

# MATH 4310 Lecture Notes (Dylan Tom)

## Introduction & Fields



**Question:** How do we determine the page order for a mini "google"?

1. (Simple Approach) Determine the importance by the number of back links (we expect page 3 should be the top\*)
2. (Weighted Approach) Back links from "important" pages should weigh more. Let the "score" of a page be the sum of the scores of its back links.
3. Prevent undue influence by one page linking to too many other pages. If page  $j$  contains  $n_j$  links, one of which is page  $k$ , then boost the score of page  $k$  by  $\frac{x_j}{n_j}$  where  $x_j$  is the score of page  $j$

In our example,

$$\begin{aligned}x_1 &= \frac{1}{1}x_3 + \frac{1}{2}x_4 \\x_2 &= \frac{1}{3}x_1 \\x_3 &= \frac{1}{3}x_1 + \frac{1}{2}x_2 + \frac{1}{2}x_4 \\x_4 &= \frac{1}{3}x_1 + \frac{1}{2}x_2\end{aligned}$$

Answer:  $x_1 = \frac{12}{31}$   $x_2 = \frac{4}{31}$   $x_3 = \frac{9}{31}$   $x_4 = \frac{6}{31}$

\*We have shown that page 1 should be ranked higher than 3, so our intuition wasn't correct.

**Question:** What are some properties of the set of real numbers with addition and multiplication?

1. There is a  $0 \in S$  such that  $0 + a = a$  for all  $a \in S$
2. There is a  $1 \in S$  such that  $1 \cdot a = a$  for all  $a \in S$
3. commutativity, associativity, distributivity
4. There exists a  $(-a) \in S$  such that  $a + (-a) = 0$  for all  $a \in S$

5. There exists a  $a^{-1} \in S$  such that  $aa^{-1} = 1$  for all  $a \in S$
6.  $a - b = a + (-b)$  and  $\frac{a}{b} = a \cdot b^{-1}$

**Question:** What sets have these properties?

$$\mathbb{R}, \mathbb{Q}, \mathbb{C}, \mathbb{F}_p = \mathbb{Z}/p$$

**Question:** What sets do not satisfy these properties?

$$\mathbb{Z}, \mathbb{N}, \mathbb{M}_{2 \times 2}$$

**Definition.** A **field**,  $\mathbb{F}$ , is a set on which addition (+) and multiplication ( $\cdot$ ) are defined so that the following properties hold for all  $a, b, c \in \mathbb{F}$ .

1.  $a + b = b + a$     $a \cdot b = b \cdot a$  (commutativity)
2.  $(a + b) + c = a + (b + c)$     $(a \cdot b) \cdot c = a \cdot (b \cdot c)$  (associativity)
3. There exists *distinct* elements  $0, 1$  such that  $0 + a = a$  and  $1 \cdot a = a$  (identity)
4. There exists  $c, d \in \mathbb{F}$  such that  $a + c = 0$  and  $bd = 1$  where  $d \neq 0$  (invertibility).  
Define  $c = -a$  and  $d = b^{-1}$  (see uniqueness below)
5.  $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$  (distributivity)

**Example:** Some fields are  $\mathbb{Q}, \mathbb{R}, \mathbb{C}, \mathbb{F} = \{a + b\sqrt{2} | a, b \in \mathbb{Q}\}, \mathbb{F}_2 = \{0, 1\}$

**Example:** Cancellation Laws

1.  $a + b = a + c \Rightarrow b = c$

*Proof.* Let's assume  $a + b = a + c$ . By (4), there is some  $x$  such that  $x + a = 0$ . Now  $x + (a + b) = x + (a + c)$ . By (2),  $(x + a) + b = (x + a) + c \Rightarrow 0 + b = 0 + c$ . By (3),  $b = c$ .  $\square$

2.  $a \cdot b = a \cdot c$  and  $a \neq 0 \Rightarrow b = c$

*Proof.* Let's assume  $a \cdot b = a \cdot c$  and  $a \neq 0$ . By (4), there is some  $x$  such that  $ax = 1$ . Now  $x(ab) = x(ac)$ . By (2),  $(xa)b = (xa)c \Rightarrow 1b = 1c$ . By (3),  $b = c$ .  $\square$

