Practice Exam Questions - Tutorial 6

1. Let $\mathbf{y} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$ and $\mathbf{u} = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$. Write $\mathbf{y}$ as the sum of a vector in Span$\{\mathbf{u}\}$ and a vector orthogonal to $\mathbf{u}$.

2. Prove or disprove the following statement.

   Let $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^3$ be two vectors that have the same length ($||\mathbf{x}|| = ||\mathbf{y}||$) and define $\mathbf{u} = \mathbf{x} + \mathbf{y}$ and $\mathbf{v} = \mathbf{x} - \mathbf{y}$. Then, $\mathbf{u}$ and $\mathbf{v}$ are orthogonal to each other.

3. True or False? For the statement below: state whether the statement is true or false.

   Two orthogonal vectors are automatically also linearly independent.

4. True or False? For the statement below: state whether the statement is true or false.

   Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$. If $\mathbf{x} \cdot \mathbf{y} = 0$, then there does not exist a plane in $\mathbb{R}^3$ that contains both $\mathbf{x}$ and $\mathbf{y}$.

5. Consider the following matrix A:

$$A = \begin{bmatrix} 1 & -4 & 9 & -7 \\ -1 & 2 & -4 & 1 \\ 5 & -6 & 10 & 7 \end{bmatrix}.$$

   Find two linearly independent vectors that are orthogonal to Nul A .

6. Consider the following matrix A:

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 1 & 1 \end{bmatrix}.$$

   Determine a vector $\mathbf{u}$ in Row A such that $\mathbf{u} - \begin{bmatrix} 3 \\ 2 \\ 9 \end{bmatrix}$ is orthogonal to Row A.

7. Determine two distinct vectors in $\mathbb{R}^3$ with length 1 that are orthogonal to both $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$.

8. Determine two distinct vectors in Span $\left\{ \begin{bmatrix} 1 \\ 0 \\ -2 \\ 3 \end{bmatrix} \right\}$ with length 1.

9. Let $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. Which one of the following subsets of $\mathbb{R}^3$ is a subspace of $\mathbb{R}^3$?

a. $\{\mathbf{x} \in \mathbb{R}^3 \mid \mathbf{x} + \mathbf{u} = \mathbf{0}\}$

b. $\{\mathbf{x} \in \mathbb{R}^3 \mid \mathbf{x} \cdot \mathbf{u} = 0\}$

c. $\{\mathbf{x} \in \mathbb{R}^3 \mid \mathbf{x} \cdot \mathbf{x} = 1\}$

d. $\{\mathbf{x} \in \mathbb{R}^3 \mid \mathbf{x} \cdot \mathbf{u} = 1\}$

e. None of the above.

10. Let A be a 6×4 matrix. Is the following statement true or false?

    The column space of A and the row space of A are orthogonal to each other.

11. Let A be a 6×4 matrix. Is the following statement true or false?

    The row space of A and the null space of A are orthogonal to each other.

12. Let A be a 6×4 matrix. Is the following statement true or false?

    The column space of A and the null space of A are orthogonal to each other.

13. What is the dot product (inner product) of $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 4 \\ 2 \\ -3 \end{bmatrix}$?

    a. $\begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}$

    b. $\begin{bmatrix} 4 \\ 3 \\ -4 \end{bmatrix}$

    c. 0

    d. 5

    e. The dot product cannot be computed for these vectors.

    f. None of the above.

14. If $\mathbf{y} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ and $\mathbf{u} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, then the orthogonal projection of $\mathbf{y}$ onto Span $\{\mathbf{u}\}$ is

    a. $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

    b. $\begin{bmatrix} \frac{3}{2} \\ \frac{-1}{2} \end{bmatrix}$

    c. $\begin{bmatrix} 10 \\ 5 \end{bmatrix}$

    d. $\begin{bmatrix} \frac{1}{10} \\ \frac{3}{10} \end{bmatrix}$

    e. None of the above.

15. Consider the following matrix A:

$$A = \begin{bmatrix} -1 & 1 & -1 \\ 1 & -1 & -1 \\ -1 & -1 & -1 \end{bmatrix}.$$

It is given that the matrix A has eigenvalues 1 and -2.

a) Show that $\left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \right\}$ forms a basis for the eigenspace corresponding to the eigenvalue 1.

b) Show that $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$ forms a basis for the eigenspace corresponding to the eigenvalue -2.

c) Orthogonally diagonalize matrix A, i.e. find an orthogonal matrix P and a diagonal matrix D such that $A = PDP^T$.

16. Let $A = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}$.

a) Show that the matrix A has eigenvalues 3 and 5.

b) Orthogonally diagonalize matrix A, i.e. find an orthogonal matrix P and a diagonal matrix D such that $A = PDP^T$.