Numerical Linear Algebra

Tutorial homework questions: 8a(ii); 19c.

H By Hand; **C** Computer; **T** Theory; **E** Extra; **A** Advanced.

Recommended: T1; C2.*; C3; T5; H8.a; C9; 10.*; C11; H12.b; C14; 15.b; C17.*; 18.b; C21.a; C27.b; T28.*.

T1. If Q is an orthogonal matrix, show that $||Q\mathbf{v}||_2 = ||\mathbf{v}||_2$ for any vector $\mathbf{v}$, and hence determine $||Q||_2$.

Linear equations

C2. Consider the system of linear algebraic equations

$$\begin{aligned} 2x_1 + \ \ 5x_2 &= 4, \\ 3x_1 + 11x_2 &= 9. \end{aligned}$$

- **a.** Use builtin MATLAB commands to solve the system, displaying your answer `x` using `format long`.
- **b.** Compute the residuals $A\mathbf{x} - \mathbf{b}$. What do you notice?
- **c.** Solve the systems in single precision using `As=single(A); bs=single(b)`, and convert back to double precision using `xs=double(xs)`. Compare the accuracy of your answer with that of the double-precision value.

C3. Let

$$A = \begin{pmatrix} 2 & 4 & -1 & 2 \\ 5 & -2 & 1 & 9 \\ -1 & 3 & 1 & 4 \\ -1 & -5 & 3 & 7 \end{pmatrix}.$$

- **a.** Compute the LU-factorisation of A using the Matlab command `[L,U]=lu(A)`. Check that $A - LU = O$. Is the form of `L` unit-lower-triangular?
- **b.** Compute the LU-factorisation with row pivoting using the Matlab command `[L,U,P]=lu(A)`. Compare P, P^T and P^{-1}. Check that $A = P^{-1}LU$, or equivalently, $PA = LU$ or $LU - PA = O$.

4. Let

$$A = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 & 0 \\ 0 & 1 & 4 & 1 & 0 \\ 0 & 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix}$$

1. Compute the LU-factorisation of A. What do you notice about the zeros of L and U?
2. Compute A^{-1}, L^{-1} and U^{-1}. What happens to the zeros in the inverses?
3. How many operations are needed to solve the matrix equations $Ly = b$ and $Ux = y$? How does this compare with the work of computing $y = L^{-1}b$ and $x = U^{-1}y$?
4. Explain why it is often undesirable to explicitly invert a matrix A in order to solve the linear equation $Ax = b$.

T5. Consider the system of linear algebraic equations

$$\begin{pmatrix} 1 & 2 \\ 1+\epsilon & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3+\delta \\ 3 \end{pmatrix}$$

for small values of δ, ϵ. Write down the analytic solution. What happens for $\delta = 10^{-2}$ as ϵ changes from 10^{-3} to -10^{-3}, and why?

6. The $n \times n$ *Hilbert matrix* $H_{ij}^{(n)}$ is defined by

$$H_{ij}^{(n)} = \frac{1}{i+j-1}, \quad 1 \le i, j \le n.$$

It arises when solving for the coefficients of least-squares polynomial approximations to a function over the interval $[0, 1]$.

a. Compute $[H^{(4)}]^{-1}$ and hence find $K_\infty(H^{(4)})$.

b. Use Matlab to solve the system of linear algebraic equations

$$H^{(4)}\mathbf{c} = \mathbf{r} \text{ where } r_i = 1/(i+5)$$

using single-precision arithmetic. Compute the error and residual, and compare the actual error to the error bound obtained from the residual using the condition number of $H^{(4)}$.

