

MATH 4310 Lecture Notes (Dylan Tom)

Introduction & Fields



Question: How do we determine the page order for a mini "google"?

1. (Simple Approach) Determine the importance by the number of back links (we expect page 3 should be the top*)
2. (Weighted Approach) Back links from "important" pages should weigh more. Let the "score" of a page be the sum of the scores of its back links.
3. Prevent undue influence by one page linking to too many other pages. If page j contains n_j links, one of which is page k , then boost the score of page k by $\frac{x_j}{n_j}$ where x_j is the score of page j

In our example,

$$\begin{aligned}x_1 &= \frac{1}{1}x_3 + \frac{1}{2}x_4 \\x_2 &= \frac{1}{3}x_1 \\x_3 &= \frac{1}{3}x_1 + \frac{1}{2}x_2 + \frac{1}{2}x_4 \\x_4 &= \frac{1}{3}x_1 + \frac{1}{2}x_2\end{aligned}$$

Answer: $x_1 = \frac{12}{31}$ $x_2 = \frac{4}{31}$ $x_3 = \frac{9}{31}$ $x_4 = \frac{6}{31}$

*We have shown that page 1 should be ranked higher than 3, so our intuition wasn't correct.

Question: What are some properties of the set of real numbers with addition and multiplication?

1. There is a $0 \in S$ such that $0 + a = a$ for all $a \in S$
2. There is a $1 \in S$ such that $1 \cdot a = a$ for all $a \in S$
3. commutativity, associativity, distributivity
4. There exists a $(-a) \in S$ such that $a + (-a) = 0$ for all $a \in S$

5. There exists a $a^{-1} \in S$ such that $aa^{-1} = 1$ for all $a \in S$
6. $a - b = a + (-b)$ and $\frac{a}{b} = a \cdot b^{-1}$

Question: What sets have these properties?

$$\mathbb{R}, \mathbb{Q}, \mathbb{C}, \mathbb{F}_p = \mathbb{Z}/p$$

Question: What sets do not satisfy these properties?

$$\mathbb{Z}, \mathbb{N}, \mathbb{M}_{2 \times 2}$$

Definition. A **field**, \mathbb{F} , is a set on which addition (+) and multiplication (\cdot) are defined so that the following properties hold for all $a, b, c \in \mathbb{F}$.

1. $a + b = b + a$ $a \cdot b = b \cdot a$ (commutativity)
2. $(a + b) + c = a + (b + c)$ $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ (associativity)
3. There exists *distinct* elements $0, 1$ such that $0 + a = a$ and $1 \cdot a = a$ (identity)
4. There exists $c, d \in \mathbb{F}$ such that $a + c = 0$ and $bd = 1$ where $d \neq 0$ (invertibility).
Define $c = -a$ and $d = b^{-1}$ (see uniqueness below)
5. $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$ (distributivity)

Example: Some fields are $\mathbb{Q}, \mathbb{R}, \mathbb{C}, \mathbb{F} = \{a + b\sqrt{2} | a, b \in \mathbb{Q}\}, \mathbb{F}_2 = \{0, 1\}$

Example: Cancellation Laws

1. $a + b = a + c \Rightarrow b = c$

Proof. Let's assume $a + b = a + c$. By (4), there is some x such that $x + a = 0$. Now $x + (a + b) = x + (a + c)$. By (2), $(x + a) + b = (x + a) + c \Rightarrow 0 + b = 0 + c$. By (3), $b = c$. \square

2. $a \cdot b = a \cdot c$ and $a \neq 0 \Rightarrow b = c$

Proof. Let's assume $a \cdot b = a \cdot c$ and $a \neq 0$. By (4), there is some x such that $ax = 1$. Now $x(ab) = x(ac)$. By (2), $(xa)b = (xa)c \Rightarrow 1b = 1c$. By (3), $b = c$. \square

Example: Uniqueness of 0, 1, additive inverse, and multiplicative inverse

Proof. (multiplicative inverse) Given $b \neq 0$, let d and d' satisfy $b \cdot d = 1$ and $b \cdot d' = 1$. Then, $b \cdot d = b \cdot d'$. So, $d = d'$ (by cancellation). Similarly, for others. \square

