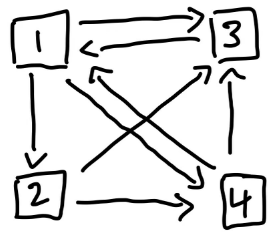
MATH 4310 Lecture Notes (Dylan Tom)

Introduction & Fields



Question: How do we determine the page order for a mini "google"?

1. (Simple Approach) Determine the importance by the number of back links (we expect page 3 should be the top*)

2. (Weighted Approach) Back links from "important" pages should weigh more. Let the "score" of a page be the sum of the scores of its back links.

3. Prevent undue influence by one page linking to too many other pages. If page j contains n_j links, one of which is page k, then boost the score of page k by $\frac{x_j}{n_j}$ where x_j is the score of page j

In our example,

$$\begin{aligned} x_1 &= \frac{1}{1}x_3 + \frac{1}{2}x_4 \\ x_2 &= \frac{1}{3}x_1 \\ x_3 &= \frac{1}{3}x_1 + \frac{1}{2}x_2 + \frac{1}{2}x_4 \\ x_4 &= \frac{1}{3}x_1 + \frac{1}{2}x_2 \end{aligned}$$

Answer: $x_1 = \frac{12}{31} \quad x_2 = \frac{4}{31} \quad x_3 = \frac{9}{31} \quad x_4 = \frac{6}{31}$

*We have shown that page 1 should be ranked higher than 3, so our intuition wasn't correct.

Question: What are some properties of the set of real numbers with addition and multiplication?

1. There is a $0 \in S$ such that $0 + a = a$ for all $a \in S$

2. There is a $1 \in S$ such that $1 \cdot a = a$ for all $a \in S$

3. commutativity, associativity, distributivity

4. There exists a $(-a) \in S$ such that $a + (-a) = 0$ for all $a \in S$

5. There exists a $a^{-1} \in S$ such that $aa^{-1} = 1$ for all $a \in S$
6. $a - b = a + (-b)$ and $\frac{a}{b} = a \cdot b^{-1}$

Question: What sets have these properties?

$$\mathbb{R}, \mathbb{Q}, \mathbb{C}, \mathbb{F}_p = \mathbb{Z}/p$$

Question: What sets do not satisfy these properties?

$$\mathbb{Z}, \mathbb{N}, \mathbb{M}_{2\times 2}$$

Definition. A **field**, $\mathbb{F}$, is a set on which addition $(+)$ and multiplication $(\cdot)$ are defined so that the following properties hold for all $a, b, c \in \mathbb{F}$.

1. $a + b = b + a \quad a \cdot b = b \cdot a$ (commutativity)
2. $(a + b) + c = a + (b + c) \quad (a \cdot b) \cdot c = a \cdot (b \cdot c)$ (associativity)
3. There exists *distinct* elements $0, 1$ such that $0 + a = a$ and $1 \cdot a = a$ (identity)
4. There exists $c, d \in \mathbb{F}$ such that $a + c = 0$ and $bd = 1$ where $d \neq 0$ (invertibility). Define $c = -a$ and $d = b^{-1}$ (see uniqueness below)
5. $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$ (distributivity)

Example: Some fields are $\mathbb{Q}, \mathbb{R}, \mathbb{C}, \mathbb{F} = \{a + b\sqrt{2} | a, b \in \mathbb{Q}\}, \mathbb{F}_2 = \{0, 1\}$

Example: Cancellation Laws

1. $a + b = a + c \Rightarrow b = c$

   Proof. Let's assume $a + b = a + c$. By (4), there is some x such that $x + a = 0$. Now $x + (a + b) = x + (a + c)$. By (2), $(x + a) + b = (x + a) + c \Rightarrow 0 + b = 0 + c$. By (3), $b = c$. □

2. $a \cdot b = a \cdot c$ and $a \neq 0 \Rightarrow b = c$

   Proof. Let's assume $a \cdot b = a \cdot c$ and $a \neq 0$. By (4), there is some x such that $ax = 1$. Now $x(ab) = x(ac)$. By (2), $(xa)b = (xa)c \Rightarrow 1b = 1c$. By (3), $b = c$. □

Example: Uniqueness of 0, 1, additive inverse, and multiplicative inverse

Proof. (multiplicative inverse) Given $b \neq 0$, let d and d' satisfy $b \cdot d = 1$ and $b \cdot d' = 1$. Then, $b \cdot d = b \cdot d'$. So, $d = d'$ (by cancellation). Similarly, for others. □