E7. Use single-precision arithmetic to approximate the solutions to the following systems of linear algebraic equations. Then use one step of iterative refinement to improve the approximation. Compare the improved approximation to the solution computed using double-precision.

a. $A = \begin{pmatrix} 0.03 & 58.9 \\ 5.31 & -6.10 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 59.2 \\ 47.0 \end{pmatrix}.$

b. $A = \begin{pmatrix} 3.3330 & 15920 & 10.333 \\ 2.2220 & 16.710 & 9.6120 \\ -1.5611 & 5.1792 & -1.6855 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 7953 \\ 0.965 \\ 2.714 \end{pmatrix}.$

Iterative methods

H8. Compute the first two iterations for (i) the Jacobi method, (ii) the Gauss-Seidel method and (iii) the successive over-relaxation method (with $\omega = 1.1$) for the following systems of linear algebraic equations, using $\mathbf{x}^{(0)} = \mathbf{0}$. Compute the errors $||\tilde{\mathbf{x}} - \mathbf{x}||$ and residuals $||A\tilde{\mathbf{x}} - \mathbf{b}||$.

a. $A = \begin{pmatrix} 2 & -1 & 1 \\ 4 & 7 & 3 \\ 3 & 1 & 6 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$

b. $A = \begin{pmatrix} 5 & 2 & 0 \\ 2 & 7 & -3 \\ 0 & -3 & 8 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 8 \\ 4 \\ 3 \end{pmatrix}$

c. $A = \begin{pmatrix} 13 & 3 & 2 & 1 \\ 3 & 8 & -6 & 2 \\ -2 & -2 & 6 & -1 \\ -1 & -1 & -3 & 4 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 4 \\ 15 \\ -6 \\ -7 \end{pmatrix}$

d. $A = \begin{pmatrix} 4 & -1 & 0 & 0 & 0 \\ -1 & 4 & -1 & 0 & 0 \\ 0 & -1 & 4 & 0 & 0 \\ 0 & 0 & 0 & 4 & -1 \\ 0 & 0 & 0 & -1 & 4 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 3 \\ 1 \\ 0 \\ -2 \\ 0 \end{pmatrix}$

How many operations are needed for each iteration of the Jacobi or Gauss-Seidel method for an $n \times n$ matrix A? How many operations are needed for an iteration if A has p nonzero elements?

C9. Write algorithms `x=jsolve(A,b,tol)`, `x=gssolve(A,b,tol)` and `x=sorsolve(A,b,omega,tol)` to implement the Jacobi, the Gauss-Seidel and the successive over-relaxation iterations for solving a system of linear algebraic equations with error tolerance tol. Try out your result on the examples of Question 8 to validate your code is correct. Which method is easiest to implement? Which converges fastest?

10. Consider the system of linear algebraic equations $A\mathbf{x} = \mathbf{b}$ where

$$A = \begin{pmatrix} 2 & -1 & 1 \\ 3 & 2 & 2 \\ -2 & 1 & 3 \end{pmatrix}; \quad \mathbf{b} = \begin{pmatrix} -1 \\ 3 \\ -4 \end{pmatrix}.$$

a. Show that the Jacobi method applied to this example fails to converge.

b. Show that the Gauss-Seidel method does converge, and approximate the solution to the equations to within 10^{-3} in the l_∞ norm.

Now set $A_{3,1} = -1$.

c. Show that both methods now converge, and compute the answer using each method to within 10^{-3}.

d. Which method gives the more accurate result after a few steps? Explain your answers in terms of the spectral radius of the iteration matrices $T_{\mathrm{Jac}} = D^{-1}(U + L)$ and $T_{\mathrm{GS}} = (L + D)^{-1}U$, where $A = L + D + U$.

Note: You can use the MATLAB commands `tril(A,-1)` and `triu(A,+1)` to extract (strictly) lower/upper-triangular parts of a matrix.

The conjugate gradient method

C11. Use the MATLAB conjugate-gradient code `pcg` to perform two steps of the conjugate gradient method with $C = C^{-1} = I$ on the positive-definite systems of linear algebraic equations in Question 8(**b**,**d**). Compare the results to those obtained using the iterative methods.

H12. Solve each of the following systems of linear algebraic equations $A\mathbf{x} = \mathbf{b}$ using the conjugate-gradient method:

a.

$$A = \begin{pmatrix} 7 & 3 \\ 3 & 2 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 17 \\ 11 \end{pmatrix}$$

b.

$$A = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} \frac{1}{4} \\ \frac{1}{5} \\ \frac{1}{6} \end{pmatrix}$$

13. For each of the systems of linear algebraic equations of Question 12,

a. Solve the equations using Gaussian elimination with single-precision arithmetic.

b. Solve the equations using the conjugate gradient method ($C = C^{-1} = I$) with single-precision arithmetic.

c. Which method gives the better answer?

d. Choose $C^{-1} = D^{-1/2}$. Does this choice improve the conjugate gradient method?