Example: Some more properties of fields

1. $a \cdot 0 = 0$

Proof. $(a \cdot 0) + 0 = a \cdot 0 = a \cdot (0 + 0) = a \cdot 0 + a \cdot 0 \Rightarrow 0 = a \cdot 0$ \square

$$2. (-a) \cdot b = a \cdot (-b) = -(a \cdot b)$$

$$\text{Proof. } [(-a) \cdot b] + [a \cdot b] = b \cdot (a + (-a)) = b \cdot 0 = 0$$

$$[a \cdot (-b)] + [a \cdot b] = a \cdot (b + (-b)) = a \cdot 0 = 0 \quad \square$$

$$3. (-a) \cdot (-b) = a \cdot b$$

$$\text{Proof. } (-a) \cdot (-b) = -[a \cdot (-b)] = -[-(a \cdot b)] = a \cdot b \quad \square$$

Properties of Relations:

1. Reflexive: $\forall a \in S, a \sim a$
2. Symmetric: $\forall a, b \in S$, if $a \sim b$, then $b \sim a$
3. Transitive: $\forall a, b, c \in S$, if $a \sim b$ and $b \sim c$, then $a \sim c$

An equivalence relation satisfies all 3 of these properties

Example: Define $S = \{\text{all humans}\}$. $a \sim b$ if a and b share a parent. It is reflexive, symmetric, but not transitive.

Definition. The class of a is all elements related to a , denoted by $[a]$. There can be no intersection between two classes.

Example: Define $S = \mathbb{Z}$. $a \sim b$ if $a - b$ is even. This is an equivalence relation. We can partition \mathbb{Z} into even and odd, $[0]$ and $[1]$. We call this $\mathbb{Z}_2 = \mathbb{F}_2$.

In general, fix $d \geq 1$. Define $a \sim b$ if $a - b$ is divisible by d . In \mathbb{Z}_d ,

1. $[a] + [b] = [(a + b) \bmod d]$
2. $[a] \cdot [b] = [(a \cdot b) \bmod d]$

Question: When is \mathbb{Z}_d a field? Only if d is prime.

Vector Spaces

Definition. Let \mathbb{F} be a field. A vector (linear) space, V over \mathbb{F} is a set with two operations, addition $(+): V \times V \rightarrow V$ and scalar multiplication $(\cdot): \mathbb{F} \times V \rightarrow V$. For all vectors, $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$ and $a, b \in \mathbb{F}$.

- | | |
|---|--|
| <ul style="list-style-type: none"> • $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$ • $\mathbf{x} + (\mathbf{y} + \mathbf{z}) = (\mathbf{x} + \mathbf{y}) + \mathbf{z}$ • There is a 0 such that $0 + \mathbf{x} = \mathbf{x}$ • There is a \mathbf{y} such that $\mathbf{x} + \mathbf{y} = 0$ | <ul style="list-style-type: none"> • There is a 1 such that $1 \cdot \mathbf{x} = \mathbf{x}$ • $(ab)\mathbf{x} = a(b\mathbf{x})$ • $a \cdot (\mathbf{x} + \mathbf{y}) = (a \cdot \mathbf{x}) + (a \cdot \mathbf{y})$ • $(a + b)\mathbf{x} + a\mathbf{x} + b\mathbf{x}$ |
|---|--|

Question: Are the following vector spaces?

1. $D(\mathbb{R}, \mathbb{R})$, the set of all differentiable functions, $f : \mathbb{R} \rightarrow \mathbb{R}$
Yes, we can show that this set is closed under addition and scalar multiplication.
2. S , the set of all polynomials of degree n with coefficients over the field, \mathbb{F}
No, take $p(x) = x^n$ and $q(x) = -x^n$ so $p(x) + q(x) = 0$, which is not a polynomial of degree n . It is not closed under addition. **Be careful**, polynomials of degree less than or equal to n form a vector space.

Claim: The zero vector is unique.

