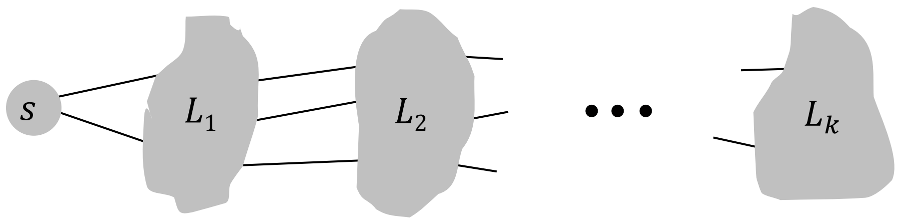
Basic Graph Algorithms

Processing Graphs

- Graphs model many scenarios
  - Many problems are presented as graph problems
  - Can then use known general graph algorithms to solve those problems
- Data is inputted as adjacency matrix or, more commonly, an adjacency lists
- To start processing the data, we often need some way to derive structure from this input
- Breadth First Search and Depth First Search are the most common simple ways of imposing structure.

Breadth First Search

BFS idea. Explore outward from s in all possible directions, adding nodes one "layer" at a time.



BFS.

- $L_0 = \{s\}$.
- $L_1 =$ all neighbors of L_0.
- $L_2 =$ all nodes that do not belong to L_0 or L_1, and that have an edge to a node in L_1.
- $L_{i+1} =$ all nodes that do not belong to an earlier layer, and that have an edge to a node in L_i.

Def: The distance from u to v is the number of edges on the shortest path from u to v.

Theorem. For each i, L_i consists of all nodes at distance exactly i from s. There is a path from s to t iff t appears in some layer.

BFS Algorithm

```
BFS(G,s):
for each vertex u ∈ V − {s}
        u.color ← white
        u.d ← ∞
        u.p ← nil
s.color ← gray
s.d ← 0
initialize an empty queue Q
Enqueue(Q,s)
...............
```

Color: indicates status

- white: (initial value) undiscovered
- gray: discovered, but neighbors not fully processed
- black: discovered and neighbors fully processed

Every node stores a color, a distance and a parent

Distance (d): the length of shortest path from s to u

Parent (p): u's predecessor on the shortest path from s to u

Note: Assume, initially, that G is connected (will fix later)

BFS Algorithm Complete

BFS(G,s):

for each vertex $u \in V - \{s\}$
  $u.color \leftarrow white$
  $u.d \leftarrow \infty$
  $u.p \leftarrow nil$
$s.color \leftarrow gray$
$s.d \leftarrow 0$
1. **initialize an empty queue** Q
2. **Enqueue**(Q,s)
3. **while** $Q \neq \emptyset$ **do**
4. $u \leftarrow$ **Dequeue**(Q)
5.   **for each** $v \in Adj[u]$
6.   **if** $v.color = white$ **then**
7. $v.color \leftarrow gray$
8. $v.d \leftarrow u.d + 1$
9. $v.p \leftarrow u$
10.   **Enqueue**(Q,v)
11. $u.color \leftarrow black$

- Algorithm keeps current active nodes in a (FIFO) Queue Q
- Starts by inserting s in Q (2)
- At each step takes node u off Q (4)
  - Checks all neighbors v of u (5)
  - If v has not been seen yet (6)
    - Marks v as seen (gray) (7)
    - Says that distance from s to v is $1 + \text{dist to } u$ (8)
    - Makes u the parent of v (9)
    - inserts v in queue (10)
  - Marks u as being fully processed (11)

Note: Nodes in Queue Q

- Are ones that have been seen but are unprocessed (gray)

BFS Algorithm Complete

```
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        u.color ← white
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s.color ← gray
s.d ← 0
initialize an empty queue Q
Enqueue(Q,s)
while Q ≠ ∅ do
        u ← Dequeue(Q)
        for each v ∈ Adj[u]
                if v.color = white then
                        v.color ← gray
                        v.d ← u.d + 1
                        v.p ← u
                        Enqueue(Q,v)
        u.color ← black
```

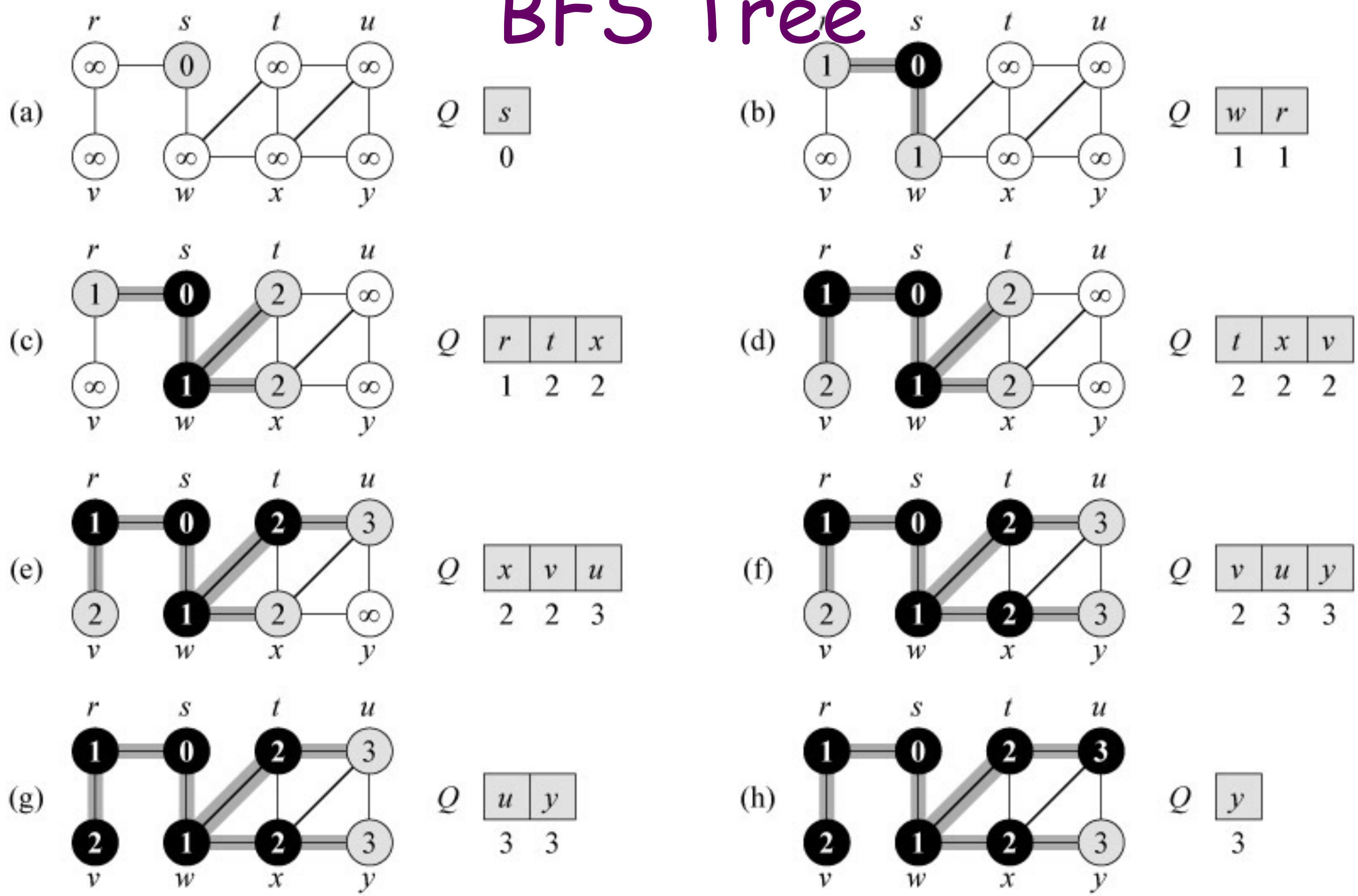
Parent pointers:

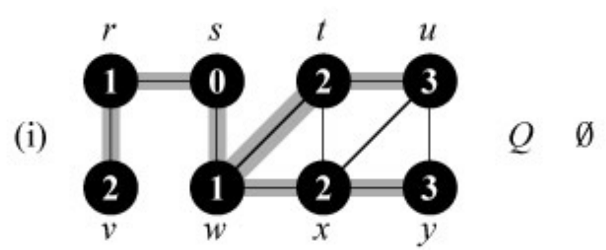
- Pointing to the node that leads to its discovery
- Parent must be in L_{i-1}
- Can follow parent pointers to find the actual shortest path
- The pointers form a BFS tree, rooted at s

Running time:

$\sum_u(1 + \deg(u)) = \Theta(|V| + |E|)$, which is $\Theta(|E|)$ if the graph is connected.