**Example:** Uniqueness of 0, 1, additive inverse, and multiplicative inverse

*Proof.* (multiplicative inverse) Given  $b \neq 0$ , let  $d$  and  $d'$  satisfy  $b \cdot d = 1$  and  $b \cdot d' = 1$ . Then,  $b \cdot d = b \cdot d'$ . So,  $d = d'$  (by cancellation). Similarly, for others.  $\square$

**Example:** Some more properties of fields

1.  $a \cdot 0 = 0$

*Proof.*  $(a \cdot 0) + 0 = a \cdot 0 = a \cdot (0 + 0) = a \cdot 0 + a \cdot 0 \Rightarrow 0 = a \cdot 0$   $\square$

$$2. (-a) \cdot b = a \cdot (-b) = -(a \cdot b)$$

$$\begin{aligned} \text{Proof. } [(-a) \cdot b] + [a \cdot b] &= b \cdot (a + (-a)) = b \cdot 0 = 0 \\ [a \cdot (-b)] + [a \cdot b] &= a \cdot (b + (-b)) = a \cdot 0 = 0 \end{aligned}$$

□

$$3. (-a) \cdot (-b) = a \cdot b$$

$$\text{Proof. } (-a) \cdot (-b) = -[a \cdot (-b)] = -[-(a \cdot b)] = a \cdot b$$

□

### Properties of Relations:

1. Reflexive:  $\forall a \in S, a \sim a$
2. Symmetric:  $\forall a, b \in S$ , if  $a \sim b$ , then  $b \sim a$
3. Transitive:  $\forall a, b, c \in S$ , if  $a \sim b$  and  $b \sim c$ , then  $a \sim c$

An equivalence relation satisfies all 3 of these properties

**Example:** Define  $S = \{\text{all humans}\}$ .  $a \sim b$  if  $a$  and  $b$  share a parent. It is reflexive, symmetric, but not transitive.

**Definition.** The class of  $a$  is all elements related to  $a$ , denoted by  $[a]$ . There can be no intersection between two classes.

**Example:** Define  $S = \mathbb{Z}$ .  $a \sim b$  if  $a - b$  is even. This is an equivalence relation. We can partition  $\mathbb{Z}$  into even and odd,  $[0]$  and  $[1]$ . We call this  $\mathbb{Z}_2 = \mathbb{F}_2$ .

In general, fix  $d \geq 1$ . Define  $a \sim b$  if  $a - b$  is divisible by  $d$ . In  $\mathbb{Z}_d$ ,

1.  $[a] + [b] = [(a + b) \bmod d]$
2.  $[a] \cdot [b] = [(a \cdot b) \bmod d]$

**Question:** When is  $\mathbb{Z}_d$  a field? Only if  $d$  is prime.

## Vector Spaces

**Definition.** Let  $\mathbb{F}$  be a field. A vector (linear) space,  $V$  over  $\mathbb{F}$  is a set with two operations, addition  $(+) : V \times V \rightarrow V$  and scalar multiplication  $(\cdot) : \mathbb{F} \times V \rightarrow V$ . For all vectors,  $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$  and  $a, b \in \mathbb{F}$ .

- |   |  |
|---|--|
| <ul style="list-style-type: none"> <li>• <math>\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}</math></li> <li>• <math>\mathbf{x} + (\mathbf{y} + \mathbf{z}) = (\mathbf{x} + \mathbf{y}) + \mathbf{z}</math></li> <li>• There is a <math>0</math> such that <math>0 + \mathbf{x} = \mathbf{x}</math></li> <li>• There is a <math>\mathbf{y}</math> such that <math>\mathbf{x} + \mathbf{y} = 0</math></li> </ul> | <ul style="list-style-type: none"> <li>• There is a <math>1</math> such that <math>1 \cdot \mathbf{x} = \mathbf{x}</math></li> <li>• <math>(ab)\mathbf{x} = a(b\mathbf{x})</math></li> <li>• <math>a \cdot (\mathbf{x} + \mathbf{y}) = (a \cdot \mathbf{x}) + (a \cdot \mathbf{y})</math></li> <li>• <math>(a + b)\mathbf{x} + a\mathbf{x} + b\mathbf{x}</math></li> </ul> |
|---|--|

**Question:** Are the following vector spaces?

1.  $D(\mathbb{R}, \mathbb{R})$ , the set of all differentiable functions,  $f : \mathbb{R} \rightarrow \mathbb{R}$   
Yes, we can show that this set is closed under addition and scalar multiplication.
2.  $S$ , the set of all polynomials of degree  $n$  with coefficients over the field,  $\mathbb{F}$   
No, take  $p(x) = x^n$  and  $q(x) = -x^n$  so  $p(x) + q(x) = 0$ , which is not a polynomial of degree  $n$ . It is not closed under addition. **Be careful**, polynomials of degree less than or equal to  $n$  form a vector space.