Eigenvalues and eigenvectors

C14. Use the MATLAB command `[Q,R]=qr(A)` to compute the QR-factorisation of the matrix

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 5 & -1 & 3 \\ -1 & 3 & 1 \\ 3 & -2 & -1 \end{pmatrix}$$

Check your answer by computing $QR - A$. Hence give an orthogonal basis for the vector space spanned by the columns of A.

15. Compute the QR-factorisation of the following matrices:

a. $\begin{pmatrix} 3 & -2 \\ 4 & 1 \end{pmatrix}$ **b.** $\begin{pmatrix} 3 & -2 & 1 & 4 \\ 4 & 1 & 0 & 3 \end{pmatrix}$

16. Use the Geršgorin circle theorem to determine bounds for (i) the eigenvalues and (ii) the spectral radius of the following matrix:

$$\begin{pmatrix} 6.5 & 0.6 & 0 \\ 0.6 & 3.8 & 0.9 \\ 0 & 0.9 & 2.7 \end{pmatrix}.$$

C17. For the matrix

$$A = \begin{pmatrix} 1 & -6 & -1 \\ -2 & -3 & 0 \\ 3 & -3 & 2 \end{pmatrix},$$

a. Use the MATLAB command `lambda=eig(A)` to compute the eigenvalues of A.

b. Use `[V,D]=eig(A)` to compute a matrix V whose columns are the eigenvectors of A, and a diagonal matrix D whose entries are the eigenvalues. Verify that $A = VDV^{-1}$, or equivalently, that $AV - VD = O$.

c. Use the MATLAB command `eig` to compute the eigenvalues and eigenvectors of the matrices below. What do you notice?

$$A_1 = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \qquad A_2 = \begin{pmatrix} 3 & -2 \\ 2 & 3 \end{pmatrix}$$

The Power Method

18. Compute the first 3 iterations of the power method applied to the following matrices A with the given initial vectors $\mathbf{x}^{(0)} = \mathbf{x}_0$. Hence give estimates of the maximum eigenvalue λ and corresponding eigenvector $\mathbf{v}$.

a. $\begin{pmatrix} 2 & 5 \\ 1 & -4 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ **b.** $\begin{pmatrix} 1 & 1 & 3 \\ 2 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ **c.** $\begin{pmatrix} 1 & -1 & 0 \\ -2 & 4 & -2 \\ 0 & -1 & 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}$

19. Repeat Question 18 using the inverse power method with initial eigenvalue estimate $\mu = (\mathbf{x}_0^T A\mathbf{x}_0) \,/\, (\mathbf{x}_0^T \mathbf{x}_0)$. Compare your results with those using the power method by computing $(A - \lambda^{(3)})\mathbf{x}^{(3)}$.

A20. Let

$$A = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

a. Use the Geršgorin circle theorem to determine a region in the complex plain containing all the eigenvalues of A.

b. Use the power method to determine the maximum eigenvalue of the matrix and its corresponding eigenvector.

c. Use deflation to obtain a 3×3 matrix B with the same eigenvalues λ_2, λ_3 and λ_4 as A.

d. Use the inverse power method to find λ_4, the only real eigenvalue of B.

e. Find the two additional eigenvalues of A, which form a complex-conjuage pair.

C21. Write a MATLAB function `[l,v]=inverse_power(A,q,e)` to compute the eigenvalue of a matrix closest to q to a tolerance of e, and the corresponding eigenvector. Apply your method to the following matrices, taking $q = \mathbf{x}^T A\mathbf{x} \,/\, \mathbf{x}^T\mathbf{x}$ for the given vector(s).

a. $\begin{pmatrix} 2 & -1 & 1 \\ 2 & 2 & 0 \\ 1 & 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ **b.** $\begin{pmatrix} 5 & 1 & -1 & 1 \\ 1 & 1 & -2 & 1 \\ -1 & -2 & 3 & -1 \\ 1 & 1 & -1 & 4 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \\ 1 \end{pmatrix}$

Householder's Method (Advanced)