BFS Tree

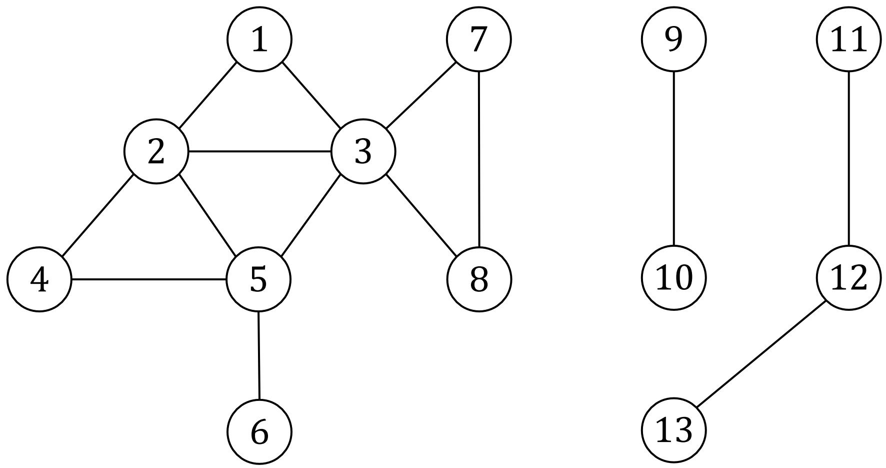




Note: BFS finds the shortest path from s to every other node.

Connected Components

Connected component containing s. All nodes reachable from s.



Connected component containing node $1 = \{1, 2, 3, 4, 5, 6, 7, 8\}$.

BFS starting from s finds the connected component containing s.

Repeatedly running BFS from an undiscovered node finds all the connected components.

Modification for Finding Connected Components

```
BFS(G):
for each vertex u ∈ V do
      u.color ← white
      u.d ← ∞
      u.p ← nil
for each vertex u ∈ V do
      if u.color = white then
            BFS-Visit(u)
```

```
BFS-Visit(G,s):
/*Assumes s is white*/

s.color ← gray
s.d ← 0
initialize an empty queue Q
Enqueue(Q,s)
while Q ≠ Ø do
      u ← Dequeue(Q)
      for each v ∈ Adj[u]
            if v.color = white then
                  v.color ← gray
                  v.d ← u.d + 1
                  v.p ← u
                  Enqueue(Q,v)
      u.color ← black
```

The old BFS(G,s) algorithm is renamed BFS-Visit(G,s).

A new upper-level BFS(G) is created.

BFS(G) initializes all vertices to white (unvisited)
It then calls all vertices s, passing them to BFS-visit(s), **if s was not already seen while traversing a previously visited connected component.**

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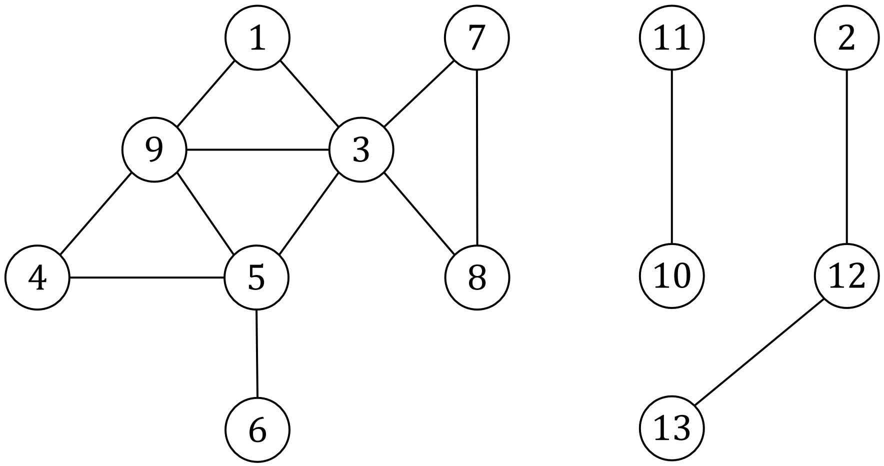
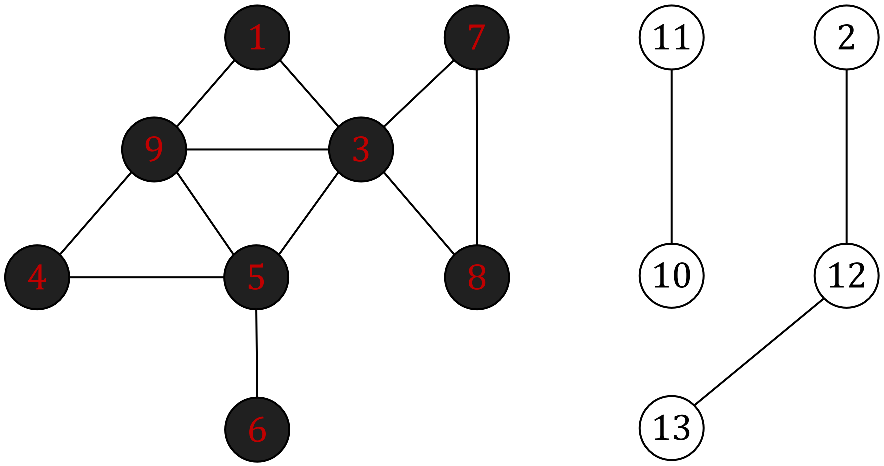


BFS-Visit(1) would turn all nodes in leftmost component black

Connected Components

Connected component containing s. All nodes reachable from s.

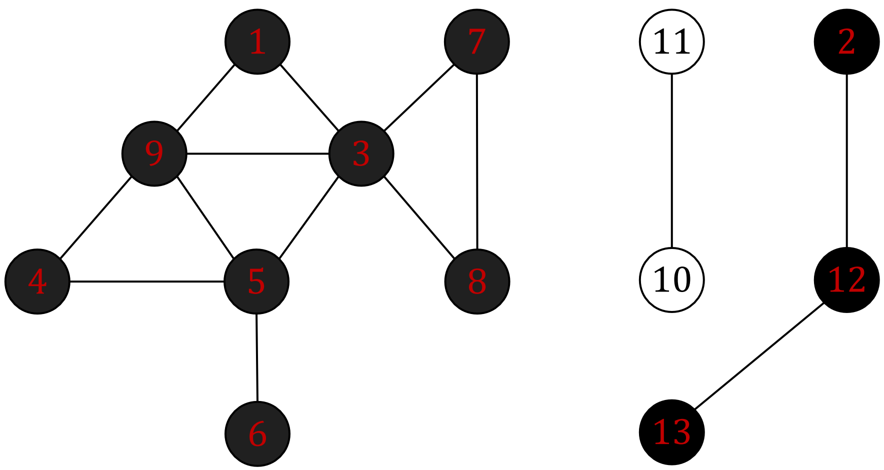


BFS-Visit(1) would turn all nodes in leftmost component black

BFS-Visit(2) would turn all nodes in rightmost component black

Connected Components

Connected component containing s. All nodes reachable from s.



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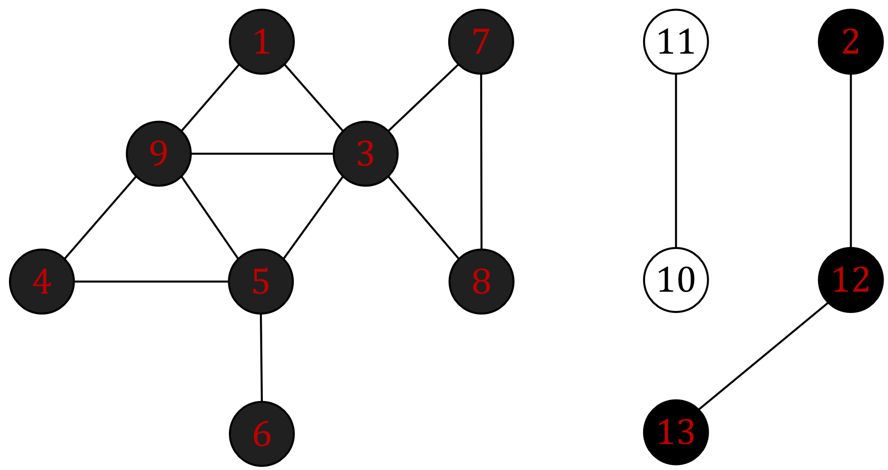


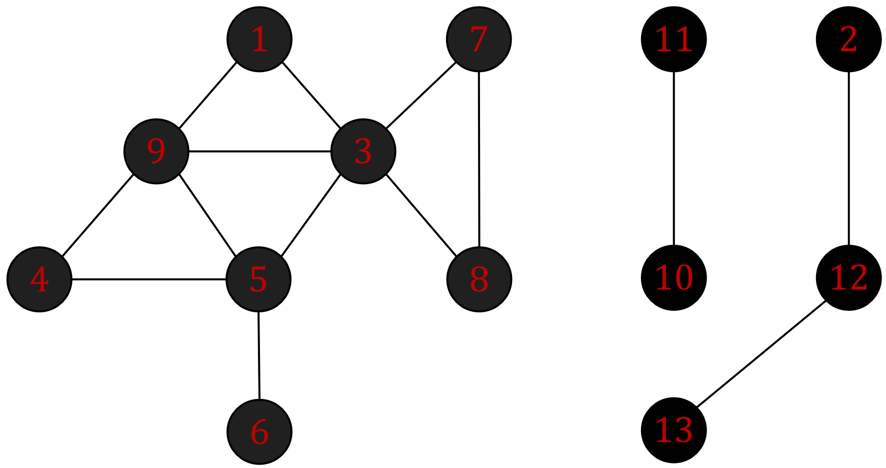
BFS-Visit(1) would turn all nodes in leftmost component black

BFS-Visit(2) would turn all nodes in rightmost component black

BFS-Visit(i) for $3 \leq i \leq 9$ would do nothing.
BFS-Visit(10) would then turn all nodes in middle component black

Connected Components

Connected component containing s. All nodes reachable from s.



