange Multipliers

1. Unconstrained: critical points of $x^3 - 3xy^2$, Hessian test.
2. Lagrange: minimize $x^2 + 3y^2$ s.t. $x + 2y = 8$; maximize xyz s.t. $x + y + z = 9, > 0$.
3. Convex: define set/function; prove e^x convex.
4. Gradient descent: one iteration for $x^2 + 4x + 4$; step-size discussion.

6. Word Sum Challenge

Solve $\text{SEND} + \text{MORE} = \text{MONEY}$ (distinct digits, no leading zeros).