DKE Numerical Mathematics 2020/2021
Exam Questions

Programme: Bachelor Data Science and Knowledge Engineering

Course codes: KEN1540

Examiners: Dr. P.J. Collins, Dr. K. Staňková

Date/time: Monday May 31st, 2021, 09.30-11.30h

Format: Closed book exam

Allowed aides: Pens, simple (non-programmable) calculator from the DKE-list of allowed calculators, formula sheet (provided).

Instructions to students:

- The exam consists of 7 questions on 2 pages (excluding the 1 cover page(s)).
- You can obtain maximally 80 points for this exam.
- Fill in your name and student ID number on the top-right corner of each side of paper you submit.
- Ensure that you properly motivate your answers.
- Do not use red pens, and write in a readable way. Answers that cannot be read easily cannot be graded and may therefore lower your grade.
- You are not allowed to have a communication device within your reach, nor to wear or use a watch.
- If you think a question is ambiguous, or even erroneous, and you cannot ask during the exam to clarify this, explain this in detail in with your answer to the question.
- If you have not registered for the exam, your answers will not be graded, and thus handled as invalid.
- **Good luck!**

The following table will be filled by the examiner:

Question:	1	2	3	4	5	6	7	Total
Points:	12	12	14	12	12	12	6	80
Score:								

1. (12 points) Use two steps of the secant method to estimate the root of $f(x) = x^3 - 2x^2 - 1$ in interval $[2, 3]$, starting at $p_0 = 3$ and $p_1 = 2$. Compute $f(p_3)$, and give the best bracket for the root you have found, and an estimate of the error.

2. (12 points) Use the two-stage Adams-Bashforth method to find the solution of the initial value problem $\dot{y} = 1 - t/y$, $y(1) = 5.0000$, $t \in [1, 2]$, with $h = \frac{1}{3}$. Bootstrap your calculation using values of $w_1 \approx y(1\frac{1}{3})$ for the most appropriate of the methods below, giving a reason for your answer.

1st-order Euler:	$w_1 = 5.60000000$
2nd-order Ralston:	$w_1 = 5.59341564$
3rd-order Heun:	$w_1 = 5.59338378$
4th-order Runge-Kutta:	$w_1 = 5.59337974$

Calculate the actual and relative error of $y(2)$, given that the exact value is 6.74733483 (8dp).

3. (14 points) Use divided differences to compute the polynomial interpolating the following data:

x_i	$2\frac{1}{3}$	$2\frac{2}{3}$	2	3
y_i	0.64	0.24	1.69	0.17

Estimate the value of y when $x = 2.5$. How does the ordering of the data points affect the interpolating polynomial? Provide a bound on the error of your estimate if the data comes from a function f satisfying $f'''(x) \leq 10$ for $x \in [2, 3]$

4. (12 points) Use the most accurate three-point formulae available to complete the following table:

x	$f(x)$	$f'(x)$	$\sqrt{1+f'(x)^2}$
1.0	−0.23925		
1.25	0.48679		
1.5	0.95160	0.41898	1.08423
1.75	0.90577	−0.57772	1.15489
2.0	0.37388	−1.54984	1.84445

Use the trapezoid rule to estimate $\int_1^2 \sqrt{1+f'(x)^2}\,dx$.

5. (12 points) Compute the first two approximations s_1, s_2 of the discrete Fourier transform for the following $n = 6$ data points over the time interval $[0, 2\pi]$:

t_j	$-\pi$	$-\frac{2}{3}\pi$	$-\frac{1}{3}\pi$	0	$\frac{1}{3}\pi$	$\frac{2}{3}\pi$
y_j	0.017	0.620	0.761	0.477	0.169	0.038

Hence estimate the value of y when $t = \frac{\pi}{2}$.

6. (12 points) Use two steps of the conjugate gradient method to estimate the solution of the equation $A\,x = b$, starting at $x^{(0)}$.

$$A = \begin{pmatrix} 3 & -1 & 1 \\ -1 & 6 & 2 \\ 1 & 2 & 7 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix}, \quad x^{(0)} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Compute the norm of the residual of your answer.

7. (6 points) In a compartmental model of an infectious disease with two variants, individuals are either susceptible (S), infections with the first variant (I_1), or with the second variant (I_2). The progress of the disease is modelled by

$$\dot{S} = -(\alpha_1 + \alpha_2)S/N, \quad \dot{I}_1 = \alpha_1 S/N - \beta_1 I_1, \quad \dot{I}_2 = \alpha_2 S/N - \beta_2 I_2.$$

Assume initally 340 individuals infected with variant 1 and 1 individual infected with variant 2, in a population with size $N = 17\,000\,000$.

Show how to solve this system of differential equations in Matlab, including writing the code you would use.