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s-t connectivity and shortest path in directed graphs

s-t connectivity (often called reachability for directed graphs). Given two nodes s and t, is there a path from s to t?

- Undirected graph: s can reach $t \Leftrightarrow t$ can reach s
- Directed graph: Not necessarily true

s-t shortest path problem. Given two node s and t, what is the length of the shortest path between s and t?

- Undirected graph: p is the shortest path from s to $t \Leftrightarrow p$ is the shortest path from t to s
- Directed graph: Not necessarily true

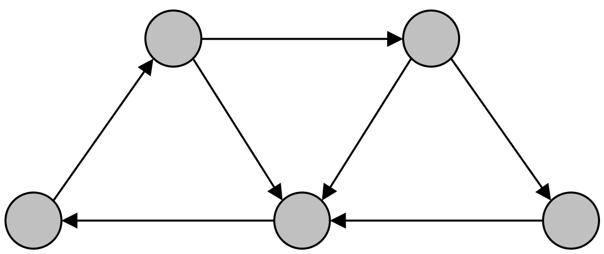
BFS on a directed graph. Same as in undirected case

- Ex: Web crawler. Start from web page s. Find all web pages linked from s, either directly or indirectly.

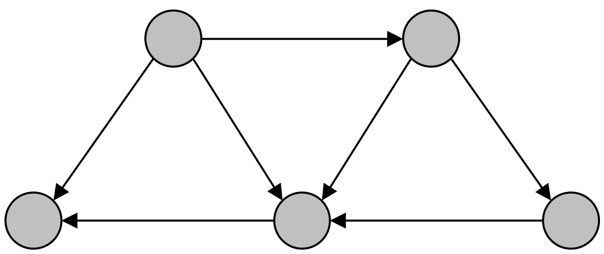
Strong Connectivity in Directed Graphs

Def. Node u and v are mutually reachable if there is a path from u to v and also a path from v to u.

Def. A graph is strongly connected if every pair of nodes is mutually reachable.



strongly connected



not strongly connected

Definition: vertex s is "strong" in Graph G if,
for every vertex t, there is a path from s to t and from t to s.

Observation 1: If graph G has a strong vertex s then
EVERY vertex in G is strong

Observation 2: A graph G is strongly connected
if and only if every vertex in G is strong

Strong Connectivity in Directed Graphs

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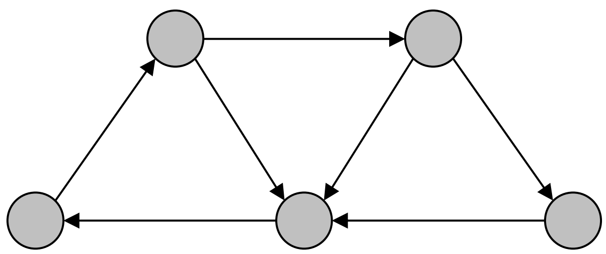

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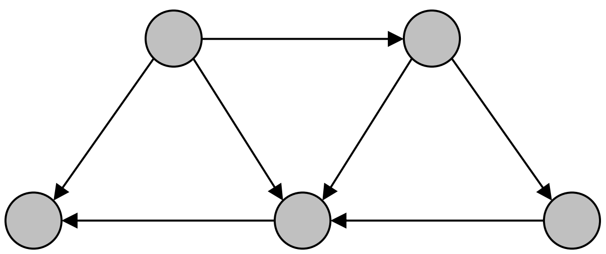
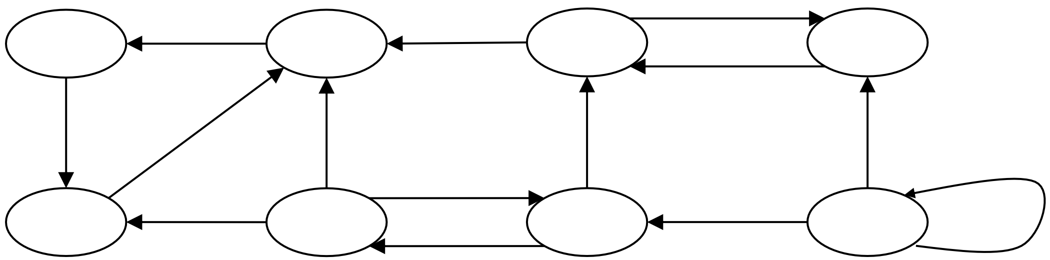


not strongly connected

Algorithm for checking strong connectivity

- Pick any node s.
- Run BFS from s in G.
- Reverse all edges in G, and run BFS from s.
- Return true iff all nodes reached in both BFS executions.

Strongly Connected Components

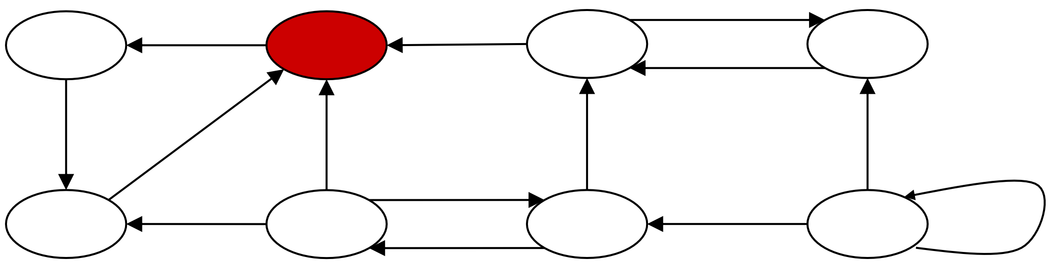


```
Strongly-Connected-Components(G):
create G^rev which is G with all edges reversed
while there are nodes left do
     u ← any node
     run BFS in G starting from u
     run BFS in G^rev starting from u
     C ← {nodes reached in both BFSs}
     output C as a strongly connected component
     remove C and its edges from G and G^rev
```

Running time: $O(|V||E|)$

See text book for a $\Theta(|V| + |E|)$ algorithm (not required)

Strongly Connected Components



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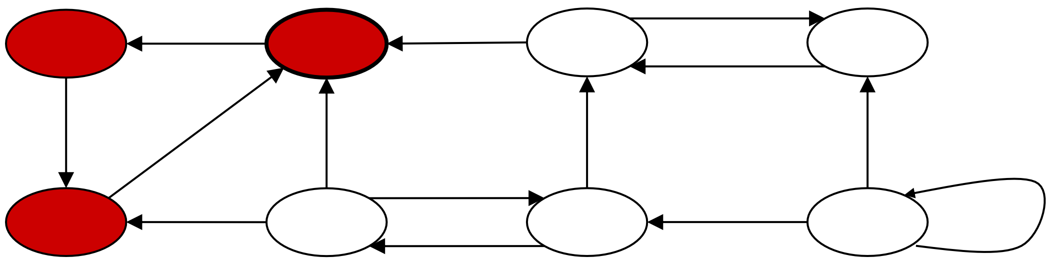

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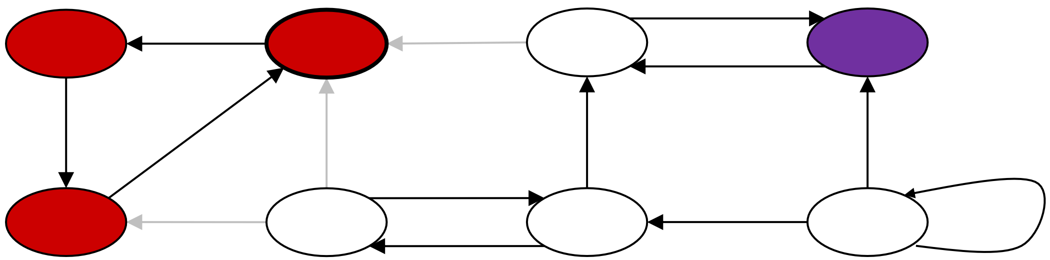

