

# MATH 4310 Lecture Notes (Dylan Tom)

## Introduction & Fields



**Question:** How do we determine the page order for a mini "google"?

1. (Simple Approach) Determine the importance by the number of back links (we expect page 3 should be the top\*)
2. (Weighted Approach) Back links from "important" pages should weigh more. Let the "score" of a page be the sum of the scores of its back links.
3. Prevent undue influence by one page linking to too many other pages. If page  $j$  contains  $n_j$  links, one of which is page  $k$ , then boost the score of page  $k$  by  $\frac{x_j}{n_j}$  where  $x_j$  is the score of page  $j$

In our example,

$$\begin{aligned}x_1 &= \frac{1}{1}x_3 + \frac{1}{2}x_4 \\x_2 &= \frac{1}{3}x_1 \\x_3 &= \frac{1}{3}x_1 + \frac{1}{2}x_2 + \frac{1}{2}x_4 \\x_4 &= \frac{1}{3}x_1 + \frac{1}{2}x_2\end{aligned}$$

Answer:  $x_1 = \frac{12}{31}$   $x_2 = \frac{4}{31}$   $x_3 = \frac{9}{31}$   $x_4 = \frac{6}{31}$

\*We have shown that page 1 should be ranked higher than 3, so our intuition wasn't correct.

**Question:** What are some properties of the set of real numbers with addition and multiplication?

1. There is a  $0 \in S$  such that  $0 + a = a$  for all  $a \in S$
2. There is a  $1 \in S$  such that  $1 \cdot a = a$  for all  $a \in S$
3. commutativity, associativity, distributivity
4. There exists a  $(-a) \in S$  such that  $a + (-a) = 0$  for all  $a \in S$

5. There exists a  $a^{-1} \in S$  such that  $aa^{-1} = 1$  for all  $a \in S$
6.  $a - b = a + (-b)$  and  $\frac{a}{b} = a \cdot b^{-1}$

**Question:** What sets have these properties?

$$\mathbb{R}, \mathbb{Q}, \mathbb{C}, \mathbb{F}_p = \mathbb{Z}/p$$

**Question:** What sets do not satisfy these properties?

$$\mathbb{Z}, \mathbb{N}, \mathbb{M}_{2 \times 2}$$

**Definition.** A **field**,  $\mathbb{F}$ , is a set on which addition (+) and multiplication ( $\cdot$ ) are defined so that the following properties hold for all  $a, b, c \in \mathbb{F}$ .

1.  $a + b = b + a$     $a \cdot b = b \cdot a$  (commutativity)
2.  $(a + b) + c = a + (b + c)$     $(a \cdot b) \cdot c = a \cdot (b \cdot c)$  (associativity)
3. There exists *distinct* elements  $0, 1$  such that  $0 + a = a$  and  $1 \cdot a = a$  (identity)
4. There exists  $c, d \in \mathbb{F}$  such that  $a + c = 0$  and  $bd = 1$  where  $d \neq 0$  (invertibility).  
Define  $c = -a$  and  $d = b^{-1}$  (see uniqueness below)
5.  $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$  (distributivity)

**Example:** Some fields are  $\mathbb{Q}, \mathbb{R}, \mathbb{C}, \mathbb{F} = \{a + b\sqrt{2} | a, b \in \mathbb{Q}\}, \mathbb{F}_2 = \{0, 1\}$

**Example:** Cancellation Laws

1.  $a + b = a + c \Rightarrow b = c$

*Proof.* Let's assume  $a + b = a + c$ . By (4), there is some  $x$  such that  $x + a = 0$ . Now  $x + (a + b) = x + (a + c)$ . By (2),  $(x + a) + b = (x + a) + c \Rightarrow 0 + b = 0 + c$ . By (3),  $b = c$ .  $\square$

2.  $a \cdot b = a \cdot c$  and  $a \neq 0 \Rightarrow b = c$

*Proof.* Let's assume  $a \cdot b = a \cdot c$  and  $a \neq 0$ . By (4), there is some  $x$  such that  $ax = 1$ . Now  $x(ab) = x(ac)$ . By (2),  $(xa)b = (xa)c \Rightarrow 1b = 1c$ . By (3),  $b = c$ .  $\square$

**Example:** Uniqueness of 0, 1, additive inverse, and multiplicative inverse

*Proof.* (multiplicative inverse) Given  $b \neq 0$ , let  $d$  and  $d'$  satisfy  $b \cdot d = 1$  and  $b \cdot d' = 1$ . Then,  $b \cdot d = b \cdot d'$ . So,  $d = d'$  (by cancellation). Similarly, for others.  $\square$

**Example:** Some more properties of fields

1.  $a \cdot 0 = 0$

*Proof.*  $(a \cdot 0) + 0 = a \cdot 0 = a \cdot (0 + 0) = a \cdot 0 + a \cdot 0 \Rightarrow 0 = a \cdot 0$   $\square$

$$2. (-a) \cdot b = a \cdot (-b) = -(a \cdot b)$$

$$\begin{aligned} \text{Proof. } [(-a) \cdot b] + [a \cdot b] &= b \cdot (a + (-a)) = b \cdot 0 = 0 \\ [a \cdot (-b)] + [a \cdot b] &= a \cdot (b + (-b)) = a \cdot 0 = 0 \end{aligned}$$

□

$$3. (-a) \cdot (-b) = a \cdot b$$

$$\text{Proof. } (-a) \cdot (-b) = -[a \cdot (-b)] = -[-(a \cdot b)] = a \cdot b$$

□

### Properties of Relations:

1. Reflexive:  $\forall a \in S, a \sim a$
2. Symmetric:  $\forall a, b \in S$ , if  $a \sim b$ , then  $b \sim a$
3. Transitive:  $\forall a, b, c \in S$ , if  $a \sim b$  and  $b \sim c$ , then  $a \sim c$

An equivalence relation satisfies all 3 of these properties

**Example:** Define  $S = \{\text{all humans}\}$ .  $a \sim b$  if  $a$  and  $b$  share a parent. It is reflexive, symmetric, but not transitive.

**Definition.** The class of  $a$  is all elements related to  $a$ , denoted by  $[a]$ . There can be no intersection between two classes.

**Example:** Define  $S = \mathbb{Z}$ .  $a \sim b$  if  $a - b$  is even. This is an equivalence relation. We can partition  $\mathbb{Z}$  into even and odd,  $[0]$  and  $[1]$ . We call this  $\mathbb{Z}_2 = \mathbb{F}_2$ .

