Least-Squares Approximation

Tutorial homework questions: 7b; 16a.

H By Hand; **C** Computer; **T** Theory; **E** Extra; **A** Advanced.

Recommended: C1; H2.a; C3; H4; 6.a; H7.a; H8; C9; C13; H14.b; H16.a; C17.b.

Polynomial Approximation

C1. Use Matlab's `polyfit` function to compute least-squares approximations of degrees 1, 2 and 3 to the following data:

x	0.5	1.0	2.0	2.5	3.0	3.5	4.0
y	1.61	2.13	2.56	2.47	2.26	2.05	1.96

For each case, compute the root-mean-square error of the approximation over the data points, and estimate the value of y when $x = 1.5$.

H2. Find the linear least-squares approximation for the following data, compute the root-mean-square error, and estimate the missing value. Graph the data and the approximation.

a.

x_i	0.00	0.25	050	0.75	1.00	0.4
y_i	0.876	1.060	1.525	1.993	2.594	

b.

x_i	1.0	1.1	1.3	1.5	1.9	2.1	2.0
y_i	1.33	1.45	1.70	1.94	2.43	2.67	

C3. Write a Matlab function to compute the coefficients a_0, a_1 of the linear least-squares approximation to data (x_i, y_i), $i = 1, \ldots, n$.

Use your function to compute the linear least-squares approximation to the data in Question 1.

H4. Find the least-squares polynomial approximations of degrees 2 and 3 for the data in Question 2, compute the root-mean-square error, and estimate the missing value.

C5. Write a Matlab function to compute the coefficients $a_0, a_1, \ldots, a_k$ of the polynomial approximation to the linear least-squares approximation to data (x_i, y_i), $i = 1, \ldots, n$.

Use your function to compute the least-squares polynomial approximations of degrees up to 4 for the data in Question 1.

6. Find the linear and quadratic least squares polynomial approximations to f on the indicated interval:

a. $f(x) = x^2 - 2x + 3$ on $[0, 1]$; **b.** $f(x) = e^x$ on $[0, 2]$; **c.** $f(x) = \ln x$ on $[1, 3]$.

H7. Compute the coefficients of Legendre basis polynomials for the least squares polynomial approximations of degrees 1, 2 and 3 on the interval $[-1, 1]$ for the following functions:

a. $f(x) = x^3$ **b.** $f(x) = e^x$ **c.** $f(x) = \ln(x + 2)$

Compute the total square errors. How does the difference in the errors relate to the coefficients?

H8. The (weighted) least-squares approximations in Legendre and Chebyshev bases for a function f are given by:

a. $p_5(x) = 0.3397 + 1.1505P_1(x) + 0.1711P_2(x) - 1.2405P_3(x) - 0.5983P_4(x) + 0.0816P_5(x)$.

b. $q_5(x) = 0.3069T_0(x) + 0.7059T_1(x) + -0.0405T_2(x) + -0.7514T_3(x) - 0.3054T_4(x) + 0.0421T_5(x)$.

Use the recurrence relations to compute $p_5(x)$ and $q_5(x)$ at $x = 0.7$.

C9. Plot the Legendre polynomials P_k for $k = 0, \ldots, 8$.

Compute the coefficients c_k of P_k in the best eighth-degree least-squares approximation to the functions given in Question 7, and plot the resulting polynomials p.

Hint: The functions P_k should be computed using the recurrence relation
$P_k(x) = \big((2 - \frac{1}{k})xP_{k-1}(x) - (1 - \frac{1}{k})P_{k-2}(x)\big)$.
For fixed x, you can use this to compute $P_k(x)$ for $k = 0, \ldots, 8$ for use in e.g. Romburg integration.

C10. Plot the Chebyshev polynomials T_k for $k = 0, \ldots, 8$.

Compute the coefficients c_k of T_k in the best eighth-degree least-squares approximation using weight $w(x) = 1/\sqrt{1 - x^2}$ to the functions given in Question 7, and plot the resulting polynomials p.

Hint: The functions T_k should be computed using the recurrence relation
$T_k(x) = 2xT_{k-1}(x) - T_{k-2}(x)$.

Plot and compare the linear and quadratic Legendre and Chebyshev approximations. Which give the best *uniform* approximations?

AT11. Construct the *Laguerre polynomials* L_k of degrees up to 4, which are orthogonal on the interval $[0, \infty)$ with respect to the weight function $w(x) = e^{-x}$.