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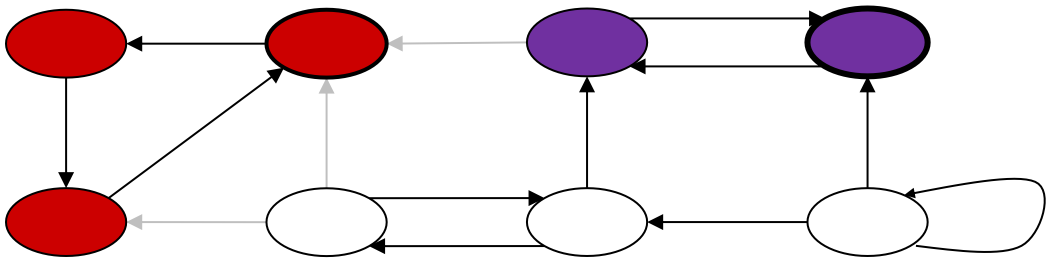

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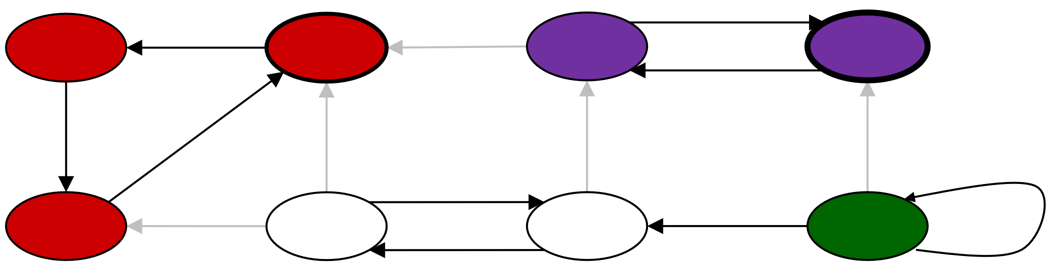

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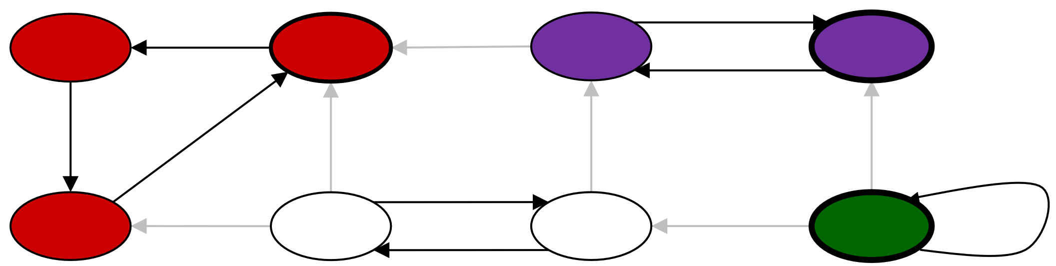

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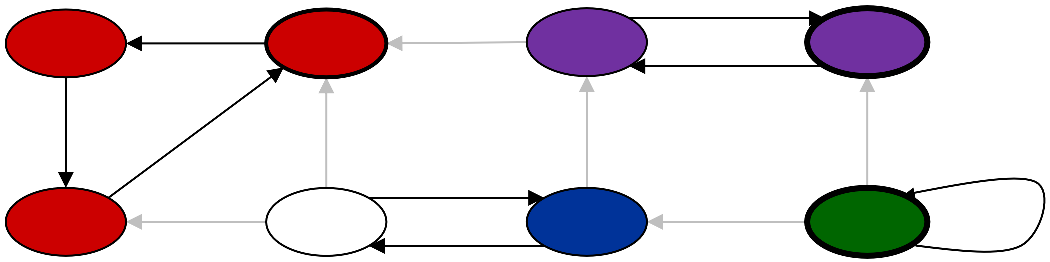

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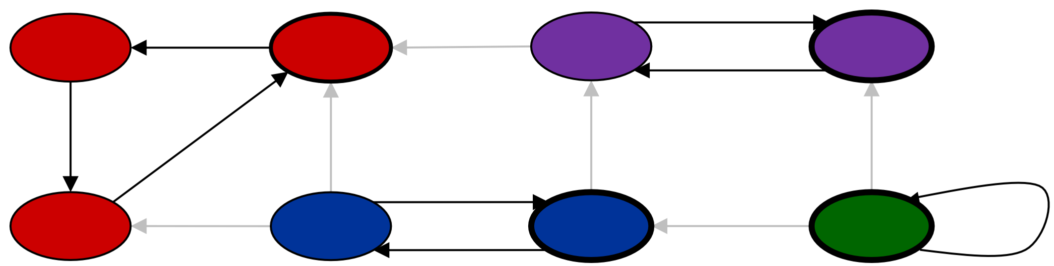

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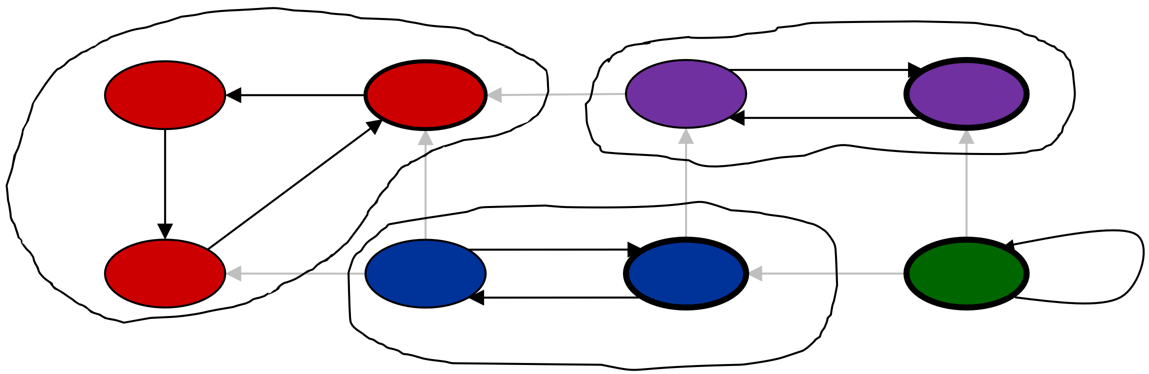
Running time: $O(|V||E|)$

See text book for a $\Theta(|V|+|E|)$ algorithm (not required)