In general, fix  $d \geq 1$ . Define  $a \sim b$  if  $a - b$  is divisible by  $d$ . In  $\mathbb{Z}_d$ ,

1.  $[a] + [b] = [(a + b) \bmod d]$
2.  $[a] \cdot [b] = [(a \cdot b) \bmod d]$

**Question:** When is  $\mathbb{Z}_d$  a field? Only if  $d$  is prime.

## Vector Spaces

**Definition.** Let  $\mathbb{F}$  be a field. A vector (linear) space,  $V$  over  $\mathbb{F}$  is a set with two operations, addition  $(+): V \times V \rightarrow V$  and scalar multiplication  $(\cdot): \mathbb{F} \times V \rightarrow V$ . For all vectors,  $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$  and  $a, b \in \mathbb{F}$ .

- |   |  |
|---|--|
| <ul style="list-style-type: none"> <li>• <math>\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}</math></li> <li>• <math>\mathbf{x} + (\mathbf{y} + \mathbf{z}) = (\mathbf{x} + \mathbf{y}) + \mathbf{z}</math></li> <li>• There is a <math>0</math> such that <math>0 + \mathbf{x} = \mathbf{x}</math></li> <li>• There is a <math>\mathbf{y}</math> such that <math>\mathbf{x} + \mathbf{y} = 0</math></li> </ul> | <ul style="list-style-type: none"> <li>• There is a <math>1</math> such that <math>1 \cdot \mathbf{x} = \mathbf{x}</math></li> <li>• <math>(ab)\mathbf{x} = a(b\mathbf{x})</math></li> <li>• <math>a \cdot (\mathbf{x} + \mathbf{y}) = (a \cdot \mathbf{x}) + (a \cdot \mathbf{y})</math></li> <li>• <math>(a + b)\mathbf{x} + a\mathbf{x} + b\mathbf{x}</math></li> </ul> |
|---|--|

**Question:** Are the following vector spaces?

1.  $D(\mathbb{R}, \mathbb{R})$ , the set of all differentiable functions,  $f : \mathbb{R} \rightarrow \mathbb{R}$   
Yes, we can show that this set is closed under addition and scalar multiplication.
2.  $S$ , the set of all polynomials of degree  $n$  with coefficients over the field,  $\mathbb{F}$   
No, take  $p(x) = x^n$  and  $q(x) = -x^n$  so  $p(x) + q(x) = 0$ , which is not a polynomial of degree  $n$ . It is not closed under addition. **Be careful**, polynomials of degree less than or equal to  $n$  form a vector space.

*Claim:* The zero vector is unique.

*Proof.* Assume that  $\mathbf{0}_1$  and  $\mathbf{0}_2$  are two zero vectors. Then,  $\mathbf{0}_1 = \mathbf{0}_1 + \mathbf{0}_2 = \mathbf{0}_2$ . □

*Claim:* Given  $\mathbf{x} \in V$ , there exists a unique  $\mathbf{y} \in V$  such that  $\mathbf{x} + \mathbf{y} = \mathbf{0}$

*Proof.* Let  $\mathbf{y}_1$  and  $\mathbf{y}_2$  be two such vectors. Then,  $\mathbf{y}_1 = \mathbf{y}_1 + \mathbf{0} = \mathbf{y}_1 + (\mathbf{x} + \mathbf{y}_2) = (\mathbf{y}_1 + \mathbf{x}) + \mathbf{y}_2 = \mathbf{0} + \mathbf{y}_2 = \mathbf{y}_2$ . □

**Bold face for vectors will be dropped unless it needs to be distinguished.**

*Claim:* Let  $u, v, w \in V$ , if  $u + v = u + w$ , then  $v = w$ .

$$\begin{aligned} u + v &= u + w \\ (-u) + (u + v) &= (-u) + (u + w) \\ (-u + u) + v &= (-u + u) + w \\ \mathbf{0} + v &= \mathbf{0} + w \\ v &= w \end{aligned}$$

*Claim:*  $a \cdot \mathbf{0} = \mathbf{0}$

$$\begin{aligned} a \cdot \mathbf{0} &= a \cdot (\mathbf{0} + \mathbf{0}) = a \cdot \mathbf{0} + a \cdot \mathbf{0} \\ a \cdot \mathbf{0} &= a \cdot \mathbf{0} + \mathbf{0} \end{aligned}$$

By cancellation,  $a \cdot \mathbf{0} = \mathbf{0}$

*Claim:*  $\mathbf{0} \cdot a = \mathbf{0}$

$$\mathbf{0} \cdot a + \mathbf{0} \cdot a = (\mathbf{0} + \mathbf{0}) \cdot a = \mathbf{0} \cdot a = \mathbf{0} \cdot a + \mathbf{0}$$

By cancellation,  $\mathbf{0} \cdot a = \mathbf{0}$ .

*Claim:* Define  $-x = (-1) \cdot x$ . Show that this is the additive inverse of  $x$ .

*Proof.*  $(-1)x + x = (-1)x + 1x = (-1 + 1)x = 0x = \mathbf{0}$  □

**Example:** Vector Spaces

- $\mathbb{F}^n$  is the space of  $n$ -tuples
- $\mathcal{F}(S, \mathbb{F}) = \{f : S \rightarrow \mathbb{F}\}$
- $\mathbb{M}_{2 \times 3}(\mathbb{F})$  space of  $2 \times 3$  matrices
- $\mathcal{P}(\mathbb{F})$  is space of all polynomials

**Aside:** As a vector space over  $\mathbb{F}$ ,  $\mathbb{F}^n$  is equivalent to  $\mathbb{M}_{2 \times 3}(\mathbb{F})$

**Example:** Non Vector Spaces

- $\{(x, y) \in \mathbb{R}^2 | x, y \geq 0\}$
- $\mathbb{R}^2; (a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 - b_2)$
- $\mathbb{Q}(\sqrt{2}) = \{a + b\sqrt{2}\}$

**Definition.** Let  $V$  be a vector space over  $\mathbb{F}$ . A subset  $W \subset V$  is called a subspace if

1.  $0 \in W$
2. If  $x, y \in W$ , then  $x + y \in W$
3. If  $x \in W$ , then  $cx \in W$  for all  $c \in \mathbb{F}$

**Example:** Subspaces

- $V = \mathbb{F}^{n \times 1}; W = \{x \in \mathbb{F}^{n \times 1} : Ax = 0\}$

*Proof.*  $0_{n \times 1} \in W$  because  $A0 = 0$ .  $W$  is closed under addition because  $x, y \in W$ ,

$$Ax = 0, Ay = 0 \implies A(x + y) = Ax + Ay = 0 + 0 = 0$$

$W$  is closed under scalar multiplication because  $x \in W, c \in \mathbb{F}$ ,

$$A(cx) = c(Ax) = c0 = 0$$

□

- $V = M_{m \times n}(\mathbb{F})$ 
  - $W = \{A \in M_{m \times n}(\mathbb{F}) : AT = A\}$
  - $W =$  diagonal  $m \times n$  matrices
  - $W =$  space of all upper triangular matrices
  - $W = \{A \in M_n(\mathbb{F}) | tr(A) = 0\}$

**Definition.** Let  $V$  be a vector space over a field of scalars  $\mathbb{F}$  and let  $S$  be a nonempty subset of  $V$ . We say that  $v \in V$  is in the span of  $S$ , if  $v$  is a linear combination of a finite number of elements in  $S$ .