Example: Some more properties of fields

1. $a \cdot 0 = 0$

   Proof. $(a \cdot 0) + 0 = a \cdot 0 = a \cdot (0 + 0) = a \cdot 0 + a \cdot 0 \Rightarrow 0 = a \cdot 0$ □

2. $(-a) \cdot b = a \cdot (-b) = -(a \cdot b)$

   Proof. $[(-a) \cdot b] + [a \cdot b] = b \cdot (a + (-a)) = b \cdot 0 = 0$
   $[a \cdot (-b)] + [a \cdot b] = a \cdot (b + (-b)) = a \cdot 0 = 0$ □

3. $(-a) \cdot (-b) = a \cdot b$

   Proof. $(-a) \cdot (-b) = -[a \cdot (-b)] = -[-(a \cdot b)] = a \cdot b$ □

Properties of Relations:

1. Reflexive: $\forall a \in S$, $a \sim a$
2. Symmetric: $\forall a, b \in S$, if $a \sim b$, then $b \sim a$
3. Transitive: $\forall a, b, c \in S$, if $a \sim b$ and $b \sim c$, then $a \sim c$

An equivalence relation satisfies all 3 of these properties

Example: Define $S = \{\text{all humans}\}$. $a \sim b$ if a and b share a parent. It is reflexive, symmetric, but not transitive.

Definition. The class of a is all elements related to a, denoted by $[a]$. There can be no intersection between two classes.

Example: Define $S = \mathbb{Z}$. $a \sim b$ if $a - b$ is even. This is an equivalence relation. We can partition $\mathbb{Z}$ into even and odd, $[0]$ and $[1]$. We call this $\mathbb{Z}_2 = \mathbb{F}_2$.

In general, fix $d \geq 1$. Define $a \sim b$ if $a - b$ is divisible by d. In $\mathbb{Z}_d$,

1. $[a] + [b] = [(a + b) \mod d]$
2. $[a] \cdot [b] = [(a \cdot b) \mod d]$

Question: When is $\mathbb{Z}_d$ a field? Only if d is prime.

Vector Spaces

Definition. Let $\mathbb{F}$ be a field. A vector (linear) space, V over $\mathbb{F}$ is a set with two operations, addition $(+): V \times V \to V$ and scalar multiplication $(\cdot): \mathbb{F} \times V \to V$. For all vectors, $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$ and $a, b \in \mathbb{F}$.

- $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$
- $\mathbf{x} + (\mathbf{y} + \mathbf{z}) = (\mathbf{x} + \mathbf{y}) + \mathbf{z}$
- There is a 0 such that $0 + \mathbf{x} = \mathbf{x}$
- There is a $\mathbf{y}$ such that $\mathbf{x} + \mathbf{y} = 0$
- There is a 1 such that $1 \cdot \mathbf{x} = \mathbf{x}$
- $(ab)\mathbf{x} = a(b\mathbf{x})$
- $a \cdot (\mathbf{x} + \mathbf{y}) = (a \cdot \mathbf{x}) + (a \cdot \mathbf{y})$
- $(a + b)\mathbf{x} + a\mathbf{x} + b\mathbf{x}$

Question: Are the following vector spaces?

1. $D(\mathbb{R}, \mathbb{R})$, the set of all differentiable functions, $f : \mathbb{R} \to \mathbb{R}$
   Yes, we can show that this set is closed under addition and scalar multiplication.

2. S, the set of all polynomials of degree n with coefficients over the field, $\mathbb{F}$
   No, take $p(x) = x^n$ and $q(x) = -x^n$ so $p(x) + q(x) = 0$, which is not a polynomial of degree n. It is not closed under addition. **Be careful**, polynomials of degree less than or equal to n form a vector space.

Claim: The zero vector is unique.

Proof. Assume that $\mathbf{0}_1$ and $\mathbf{0}_2$ are two zero vectors. Then, $\mathbf{0}_1 = \mathbf{0}_1 + \mathbf{0}_2 = \mathbf{0}_2$. □

Claim: Given $\mathbf{x} \in V$, there exists a unique $\mathbf{y} \in V$ such that $\mathbf{x} + \mathbf{y} = \mathbf{0}$

Proof. Let $\mathbf{y}_1$ and $\mathbf{y}_2$ be two such vectors. Then, $\mathbf{y}_1 = \mathbf{y}_1 + \mathbf{0} = \mathbf{y}_1 + (\mathbf{x} + \mathbf{y}_2) = (\mathbf{y}_1 + \mathbf{x}) + \mathbf{y}_2 = \mathbf{0} + \mathbf{y}_2 = \mathbf{y}_2$. □

Bold face for vectors will be dropped unless it needs to be distinguished.

