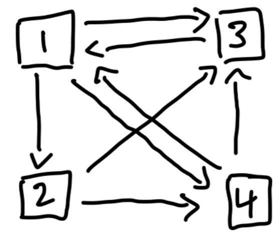
MATH 4310 Lecture Notes (Dylan Tom)

Introduction & Fields



Question: How do we determine the page order for a mini "google"?

1. (Simple Approach) Determine the importance by the number of back links (we expect page 3 should be the top*)

2. (Weighted Approach) Back links from "important" pages should weigh more. Let the "score" of a page be the sum of the scores of its back links.

3. Prevent undue influence by one page linking to too many other pages. If page j contains n_j links, one of which is page k, then boost the score of page k by $\frac{x_j}{n_j}$ where x_j is the score of page j

In our example,

$$
\begin{aligned}
x_1 &= \frac{1}{1}x_3 + \frac{1}{2}x_4 \\
x_2 &= \frac{1}{3}x_1 \\
x_3 &= \frac{1}{3}x_1 + \frac{1}{2}x_2 + \frac{1}{2}x_4 \\
x_4 &= \frac{1}{3}x_1 + \frac{1}{2}x_2
\end{aligned}
$$

Answer: $x_1 = \frac{12}{31} \quad x_2 = \frac{4}{31} \quad x_3 = \frac{9}{31} \quad x_4 = \frac{6}{31}$

*We have shown that page 1 should be ranked higher than 3, so our intuition wasn't correct.

Question: What are some properties of the set of real numbers with addition and multiplication?

1. There is a $0 \in S$ such that $0 + a = a$ for all $a \in S$

2. There is a $1 \in S$ such that $1 \cdot a = a$ for all $a \in S$

3. commutativity, associativity, distributivity

4. There exists a $(-a) \in S$ such that $a + (-a) = 0$ for all $a \in S$

5. There exists a $a^{-1} \in S$ such that $aa^{-1} = 1$ for all $a \in S$
6. $a - b = a + (-b)$ and $\frac{a}{b} = a \cdot b^{-1}$

Question: What sets have these properties?

$$\mathbb{R}, \mathbb{Q}, \mathbb{C}, \mathbb{F}_p = \mathbb{Z}/p$$

Question: What sets do not satisfy these properties?

$$\mathbb{Z}, \mathbb{N}, \mathbb{M}_{2\times 2}$$

Definition. A **field**, $\mathbb{F}$, is a set on which addition $(+)$ and multiplication $(\cdot)$ are defined so that the following properties hold for all $a, b, c \in \mathbb{F}$.

1. $a + b = b + a \quad a \cdot b = b \cdot a$ (commutativity)
2. $(a + b) + c = a + (b + c) \quad (a \cdot b) \cdot c = a \cdot (b \cdot c)$ (associativity)
3. There exists *distinct* elements $0, 1$ such that $0 + a = a$ and $1 \cdot a = a$ (identity)
4. There exists $c, d \in \mathbb{F}$ such that $a + c = 0$ and $bd = 1$ where $d \neq 0$ (invertibility).
   Define $c = -a$ and $d = b^{-1}$ (see uniqueness below)
5. $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$ (distributivity)

Example: Some fields are $\mathbb{Q}, \mathbb{R}, \mathbb{C}, \mathbb{F} = \{a + b\sqrt{2} | a, b \in \mathbb{Q}\}, \mathbb{F}_2$

Cancellation Laws:

1. $a + b = a + c \Rightarrow b = c$

   Proof. Let's assume $a + b = a + c$. By (4), there is some x such that $x + a = 0$. Now $x + (a + b) = x + (a + c)$. By (2), $(x + a) + b = (x + a) + c \Rightarrow 0 + b = 0 + c$. By (3), $b = c$. ☐

2. $a \cdot b = a \cdot c$ and $a \neq 0 \Rightarrow b = c$

   Proof. Let's assume $a \cdot b = a \cdot c$ and $a \neq 0$. By (4), there is some x such that $ax = 1$. Now $x(ab) = x(ac)$. By (2), $(xa)b = (xa)c \Rightarrow 1b = 1c$. By (3), $b = c$. ☐

Example: Uniqueness of 0, 1, additive inverse, and multiplicative inverse

Proof. (multiplicative inverse) Given $b \neq 0$, let d and d' satisfy $b \cdot d = 1$ and $b \cdot d' = 1$. Then, $b \cdot d = b \cdot d'$. So, $d = d'$ (by cancellation). Similarly, for others. ☐