$\text{span}(S)$  is the set of all linear combinations of vectors in  $S$

$$\text{span}(\emptyset) = \{0\}$$

$S$  generates  $V$  if  $\text{span}(S) = V$

**Definition.** Let  $V$  be a vector space over  $\mathbb{F}$ . A subset  $S \subset V$  is called linearly dependent if there is a finite number of distinct vectors  $u_1, \dots, u_n \in S$  and scalars  $a_1, \dots, a_n$ , not all zero, such that

$$a_1 u_1 + \dots + a_n u_n = 0$$

$S$  is linearly independent if it is not linearly dependent.

A trivial linear combination for the vector 0 would be setting all the coefficients to 0

*Claim:* A subset  $S$  of  $V$  is linearly independent iff the only linear combination for 0 in  $\text{span}(S)$  is the trivial one.

*Proof.* Assume that  $S$  is linearly independent. Cannot form 0 vector using nonzero scalars for  $u_1, \dots, u_n \in S$ . Since  $\text{span}(S)$  is the set of all linear combinations of vectors in  $S$ , it must be the trivial solution. To prove the opposite direction, use the contrapositive. If  $S \subset V$  is linearly dependent, then there exists a nontrivial linear combination for 0 in  $\text{span}(S)$ .  $\square$

$Ax = 0$  has a unique solution iff the set is  $\{u_1, \dots, u_n\}$  (which forms  $A$ ) is linearly independent.

Assume  $S_1 \subseteq S_2$ . If  $S_2$  is linearly independent, then  $S_1$  is linearly independent. If  $S_1$  is linearly dependent, then  $S_2$  is linearly dependent.

**Definition.** A **basis** for a vector space  $V$  is a linearly independent subset of  $V$  that generates  $V$ .

**Observe:**

1.  $S = \emptyset$  is linearly independent. Then  $\emptyset$  is a basis for  $v = \{0\}$
2.  $S = \{u\}$  is linearly independent if and only if  $u \neq 0$

*Proof.* Since  $S$  is linearly independent, we only have the trivial combination for 0. But if  $u = 0$ , then  $1 \cdot u = 0$  would be a nontrivial combination for 0, so a contradiction. Assume that  $u \neq 0$ . Let  $a \cdot u = 0$  because  $u \neq 0$ . We have  $a = 0$ , so no dependence exists. Therefore  $S$  is linearly independent.  $\square$

3.  $v = \mathcal{P}(\mathbb{R})$   
 $S = \{1, x, x^2, \dots\}$ .  $S$  is spanning and linearly independent because  $a_0 + a_1x + a_2x^2 + \dots + a_nx^n = 0$  if and only if  $a_0 = a_1 = \dots = a_n = 0$ . So, there is no dependence relations in  $S$ .
4.  $P(\mathbb{R}) = \mathbb{R} \oplus \mathbb{N}$  is the set of sequences where the tail is all zeros
5.  $V = \mathbb{F}^n$   
This is the space of  $n$ -tuples with entries in  $\mathbb{F}$ .

$$B = \begin{cases} e_1 = (1, 0, \dots, 0) \\ e_2 = (0, 1, \dots, 0) \\ \vdots \\ e_n = (0, 0, \dots, 1) \end{cases}$$

We say that  $B$  is the standard basis of  $\mathbb{F}$ .  $B$  is spanning because any  $a = (a_1, a_2, \dots, a_n) = a_1e_1 + a_2e_2 + \dots + a_ne_n$ .  $B$  is linearly independent because  $a_1e_1 + \dots + a_ne_n = (0, \dots, 0)$  if and only if  $(a_1, a_2, \dots, a_n) = (0, 0, \dots, 0)$ .

6.  $V = \mathbb{R}^2$   
Let  $B_1 = \{e_1, e_2\}$  and  $B_2 = \{e_1 + e_2, e_1 - e_2\}$ . Both are bases for  $V$ .

*Proof.*  $B_2$  is spanning.  $v = (a, b) = x_1 v_1 + x_2 v_2$  implies that  $x_1 = \frac{a+b}{2}$  and  $x_2 = \frac{a-b}{2}$ . It is also linearly independent.  $x_1 v_1 + x_2 v_2 = 0$  if and only iff  $(x_1, x_2) = (0, 0)$ .  $\square$

7. If  $\mathbb{F}$  is an arbitrary field of scalars, then  $\{e_1 + e_2, e_1 - e_2\}$  is a basis for  $\mathbb{F}^2$  if and only if  $\text{char}(\mathbb{F}) \neq 2$ .

8.  $V = P_n(\mathbb{F})$  is the set of polynomials of degree less than or equal to  $n$  with coefficients in  $\mathbb{F}$

**Important:** Let  $V$  be a vector space over  $\mathbb{F}$  and let  $u_1, u_2, \dots, u_n \in V$ .  $B = \{u_1, u_2, \dots, u_n\}$  is a basis for  $V$  if and only if each  $v \in V$  can be uniquely expressed as a combination of  $u_1, u_2, \dots, u_n$ .

*Proof.* Assume  $B$  is a basis meaning  $V = \text{span}(B)$ . So, for all  $v \in V$  it can be expressed as a linear combination. Assume that  $v$  can be written in two ways.  $v = c_1 u_1 + \dots + c_n u_n = d_1 u_1 + \dots + d_n u_n$ .  $0 = v - v = (c_1 - d_1)u_1 + \dots + (c_n - d_n)u_n$ . Since the basis is linearly independent, we must have  $c_1 = d_1, \dots, c_n = d_n$ . Assume that  $B$  can be uniquely expressed. Then, the only combination for 0 is the trivial solution and the vectors span  $V$ . So,  $B$  is a basis for  $V$ .  $\square$

*Claim:* If  $V$  is generated by a finite sets, then some subset of  $S$  is a basis for  $V$ .

*Proof.* If  $S = \{0\}$ . Otherwise, take some nonzero vector  $u_1 \in S$ . Then  $\{u_1\}$  is linearly independent. If possible, choose some vectors  $u_2, \dots \in S$  such that the set  $\{u_1, u_2, \dots, u_k\}$  is linearly independent for all  $k \geq 1$ . By finiteness of  $S$ , this process ends and we call the resulting linearly independent subset of  $S$  as  $B = \{u_1, u_2, \dots, u_n\}$ . We claim that this  $B$  generates  $V$ . Note that either  $B = S$  so  $B$  is a basis or  $B \subset S$ . Then  $S \subset B \implies \text{span}(S) \subset \text{span}(B) \implies V = \text{span}(B)$ . Therefore,  $B$  is a basis.  $\square$

*Claim:* (Replacement Theorem) Let  $V$  be any vector space over  $\mathbb{F}$  generated by a set  $G$  containing exactly  $n$  vectors. If  $L$  is a linearly independent subset of  $V$  containing exactly  $m$  vectors, then  $m \leq n$ . Also there is a subset  $H$  of  $G$  containing exactly  $n - m$  vectors such that  $L \cup H$  generates  $V$ .

