Exercises

1	2	3	4	5	6	7	8

1	1	1	1	1	1	1
2	2	2	2	2	2	2
3	3	3	3	3	3	3
4	4	4	4	4	4	4
5	5	5	5	5	5	5
6	6	6	6	6	6	6
7	7	7	7	7	7	7
8	8	8	8	8	8	8
9	9	9	9	9	9	9
0	0	0	0	0	0	0

Surname, First name

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Numerical Mathematics (KEN1540)
Resit

Program: Bachelor Data Science and Knowledge Engineering
Course code: KEN1540
Examiners: Pieter Collins & Martijn Boussé
Date/time: Friday 1st July 2022; 13:00-15:00
Format: Closed Book Exam
Allowed aids: DKE-approved calculator; Formula sheet (provided)
Instructions to students:

- The exam consists of 7 questions on 20 pages.
- Fill in your name and student ID number on the cover page and tick the corresponding numerals of your student number in the table (top right cover page).
- Answer every question in the reserved space below the question. **Do not write outside the reserved space or on the back of pages, this will not be scanned and will NOT be graded!** As a last resort if you run out of space, use the extra answer space at the end of the exam.
- *In no circumstance write on or near the QR code at the bottom of the page!*
- Ensure that you properly motivate your answers.
- Only use black or dark blue pens, and write in a readable way. Do not use pencils.
- Answers that cannot be read easily cannot be graded and may therefore lower your grade.
- If you think a question is ambiguous, or even erroneous, and you cannot ask during the exam to clarify this, explain this in detail in the space reserved for the answer to the question.
- If you have not registered for the exam, your answers will not be graded, and thus handled as invalid.
- You are not allowed to have a communication device within your reach, nor to wear or use a watch.
- You have to return all pages of the exam. You are not allowed to take any sheets, even blank, home.
- Good luck!

Algebraic Equations

14p **1** Explain the difference between the error and the residual of an approximate solution $\tilde{x}$ to an equation of the form $f(x) = 0$.

Next, use two steps of the secant method to estimate the root of $f(x) = e^x - 3x - 1$ in the interval $[1, 2]$, starting at $p_0 = 1$ and $p_1 = 2$. Afterwards, compute $f(p_3)$, give the best bracket for the root that you have found, and give an estimate of the error.

Differential Equations

14p **2** Use two steps of Heun's third order method to estimate the solution of the initial value problem $\dot{y} = t - y^2$, $y(0.5) = 2.500$ up to time $t = 1.0$. You should aim to compute the solution using Heun's method to an accuracy of at least 3 decimal places, and use sufficient precision in your working to do this.

Compare your answer with the exact solution, which has $y(1.0) = 1.32733490\,(8\text{dp})$. What would you expect the absolute error to be if you were to use Heun's method with 10 steps?

Polynomial Interpolation

12p **3** Use divided differences to compute the cubic polynomial interpolating the following data:

i	0	1	2	3
x_i	2.3	2.7	2.0	3.0
y_i	0.60	0.23	0.98	0.06

Estimate the value of y when $x = 2.5$. How does the ordering of the data points affect the interpolating polynomial?

Differentiation & Integration

12p **4** Use the most accurate three-point formulae available to complete the following table:

x	$f(x)$	$f'(x)$	$\sqrt{1+f'(x)^2}$
2.0	−0.57844		
2.2	−0.96792	−0.8093	1.2865
2.4	−0.90217	1.3895	1.7119
2.6	−0.41212	2.9348	3.1005
2.8	0.27176		
3.0	0.82783		

Use the trapezoid rule to estimate $\int_2^3 \sqrt{1+f'(x)^2}\,dx$.

Least-Squares Approximation

10p **5** The least-squares approximation to a function f is given by $q_n(x) = \sum_{k=0}^{n} c_k P_k(x)$ where the P_k are the Legendre polynomials, and the c_k are given by

k	0	1	2	3	4	5
c_k	0.2267	-0.1341	-0.0570	0.0620	-0.0077	-0.0122

Use the recurrence relation to evaluate $P_k(x)$ for $k = 0, 1, \ldots, 4$ for $x = 0.3$, and hence compute $q_4(x)$.

Compute the total square error $\int_{-1}^{+1} (q_2(x) - f(x))^2 \, dx$, assuming $\int_{-1}^{+1} f(x)^2 dx = 0.117242 \, (6\mathrm{dp})$.

Linear Algebra

12p **6** Let

$$A = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 2 \end{pmatrix}.$$

After applying one step of the QR method we obtain

$$A = QR = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -0.4472 & -0.8944 \\ 0 & -0.8944 & 0.4472 \end{pmatrix} \begin{pmatrix} 3 & 0 & 0 \\ 0 & -2.2361 & -2.6833 \\ 0 & 0 & -0.8944 \end{pmatrix}.$$

and

$$A^{(1)} = R^{(0)}Q^{(0)} = RQ = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3.4000 & 0.8000 \\ 0 & 0.8000 & -0.4000 \end{pmatrix}.$$

Apply another step of the QR method to estimate the eigenvalues of A. Give bounds on the eigenvalues using the Gersgorin circle theorem.

Modelling

6p **7** In a compartmental model of an infectious disease with two variants, individuals are either susceptible (S), infections with the first variant (I_1), or with the second variant (I_2). The progress of the disease is modelled by

$$\dot{S} = -(\alpha_1 I_1 + \alpha_2 I_2)S/N, \quad \dot{I}_1 = \alpha_1 I_1 S/N - \beta_1 I_1, \quad \dot{I}_2 = \alpha_2 I_2 S/N - \beta_2 I_2.$$

Assume initially 340 individuals infected with variant 1 and 1 individual infected with variant 2, in a population with size $N = 17\,000\,000$.

Show how to solve this system of differential equations in Matlab, including writing the code you would use.

Extra Paper

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