C22. Use the MATLAB command `[P,H]=hess(A)` to compute an orthogonal matrix P such that $H = P^T AP$ is in upper-Hessenberg form for the following matrices:

a. $\begin{pmatrix} 5 & 1 & -1 & 1 \\ 1 & 1 & -2 & 1 \\ -1 & -2 & 3 & -1 \\ 1 & 1 & -1 & 4 \end{pmatrix}$ **b.** $\begin{pmatrix} 2 & 5 & 6 & -1 \\ 6 & 0 & 1 & 2 \\ 5 & 4 & 3 & 0 \\ 3 & 2 & 7 & -3 \end{pmatrix}$

A23. Use Householder's method to find a matrix in upper-Hessenberg form similar to the given matrix.

a. $\begin{pmatrix} 12 & 3 & 4 \\ 3 & 5 & -3 \\ 4 & -3 & 2 \end{pmatrix}$ **b.** $\begin{pmatrix} 1 & -1 & 2 \\ 2 & 1 & 0 \\ 1 & 3 & 1 \end{pmatrix}$

C24. Write a MATLAB function to compute an orthogonal matrix P such that $H = P^T AP$ is in upper-Hessenberg form. Apply your method to the matrices of Question 22. For a symmetric matric, check that your result is *exactly* tridiagonal.

The QR Method

C25. Use the QR method to determine all the eigenvalues for the matrices given below. Continue until the Geršgorin circles are disjoint. Then use the inverse power method to refine your bounds.

a. $\begin{pmatrix} 4 & 3 & 0 \\ 3 & 6 & 1 \\ 0 & 1 & 2 \end{pmatrix}$ **b.** $\begin{pmatrix} 4 & 3 & -2 \\ 3 & 6 & 1 \\ 0 & 1 & 2 \end{pmatrix}$

c. $\begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix}$ **d.** $\begin{pmatrix} 5 & 0 & -2 & 2 \\ -1 & 4 & 3 & -2 \\ 0 & 1 & 3 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix}$

Note: If A is symmetric with distinct eigenvalues, then repeated application of the QR method always eventually yields a matrix for which the Gershgorin circles are disjoint. This need not be the case for a non-symmetric matrix, even if the eigenvalues are real and distinct.

C26. Apply the QR method directly to the matrices of Question 22, and to the upper-Hessenberg form of the matrices. Which method converges fastest? For the 4×4 case, compare the amount of work needed per step.

Singular Value Decomposition

C27. Use the MATLAB command `[U,S,V]=svd(A)` to determine the singular value decomposition of the following matrices. Compare your answer with that which you obtain by computing eigenvalues of $A^T A$.

a. $\begin{pmatrix} 2 & -1 & 1 \\ 1 & 3 & 0 \end{pmatrix}$ **b.** $\begin{pmatrix} 2 & 1 \\ -1 & 3 \\ 1 & 0 \end{pmatrix}$ **c.** $\begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & -1 & 1 & 0 \end{pmatrix}$

d. $\begin{pmatrix} 5 & -1 & 0 \\ -1 & 4 & 2 \\ 0 & 2 & 1 \end{pmatrix}$ **e.** $\begin{pmatrix} 1 & -2 & 0 \\ 3 & -1 & 4 \\ 0 & 1 & -1 \end{pmatrix}$

T28. Suppose that A is an $m \times n$ matrix. Show that

- **a.** The matrices $A^T A$ and AA^T are both symmetric.
- **b.** The eigenvalues of $A^T A$ are real and nonnegative.
- **c.** If $\mathbf{v}$ is an eigenvector corresponding to the *nonzero* eigenvalue λ of $A^T A$, then $A\mathbf{v}$ is an eigenvector of AA^T, and give the corresponding eigenvalue.
- **d.** If A is symmetric, then the singular values of A are the absolute values of the eigenvalues of A.

T29. Suppose that A is an $n \times n$ matrix, and that the singular values of A are $s_1 \geq s_2 \geq \cdots \geq s_n$.

- **a.** Show that A is invertible if, and only if, all singular values of A are nonzero. Write down the singular value decomposition of A^{-1}.
- **b.** Show that $||A||_2 = s_1$ and that $||A^{-1}||_2 = 1/s_n$. Hence show that $K_2(A) = s_1/s_n$.