Strongly Connected Components



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     remove C and its edges from G and G^rev
```

Running time: $O(|V||E|)$

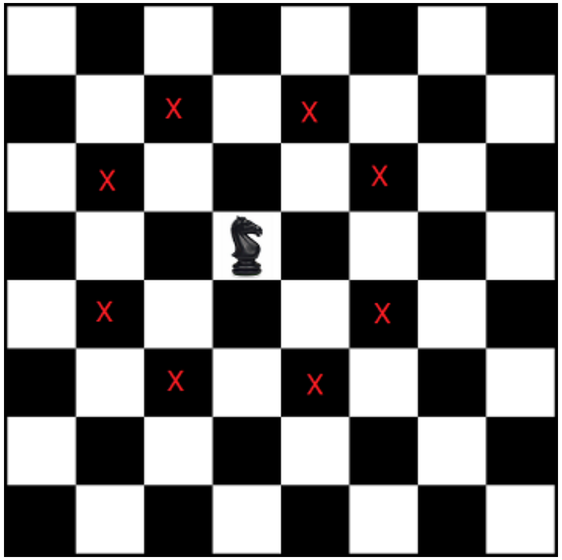
See text book for a $\Theta(|V| + |E|)$ algorithm (not required)

Exercise on Chess

Find the minimum number of steps taken by a knight to reach a destination (x', y') from an input position (x, y) on a chess board.

From some position (x, y) the knight can move to the following positions provided that they are within the board limits:

$(x + 2, y - 1)$
$(x + 2, y + 1)$
$(x - 2, y + 1)$
$(x - 2, y - 1)$
$(x + 1, y + 2)$
$(x + 1, y - 2)$
$(x - 1, y + 2)$
$(x - 1, y - 2)$



Start from position (x, y) and apply BFS, considering that the neighbors of (x, y) are all the positions that can be reach with one move (as above).

Continue this process for each neighbor

The first time that you reach (x', y') corresponds to the minimum number of steps.

Exercise on Binary Maze

Given a binary rectangular maze, find the shortest path's length from a position (x, y) to position (x', y').

The path can only contain cells having value 1, and at position (x, y) the valid moves are:
Go Top: $(x-1, y)$
Go Left: $(x, y-1)$
Go Down: $(x+1, y)$
Go Right: $(x, y+1)$

[**1** **1** 0 1 0]
[0 **1** **1** 1 0]
[0 1 **1** 0 **1**]
[1 0 **1** **1** **1**]

Start from position (x, y) and apply BFS, considering that the neighbors of (x, y) are all the positions that can be reach with one move (as above).

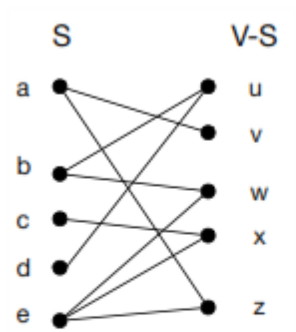
Continue this process for each neighbor

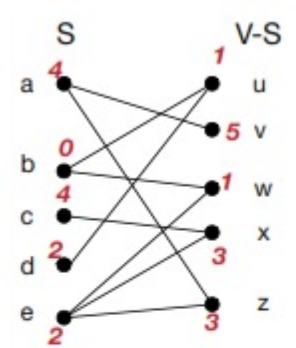
The first time that you reach (x', y') corresponds to the minimum number of steps.

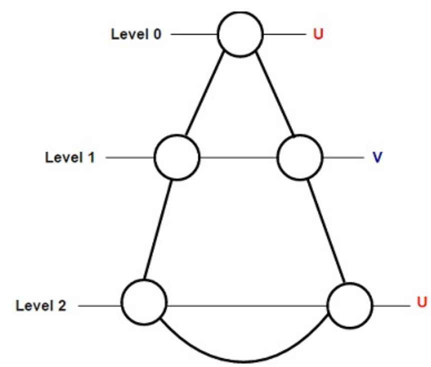
Exercise on Bipartite Graphs

A bipartite graph is a graph whose vertices can be divided into two disjoint sets U and V such that every edge connects a vertex in U to one in V.

Problem: Given a connected undirected graph determine whether it is bipartite or not.







- Run BFS from any vertex: $d[v]$ stores the shortest distance from the root to v. Set S to be the set of all vertices with even $d[v]$, and $V - S$ all vertices with odd $d[v]$.
- G is bipartite if and only if all edges $\{u, v\}$ in the graph satisfy that the parity of $d[v]$ and $d[u]$ are not the same, i.e., $d[v]$ is odd and $d[u]$ is even or vice versa.
- Alternatively: If a graph contains an odd cycle, we cannot divide the graph such that every adjacent vertex has a different parity. To check if a given graph contains an odd-cycle or not, do a breadth-first search starting from an arbitrary vertex v. If in the BFS, we find an edge, both of whose endpoints are at the same level, then the graph is not Bipartite, and an odd-cycle is found.

Depth First Search and DFS Tree

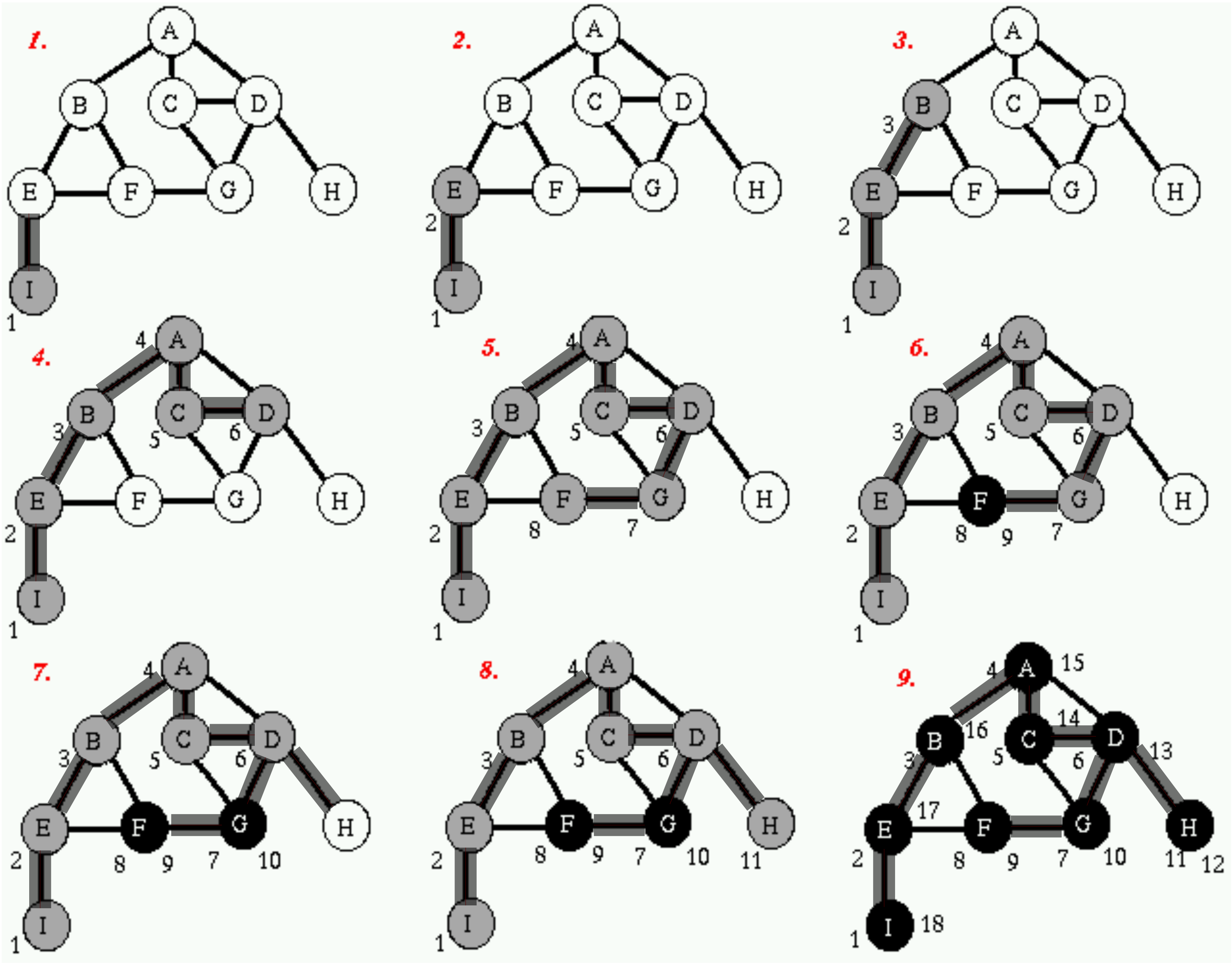
- Breadth first search is "Broad".
  - It builds a wide tree, connecting a node to ALL of the neighbors that have not yet been processed.
  - Once a node starts being processed, it sees ALL of its neighbors before any other node is processed

- There is another procedure, called DEPTH first search.
  - Instead of going broad, it goes DEEP
  - It recursively searches deep into the tree

  - When a node u is processed, it looks at each of its neighbors in order
    - At the time u checks a neighbor v, DFS starts processing v (which starts processing it's children, which start processing their children, etc.).

    - Only after all descendants of v have been processed does u go on to process its next neighbor

Depth First Search and DFS Tree



DFS Algorithm

```
DFS(G):
for each vertex u ∈ V do
      u.color ← white
      u.p ← nil
for each vertex u ∈ V do
      if u.color = white then
            DFS-Visit(u)
………….
```

Colors:

- **White**: undiscovered
- **Gray**: discovered, but neighbors not fully explored (on recursion stack)
- **Black**: discovered and neighbors fully explored

Parent pointers:

- Pointing to the node that leads to its discovery
- The pointers form a tree, rooted at s

- DFS(G) calls the DFS-visit search on each vertex u
- Before DFS-Visit(u) returns, all nodes in the connected component containing u are turned black (will see later)
- So DFS-Visit will only be called once for each connected component in G

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DFS-Visit(u):
u.color ← gray
for each v ∈ Adj[u] do
      if v.color = white then
            v.p ← u
            DFS-Visit(v)
u.color ← black
```

Running time: $\Theta(|V| + |E|)$

Colors:

- **White:** undiscovered
- **Gray:** discovered, but neighbors not fully explored (on recursion stack)
- **Black:** discovered and neighbors fully explored

Parent pointers:

- Pointing to the node that leads to its discovery
- The pointers form a tree, rooted at s

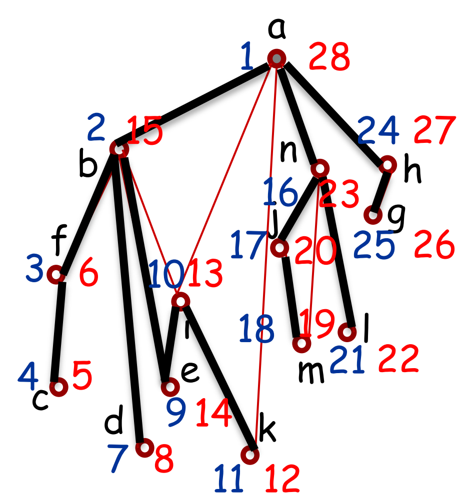
We can add starting and finishing time for each u:

Starting time when u.color ← gray

Finishing time when u.color ← black

DFS Worked Example

Adjacency Lists:
a: b, i, k, n, h
b: a, f, d, e, i
c: f
d: b
e: b, i
f: c, b
g: h
h: a, g
i: e, b, k, a
j: n, m
k: i, a
l: n
m: j, n
n: a, j, m, l



- The starting and finishing times are useful for some applications (to be discussed later)
- The bold edges form the DFS tree.
- The rest of the edges (light red) point to ancestors in the tree, and are called back-edges.
- Back edges are also useful for some applications.