Claim: Let $u, v, w \in V$, if $u + v = u + w$, then $v = w$.

$$\begin{aligned} u + v &= u + w \\ (-u) + (u + v) &= (-u) + (u + w) \\ (-u + u) + v &= (-u + u) + w \\ 0 + v &= 0 + w \\ v &= w \end{aligned}$$

Claim: $a \cdot \mathbf{0} = \mathbf{0}$

$$\begin{aligned} &a \cdot \mathbf{0} = a \cdot (\mathbf{0} + \mathbf{0}) = a \cdot \mathbf{0} + a \cdot \mathbf{0} \\ &a \cdot \mathbf{0} = a \cdot \mathbf{0} + \mathbf{0} \end{aligned}$$

By cancellation, $a \cdot \mathbf{0} = \mathbf{0}$

Claim: $0 \cdot a = \mathbf{0}$

$$0 \cdot a + 0 \cdot a = (0 + 0) \cdot a = 0 \cdot a = 0 \cdot a + 0$$

By cancellation, $0 \cdot a = \mathbf{0}$.

Claim: Define $-x = (-1) \cdot x$. Show that this is the additive inverse of x.

Proof. $(-1)x + x = (-1)x + 1x = (-1 + 1)x = 0x = \mathbf{0}$ □

Example: Vector Spaces

- $\mathbb{F}^n$ is the space of n-tuples
- $\mathbb{M}_{2\times 3}(\mathbb{F})$ space of 2×3 matrices
- $\mathcal{F}(S, \mathbb{F}) = \{f : S \to \mathbb{F}\}$
- $\mathcal{P}(\mathbb{F})$ is space of all polynomials

Aside: As a vector space over $\mathbb{F}$, $\mathbb{F}^n$ is equivalent to $\mathbb{M}_{2\times 3}(\mathbb{F})$

Example: Non Vector Spaces

- $\{(x, y) \in \mathbb{R}^2 | x, y \geq 0\}$
- $\mathbb{R}^2$; $(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 - b_2)$
- $\mathbb{Q}(\sqrt{2}) = \{a + b\sqrt{2}\}$

Definition. Let V be a vector space over $\mathbb{F}$. A subset $W \subset V$ is called a subspace if

1. $0 \in W$
2. If $x, y \in W$, then $x + y \in W$
3. If $x \in W$, then $cx \in W$ for all $c \in \mathbb{F}$

Example: Subspaces

- $V = \mathbb{F}^{n\times 1}$; $W = \{x \in \mathbb{F}^{n\times 1} : Ax = 0\}$

  Proof. $0_{n\times 1} \in W$ because $A0 = 0$. W is closed under addition because $x, y \in W$,

  $$Ax = 0, Ay = 0 \implies A(x + y) = Ax + Ay = 0 + 0 = 0$$

  W is closed under scalar multiplication because $x \in W$, $c \in \mathbb{F}$,

  $$A(cx) = c(Ax) = c0 = 0$$

  $\square$

- $V = \mathbb{M}_{m\times n}(\mathbb{F})$
  - $W = \{A \in M_{m\times n}(\mathbb{F}) := AT = A\}$
  - W = diagonal $m \times n$ matrices
  - W = space of all upper triangular matrices
  - $W = \{A \in M_n(\mathbb{F}) | tr(A) = 0\}$

Definition. Let V be a vector space over a field of scalars $\mathbb{F}$ and let S be a nonempty subset of V. We say that $v \in V$ is in the span of S, if v is a linear combination of a finite number of elements in S.

span(S) is the set of all linear combinations of vectors in S

$$\text{span}(\emptyset) = \{0\}$$

S generates V if span(S) $= V$

Definition. Let V be a vector space over $\mathbb{F}$. A subset $S \subset V$ is called linearly dependent if there is a finite number of distinct vectors $u_1, ..., u_n \in S$ and scalars $a_1, ..., a_n$, not all zero, such that

$$a_1u_1 + ... + a_nu_n = 0$$

S is linearly independent if it is not linearly dependent.