Proof. Assume that $\mathbf{0}_1$ and $\mathbf{0}_2$ are two zero vectors. Then, $\mathbf{0}_1 = \mathbf{0}_1 + \mathbf{0}_2 = \mathbf{0}_2$. □

Claim: Given $\mathbf{x} \in V$, there exists a unique $\mathbf{y} \in V$ such that $\mathbf{x} + \mathbf{y} = \mathbf{0}$

Proof. Let \mathbf{y}_1 and \mathbf{y}_2 be two such vectors. Then, $\mathbf{y}_1 = \mathbf{y}_1 + \mathbf{0} = \mathbf{y}_1 + (\mathbf{x} + \mathbf{y}_2) = (\mathbf{y}_1 + \mathbf{x}) + \mathbf{y}_2 = \mathbf{0} + \mathbf{y}_2 = \mathbf{y}_2$. □

Bold face for vectors will be dropped unless it needs to be distinguished.

Claim: Let $u, v, w \in V$, if $u + v = u + w$, then $v = w$.

$$\begin{aligned} u + v &= u + w \\ (-u) + (u + v) &= (-u) + (u + w) \\ (-u + u) + v &= (-u + u) + w \\ \mathbf{0} + v &= \mathbf{0} + w \\ v &= w \end{aligned}$$

Claim: $a \cdot \mathbf{0} = \mathbf{0}$

$$\begin{aligned} a \cdot \mathbf{0} &= a \cdot (\mathbf{0} + \mathbf{0}) = a \cdot \mathbf{0} + a \cdot \mathbf{0} \\ a \cdot \mathbf{0} &= a \cdot \mathbf{0} + \mathbf{0} \end{aligned}$$

By cancellation, $a \cdot \mathbf{0} = \mathbf{0}$

Claim: $\mathbf{0} \cdot a = \mathbf{0}$

$$\mathbf{0} \cdot a + \mathbf{0} \cdot a = (\mathbf{0} + \mathbf{0}) \cdot a = \mathbf{0} \cdot a = \mathbf{0} \cdot a + \mathbf{0}$$

By cancellation, $\mathbf{0} \cdot a = \mathbf{0}$.

Claim: Define $-x = (-1) \cdot x$. Show that this is the additive inverse of x .

Proof. $(-1)x + x = (-1)x + 1x = (-1 + 1)x = 0x = \mathbf{0}$ □

Example: Vector Spaces

- \mathbb{F}^n is the space of n -tuples
- $\mathcal{F}(S, \mathbb{F}) = \{f : S \rightarrow \mathbb{F}\}$
- $\mathbb{M}_{2 \times 3}(\mathbb{F})$ space of 2×3 matrices
- $\mathcal{P}(\mathbb{F})$ is space of all polynomials

Aside: As a vector space over \mathbb{F} , \mathbb{F}^n is equivalent to $\mathbb{M}_{2 \times 3}(\mathbb{F})$

Example: Non Vector Spaces

- $\{(x, y) \in \mathbb{R}^2 | x, y \geq 0\}$
- $\mathbb{R}^2; (a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 - b_2)$
- $\mathbb{Q}(\sqrt{2}) = \{a + b\sqrt{2}\}$

Definition. Let V be a vector space over \mathbb{F} . A subset $W \subset V$ is called a subspace if

1. $0 \in W$
2. If $x, y \in W$, then $x + y \in W$
3. If $x \in W$, then $cx \in W$ for all $c \in \mathbb{F}$

Example: Subspaces

- $V = \mathbb{F}^{n \times 1}; W = \{x \in \mathbb{F}^{n \times 1} : Ax = 0\}$

Proof. $0_{n \times 1} \in W$ because $A0 = 0$. W is closed under addition because $x, y \in W$,

$$Ax = 0, Ay = 0 \implies A(x + y) = Ax + Ay = 0 + 0 = 0$$

W is closed under scalar multiplication because $x \in W, c \in \mathbb{F}$,

$$A(cx) = c(Ax) = c0 = 0$$

□

- $V = M_{m \times n}(\mathbb{F})$
 - $W = \{A \in M_{m \times n}(\mathbb{F}) : AT = A\}$
 - $W =$ diagonal $m \times n$ matrices
 - $W =$ space of all upper triangular matrices
 - $W = \{A \in M_n(\mathbb{F}) | \text{tr}(A) = 0\}$

Definition. Let V be a vector space over a field of scalars \mathbb{F} and let S be a nonempty subset of V . We say that $v \in V$ is in the span of S , if v is a linear combination of a finite number of elements in S .