Application: Cycle Detection

Problem: Given an undirected graph $G = (V, E)$, check if it contains a cycle.

Idea:

- A tree (connected and acyclic) contains **exactly** $|V| - 1$ edges.
- If it has fewer edges, it cannot be connected.
- If it has more edges, it must contain a cycle.

Algorithm:

- Run BFS/DFS to find all the connected components of G.
- For each connected component, count the number of edges.
- If # edges $\geq$ # vertices, return "cycle detected".

Running time: $\Theta(|V| + |E|)$

Q: What if we also want to **find** a cycle (any is OK) if it exists?

Tree edges, back edges, and cross edges

After running BFS or DFS on an undirected graph, all edges can be classified into one of 3 types:

- Tree edges: traversed by the BFS/DFS.
- Back edges: connecting a node with one of its ancestors in the BFS/DFS-tree (other than its parent).
- Cross edges: connecting two nodes with no ancestor/descendent relationship.

Theorem: In a DFS on an **undirected** graph, there are no cross edges.

Pf: Consider any edge (u, v) in G.

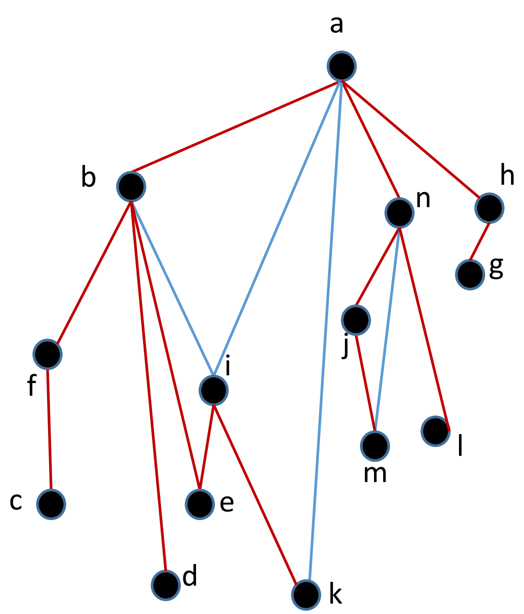
- Without loss of generality, assume u is discovered before v.
- Then v is discovered while u is gray (why?).
- Hence v is in the DFS subtree rooted at u.
  - If $v.p = u$, then (u, v) is a tree edge.
  - If $v.p \neq u$, then (u, v) is a back edge.

Theorem: In a BFS on an **undirected** graph, there are no back edges.
(Not proven)

DFS for cycle detection

Idea: Run DFS on each connected component of G.

- If (u, v) is a back edge.
  - => v is an ancestor (but not parent) of u in the DFS trees.
    =>There is thus a path from v to u in the DFS-tree and
  - => v to u plus back edge (u, v) creates a cycle.
- If no back edge exists then only contains (DFS) tree edges
  - => the graph is a forest, and hence is acyclic.



- In DFS starting at a,
  (i, b) was first back edge found
- => b was ancestor (not parent) of i in tree
- => tree contains path (b, e, i) from b to i
- => this path plus edge (i, b) is the cycle
  (b, e, i, b)

DFS for cycle detection

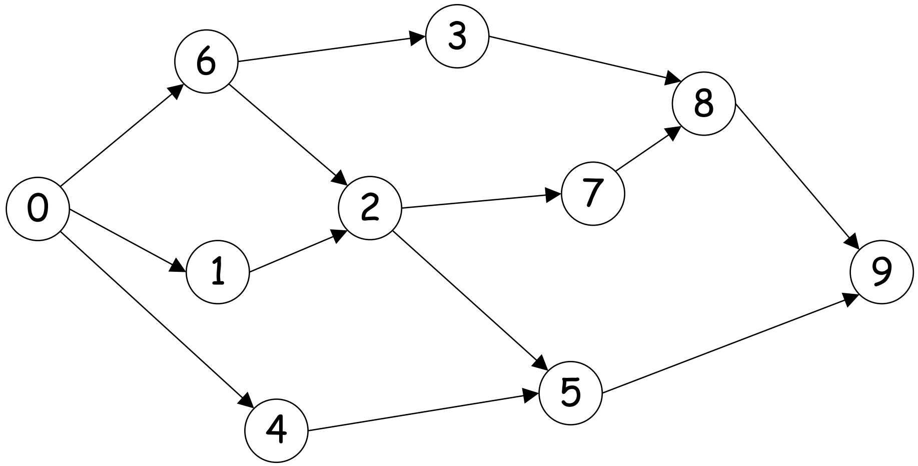
```
CycleDetection(G):
for each vertex u ∈ V do
    u.color ← white
    u.p ← nil
for each vertex u ∈ V do
    if u.color = white then DFS-Visit(u)
return "No cycle"

DFS-Visit(u):
u.color ← gray
for each v ∈ Adj[u] do
    if v.color = white then
        v.p ← u
        DFS-Visit(v)
    else if v ≠ u.p then //back edge (u,v)
        output "Cycle found:"
        while u ≠ v do
            output u
            u ← u.p
        output v
        return
u.color ← black
```

Running time: $\Theta(|V|)$

- Only traverse DFS-tree edges, until the first non-tree edge is found
- At most $|V| - 1$ tree edges

Directed Graph

A directed graph distinguishes between edge (u, v) and edge (v, u). Directed graphs are often used to represent order-dependent tasks

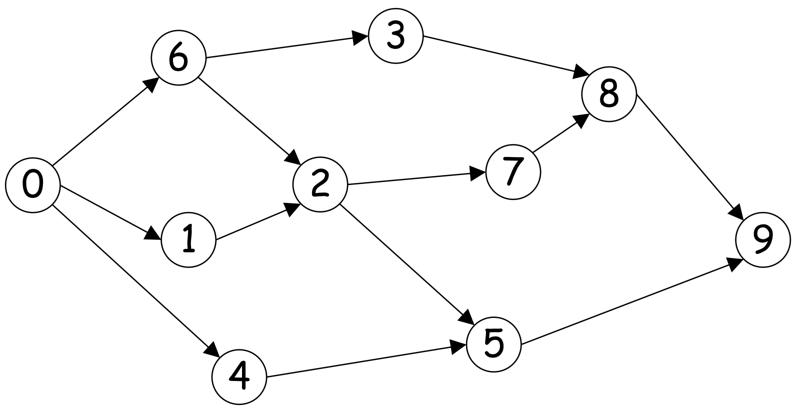


- out-degree of vertex v is the number of edges leaving v
- in-degree of vertex v is the number of edges entering v
- Each edge (u, v) contributes one to the out-degree of u and one to the in-degree of v, so

$$\sum_{v \in V} \text{outdegree}(v) = \sum_{v \in V} \text{indegree}(v) = |E|$$

Topological Sort

- Directed Acyclic Graph (DAG): Directed graph with no cycles.
- A Topological ordering of a graph is a linear ordering of the vertices of a DAG such that if (u, v) is in the graph, u appears before v in the linear ordering



- Topological ordering may not be unique
- The graph above has many topological orderings
  - 0, 6, 1, 4, 3, 2, 5, 7, 8, 9
  - 0, 4, 1, 6, 2, 5, 3, 7, 8, 9
  - ...

Topological Sort Algorithm

- Observations
  - A DAG must contain at least one vertex with in-degree zero
- Algorithm: Topological Sort (TS)
  1. Output a vertex u with in-degree zero in current graph.
  2. Remove u and all edges (u, v) from current graph.
  3. If graph is not empty, goto step 1.