A trivial linear combination for the vector 0 would be setting all the coefficients to 0

Claim: A subset S of V is linearly independent iff the only linear combination for 0 in $span(S)$ is the trivial one.

Proof. Assume that S is linearly independent. Cannot form 0 vector using nonzero scalars for $u_1, ..., u_n \in S$. Since $span(S)$ is the set of all linear combinations of vectors in S, it must be the trivial solution. To prove the opposite direction, use the contrapositive. If $S \subset V$ is linearly dependent, then there exists a nontrivial linear combination for 0 in $span(S)$. □

$Ax = 0$ has a unique solution iff the set is $\{u_1, ..., u_n\}$ (which forms A) is linearly independent.

Assume $S_1 \subseteq S_2$. If S_2 is linearly independent, then S_1 is linearly independent. If S_1 is linearly dependent, then S_2 is linearly dependent.

Definition. A **basis** for a vector space V is a linearly independent subset of V that generates V.

Observe:

1. $S = \emptyset$ is linearly independent. Then $\emptyset$ is a basis for $v = \{0\}$

2. $S = \{u\}$ is linearly independent if and only if $u \neq 0$

   Proof. Since S is linearly independent, we only have the trivial combination for 0. But if $u = 0$, then $1 \cdot u = 0$ would be a nontrivial combination for 0, so a contradiction. Assume that $u \neq 0$. Let $a \cdot u = 0$ because $u \neq 0$. We have $a = 0$, so no dependence exists. Therefore S is linearly independent. □

3. $v = \mathcal{P}(\mathbb{R})$
   $S = \{1, x, x^2, ...\}$. S is spanning and linearly independent because $a_0 + a_1x + a_2x^2 + ... + a_nx^n = 0$ if and only if $a_0 = a_1 = ... = a_n = 0$. So, there is no dependence relations in S.

4. $P(\mathbb{R}) = \mathbb{R}^{\bigoplus \mathbb{N}}$ is the set of sequences where the tail is all zeros

5. $V = \mathbb{F}^n$
   This is the space of n-tuples with entries in $\mathbb{F}$.

$$B = \begin{cases} e_1 = (1, 0, ..., 0) \\ e_2 = (0, 1, ..., 0) \\ \vdots \\ e_n = (0, 0, ..., 1) \end{cases}$$

   We say that B is the standard basis of $\mathbb{F}$. B is spanning because any $a = (a_1, a_2, ..., a_n) = a_1e_1 + a_2e_2 + ... + a_ne_n$. B is linearly independent because $a_1e_1 + ... + a_ne_n = (0, ..., 0)$ if and only if $(a_1, a_2, ..., a_n) = (0, 0..., 0)$.

6. $V = \mathbb{R}^2$
   Let $B_1 = \{e_1, e_2\}$ and $B_2 = \{e_1 + e_2, e_1 - e_2\}$. Both are bases for V.

Proof. B_2 is spanning. $v = (a, b) = x_1v_1 + x_2v_2$ implies that $x_1 = \frac{a+b}{2}$ and $x_2 = \frac{a-b}{2}$. It is also linearly indepedent. $x_1v_1 + x_2v_2 = 0$ if and only iff $(x_1, x_2) = (0, 0)$. $\square$

7. If $\mathbb{F}$ is an arbitrary field of scalars, then $\{e_1 + e_2, e_1 - e_2\}$ is a basis for $\mathbb{F}^2$ if and only if $char(\mathbb{F}) \neq 2$.

8. $V = P_n(\mathbb{F})$ is the set of polynomials of degree less than or equal to n with coefficients in $\mathbb{F}$

Important: Let V be a vector space over $\mathbb{F}$ and let $u_1, u_2, ..., u_n \in V$. $B = \{u_1, u_2, ..., u_n\}$ is a basis for V if and only if each $v \in V$ can be uniquely expressed as a combination of $u_1, u_2, ..., u_n$.

Proof. Assume B is a basis meaning $V = span(B)$. So, for all $v \in V$ it can be expressed as a linear combination. Assume that v can be written in two ways. $v = c_1u_1 + ... + c_nu_n = d_1u_1 + ... + d_nu_n$. $0 = v - v = (c_1 - d_1)u_1 + ... + (c_n - d_n)u_n$. Since the basis is linearly independent, we must have $c_1 = d_1, ..., c_n = d_n$. Assume that B can be uniquely expressed. Then, the only combination for 0 is the trivial solution and the vectors span V. So, B is a basis for V. $\square$