Numerical Mathematics 2017/2018
Exam Questions

— Do not turn this page before the official start of the exam! —

First Name, Surname: ____________________

Student ID: ____________________

Program: Bachelor Data Science and Knowledge Engineering

Course code: KEN1540

Examiner: Dr. P.J. Collins

Date/time: Monday June 4^{th}, 2018, 9.00-12.00h

Format: Closed book exam

Allowed aides: Pens, simple (non-programmable) calculator from the DKE-list of allowed calculators, formula sheet (provided).

Instructions to students:

- The exam consists of 7 questions on 14 pages (excluding the 1 cover page(s)).
- Fill in your name and student ID number on each page, including the cover page.
- Answer every question at the reserved space below the questions. If you run out of space, continue on the back side, and if needed, use the extra blank page.
- Ensure that you properly motivate your answers.
- Do not use red pens, and write in a readable way. Answers that cannot be read easily cannot be graded and may therefore lower your grade.
- You are not allowed to have a communication device within your reach, nor to wear or use a watch.
- You have to return all pages of the exam. You are not allowed to take any sheets, even blank, home.
- If you think a question is ambiguous, or even erroneous, and you cannot ask during the exam to clarify this, explain this in detail in the space reserved for the answer to the question.
- If you have not registered for the exam, your answers will not be graded, and thus handled as invalid.
- **Good luck!**

The following table will be filled by the examiner:

Question:	1	2	3	4	5	6	7	Total
Points:	14	12	12	12	12	10	8	80
Score:								

1. (14 points) Briefly describe the double-precision floating-point number system. Explain why the result of the computation $4.1 - 0.1$ need not exactly equal 4. Suggest how decimal arithmetic with 3 significant figures can give an insight into double-precision arithmetic.

   Use decimal arithmetic with 3 significant figures and the standard quadratic formula to estimate the smaller solution of
$$x^2 - 5x + 0.03 = 0\ .$$
   Compare your results to the exact answer 0.0060072 (5sf), computing the absolute and relative error. Use the quadratic formula
$$x = \frac{-2c}{b \mp \sqrt{b^2 - 4ac}}$$
   to obtain better results. Explain why this formula is better for this particular problem.

2. (12 points) Let
$$f(x) = e^x - 4x^2 \, .$$
Show that f has a root in the interval $I = [1, 5]$.

Apply one step of Newton's method starting at the midpoint of I. What happens?

Use one step of the bisection method to determine a smaller interval containing the root. Use Newton's method to starting at the midpoint of the interval you have found to estimate the root of $f(x)$ in I to an accuracy of approximately 10^{-1}.

3. (12 points) Compute the divided differences $f[x_i, \ldots, x_j]$ for the following data.

i	0	1	2	3
x	1.25	1.5	1.0	1.75
y	0.18	0.10	0.30	0.06

Hence write down the cubic polynomial interpolation p, giving your answer in nested form.

Show that if $p(x) = a_0 + (x - x_0)q(x)$, then $p'(x_0) = q(x_0)$. Hence estimate $f'(1.25)$.

How would you use builtin Matlab commands to compute the polynomial interpolation to the data, and to evaluate the interpolant?

4. (12 points) Use the trapezoid rule with $n = 4$ subdivisions to estimate the integral

$$\int_0^1 \frac{\cos(x)}{x+2} dx.$$

Use the error estimate formula for the trapezoid rule to estimate the error on $[0, \frac{1}{2}]$ and on $[\frac{1}{2}, 1]$. Obtain an improved value for the integral by subdividing the interval on which you expect the result to be less accurate, and provide an estimate of the error.

5. (12 points) A 3rd-order Runge-Kutta method for solving differential equations is given below:

$$\begin{aligned} k_{i,1} &= h_i f(t_i, w_i); \\ k_{i,2} &= h_i f(t_i + h_i, w_i + k_{i,1}); \\ k_{i,3} &= h_i f(t_i + \tfrac{1}{2}h_i, w_i + \tfrac{1}{4}k_{i,1} + \tfrac{1}{4}k_{i,2}); \\ w_{i+1} &= w_i + \left(\tfrac{1}{6}k_{i,1} + \tfrac{1}{6}k_{i,2} + \tfrac{2}{3}k_{i,3}\right). \end{aligned}$$

Consider the initial-value problem

$$\frac{dy}{dt} = t/y - 1; \quad y(0) = \tfrac{4}{5}.$$

Use two steps of the method to approximate y at $t = 1.0$.

Compare your answer to the exact value 0.66916744 (8dp). Roughly what would you expect the error to be if you were to use 10 steps instead of 2?

6. (10 points) The least-squares appoximation of degree 6 to a function f has Legendre coeffients c_k given by

k	0	1	2	3	4	5	6
c_k	0.619	0	−0.587	0	0.189	0	−0.044

Using the recurrence relation for Legendre polynomials, compute the values of $P_k(0.4)$ for $k = 0, \ldots, 6$, and estimate $f(0.4)$.

Using the Legendre coefficients, give an estimate of $\int_{-1}^{+1} f(x)^2 \, dx$.

7. (8 points) The *forced Duffing equation* is given by

$$\ddot{x} + \delta\dot{x} + \alpha x + \beta x^3 = \rho\cos(\omega t),$$

where $\alpha, \beta, \delta, \rho, \omega$ are parameters. Describe how to simulate this system using Matlab, with initial condition $x(0) = 2$ and $\dot{x}(0) = -3$, including giving the code you would write.