$\text{span}(S)$ is the set of all linear combinations of vectors in S

$$\text{span}(\emptyset) = \{0\}$$

S generates V if $\text{span}(S) = V$

Definition. Let V be a vector space over \mathbb{F} . A subset $S \subset V$ is called linearly dependent if there is a finite number of distinct vectors $u_1, \dots, u_n \in S$ and scalars a_1, \dots, a_n , not all zero, such that

$$a_1 u_1 + \dots + a_n u_n = 0$$

S is linearly independent if it is not linearly dependent.

A trivial linear combination for the vector 0 would be setting all the coefficients to 0

Claim: A subset S of V is linearly independent iff the only linear combination for 0 in $\text{span}(S)$ is the trivial one.

Proof. Assume that S is linearly independent. Cannot form 0 vector using nonzero scalars for $u_1, \dots, u_n \in S$. Since $\text{span}(S)$ is the set of all linear combinations of vectors in S , it must be the trivial solution. To prove the opposite direction, use the contrapositive. If $S \subset V$ is linearly dependent, then there exists a nontrivial linear combination for 0 in $\text{span}(S)$. \square

$Ax = 0$ has a unique solution iff the set is $\{u_1, \dots, u_n\}$ (which forms A) is linearly independent.

Assume $S_1 \subseteq S_2$. If S_2 is linearly independent, then S_1 is linearly independent. If S_1 is linearly dependent, then S_2 is linearly dependent.

Definition. A **basis** for a vector space V is a linearly independent subset of V that generates V .

Observe:

1. $S = \emptyset$ is linearly independent. Then \emptyset is a basis for $v = \{0\}$
2. $S = \{u\}$ is linearly independent if and only if $u \neq 0$

Proof. Since S is linearly independent, we only have the trivial combination for 0. But if $u = 0$, then $1 \cdot u = 0$ would be a nontrivial combination for 0, so a contradiction. Assume that $u \neq 0$. Let $a \cdot u = 0$ because $u \neq 0$. We have $a = 0$, so no dependence exists. Therefore S is linearly independent. \square

3. $v = \mathcal{P}(\mathbb{R})$
 $S = \{1, x, x^2, \dots\}$. S is spanning and linearly independent because $a_0 + a_1x + a_2x^2 + \dots + a_nx^n = 0$ if and only if $a_0 = a_1 = \dots = a_n = 0$. So, there is no dependence relations in S .
4. $P(\mathbb{R}) = \mathbb{R} \oplus \mathbb{N}$ is the set of sequences where the tail is all zeros
5. $V = \mathbb{F}^n$
This is the space of n -tuples with entries in \mathbb{F} .

$$B = \begin{cases} e_1 = (1, 0, \dots, 0) \\ e_2 = (0, 1, \dots, 0) \\ \vdots \\ e_n = (0, 0, \dots, 1) \end{cases}$$

We say that B is the standard basis of \mathbb{F} . B is spanning because any $a = (a_1, a_2, \dots, a_n) = a_1e_1 + a_2e_2 + \dots + a_ne_n$. B is linearly independent because $a_1e_1 + \dots + a_ne_n = (0, \dots, 0)$ if and only if $(a_1, a_2, \dots, a_n) = (0, 0, \dots, 0)$.

6. $V = \mathbb{R}^2$
Let $B_1 = \{e_1, e_2\}$ and $B_2 = \{e_1 + e_2, e_1 - e_2\}$. Both are bases for V .

Proof. B_2 is spanning. $v = (a, b) = x_1 v_1 + x_2 v_2$ implies that $x_1 = \frac{a+b}{2}$ and $x_2 = \frac{a-b}{2}$. It is also linearly independent. $x_1 v_1 + x_2 v_2 = 0$ if and only iff $(x_1, x_2) = (0, 0)$. \square

7. If \mathbb{F} is an arbitrary field of scalars, then $\{e_1 + e_2, e_1 - e_2\}$ is a basis for \mathbb{F}^2 if and only if $\text{char}(\mathbb{F}) \neq 2$.

8. $V = P_n(\mathbb{F})$ is the set of polynomials of degree less than or equal to n with coefficients in \mathbb{F}

Important: Let V be a vector space over \mathbb{F} and let $u_1, u_2, \dots, u_n \in V$. $B = \{u_1, u_2, \dots, u_n\}$ is a basis for V if and only if each $v \in V$ can be uniquely expressed as a combination of u_1, u_2, \dots, u_n .