*Proof.* (by induction) If  $m = 0$ ,  $L = \emptyset$ , then  $H = G$ . Next assume the statement holds for  $m \geq 0$ .  $\mathcal{L} = \{v_1, v_2, \dots, v_{m+1}\}$ . Then,  $\{v_1, v_2, \dots, v_m\}$  is linearly independent. By IH, we know  $m \leq n$  and there is a subset  $\{u_1, u_2, \dots, u_{n-m}\}$  of  $G$  such that  $\{v_1, v_2, \dots, v_m\} \cup \{u_1, u_2, \dots, u_{n-m}\}$  generates  $V$ . Then,  $v_{m+1} = a_1 v_1 + a_2 v_2 + \dots + a_m v_m + b_1 u_1 + \dots + b_{n-m} u_{n-m}$  for some  $a_i, b_j \in \mathbb{F}$ . We already know  $n - m \geq 0$  but we claim  $n - m \neq 0$  because otherwise  $L$  would not be linearly independent. So  $n \geq m + 1$  and some  $b_j \neq 0$ . Without loss of generality, consider  $b_1 \neq 0$ .

$$u_1 = \left(-\frac{a_1}{b_1}\right) v_1 + \dots + \left(-\frac{a_m}{b_1}\right) v_m + \left(\frac{1}{b_1}\right) v_{m+1} + \left(-\frac{b_2}{b_1}\right) u_2 + \dots + \left(-\frac{b_{n-m}}{b_1}\right) u_{n-m}$$

Let  $H = \{u_2, u_3, \dots, u_{n-m}\}$ . Then  $u_1 \in \text{span}(L \cup H)$ . So,  $V = \text{span}(L \cup H)$ .  $\square$

*Claim:* Let  $B$  have a finite basis. Then all bases for  $B$  are finite and all have the same number of vectors.

*Proof.* Let  $B$  be a basis with cardinality of  $V$ .  $|B| = n$  and let  $C$  be any other basis. If  $|C| > n$ , then choose a subset  $S \in C$  with exactly  $n + 1$  vectors. But then I have  $n + 1$  linearly independent vectors in something that is spanned by  $n$  vectors, which is a contradiction so  $C$  is finite and  $m \leq n$ . Similarly,  $n \leq m$  and  $n = m$ .  $\square$

**Definition.** Let  $V$  be any vector space over  $\mathbb{F}$ . We call  $V$  finite dimensional if it has a finite basis. We then call the number of vectors in any basis the dimension of  $V$ . If  $V$  is not finite dimensional, we call it infinite dimensional.

**Examples:**

1.  $V = \{0\}$ ;  $\dim V = 0$  because  $\{\}$  is a basis
2.  $F = \mathbb{C}, V = \mathbb{C}$ ;  $\dim_{\mathbb{C}} \mathbb{C} = 1$  because  $\{1\}$  is a basis
3.  $F = \mathbb{R}, V = \mathbb{C}$ ;  $\dim_{\mathbb{R}} \mathbb{C} = 2$  because  $\{1, i\}$  is a basis
4.  $V = P(\mathbb{F})$  is infinite dimensional

**Important:** A vector space always has a basis

*Existence of Basis and Axiom of Choice*

*Proof.* Take nonempty set  $A_\alpha \neq \emptyset$

□

## Linear Transformations

**Definition.** Let  $V$  and  $W$  be vector spaces over  $\mathbb{F}$ . A transformation  $T : V \rightarrow W$  is called linear if the following hold

1.  $T(x + y) = T(x) + T(y)$  for all  $x, y \in V$
2.  $T(cx) = cT(x)$  for all  $c \in \mathbb{F}$  and  $x \in V$

Observe that if  $T$  is linear then

1.  $T(0_V) = 0_W$
2.  $T(x - y) = T(x) - T(y)$
3.  $T(\sum_i c_i x_i) = \sum_i c_i T(x_i)$

**Definition.** Given  $V, W/\mathbb{F}$  and  $T : V \rightarrow W$  linear, define null space (or kernel) of  $T$ :

$$N(T) = \{x \in V : T(x) = 0\}$$

and the range (or image) of  $T$ :

$$R(T) = \{T(x) : x \in V\}$$

**Example**

1.  $T : V \rightarrow V$  ( $x \rightarrow x$ ); Range is all of  $V$ , Null space is  $\{0\}$
2.  $T : V \rightarrow W$  ( $x \rightarrow 0$ )



*Theorem:* Let  $T \in \mathcal{L}(V, W)$ , then  $R(T)$  is a subspace of  $W$ .

*Theorem:* Let  $T \in \mathcal{L}(V, W)$ , then  $N(T)$  is a subspace of  $V$ .

*Theorem:* Let  $T \in \mathcal{L}(V, W)$ , then  $T$  is injective if and only if  $N(T) = \{0\}$ .

*Proof.* Assume that  $T$  is injective. Assume  $T(v) = 0_W$ . By injectivity,  $v = 0_V$ . So,  $N(T) = \{0\}$ . Assume  $N(T) = \{0\}$ . Take any  $v_1 \neq v_2 \in V$ . Assume that  $T(v_1) = T(v_2)$ . Then  $T(v_1) - T(v_2) = T(v_1 - v_2) = 0$ .  $\square$

**Definition.** Given  $V, W$  as vector spaces over  $\mathbb{F}$  and a linear  $T : V \rightarrow W$ . If  $N(T)$  and  $R(T)$  are finite dimensional, then define **nullity** of  $T$  as  $\dim N(T)$  and **rank** of  $T$  as  $\dim R(T)$ .

**Rank-Nullity Theorem:** If  $V$  is finite dimensional, then  $\dim V = \text{nullity of } T + \text{rank of } T$ .

*Proof.* Let  $\dim V = n$  and  $\dim N(T) = k$ .  $k \leq n$  because  $N(T)$  is a subspace of  $V$ . Choose basis  $\{v_1, \dots, v_k\}$  for the nullspace. Extend it to  $B$  for  $V$ , where  $B = \{v_1, \dots, v_k, v_{k+1}, \dots, v_n\}$ . We claim  $\{T(v_{k+1}), \dots, T(v_n)\}$  is a basis for the range.  $T$  on the basis  $\{v_1, \dots, v_k\}$  will be 0.  $\square$

**Example:** If  $V$  and  $W$  are finite dimensional with equal dimension. For,  $T : V \rightarrow W$ , (a)  $T$  is injective, (b)  $T$  is surjective, and (c)  $\text{rank}(T) = \dim V = \dim W$

*Claim:* Let  $V, B$  be vector spaces over  $\mathbb{F}$ .  $\{v_1, \dots, v_n\}$  is a basis for  $V$ . Let  $w_1, \dots, w_n \in W$ . Then  $\exists! T : V \rightarrow W$  such that  $T(v_i) = w_i$  for all  $1 \leq i \leq n$ .