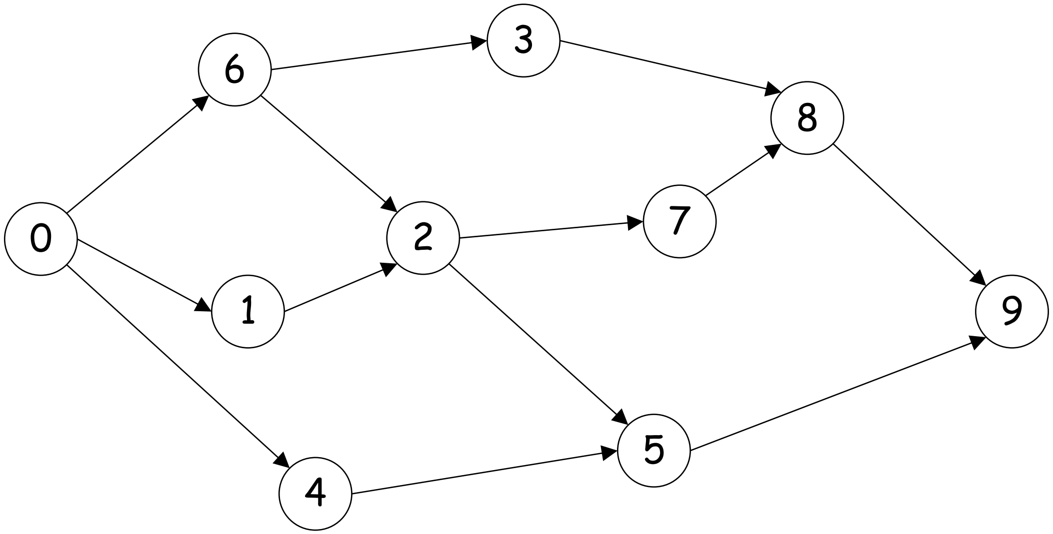
- Correctness
  - At every stage, current graph remains a DAG (why?)
  - Because current graph is always a DAG, TS can always output some vertex. So algorithm outputs all vertices.
  - Suppose output order is **not** a topological order.
    => Then there is some edge (u, v) such that v appears before u in the order. This is impossible, though, because v can not be output until edge (u, v) is removed!

Topological Sort Algorithm

Topological Sort(G)

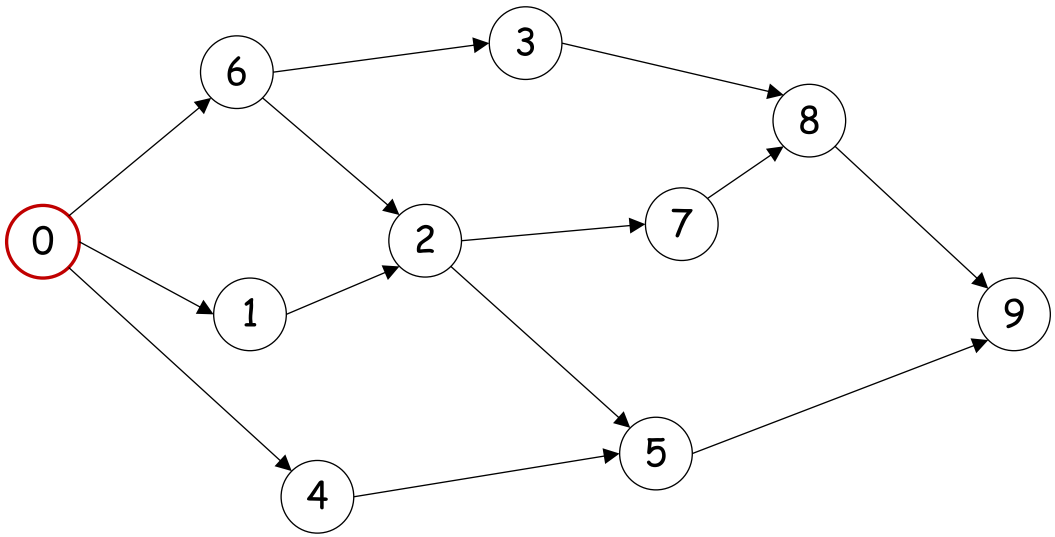
```
Initialize Q to be an empty queue;
foreach u in V do
    If indegree(u) = 0 then
        // Find all starting vertices
        Enqueue(Q, u);
    end
end
while Q is not empty do
    u = Dequeue(Q);
    Output u;
    foreach v in Adj(u) do
        // remove u's outgoing edges
        indegree(v) = indegree(v) − 1
        if indegree(v) = 0 then
            Enqueue(Q, v);
        end
    end
end
```