Proof. Assume B is a basis meaning $V = \text{span}(B)$. So, for all $v \in V$ it can be expressed as a linear combination. Assume that v can be written in two ways. $v = c_1 u_1 + \dots + c_n u_n = d_1 u_1 + \dots + d_n u_n$. $0 = v - v = (c_1 - d_1)u_1 + \dots + (c_n - d_n)u_n$. Since the basis is linearly independent, we must have $c_1 = d_1, \dots, c_n = d_n$. Assume that B can be uniquely expressed. Then, the only combination for 0 is the trivial solution and the vectors span V . So, B is a basis for V . \square

Claim: If V is generated by a finite sets, then some subset of S is a basis for V .

Proof. If $S = \{0\}$. Otherwise, take some nonzero vector $u_1 \in S$. Then $\{u_1\}$ is linearly independent. If possible, choose some vectors $u_2, \dots \in S$ such that the set $\{u_1, u_2, \dots, u_k\}$ is linearly independent for all $k \geq 1$. By finiteness of S , this process ends and we call the resulting linearly independent subset of S as $B = \{u_1, u_2, \dots, u_n\}$. We claim that this B generates V . Note that either $B = S$ so B is a basis or $B \subset S$. Then $S \subset B \implies \text{span}(S) \subset \text{span}(B) \implies V = \text{span}(B)$. Therefore, B is a basis. \square

Claim: (Replacement Theorem) Let V be any vector space over \mathbb{F} generated by a set G containing exactly n vectors. If L is a linearly independent subset of V containing exactly m vectors, then $m \leq n$. Also there is a subset H of G containing exactly $n - m$ vectors such that $L \cup H$ generates V .

Proof. (by induction) If $m = 0$, $L = \emptyset$, then $H = G$. Next assume the statement holds for $m \geq 0$. $\mathcal{L} = \{v_1, v_2, \dots, v_{m+1}\}$. Then, $\{v_1, v_2, \dots, v_m\}$ is linearly independent. By IH, we know $m \leq n$ and there is a subset $\{u_1, u_2, \dots, u_{n-m}\}$ of G such that $\{v_1, v_2, \dots, v_m\} \cup \{u_1, u_2, \dots, u_{n-m}\}$ generates V . Then, $v_{m+1} = a_1 v_1 + a_2 v_2 + \dots + a_m v_m + b_1 u_1 + \dots + b_{n-m} u_{n-m}$ for some $a_i, b_j \in \mathbb{F}$. We already know $n - m \geq 0$ but we claim $n - m \neq 0$ because otherwise L would not be linearly independent. So $n \geq m + 1$ and some $b_j \neq 0$. Without loss of generality, consider $b_1 \neq 0$.

$$u_1 = \left(-\frac{a_1}{b_1}\right) v_1 + \dots + \left(-\frac{a_m}{b_1}\right) v_m + \left(\frac{1}{b_1}\right) v_{m+1} + \left(-\frac{b_2}{b_1}\right) u_2 + \dots + \left(-\frac{b_{n-m}}{b_1}\right) u_{n-m}$$

Let $H = \{u_2, u_3, \dots, u_{n-m}\}$. Then $u_1 \in \text{span}(L \cup H)$. So, $V = \text{span}(L \cup H)$. \square

Claim: Let B have a finite basis. Then all bases for B are finite and all have the same number of vectors.

Proof. Let B be a basis with cardinality of V . $|B| = n$ and let C be any other basis. If $|C| > n$, then choose a subset $S \in C$ with exactly $n + 1$ vectors. But then I have $n + 1$ linearly independent vectors in something that is spanned by n vectors, which is a contradiction so C is finite and $m \leq n$. Similarly, $n \leq m$ and $n = m$. \square

Definition. Let V be any vector space over \mathbb{F} . We call V finite dimensional if it has a finite basis. We then call the number of vectors in any basis the dimension of V . If V is not finite dimensional, we call it infinite dimensional.