*Proof.* Existence: Define  $T(\sum_{i=1}^n c_i v_i) := \sum_{i=1}^n c_i w_i$ . This  $T$  defines a linear transformation  $T(v_i) = w_i$ .

Uniqueness: Assume there exists another  $S : V \rightarrow W$  satisfying the property. Then,

$$S(v) = S\left(\sum c_i v_i\right) = \sum c_i S(v_i) = \sum c_i w_i = T\left(\sum c_i v_i\right) = T(v)$$

$\square$

**Definition.** Let  $V$  be a finite dimensional vector space over  $\mathbb{F}$  and let  $B = \{v_1, \dots, v_n\}$  be an ordered basis for  $V$ . For any  $x \in V$ , write  $x = \sum_{i=1}^n a_i v_i$ . We define the coordinate vector of  $x$  relative to  $B$  as the following,

$$[x]_B = (a_1 \quad a_2 \quad \cdots \quad a_n)$$

**Definition.** Let  $V$  be a finite dimensional vector space over  $\mathbb{F}$  with an ordered basis  $B = \{v_1, \dots, v_n\}$  and let  $W$  be a finite dimensional vector space over  $\mathbb{F}$  with an ordered basis  $C = \{w_1, \dots, w_m\}$ . Then, the matrix of a linear transformation  $T : V \rightarrow W$  is

$$A = [T]_B^C := [a_{ij}]_{m \times n} = ([T(v_1)]_C \quad \cdots \quad [T(v_n)]_C)$$

If  $V = W$  and  $B = C$ , then we write  $[T]_B$ .

The Kronecker-Delta function is

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

**Examples:**

1. The identity transformation  $Id : V \rightarrow V$ . If  $V$  is finite dimensional with basis  $\beta$ , then  $[T]_\beta = [\delta_{ij}]_{n \times n} = I$
2.  $T : V \rightarrow W$ , where every  $v \in V$  is mapped to  $0 \in W$ , then  $[T] = [0]_{m \times n}$
3. Isomorphism: Let  $V$  and  $W$  be vector spaces over  $\mathbb{F}$ . Set  $\mathcal{L}(V, W) := \{ \text{linear transformations } V \rightarrow W \}$ .  $\mathcal{L}(V, W)$  is a vector space over  $\mathbb{F}$ . If  $V = W$ , we write  $\mathcal{L}(V) := \mathcal{L}(V, V)$ . When  $\dim V = n < \infty$  and  $\dim W = m < \infty$ , then there is a bijection between  $\mathcal{L}(V, W)$  and  $M_{m \times n}(\mathbb{F})$ . This is an isomorphism because

$$[T + U]_\beta^{\mathcal{C}} = [T]_\beta^{\mathcal{C}} + [U]_\beta^{\mathcal{C}} \quad [aT]_\beta^{\mathcal{C}} = a[T]_\beta^{\mathcal{C}}$$

Let  $V, W, Z$  be vector spaces over  $\mathbb{F}$ . Let  $T : V \rightarrow W$  and  $U : W \rightarrow Z$ . Then,  $UT = U \circ T = V \rightarrow Z$ .

*Observe:*

1.  $T \in \mathcal{L}(V, W)$  and  $U \in \mathcal{L}(W, Z)$  implies  $UT \in \mathcal{L}(V, Z)$

$$(UT)(x + y) = U(T(x + y)) = U(T(x) + T(y)) = U(T(x)) + U(T(y)) = (UT)(x) + (UT)(y)$$

$$(UT)(ax) = U(T(ax)) = U(aT(x)) = aU(T(x)) = a(UT)(x)$$

2. If  $T_1, T_2 \in \mathcal{L}(V, W)$  and  $U_1, U_2 \in \mathcal{L}(W, Z)$ , then  $U_1(T_1 + T_2) = (U_1T_1) + (U_1T_2)$  and  $(U_1 + U_2)(T_1) = (U_1T_1) + (U_2T_1)$ .
3. Composition is associative
4.  $T \in \mathcal{L}(V, W)$  implies  $T = TI_V = I_WT$
5.  $a(UT) = (aU)T = U(aT)$
6. When  $T \in \mathcal{L}(V)$ ,  $T^0 = I_V, T^1 = T, T^2 = TT, T^3 = TTT, \dots$
7. Assume that  $T : V \rightarrow W, U : W \rightarrow Z$ ,  $V$  has basis  $\mathcal{B}$ ,  $W$  has basis  $\mathcal{C}$ , and  $Z$  has basis  $\mathcal{D}$ . Also,  $\dim V = n, \dim W = m, \dim Z = p$ . So,  $[T]_\mathcal{B}^\mathcal{C} = B_{m \times n}$  and  $[U]_\mathcal{B}^\mathcal{D} = A_{p \times m}$ . Then,  $[UT]_\mathcal{B}^\mathcal{D} = C_{p \times n}$ . This leads to the definition of matrix multiplication.

$$(UT)(v_j) = U(T(v_j)) = U\left(\sum_k B_{kj}w_k\right) = \sum_k B_{kj}U(w_k) = \sum_k B_{kj}\left(\sum_i A_{ik}z_i\right) = \sum_i C_{ij}z_i$$

So,  $C = AB$  with  $C_{ij} = A_{i1}B_{1j} + \dots + A_{ip}B_{pj}$ .

