Ordinary Differential Equations

Tutorial homework questions: 2c(ii); 4c due at the beginning of the corresponding tutorial class..

There is no need to use every method suggested, but for each question, you should at the very least use the most accurate method suggested for at least one example, and continue until you are comfortable with the methods.

H By Hand; **C** Computer; **T** Theory; **E** Extra; **A** Advanced.

Recommended: C1; H2.a; H3.a; H4.a; T5; C6; C7.a; C8; C9.b; T12.

C1. Consider the differential equation

$$\frac{dy}{dt} = e^{-t} - y^2; \quad y(0) = 0\,. \tag{†}$$

Use Matlab's builtin commands `ode23` and `ode45` to compute an approximation to the solution over the time interval $[0, 10]$. Compare your answers with the exact solution, which has $y(1) = 0.503346658225$, $y(2) = 0.478421766451$, $y(5) = 0.237813428537$ and $y(10) = 0.110790590981$.

Hint: To specify the requested time values, use `[0,1,2,5,10]` as the `tspan` argument. To obtain the y-values, use `[ts,ys]=odeXX(f,tspan,y0)`.

H2. Use (i) Euler's method, (ii) Ralston's method, and (iii) the fourth-order Runge-Kutta method to solve each of the problems below. Compute the actual error after the first step and after the final step, and compare the accuracy of the different methods.

a. $\dot{y} = \cos(t) - y/3$, $y(0) = 0$ up to $t = 2$ with $h = 1.0$ and $h = 0.5$;
solution $y(t) = \frac{9}{10}\sin(t) + \frac{3}{10}(\cos(t) - \exp(-t/3))$.

b. $\dot{y} = y/t - 2$, $y(1) = 3$ up to $t = 2$ with $h = 0.5$, $h = 0.25$; solution $y(t) = t(3 - 2\log t)$.

c. $\dot{y} = (y + t)^2 - 1$ up to $t = 1$ with $y(0) = 2/3$ with $h = 0.5$ and $h = 0.25$; solution $y(t) = 2/(3 - 2t) - t$.

H3. Use (i) the two-stage and (ii) three-stage Adams-Bashforth methods to approximate the solutions to the initial-value problems in Question 2. Use an appropriate Runge-Kutta method to determine the starting values. Compare the results to the actual values.

H4. Use the Adams-Bashforth two-stage method as a predictor and the Adams-Moulton two-stage method as a corrector to approximate the solutions to the inital-value problems in Question 2. Use an appropriate Runge-Kutta method to determine the starting values.

T5. How many function evaluations are needed to solve the initial value problem (†) using a fourth-order Runge-Kutta method with a step size of 0.01, compared with a four-stage Adams-Bashforth method, or the three-stage Adams-Moulton method? Which method provides the best accuracy for the *same* number of function evaluations?

C6. A second-order differential equation $\ddot{x} = g(t, x, \dot{x})$ can be converted into a system of first-order differential equations

$$\dot{x} = v; \qquad \dot{v} = g(t, x, v).$$

The resulting state vector $\mathbf{y} = (x; v)$ satisfies

$$\dot{\mathbf{y}} = \begin{pmatrix} \dot{x} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} v \\ g(t, x, v) \end{pmatrix} = \begin{pmatrix} y_2 \\ g(t, y_1, y_2) \end{pmatrix} =: \mathbf{f}(t, \mathbf{y}).$$

Use Matlab's `ode45` method to solve the *Van der Pol equation*

$$\ddot{x} - (1 - x^2)\dot{x} + x = \cos(t) \text{ with } x(0) = 1 \text{ and } \dot{x}(0) = 0.$$

What happens to the solution as $t \to \infty$?

C7. Write Matlab functions implementing (i) Euler's method, (ii) Ralston's method (iii) Heun's third-order method, and (iv) the fourth-order Runge-Kutta method. At each step, display t_i, w_i, $f(t_i, w_i)$ and any $k_{i,j}$.

Use the methods you have developed to approximate the solutions for each of the following initial-value problems, and compare the results to the actual values.

- **a.** $\dot{y} = \cos(t) - y/3$, $y(0) = 0$ up to $t = 2$ with $h = 0.2$; solution $y(t) = \frac{9}{10}\sin(t) + \frac{3}{10}(\cos(t) - \exp(-t/3))$.
- **b.** $\dot{y} = y/t - 2$, $y(1) = 3$ up to $t = 2$ with $h = 0.1$; solution $y(t) = t(3 - 2\log t)$.
- **c.** $\dot{y} = 5t/y - ty$, $y(0) = 1$ up to $t = 1$ with $h = 0.1$; solution $y(t) = \sqrt{5 - 4e^{-t^2}}$.
- **d.** $\dot{y} = t - y^2$, $y(1) = 0$ up to $t = 3$ with $h = 0.2$. (Compute a very accurate approximation to the actual solution using the fourth-order Runge-Kutta method with $h = 0.001$.)