Examples:

1. $V = \{0\}$; $\dim V = 0$ because $\{\}$ is a basis
2. $F = \mathbb{C}, V = \mathbb{C}$; $\dim_{\mathbb{C}} \mathbb{C} = 1$ because $\{1\}$ is a basis
3. $F = \mathbb{R}, V = \mathbb{C}$; $\dim_{\mathbb{R}} \mathbb{C} = 2$ because $\{1, i\}$ is a basis
4. $V = P(\mathbb{F})$ is infinite dimensional

Important: A vector space always has a basis

Existence of Basis and Axiom of Choice

Proof. Take nonempty set $A_\alpha \neq \emptyset$

□

Linear Transformations

Definition. Let V and W be vector spaces over \mathbb{F} . A transformation $T : V \rightarrow W$ is called linear if the following hold

1. $T(x + y) = T(x) + T(y)$ for all $x, y \in V$
2. $T(cx) = cT(x)$ for all $c \in \mathbb{F}$ and $x \in V$

Observe that if T is linear then

1. $T(0_V) = 0_W$
2. $T(x - y) = T(x) - T(y)$
3. $T(\sum_i c_i x_i) = \sum_i c_i T(x_i)$

Definition. Given $V, W/\mathbb{F}$ and $T : V \rightarrow W$ linear, define null space (or kernel) of T :

$$N(T) = \{x \in V : T(x) = 0\}$$

and the range (or image) of T :

$$R(T) = \{T(x) : x \in V\}$$

Example

1. $T : V \rightarrow V$ ($x \rightarrow x$); Range is all of V , Null space is $\{0\}$
2. $T : V \rightarrow W$ ($x \rightarrow 0$)

Theorem: Let $T \in \mathcal{L}(V, W)$, then $R(T)$ is a subspace of W .

Theorem: Let $T \in \mathcal{L}(V, W)$, then $N(T)$ is a subspace of V .

Theorem: Let $T \in \mathcal{L}(V, W)$, then T is injective if and only if $N(T) = \{0\}$.

Proof. Assume that T is injective. Assume $T(v) = 0_W$. By injectivity, $v = 0_V$. So, $N(T) = \{0\}$. Assume $N(T) = \{0\}$. Take any $v_1 \neq v_2 \in V$. Assume that $T(v_1) = T(v_2)$. Then $T(v_1) - T(v_2) = T(v_1 - v_2) = 0$. \square

Definition. Given V, W as vector spaces over \mathbb{F} and a linear $T : V \rightarrow W$. If $N(T)$ and $R(T)$ are finite dimensional, then define **nullity** of T as $\dim N(T)$ and **rank** of T as $\dim R(T)$.

Rank-Nullity Theorem: If V is finite dimensional, then $\dim V = \text{nullity of } T + \text{rank of } T$.

Proof. Let $\dim V = n$ and $\dim N(T) = k$. $k \leq n$ because $N(T)$ is a subspace of V . Choose basis $\{v_1, \dots, v_k\}$ for the nullspace. Extend it to B for V , where $B = \{v_1, \dots, v_k, v_{k+1}, \dots, v_n\}$. We claim $\{T(v_{k+1}), \dots, T(v_n)\}$ is a basis for the range. T on the basis $\{v_1, \dots, v_k\}$ will be 0. \square

Example: If V and W are finite dimensional with equal dimension. For, $T : V \rightarrow W$, (a) T is injective, (b) T is surjective, and (c) $\text{rank}(T) = \dim V = \dim W$

Claim: Let V, B be vector spaces over \mathbb{F} . $\{v_1, \dots, v_n\}$ is a basis for V . Let $w_1, \dots, w_n \in W$. Then $\exists! T : V \rightarrow W$ such that $T(v_i) = w_i$ for all $1 \leq i \leq n$.

Proof. Existence: Define $T(\sum_{i=1}^n c_i v_i) := \sum_{i=1}^n c_i w_i$. This T defines a linear transformation $T(v_i) = w_i$.

