Numerical Mathematics 2018/2019
Resit Exam Questions

— *Do not turn this page before the official start of the exam!* —

First Name, Surname: ______________________________

Student ID: ______________________________

Program: Bachelor Data Science and Knowledge Engineering / Maastricht Science Programme

Course code: KEN1540 / MAT3005

Examiner: Dr. P.J. Collins

Date/time: Friday 7^{th} July 2019, 9.00-12.00h

Format: Closed book exam

Allowed aides: Pens, simple (non-programmable) calculator from the DKE-list of allowed calculators, formula sheet (provided).

Instructions to students:

- The exam consists of 8 questions on 8 pages (excluding the 1 cover page(s)).
- Fill in your name and student ID number on each page, including the cover page.
- Answer every question at the reserved space below the questions. If you run out of space, continue on the back side, and if needed, use the extra blank page.
- Ensure that you properly motivate your answers.
- Do not use red pens, and write in a readable way. Answers that cannot be read easily cannot be graded and may therefore lower your grade.
- You are not allowed to have a communication device within your reach, nor to wear or use a watch.
- You have to return all pages of the exam. You are not allowed to take any sheets, even blank, home.
- If you think a question is ambiguous, or even erroneous, and you cannot ask during the exam to clarify this, explain this in detail in the space reserved for the answer to the question.
- If you have not registered for the exam, your answers will not be graded, and thus handled as invalid.
- **Good luck!**

The following table will be filled by the examiner:

Question:	1	2	3	4	5	6	7	8	Total
Points:	8	12	8	8	12	10	12	10	80
Score:									

1. (8 points) Briefly describe the double-precision floating-point number system. Suggest how decimal arithmetic with 3 significant figures can give an insight into double-precision arithmetic.

   Use decimal arithmetic rounded to 3 significant figures (i.e. round after every operation) to compute $x = 2 \div 11 - 3 \div 17$. What is the relative error of your answer? By rearranging the formula, obtain a more accurate estimate of x, still using 3-digit arithmetic.

2. (12 points) Let

$$f(x) = x^3 - 11x - 1\,.$$

Show that f has a root in the interval $I = [0, 4]$.

Apply one step of Newton's method starting at the midpoint of I. What happens?

Use one step of the bisection method to determine a smaller interval containing the root. Use (at most three steps of) Newton's method starting at the midpoint of the interval you have found to estimate the root of $f(x)$ in I to an accuracy of approximately 10^{-1}.

3. (8 points) Use the Bogacki-Shampine method to estimate the solution of the differential equation $\dot{y} = f(t, y) = 1 - t/y$ with initial condition $y(0) = 2$ at time $t = 1$ using a step-size $h = 0.5$.

   DKE Students Note that there is no need to calculate $k_{i,4}$ or $\hat{w}_{i+1}$.

   (5 points) **MSP Students Only** Provide an estimate of the error of the first step. Show that $h = 0.5$ is not suitable if you require an error of at most 10^{-4} at time $t = 1$. What step size would you recommend instead?

4. (8 points) The *forced van der Pol* system is given by

$$\ddot{x} + \mu(1 - x^2)\dot{x} + \alpha x = \rho \cos(\omega t),$$

where $\alpha, \mu, \rho, \omega$ are parameters. Describe how to simulate this system using Matlab, with initial condition $x(0) = 2$ and $\dot{x}(0) = -3$, including giving the code you would write.

For $\mu \gg 1$, which integration method would you prefer? Give a reason for your answer.

5. (12 points) Consider the data

x	0.4	0.6	0.2	0.8
y	0.22	0.60	0.28	0.46

Use divided differences to compute the cubic polynomial interpolating the data, writing your answer in nested form. Hence estimate y when $x = 0.5$.

Assuming that each data point has a measurement error of ± 0.01, how accurately would you expect to be able to estimate the values of $y'(x_i)$? Using the fact that if $f(x) = a_0 + (x - x_0)g(x)$, then $f'(x_0) = g(x_0)$, estimate $y'(0.4)$, giving your answer to appropriate precision.

6. (10 points) Compute the Romburg estimate $R_{3,3}$ for the integral

$$\int_1^3 f(x)\,dx$$

based on the following data:

x	0.5	0.75	1.0	1.25	1.5	1.75	2.0	2.25	2.5	2.75	3.0
y	0.6544	0.9869	0.9719	0.7726	0.4821	0.1734	−0.0953	−0.2835	−0.3716	−0.3610	−0.2711

You should compute your intermediate $R_{i,j}$ using at least 6 decimal places of precision, and give your final answer to a precision appropriate for its accuracy.

7. (12 points) The *shifted Legendre polynomials* are defined by the recurrence relation

$$Q_0(x) = 1,\ Q_1(x) = 2x - 1,\ Q_k(x) = \big((2k-1)(2x-1)Q_{k-1}(x) - (k-1)Q_{k-2}(x)\big)/k.$$

They are orthogonal over the interval $[0, 1]$, i.e. for $i \neq j$, $\int_0^1 Q_i(x)Q_j(x)dx = 0$ and satisfy the normalisation condition $\int_0^1 Q_k(x)^2 dx = 1/(2k+1)$

The coefficients of function f with respect to the shifted Legendre basis are

i	0	1	2	3	4	5
c_i	0.5981	−0.4974	−0.0232	0.0684	−0.0080	−0.0034

Compute $Q_k(x)$ for $k = 0, 1, 2, 3, 4$ using the recurrence relation for $x = 0.2$, and hence estimate the value of $g_4(0.3)$, where g_4 is the quartic polynomial least-squares approximation to f. Assuming $c_5 \gg c_k$ for $k > 5$, give an approximation to the square error $E_4 = \int_0^1 \big(f(x) - g_4(x)\big)^2 dx$.

(5 points) **MSP Students Only** With respect to the basis function $1, x, x^2, x^3, x^4$, the coefficients a_k of the least-squares interplating polynomial satisfy $H\mathbf{a} = \mathbf{b}$, where

$$H_{i,j} = \frac{1}{i+j+1} \text{ for } i, j = 0, 1, \ldots, 4;$$

$$H^{-1} = \begin{pmatrix} 25 & -300 & 1050 & -1400 & 630 \\ -300 & 4800 & -18900 & 26880 & -12600 \\ 1050 & -18900 & 79380 & -117600 & 56700 \\ -1400 & 26880 & -117600 & 179200 & -88200 \\ 630 & -12600 & 56700 & -88200 & 44100 \end{pmatrix},\ \mathbf{b} \approx \tilde{\mathbf{b}} = \begin{pmatrix} 0.5981 \\ 0.2162 \\ 0.1157 \\ 0.0742 \\ 0.0529 \end{pmatrix}$$

Compute the condition number $K_\infty(H)$. Hence estimate the absolute and relative errors of the approximation $\tilde{\mathbf{a}}$ assuming $\|\mathbf{b} - \tilde{\mathbf{b}}\| \leq 10^{-4}$. Explain how using the shifted Legendre polynomials as basis functions can improve the accuracy of polynomial approximation.

8. (10 points) Let

$$A = \begin{pmatrix} 5 & -3 \\ -1 & 1 \end{pmatrix}, \quad \mathbf{x}_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \mu = 1.$$

Use the Gershgorin circle theorem to give bounds on the eigenvalues, assuming both are real. Use two steps of the inverse power method using the given μ to approximate the smaller of the two eigenvalues, starting with the given $\mathbf{x}_0$.