C8. Write Matlab functions implementing (i) the explicit three-stage Adams-Bashforth method and (ii) the three-stage Adams-Bashforth method as a predictor and the three-stage Adams-Moulton method as a corrector to solve the initial value problems in Question 7. In each case, bootstrap using a suitable Runge-Kutta method.

C9. The Matlab solvers `ode23s` and `ode15s` can be used to solve stiff systems. Try these solvers on the initial-value problems below, and compare your results with the built-in solvers `ode23` and `ode45` for non-stiff systems.

- **a.** $\dot{y} = 40/y - 40y$ $y(0) = \sqrt{2}$ up to $t = 1$; actual solution $y(t) = \sqrt{1 + e^{-80t}}$.
- **b.** $\dot{y} = 1 - 2ty$ $y(0) = 0$ up to $t = 10$. Approximate solution $y(t) \approx 1/2t$ for $y \gg 1$.

CT10. Solve the stiff initial-value problems of Question 9 using (i) Euler's method, (ii) the fourth-order Runge-Kutta method, (iii) the fourth-order Adams-Bashforth method, and (iv) the Adams fourth-order predictor-corrector method. Comment on the stability of the methods.

A11. Solve the stiff initial-value problems of Question 9 using (i) the backward Euler method and (ii) the implicit trapezoidal (Crank-Nicolson) method, incorporating the secant method (or Newton's method) to solve for w_{i+1}. Comment on the stability of the methods.

T12. For the differential equation

$$\dot{y} = -\lambda y,$$

find a formula for w_{i+1} in terms of w_i using (i) the backward Euler method and (ii) the implicit trapezoidal method, (iii) Euler's method, and (iv) the method obtained using Euler's method as a predictor and the implicit trapezoidal rule as a corrector with step size h. Determine the stability radius of each method.

AH13. For the following initial-value problems, compute two steps of the adaptive Bogacki-Shampine method with given tolerance ε. Compare the approximate solution to the actual solution, estimated using the fourth-order Runge-Kutta method with a small step size.

- **a.** $\dot{y} = e^t\sqrt{3 - y^2}$ with $y(0) = 1$ and $\varepsilon = 10^{-3}$.
- **b.** $\dot{y} = e^{-t} - y^2$ with $y(0) = 0$ and $\varepsilon = 10^{-2}$.

AC14. Write your own adaptive solver based on the Bogacki-Shampine method. Test your routine by computing the solution for $t \in [0, 10]$ of the differential equation

$$\dot{x} = 1 - ty; \qquad y(0) = 0.5\ . \tag{†}$$

E15. Consider the initial-value problem

$$\dot{y} = y^2/t + 2 - t,\ y(1) = 0.$$

a. Use Euler's method with both $h = 0.2$ and $h = 0.1$ to approximate the solution. In each case, use linear interpolation to approximate the values of y at the following points.

(i) $y(1.30)$ (ii) $y(1.55)$ (iii) $y(1.92)$.

How does the error depend on the step size? (Use an accurate method with a small step size to estimate the exact value.) Estimate the order of the approximation.

b. Repeat using the fourth-order Runge-Kutta method and cubic spline interpolation.

The use of interpolation to estimate the solution at times between explicitly computed time points is a common technique when solving differential equations, as it does not require additional function evaluation.

E16. Adapt the predictor-corrector approach so to use the previously-computed value $f(t_i, \tilde{w}_i)$ to predict $\tilde{w}_{i+1}$, instead of computing $f(t_i, w_i)$. Make sure you only evaluate f *once* per time step. Determine experimentally the order of the method using the three-stage Adams-Bashforth method as a predictor and three-stage Adams-Moulton method as a corrector using the problems in Question 7.

A17. Derive the following 2-stage Adams-Bashforth formulae for doubling and halving the step size. Explain how this is useful in the context of adaptive solvers.

$$w_{i+1} = w_i + h_i(2f(t_i, w_i) - f(t_{i-1}, w_{i-1})) \text{ if } h_i = 2h_{i-1};$$
$$w_{i+1} = w_i + h_i(\tfrac{5}{4}f(t_i, w_i) - \tfrac{1}{4}f(t_{i-1}, w_{i-1})) \text{ if } h_i = h_{i-1}/2.$$