**Examples:** Properties of Matrix Multiplication: For  $A_{m \times n}, B_{n \times p}, C_{n \times p}, D_{q \times m}, E_{q \times m}$  matrices,

1.  $A(B + C) = (AB) + (AC) \quad (D + E)A = (DA) + (EA)$
2.  $a(AB) = (aA)B = (A)(aB)$  for all  $a \in \mathbb{F}$
3.  $I_m A = A = A = A I_n$
4.  $AB = AC \not\Rightarrow B = C$  even if  $A \neq 0$

5. Given  $B = [v_1 \ v_2 \ \cdots \ v_p]$ ,  $AB = [Av_1 \ Av_2 \ \cdots \ Av_p]$
6. Let  $V$  and  $W$  be finite dimensional vector space over  $\mathbb{F}$  with bases  $\mathcal{B}$  and  $\mathcal{C}$ . Let  $T : V \rightarrow W$  be a linear transformation. Then,

$$[T(u)]_{\mathcal{C}} = [T]_{\mathcal{B}}^{\mathcal{C}}[u]_{\mathcal{B}}$$

7. Fix  $A_{m \times n}$ , define  $L_A : \mathbb{F}^n \rightarrow \mathbb{F}^m$  via  $L_A(x) = Ax$  (left multiplication). Consider  $\mathcal{B}$  and  $\mathcal{C}$  standard bases of  $\mathbb{F}^n$  and  $\mathbb{F}^m$  respectively. Then,
- (a)  $[LA]_{\mathcal{B}}^{\mathcal{C}} = A$
  - (b)  $L_A = L_B \iff A = B$
  - (c)  $L_{A+B} = L_A + L_B$
  - (d) For  $T : \mathbb{F}^n \rightarrow \mathbb{F}^m$ , there exists a unique  $C$  such that  $T = L_C$
  - (e) Given  $E$ , we have  $L_{AE} = L_A L_E$
  - (f) When  $m = n$ , we have  $L_{I_n} = I_{\mathbb{F}^n}$
  - (g) Left multiplication transformations from  $\mathbb{F}^n$  to  $\mathbb{F}^m$  is an isomorphism to  $m \times n$  matrices
  - (h) Matrix multiplication is associative

**Definition.** Let  $V$  and  $W$  be vector spaces over  $\mathbb{F}$ . A linear transformation,  $T : V \rightarrow W$  is invertible if  $T$  has an inverse. There exists a function  $U : W \rightarrow V$  such that (a)  $TU = I_W$  and (b)  $UT = I_V$ .

*Claim:* A linear map is invertible if and only if it is injective and bijective.

*Proof.* Suppose  $T$  is invertible. Suppose  $T(v_1) = T(v_2)$ . Because  $T$  is invertible, there exists a  $U$  such that  $UT = I_V$ . Then,  $U(T(v_1)) = U(T(v_2)) \rightarrow (UT)(v_1) = (UT)(v_2) \rightarrow I_V(v_1) = I_V(v_2) \rightarrow v_1 = v_2$ . So,  $T$  is injective. [Not complete]

Suppose  $T$  is injective and surjective. Define  $U(w)$  to be the unique element in  $V$  such that  $T(U(w)) = w$ . Take any  $w \in W$ . Since  $T$  is surjective,  $\exists U(w) \in V$  such that  $T(U(w)) = w$ . Suppose there exists another  $[U(w)]'$  satisfying  $T([U(w)]') = w$ . By injectivity,  $T(U(w)) = T([U(w)]') \rightarrow U(w) = [U(w)]'$ . So,  $U(w)$  is unique. We want to show that  $U$  is linear. Consider  $x, y \in W$  and  $a \in \mathbb{F}$ . Then  $T(S(ax + y)) =$

Proof to be completed later □

*Observe:*

1. If an inverse exists, then it must be unique. Denote the inverse of a linear transformation as  $T^{-1}$ .

*Proof.* If  $U_1$  and  $U_2$  are both inverses to  $T$ ,  $U_1 = U_1 I_W = U_1 (TU_2) = (U_1 T)(U_2) = I_V U_2 = U_2$ . □

2. If  $T, U$  are invertible, then  $TU$  is invertible and  $(TU)^{-1} = U^{-1}T^{-1}$ .

*Proof.*

$$(TU)(U^{-1}T^{-1}) = T(UU^{-1})T^{-1} = TT^{-1} = I \quad (U^{-1}T^{-1})(TU) = U^{-1}(T^{-1}T)U = U^{-1}U = I$$

□

3.  $T$  is invertible if and only if  $(T^{-1})^{-1} = T$
4. If  $V, W$  are finite dimensional with  $\dim V = \dim W$ , then  $T : V \rightarrow W$  is invertible if and only if  $\text{rank}(T) = \dim W$ .

*Proof.* content...

□

5. Let  $T$  be an invertible linear transformation. Then  $V$  is finite dimensional if and only if  $W$  is finite dimensional.

*Proof.* content...

□

**Definition.** A matrix  $A_{n \times n}$  is **invertible** if there exists a  $B_{n \times n}$  such that  $AB = BA = I_n$ .

1. If  $A$  is invertible, the inverse is unique denoted by  $A^{-1}$ .
2. If  $V$  and  $W$  are finite dimensional with ordered bases and  $T : V \rightarrow W$  is linear, then  $T$  is invertible if and only if  $[T]_{\mathcal{B}^C}$  is invertible. In this case,  $([T]_{\mathcal{B}^C})^{-1} = [T^{-1}]_{\mathcal{C}}^{\mathcal{B}}$ .

*Proof.* content...

□

3. Let  $V$  be finite dimensional. Let  $T \in \mathcal{L}(V)$ . Then  $T$  is invertible if and only if  $(T)_{\mathcal{B}}$  is invertible.
4.  $A_{n \times n}$  is invertible if and only if  $L_A \in \mathcal{L}(\mathbb{F}^n)$  is invertible. Special Case:  $V = \mathbb{F}^n$ , standard basis, and  $T = L_A$ .

**Definition.** Let  $V, W$  be vector spaces over  $\mathbb{F}$ .  $V$  is **isomorphic** to  $W$  if there exists an invertible linear transformation  $T : V \rightarrow W$ . Such a  $T$  is called an isomorphism from  $V$  to  $W$ .

Observe that  $T$  is not unique and "is an isomorphism to" is an equivalence relation.

1. If  $V, W$  are finite dimensional over  $\mathbb{F}$ ,  $V$  is isomorphic to  $W$  if and only if  $\dim V = \dim W$ .

*Proof.* Let  $T$  be invertible. Then,  $\dim V = \dim W$ . Now assume  $\dim V = \dim W = n$ . Choose a basis  $\mathcal{B} = \{v_1, \dots, v_n\}$  for  $V$  and  $\mathcal{C} = \{w_1, \dots, w_n\}$  for  $W$ . There exists a  $T : V \rightarrow W$  such that  $T(v_i) = w_i$  for all  $i$ . Now,  $R(T) = W$  so  $T$  is surjective. By dimension theory,  $T$  is one-to-one so  $V \sim W$ . □

2. Every  $n$ -dim vector space over  $\mathbb{F}$  is isomorphic to  $\mathbb{F}^n$ .

3. If  $V$  is finite dimensional, with dimension  $n$ , then  $\Phi_{\mathcal{B}}^{\mathcal{C}} : \mathcal{L}(V, W) \rightarrow M_{n \times n}(\mathbb{F})$ .
4. Let  $V$  be a finite dimensional vector space over  $\mathbb{F}$  with ordered basis  $\mathcal{B}$ . Then  $\Phi_{\mathcal{B}} : V \rightarrow \mathbb{F}^n$  is an isomorphism.