*Claim:* The zero vector is unique.

*Proof.* Assume that  $\mathbf{0}_1$  and  $\mathbf{0}_2$  are two zero vectors. Then,  $\mathbf{0}_1 = \mathbf{0}_1 + \mathbf{0}_2 = \mathbf{0}_2$ . □

*Claim:* Given  $\mathbf{x} \in V$ , there exists a unique  $\mathbf{y} \in V$  such that  $\mathbf{x} + \mathbf{y} = \mathbf{0}$

*Proof.* Let  $\mathbf{y}_1$  and  $\mathbf{y}_2$  be two such vectors. Then,  $\mathbf{y}_1 = \mathbf{y}_1 + \mathbf{0} = \mathbf{y}_1 + (\mathbf{x} + \mathbf{y}_2) = (\mathbf{y}_1 + \mathbf{x}) + \mathbf{y}_2 = \mathbf{0} + \mathbf{y}_2 = \mathbf{y}_2$ . □

**Bold face for vectors will be dropped unless it needs to be distinguished.**

*Claim:* Let  $u, v, w \in V$ , if  $u + v = u + w$ , then  $v = w$ .

$$\begin{aligned}
 u + v &= u + w \\
 (-u) + (u + v) &= (-u) + (u + w) \\
 (-u + u) + v &= (-u + u) + w \\
 \mathbf{0} + v &= \mathbf{0} + w \\
 v &= w
 \end{aligned}$$

*Claim:*  $a \cdot \mathbf{0} = \mathbf{0}$

$$\begin{aligned}
 a \cdot \mathbf{0} &= a \cdot (\mathbf{0} + \mathbf{0}) = a \cdot \mathbf{0} + a \cdot \mathbf{0} \\
 a \cdot \mathbf{0} &= a \cdot \mathbf{0} + \mathbf{0}
 \end{aligned}$$

By cancellation,  $a \cdot \mathbf{0} = \mathbf{0}$

*Claim:*  $\mathbf{0} \cdot a = \mathbf{0}$

$$\mathbf{0} \cdot a + \mathbf{0} \cdot a = (\mathbf{0} + \mathbf{0}) \cdot a = \mathbf{0} \cdot a = \mathbf{0} \cdot a + \mathbf{0}$$

By cancellation,  $\mathbf{0} \cdot a = \mathbf{0}$ .

*Claim:* Define  $-x = (-1) \cdot x$ . Show that this is the additive inverse of  $x$ .

*Proof.*  $(-1)x + x = (-1)x + 1x = (-1 + 1)x = 0x = \mathbf{0}$  □

**Example:** Vector Spaces

- $\mathbb{F}^n$  is the space of  $n$ -tuples
- $\mathcal{F}(S, \mathbb{F}) = \{f : S \rightarrow \mathbb{F}\}$
- $\mathbb{M}_{2 \times 3}(\mathbb{F})$  space of  $2 \times 3$  matrices
- $\mathcal{P}(\mathbb{F})$  is space of all polynomials

**Aside:** As a vector space over  $\mathbb{F}$ ,  $\mathbb{F}^n$  is equivalent to  $\mathbb{M}_{2 \times 3}(\mathbb{F})$

**Example:** Non Vector Spaces

- $\{(x, y) \in \mathbb{R}^2 | x, y \geq 0\}$
- $\mathbb{Q}(\sqrt{2}) = \{a + b\sqrt{2}\}$
- $\mathbb{R}^2; (a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 - b_2)$

**Definition.** Let  $V$  be a vector space over  $\mathbb{F}$ . A subset  $W \subset V$  is called a subspace if

1.  $0 \in W$
2. If  $x, y \in W$ , then  $x + y \in W$
3. If  $x \in W$ , then  $cx \in W$  for all  $c \in \mathbb{F}$