Example



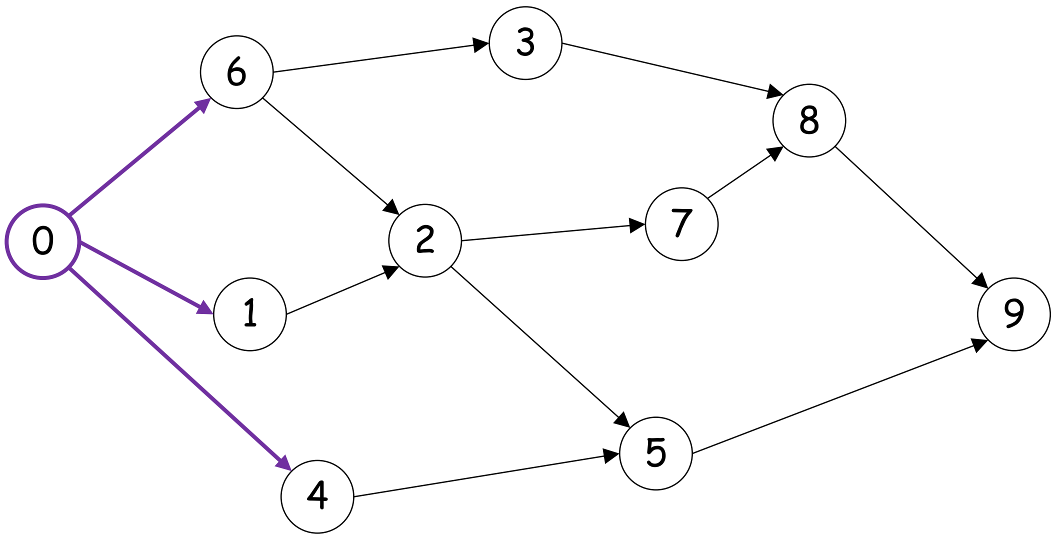
$Q = \{\}$

Example



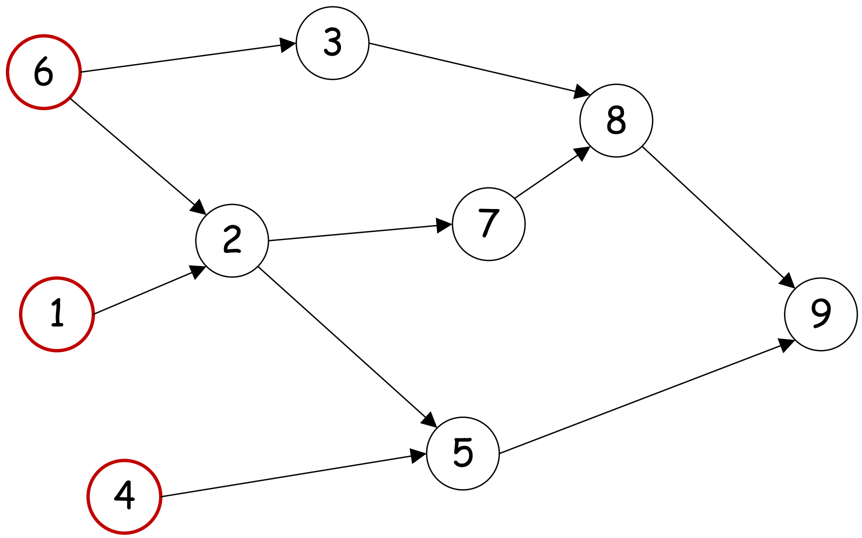
$$Q = \{0\}$$

Example



$Q = \{0\}$

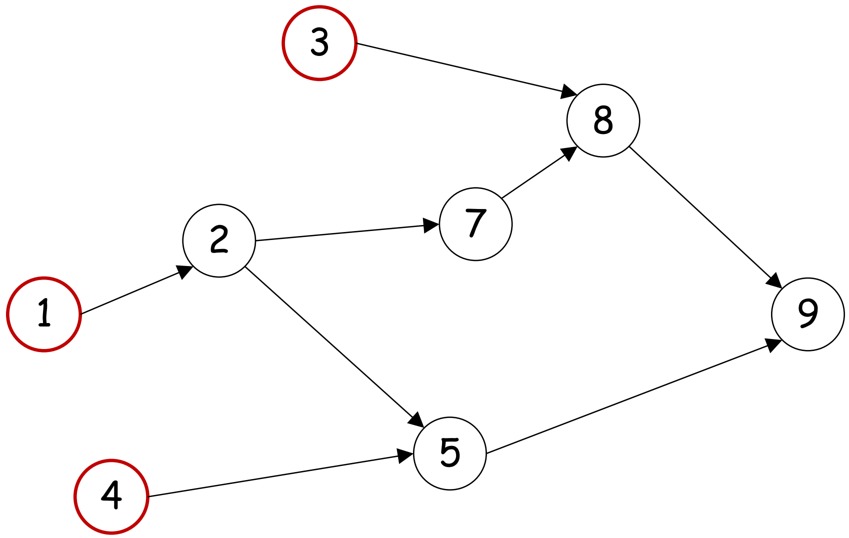
Example



$Q = \{6,1,4\}$

Output: 0

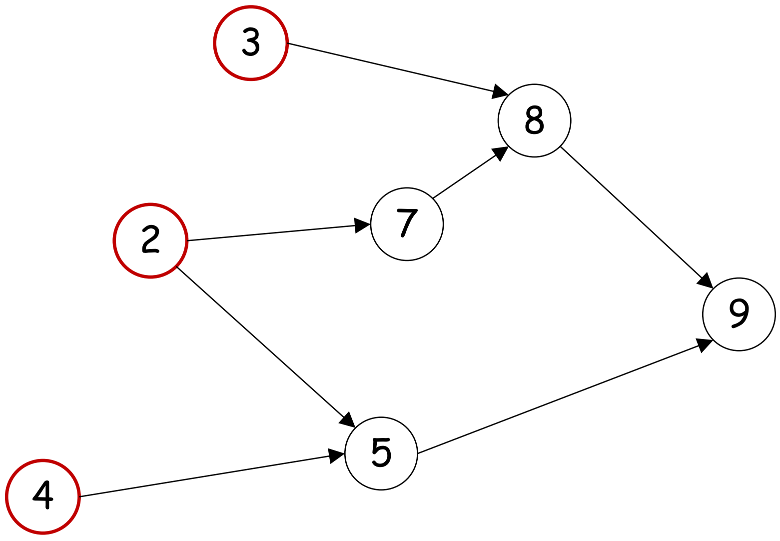
Example



$Q = \{1,4,3\}$

Output: 0,6

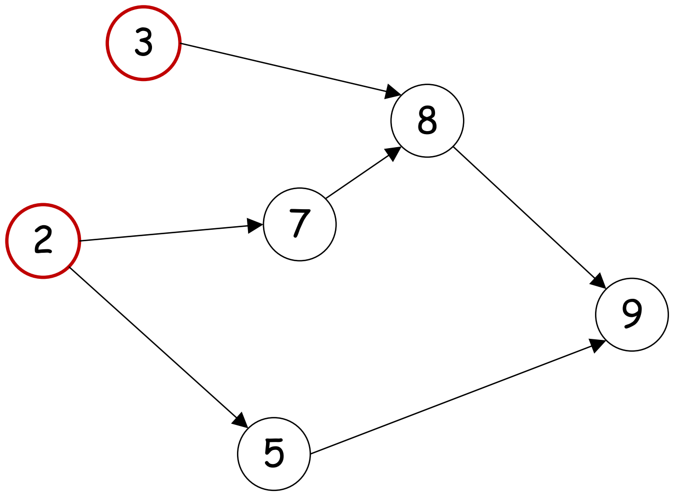
Example



$Q = \{4,3,2\}$

Output: 0,6,1

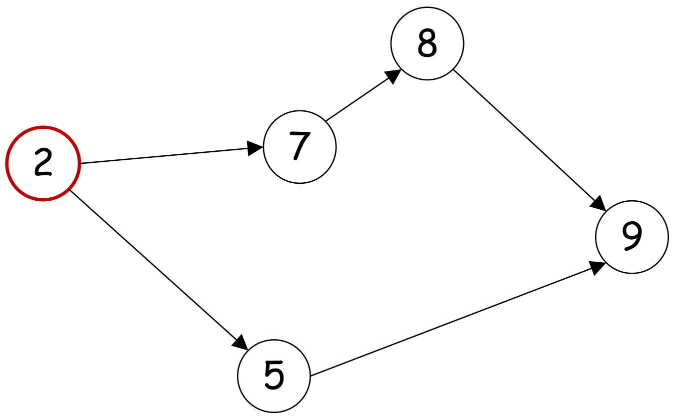
Example



$Q = \{3,2\}$

Output: 0,6,1,4

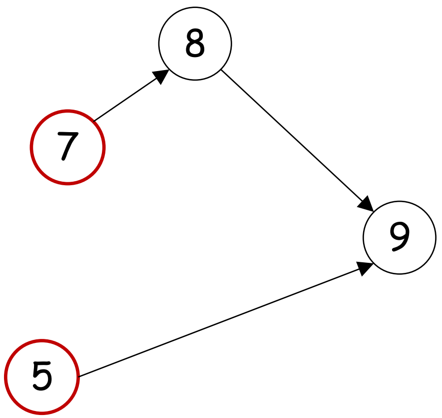
Example



$Q = \{2\}$

Output: 0,6,1,4,3

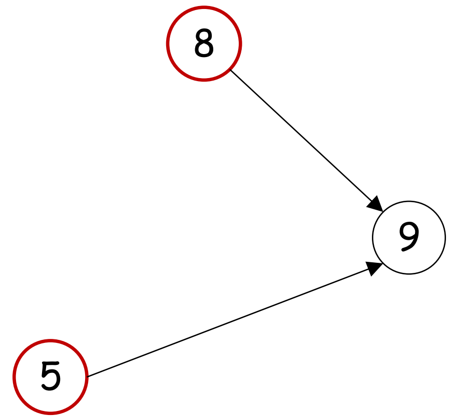
Example



$Q = \{7,5\}$

Output: 0,6,1,4,3,2

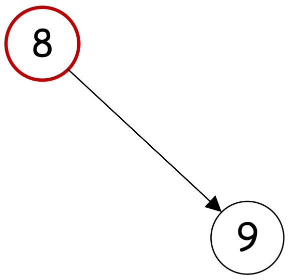
Example



$Q = \{5,8\}$

Output: 0,6,1,4,3,2,7

Example



$Q = \{8\}$

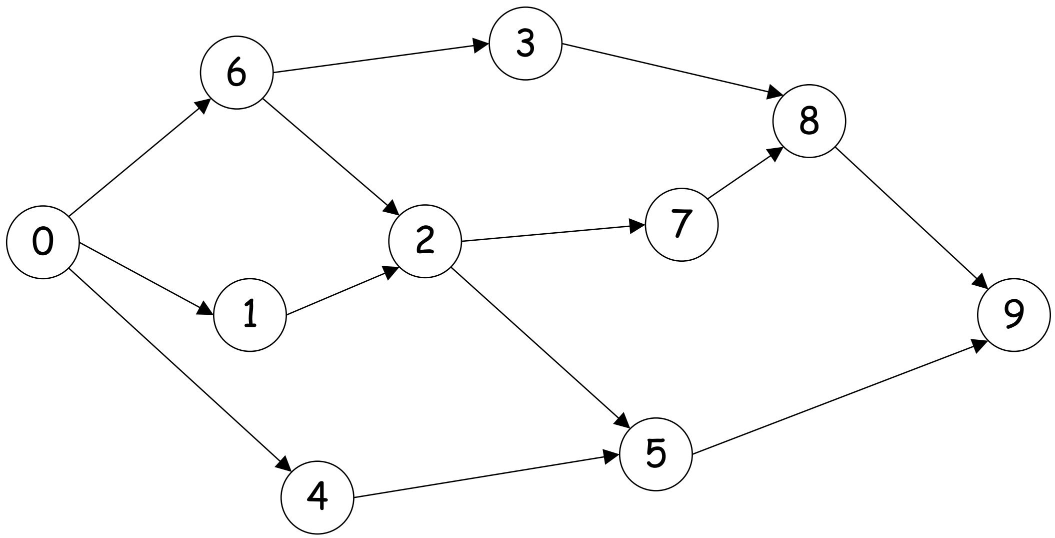
Output: 0,6,1,4,3,2,7,5

Example

$Q = \{9\}$

Output: 0,6,1,4,3,2,7,5
,8

Example



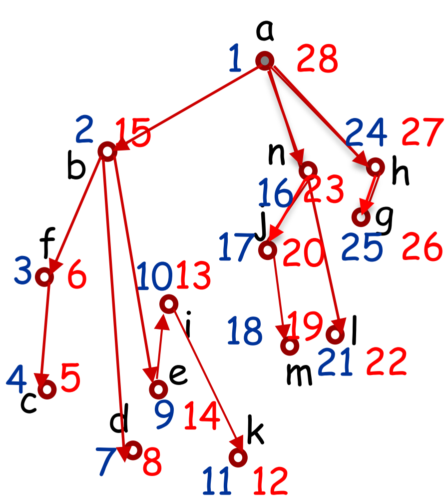
$Q = \{\}$

Output: 0,6,1,4,3,2,7,5,
8,9

Done!

Topological Sort: Complexity

- We never visit a vertex more than once
- For each vertex, we examine all outgoing edges
  - $\sum_{v \in V} \text{outdegree}(v) = |E|$
- Therefore, the running time is $O(|V| + |E|)$
- Q: Can we use DFS to implement topological sort?



- Apply DFS from a node that has in-degree 0
- Output the nodes in decreasing order of finishing time:
- Example:
- a, h, g, n, l, j, m, b, e, ...