*Question:* How does the coordinate vector  $[x]_{\mathcal{B}}$  change if  $\mathcal{B}$  changes?

Let  $\mathcal{B}_1, \mathcal{B}_2$  be two ordered bases for a finite dimensional vector space  $V$ , and let  $Q = [I_V]_{\mathcal{B}_1}^{\mathcal{B}_2}$ . Then  $Q$  is invertible, and for all  $v \in V$ ,  $[v]_{\mathcal{B}_2} = Q[v]_{\mathcal{B}_1}$ . "change of coordinates from  $\mathcal{B}_1$  to  $\mathcal{B}_2$ "

*Proof.* The identity matrix is invertible, so  $Q$  must also be invertible. □

If  $T \in \mathcal{L}(V)$ , then  $[T]_{\mathcal{B}_1} = Q^{-1}[T]_{\mathcal{B}_2}Q$ .

*Proof.* □

**Definition.** Let  $A, B \in M_{n \times n}(\mathbb{F})$ .  $A$  is similar to  $B$  if there exists an invertible matrix  $Q$  such that  $B = Q^{-1}AQ$ . Note that  $A \sim B$  is an equivalence relation.

*Note:* We can classify all  $n \times n$  matrices up to similarity (also called conjugation)

**Definition.** Given a vector space  $V$  over  $\mathbb{F}$ , we define the **dual space** of  $V$  as  $V^* := \mathcal{L}(V, \mathbb{F})$ .

*Observe:*

1. If  $V$  is finite dimensional, then  $V^*$  is also finite dimensional and

$$\dim V^* = \dim \mathcal{L}(V, \mathbb{F}) = \dim V \cdot \dim \mathbb{F} = \dim V$$

2. The double dual,  $V^{**}$ . If  $V$  is finite dimensional, then  $\dim V^{**} = \dim V^* = \dim V$ . All 3 vector spaces are isomorphic. However, there is no natural isomorphism from  $V$  to  $V^*$  or from  $V^*$  to  $V^{**}$ . There is a natural isomorphism from  $V$  to  $V^{**}$ .

**Examples:**

1.  $V = C[0, 2\pi]$  is the space of continuous functions from  $[0, 2\pi] \rightarrow \mathbb{R}$ . Fix  $g \in V$ . Define  $h(x) = \frac{1}{2\pi} \int_0^{2\pi} x(t)g(t)dt$  for all  $x \in V$ . Then,  $h \in V^*$ .

Take  $g(t) = \sin(nt)$  for any  $n \in \mathbb{Z}$ .  $h(x)$  gives the  $n$ th Fourier coefficients  $a_n$ ,

2.  $V = M_{n \times n}(\mathbb{F})$

Define  $f : V \rightarrow \mathbb{F}$ . If  $A \in V$ , then we output  $tr(A) = \sum_i A_{ii}$ . Then  $f = tr \in V^*$

We showed that if  $v \neq 0 \in V$ , then there exists a  $f \in V^*$  such that  $f(v) = 1$ . This helps us extend this to a basis.

**Definition.** Let  $V$  be a finite dimensional vector space over  $\mathbb{F}$  of dimension  $n$  with an ordered basis  $\mathcal{B} = \{x_1, x_2, \dots, x_n\}$  of  $V$ . Define  $f_i \in V^*$  via  $f_i(x) = a_i$  where  $[x]_{\mathcal{B}} = [a_1 \ a_2 \ \dots \ a_n]$ . Then,  $\mathcal{B}^* = \{f_1, f_2, \dots, f_n\}$  is an ordered basis for  $V^*$  called the **dual basis** and

$$f = \sum_{i=1}^n f(x_i)f_i \quad \forall f \in V^*$$

*Proof.* To show  $\mathcal{B}^*$  is a basis for  $V^*$  it is enough to check linear independence. First, observe  $f_i(x_j) = \delta_{ij}$ . Let  $\sum_{i=1}^n \lambda_i f_i = 0$  in  $V^*$ . Apply both sides to  $x_j$ , then

$$\sum_{i=1}^n \lambda_i f_i(x_j) = \lambda_j = 0 \quad \forall j$$

In other words,  $\mathcal{B}^*$  is linearly independent. To show  $\sum_{i=1}^n f(x_i) f_i = f$ , it is enough to apply them to each  $x_j$  and get the same scalar.

$$\left( \sum_{i=1}^n f(x_i) f_i \right) (x_j) = \sum_{i=1}^n$$

□

*Claim:* Assume that  $U \subseteq W$ . Show  $W^0 \subseteq U^0$ .

*Proof.* Given  $f \in W^0$ . We know that  $f(w) = 0 \forall w \in W$ . Since  $U \subseteq W$ , for all  $u \in U, u \in W$ . So,  $f(u) = 0$  for all  $u \in U$ . Therefore,  $f \in U^0$ . □

*Observe*

1.  $S = \{0\}$ , then  $S^0 = V^*$
2.  $S = \{e_1\}$  and  $V = \mathbb{R}^3$ , then  $S^0 = \text{span}\{f_2 = e_2^*, f_3 = e_3^*\}$
3.  $S = \{e_1, e_2\}$ , then  $S^0 = \text{span}\{f_3 = e_3^*\}$

**Definition.** Let  $V$  and  $W$  be finite dimensional vector spaces over  $\mathbb{F}$  with ordered bases  $\mathcal{B}$  and  $\mathcal{C}$ . For  $T : V \rightarrow W$  linear, define transpose of  $T$ ,  $T^t : W^* \rightarrow V^*$  via  $T^t(g) = gT$  for  $g \in W^*$

*Observe*

1.  $T^t$  is a linear transformation

*Proof.*  $T^t$  is the composition of two linear maps. Concretely,  $T^t(ag_1 + g_2) = (ag_1 + g_2)T = (ag_1)T + g_2T = a(g_1T) + g_2T = aT^t(g_1) + T^t(g_2)$ . So,  $T^t \in \mathcal{L}(W^*, V^*)$  □

2.  $[T^t]_{\mathcal{C}^*}^{\mathcal{B}^*} = ([T]_{\mathcal{B}}^{\mathcal{C}})^t$

*Proof.* To write the matrix of  $T^t$ , write  $\mathcal{B} = \{x_1, \dots, x_n\}$  for  $V$  and  $\mathcal{C} = \{y_1, \dots, y_m\}$  for  $W$ . Also write  $\mathcal{B}^* = \{f_1, \dots, f_n\}$  for  $V^*$  and  $\mathcal{C}^* = \{g_1, \dots, g_m\}$  for  $W^*$ . Then,  $A = [T]_{\mathcal{B}}^{\mathcal{C}}$  and  $B = [T^t]_{\mathcal{C}^*}^{\mathcal{B}^*}$ . Now the  $(i, j)$ th entry of  $B$  is obtained by  $T^t(g_j)(x_i)$ .  $B_{ij} = (g_jT)(x_i) = g_j(T(x_i)) = g_j\left(\sum_{k=1}^m A_{ki} y_k\right) = \sum_{k=1}^m A_{ki} g_j(y_k) = \sum_{k=1}^m A_{ki} \delta_{jk} = A_{ji}$ . Thus,  $B = A^t$ . □

3. Double Dual - Let  $V$  be a finite dimensional vector space over  $\mathbb{F}$ . Then  $\psi : V \rightarrow V^{**}$  via  $\psi(x) = \hat{x}$  where  $\hat{x}(f) = f(x)$  is an isomorphism.