Uniqueness: Assume there exists another $S : V \rightarrow W$ satisfying the property. Then,

$$S(v) = S\left(\sum c_i v_i\right) = \sum c_i S(v_i) = \sum c_i w_i = T\left(\sum c_i v_i\right) = T(v)$$

\square

Definition. Let V be a finite dimensional vector space over \mathbb{F} and let $B = \{v_1, \dots, v_n\}$ be an ordered basis for V . For any $x \in V$, write $x = \sum_{i=1}^n a_i v_i$. We define the coordinate vector of x relative to B as the following,

$$[x]_B = (a_1 \quad a_2 \quad \cdots \quad a_n)$$

Definition. Let V be a finite dimensional vector space over \mathbb{F} with an ordered basis $B = \{v_1, \dots, v_n\}$ and let W be a finite dimensional vector space over \mathbb{F} with an ordered basis $C = \{w_1, \dots, w_m\}$. Then, the matrix of a linear transformation $T : V \rightarrow W$ is

$$A = [T]_B^C := [a_{ij}]_{m \times n} = ([T(v_1)]_C \quad \cdots \quad [T(v_n)]_C)$$

If $V = W$ and $B = C$, then we write $[T]_B$.

The Kronecker-Delta function is

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Examples:

1. The identity transformation $Id : V \rightarrow V$. If V is finite dimensional with basis β , then $[T]_\beta = [\delta_{ij}]_{n \times n} = I$
2. $T : V \rightarrow W$, where every $v \in V$ is mapped to $0 \in W$, then $[T] = [0]_{m \times n}$
3. Isomorphism: Let V and W be vector spaces over \mathbb{F} . Set $\mathcal{L}(V, W) := \{ \text{linear transformations } V \rightarrow W \}$. $\mathcal{L}(V, W)$ is a vector space over \mathbb{F} . If $V = W$, we write $\mathcal{L}(V) := \mathcal{L}(V, V)$. When $\dim V = n < \infty$ and $\dim W = m < \infty$, then there is a bijection between $\mathcal{L}(V, W)$ and $M_{m \times n}(\mathbb{F})$. This is an isomorphism because

$$[T + U]_\beta^{\mathcal{C}} = [T]_\beta^{\mathcal{C}} + [U]_\beta^{\mathcal{C}} \quad [aT]_\beta^{\mathcal{C}} = a[T]_\beta^{\mathcal{C}}$$

Let V, W, Z be vector spaces over \mathbb{F} . Let $T : V \rightarrow W$ and $U : W \rightarrow Z$. Then, $UT = U \circ T = V \rightarrow Z$.

Observe:

1. $T \in \mathcal{L}(V, W)$ and $U \in \mathcal{L}(W, Z)$ implies $UT \in \mathcal{L}(V, Z)$

$$(UT)(x + y) = U(T(x + y)) = U(T(x) + T(y)) = U(T(x)) + U(T(y)) = (UT)(x) + (UT)(y)$$

$$(UT)(ax) = U(T(ax)) = U(aT(x)) = aU(T(x)) = a(UT)(x)$$

2. If $T_1, T_2 \in \mathcal{L}(V, W)$ and $U_1, U_2 \in \mathcal{L}(W, Z)$, then $U_1(T_1 + T_2) = (U_1T_1) + (U_1T_2)$ and $(U_1 + U_2)(T_1) = (U_1T_1) + (U_2T_1)$.
3. Composition is associative
4. $T \in \mathcal{L}(V, W)$ implies $T = TI_V = I_WT$
5. $a(UT) = (aU)T = U(aT)$
6. When $T \in \mathcal{L}(V)$, $T^0 = I_V, T^1 = T, T^2 = TT, T^3 = TTT, \dots$
7. Assume that $T : V \rightarrow W, U : W \rightarrow Z$, V has basis \mathcal{B} , W has basis \mathcal{C} , and Z has basis \mathcal{D} . Also, $\dim V = n, \dim W = m, \dim Z = p$. So, $[T]_\mathcal{B}^\mathcal{C} = B_{m \times n}$ and $[U]_\mathcal{B}^\mathcal{D} = A_{p \times m}$. Then, $[UT]_\mathcal{B}^\mathcal{D} = C_{p \times n}$. This leads to the definition of matrix multiplication.

$$(UT)(v_j) = U(T(v_j)) = U\left(\sum_k B_{kj}w_k\right) = \sum_k B_{kj}U(w_k) = \sum_k B_{kj}\left(\sum_i A_{ik}z_i\right) = \sum_i C_{ij}z_i$$

So, $C = AB$ with $C_{ij} = A_{i1}B_{1j} + \dots + A_{ip}B_{pj}$.