Exercises

1	2	3	4	5	6	7	8

Surname, First name

1	1	1	1	1	1	1
2	2	2	2	2	2	2
3	3	3	3	3	3	3
4	4	4	4	4	4	4
5	5	5	5	5	5	5
6	6	6	6	6	6	6
7	7	7	7	7	7	7
8	8	8	8	8	8	8
9	9	9	9	9	9	9
0	0	0	0	0	0	0

KEN1540 Numerical Mathematics
KEN1540 Exam

Program: Bachelor Data Science and Artificial Intelligence
Course code: KEN1540
Examiners: Dr. Ir. Martijn Boussé and Dr. Pieter Collins
Date/time: Monday 5 June 2023; 13:00-15:00
Format: Closed Book Exam
Allowed aids: DACS-approved calculator; Formula sheet (provided)
Instructions to students:

- The exam consists of 7 questions on 22 pages.
- Fill in your name and student ID number on the cover page and tick the corresponding numerals of your student number in the table (top right cover page).
- Answer every question in the reserved space below the question. **Do not write outside the reserved space or on the back of pages, this will not be scanned and will NOT be graded!** As a last resort if you run out of space, use the extra answer space at the end of the exam.
- *In no circumstance write on or near the QR code at the bottom of the page!*
- Ensure that you properly motivate your answers.
- Only use black or dark blue pens, and write in a readable way. Do not use pencils.
- Answers that cannot be read easily cannot be graded and may therefore lower your grade.
- If you think a question is ambiguous, or even erroneous, and you cannot ask during the exam to clarify this, explain this in detail in the space reserved for the answer to the question.
- If you have not registered for the exam, your answers will not be graded, and thus handled as invalid.
- You are not allowed to have a communication device within your reach, nor to wear or use a watch.
- You have to return all pages of the exam. You are not allowed to take any sheets, even blank, home.
- Good luck!

Computer Arithmetic and Algebraic Equations

Consider the polynomial $p(x) = 1.03x^4 - 5.34x^2 - 0.0156x + 2.78$. Direct evaluation of $p(x)$ at $x = 1.51$ using 3-digit rounded arithmetic gives the value -4.08.

5p **1a** Evaluate $p(x)$ at $x = 1.51$ in nested (Horner) form using 3-digit rounded arithmetic. Give a detailed answer.

3p **1b** Compute the absolute and relative error for direct evaluation **and** evaluation using nested (Horner) form at $x = 1.51$, given that the exact value is -4.05446831 (8 dp). Which method performs better?

3p **1c** Perform two steps of bisection to find the root of $p(x)$ in $[1.5, 2.5]$. Then, compute the residual of the estimate in the second step.

1p **1d** How many steps of bisection would you need if you want to find the root in $[1.5, 2.5]$ up to an accuracy of 10^{-3}. (You do **not** have to compute any bisection steps for this subquestion.)

Differential Equations

Consider the initial value problem $y' = \cos(ty) - t^2$ with $y(0) = 0.0000$.

9p **2a** Use a predictor-corrector method to find the solution for $t \in [0, 1.5]$ and $h = 0.5$. Use the two-stage Adams–Bashforth method for the prediction step and **the most appropriate** Adams–Moulton method for the correction step. Bootstrap your calculations using values of $w_1 \approx y(0.5)$ for **the most appropriate** of the methods given in the table below. Motivate your choices for the Adams-Moulton method and the bootstrapping.

Euler's method	$w_1 = 0.50000000$
2nd-order Runge-Kutta method	$w_1 = 0.45602090$
3rd-order Runge-Kutta method	$w_1 = 0.45614919$
4th-order Runge-Kutta method	$w_1 = 0.45545299$

1p **2b** Calculate the absolute and relative error of $y(1.5)$, given the exact value is 0.2400 3580 (8 dp).

2p **2c** Roughly, what would you expect the absolute error to be if you were to use $h = 0.25$ instead?

Polynomial Interpolation

Consider the function f on $[0, 3]$ given by $f(x) = \frac{1}{1+x}$.

The nth-order derivative of $f(x)$ is given by $f^{(n)}(x) = \frac{(-1)^n\, n!}{(x+1)^{n+1}}$.

3p **3a** Find the **cubic** Taylor polynomial $T_3(x)$ around $x_0 = 1$ for $f(x)$.

5p **3b** Use divided differences to compute the **cubic** polynomial $P_3(x)$ interpolating $f(x)$ at the values $x_0 = 0$, $x_1 = 1$, $x_2 = 2$, and $x_3 = 3$. You may assume that $f[x_0, x_1] = -0.5000$, $f[x_1, x_2] = -0.1667$, $f[x_2, x_3] = -0.0833$, $f[x_0, x_1, x_2] = 0.1667$, and $f[x_1, x_2, x_3] = 0.0417$.

4p **3c** Estimate the value of $f(x)$ at $x = 1.5$ using $T_3(x)$ and $P_3(x)$. Determine the absolute error for both cases. What can you say about the errors and why? What would you expect to happen at $x = 2.5$?

Numerical Integration & Differentiation

Consider the definite integral $\int_1^3 \sin\left(x^2\right)\ dx$.

10p **4a** Compute the Romburg estimate $R_{3,3}$ using four decimal places for the computed values. You may assume that $R_{2,0} = 0.3074$ and $R_{3,0} = 0.4281$.

2p **4b** Give an error estimate for $R_{3,2}$.

Least-Squares Approximation

The least-squares approximation to a function f over $[-1, +1]$ is given by

$$q_n(x) = \sum_{k=0}^{n} c_k P_k(x)$$

where the P_k are the Legendre polynomials.
Suppose the c_k are given by

k	0	1	2	3	4	5
c_k	0.3086	0.2193	−0.0842	−0.1233	−0.0267	0.0262

4p **5a** Use the recurrence relation to evaluate $P_k(0.7)$ for $k = 0, 1, \ldots, 4$.

2p **5b** Compute $q_4(0.7)$.

3p **5c** Explain how orthogonality of the P_k gives rise to a simple formula for the square integral

$$\int_{-1}^{+1} q_n(x)^2 \, \mathrm{d}x.$$

3p **5d** Compute the square integral $\int_{-1}^{+1} q_2(x)\,\mathrm{d}x$.

Linear Algebra

8p **6a** Use two steps of the Gauss-Seidel method to estimate the solution of the equation $A\mathbf{x} = \mathbf{b}$, starting at $\mathbf{x}^{(0)}$.

$$A = \begin{pmatrix} 5 & 2 & 0 \\ 3 & 6 & -1 \\ -2 & 3 & 1 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 2 \\ -4 \\ 1 \end{pmatrix}, \quad \mathbf{x}^{(0)} = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}.$$

2p **6b** What is the difference between the *error* and the *residual* of an approximate solution $\tilde{\mathbf{x}}$ to the linear algebraic equation $A\mathbf{x} = \mathbf{b}$?

2p **6c** Compute the norm of the residual of your answer to .

Modelling and Matlab

Consider the following equation for the forced Duffing oscillator:

$$\ddot{x} + \delta\dot{x} + \alpha x + \beta x^3 = A\cos(\omega t)$$

with parameter values

$$\delta = 0.5,\ \alpha = 0.07,\ \beta = 0.1,\ A = 2.3,\ \omega = 1.9.$$

and initial conditions $x(0) = 2,\ \dot{x}(0) = 1.$

4p **7a** Show how to re-write this second-order equation as a system of first-order equations.

4p **7b** Write Matlab code to solve the equation.

Extra paper

8