*Proof.*  $\psi$  is linear. For  $x, y \in V$ ,  $c \in \mathbb{F}$  and for all  $f \in V^*$ , we have

$$\psi(cx + y)(f) = (cx + y)(f) = f(cx + y) = cf(x) + f(y) = c\hat{x}(f) + \hat{y}(f) = (c\psi(x) + \psi(y))(f)$$

To show  $\psi$  is an isomorphism, it suffices to show that it is one-to-one. Say  $\psi(x) = 0$ . Then,  $\psi(x)(f) = 0$  for all  $f \in V^*$ . So,  $f(x) = 0 \rightarrow x = 0 \rightarrow N(\psi) = \{0\}$ . Therefore,  $\psi$  is an isomorphism.  $\square$

4. Finite dimensional assumption is crucial. In infinite dimensional case,  $V, V^*, V^{**}$  need not be isomorphic.

**Definition.** Let  $V_1, V_2, \dots, V_m$  be vector spaces over  $\mathbb{F}$ . The product  $V_1 \times V_2 \times \dots \times V_m = \{(v_1, v_2, \dots, v_m) : v_1 \in V_1, \dots, v_m \in V_m\}$ .

Suppose that  $U_1, \dots, U_m$  are subspaces of a finite dimensional vector space  $V$ . Define a linear map,

$$\Gamma : U_1 \times \dots \times U_m \rightarrow U_1 + \dots + U_m \quad (u_1, \dots, u_m) \mapsto u_1 + \dots + u_m$$

Show that  $\Gamma$  is surjective.

*Proof.* Let  $u \in U_1 + \dots + U_m$ . Then, there exists a unique combination in the product space of  $U_1 \times \dots \times U_m$  specifically  $u_1 \in U_1, \dots, u_m \in U_m$  such that  $u = u_1 + u_2 + \dots + u_m$ .  $\square$

$U_1 + \dots + U_m$  is a direct sum if and only if  $\Gamma$  is injective.

*Proof.* Assume that  $U_1 + \dots + U_m$  is a direct sum and that  $\Gamma$  is not injective, then there exists  $a \neq b \in U_1 \times \dots \times U_m$  such that  $\Gamma(a) = \Gamma(b)$ . Then there are two ways to express  $v = \Gamma(a) = \Gamma(b) \in U_1 + \dots + U_m$  as a direct sum, a contradiction.

Suppose that  $v = a_1 + \dots + a_m = b_1 + \dots + b_m$ . So,  $\Gamma(a) = \Gamma(b)$ . Therefore, because  $\Gamma$  is injective,  $a = b$ , so it is a direct sum.  $\square$

$U_1 + \dots + U_m$  is a direct sum if and only if  $\dim(U_1 + \dots + U_m) = \dim U_1 + \dots + \dim U_m$

*Proof.* From (a) and (b),  $U_1 + \dots + U_m$  is a direct sum if and only if  $\Gamma$  is bijective.

$$\dim(U_1 \times \dots \times U_m) = \dim(U_1 + \dots + U_m) \iff \dim(U_1) + \dots + \dim(U_m)$$

$\square$

**Definition.** Let  $V$  be vector spaces over  $\mathbb{F}$ . Take a subspace  $U$  and  $v \in V$ . An affine subset of  $V$  is  $v + U := \{v + u : u \in U\}$ . It is also called parallel to  $U$ .

**Definition.** Suppose  $U$  is a subspace of  $V$ , the quotient space  $V \setminus U$  is the set of all affine subsets of  $V$  parallel to  $U$

$$V \setminus U = \{v + U : v \in V\}$$

*Observe:*

1.  $V \setminus U$  becomes a vector space over  $\mathbb{F}$  under the following operations

$$\lambda(v + U) := \lambda v + U \quad (v_1 + U) + (v_2 + U) := (v_1 + v_2) + U$$

# Determinants

## Diagonalization

Given a linear operator  $T \in \mathcal{L}(V)$  where  $V$  is finite dimensional. Is there a basis  $\beta$  of  $V$  such that  $[T]_\beta$  is diagonal?

Let  $\beta = \{v_1, v_2, \dots, v_n\}$ . Then,

$$[T]_\beta = D = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix}$$

if and only if  $T(v_j) = \lambda_j v_j$  for all  $j$ . If such an ordered basis exists, then we say  $T$  is diagonalizable. We say  $A \in M_{n \times n}(\mathbb{F})$  is diagonalizable if  $L_A : \mathbb{F}^n \rightarrow \mathbb{F}^n$  is diagonalizable where  $L_A : x \mapsto Ax$ . It is diagonalizable if and only if there exists an ordered basis of  $\mathbb{F}^n$  such that  $[L_A]_\beta = D$  if and only if  $Q^{-1}AQ = D$ .

**Definition.** Let  $V$  be a vector space over  $\mathbb{F}$  and  $T \in \mathcal{L}(V)$ . A nonzero vector  $v \in V$  is an eigenvector of  $T$  if  $\exists \lambda \in \mathbb{F}$  such that  $T(v) = \lambda v$ . The scalar  $\lambda$  is called an eigenvalue corresponding to  $v$ .

1.  $T$  diagonalizable if and only if there exists an ordered basis of eigenvectors of  $T$
2.  $A$  diagonalizable if and only if there exists an ordered basis of eigenvectors of  $A$ . If  $\beta = \{v_1, \dots, v_n\}$  is such an ordered basis, then with  $Q = [v_1 \ v_2 \ \dots \ v_n]$ , we have  $Q^{-1}AQ = D$

### Example:

Let  $V = \mathbb{R}^3$  and  $\mathbb{F} = \mathbb{R}$ .

$$A = \begin{bmatrix} 5 & 8 & 16 \\ 4 & 1 & 8 \\ -4 & -4 & -11 \end{bmatrix}$$

Note

$$A \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} \quad A \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = -3 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \quad A \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} = -3 \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

**Diagonalization:** Find nonzero  $v$  such that  $Av = \lambda v$  if and only if  $(A - \lambda I)v = 0$  if and only if  $\det(A - \lambda I) = 0$ .