E12. Use the Laguerre polynomials calculated to compute the least squares polynomials of degree $1, \ldots, 4$ on the interval $[0, \infty)$ with respect to the weight function $w(x) = e^{-x}$ for the following functions.

a. $f(x) = x^2$ **b.** $f(x) = e^{-x}$ **c.** $f(x) = x^3$ **d.** $f(x) = e^{-2x}$

Fourier Series

C13. Use Matlab to compute the coefficients of the Fourier approximation $s_5(x)$ of degree $n = 5$ to the function $f(x) = (\cos x)/(2 + \sin x)$. Plot the graph of the function f and the Fourier approximations s_1, s_2, s_3.

The *energy* of the signal in the k-the mode is $a_k^2 + b_k^2$. Comment on the how energy in the k^{th} mode for $k = 1, \ldots, n$ relates to the form of the function.

Hint: You may wish to write a function `fourierval(as,bs,x)`, which evaluates the Fourier series with coefficients a_k, b_k at the point x. Alternatively, you could use arguments `(a0,as,bs,x)`.

H14. Find the Fourier approximations $s_n(x)$ on $[-\pi, \pi]$ for $n = 1, 2, 3$ and the following functions.

a. $f(x) = x \sin x$; **b.** $f(x) = e^x$.

Compute the total-square-errors and the mean-square-errors. How does the difference in the total-square-errors relate to the coefficients?

Hint: You can evaluate the integrals by parts.

E15. Determine the Fourier approximations $s_n(x)$ on the interval $[-\pi, \pi]$ for the functions given in Question 14 by computing $2m$ data points $y_j = f(x_j)$ with $x_j = j\pi/m$, using $m = 4, 8$ and $n = 3$. Compare your answers with those found previously.

H16. Find the Fourier approximations $S_2(x)$ on $[-\pi, \pi]$ for the following data:

a.

x_i	$-\pi$	$-2\pi/3$	$-\pi/3$	0	$\pi/3$	$2\pi/3$	π	$\pi/2$
y_i	0.54	0.37	0.69	1.87	2.69	1.44	0.54	

b.

x_i	$-\pi$	$-3\pi/4$	$-\pi/2$	$-\pi/4$	0	$\pi/4$	$\pi/2$	$3\pi/4$	π	$\pi/3$
y_i	0.49	0.81	0.98	0.96	0.75	0.42	0.08	0.14	0.49	

Use your approximations to estimate the missing value.

C17. Use Matlab to compute the coefficients of the Fourier approximations to the data (x_i, y_i) below:

a.

x_i	$-\pi$	$-3\pi/4$	$-\pi/2$	$-\pi/4$	0	$\pi/4$	$\pi/2$	$3\pi/4$
y_i	0.17647	0.33194	0.47059	0.88653	0.76471	0.60924	0.47059	0.05464

b.

x_i	$-\pi$	$-3\pi/4$	$-\pi/2$	$-\pi/4$	0	$\pi/4$	$\pi/2$	$3\pi/4$
y_i	0.00000	0.06678	0.21630	0.57983	1.00000	0.57983	0.21630	0.06678

Comment on your results.

E18. Compute the coefficients of the Fourier series of the *sawtooth wave*, which is the 2π-periodic function satisfying

$$f(x) = x \text{ for } x \in [-\pi, +\pi).$$

Write down the general form of the least-squares approximation s_n.

Plot s_n for $n = 1, 2, 3, \ldots$. What happens to the least-squares error as $n \to \infty$? What happens to the uniform error?

By computing $\int_0^\pi f(x)^2\,dx$ and $\int_0^\pi s(x)^2\,dx$, find a formula for $\sum_{k=1}^\infty 1/k^2$.

AC19. Write a Matlab function to compute the discrete Fourier transform using the fast Fourier transform method. Compute the trigonometric interpolating polynomial of degree 4 and 6 on $[-\pi, \pi]$ using $2m = 2^6$ and 2^8 data points for the following functions.

a. $f(x) = \pi(x - \pi)$ **b.** $f(x) = |x|$ **c.** $f(x) = x \cos x^2$

E20. Compute the Fourier coefficients b_m up to $m = 15$ for the wavelet

$$f(x) = \exp\bigl(3\cos(x)\bigr)\sin(5x).$$

Comment on how the coefficients vary. Which is the largest coefficient?