Version of 03/17/2021

Matrix Multiplication Worked Example

Input: $p_0\, p_1 \cdots p_{n-1}\, p_n$
These represent sizes of n matrices $A_1\ A_2 \cdots A_n$
Matrix A_i has dimensions $p_{i-1} \times p_i$

i	0	1	2	3	4	5
p_i	5	4	6	2	7	4

A_1: 5×4 A_2: 4×6 A_3: 6×2 A_4: 2×7 A_5: 7×4

$A_{i\ldots j}$: matrix product of $A_i\ A_{i+1} \cdots A_j$
$A_{i\ldots j}$ has dimensions $p_{i-1} \times p_j$

$m[i,j]$ = minimum number of scalar multiplications required to compute $A_{i\ldots j}$

Recurrence

$$m[i,i] = 0$$
$$m[i,j] = \min_{i \le k < j} [i,k] + m[k+1,j] + p_{i-1}\, p_k p_j$$

$s[i,j]$ = records values of k at which minimum occurs

i	0	1	2	3	4	5
p_i	5	4	6	2	7	4

$$m[i,i] = 0$$
$$m[i,j] = \min_{i \le k < j} [i,k] + m[k+1,j] + p_{i-1} p_k p_j$$

$\boldsymbol{m[i,j]}$

i \ j	1	2	3	4	5
1					
2					
3					
4					
5					

$\boldsymbol{s[i,j]}$

i \ j	1	2	3	4	5
1					
2					
3					
4					
5					

$l = 1$

$m[i, j]$

i \ j	1	2	3	4	5
1	0				
2		0			
3			0		
4				0	
5					0

$s[i, j]$

i \ j	1	2	3	4	5
1					
2					
3					
4					
5					

i	0	1	2	3	4	5
p_i	5	4	6	2	7	4

$$m[i, i] = 0$$
$$m[i, j] = \min_{i \le k < j} [i, k] + m[k + 1, j] + p_{i-1} p_k p_j$$

- $m[1, 1] = 0$ A_1
- $m[2, 2] = 0$ A_2
- $m[3, 3] = 0$ A_3
- $m[4, 4] = 0$ A_4
- $m[5, 5] = 0$ A_5

$l = 2$

i	0	1	2	3	4	5
p_i	5	4	6	2	7	4

$$m[i,i] = 0$$
$$m[i,j] = \min_{i \le k < j} [i,k] + m[k+1,j] + p_{i-1} p_k p_j$$

$\boldsymbol{m[i,j]}$

i \ j	1	2	3	4	5
1	0	120			
2		0	48		
3			0	84	
4				0	56
5					0

$\boldsymbol{s[i,j]}$

i \ j	1	2	3	4	5
1		1			
2			2		
3				3	
4					4
5					

- $m[1,2] = m[1,1] + m[2,2] + 5 * 4 * 6 = 120$ $\quad \boldsymbol{A_1 A_2}$
- $m[2,3] = m[2,2] + m[3,3] + 4 * 6 * 2 = 48$ $\quad \boldsymbol{A_2 A_3}$
- $m[3,4] = m[3,3] + m[4,4] + 6 * 2 * 7 = 84$ $\quad \boldsymbol{A_3 A_4}$
- $m[4,5] = m[4,4] + m[5,5] + 2 * 7 * 4 = 56$ $\quad \boldsymbol{A_4 A_5}$

$l = 3$

i	0	1	2	3	4	5
p_i	5	4	6	2	7	4

$$m[i,i] = 0$$
$$m[i,j] = \min_{i \le k < j} [i,k] + m[k+1,j] + p_{i-1} p_k p_j$$

$m[i,j]$

i \ j	1	2	3	4	5
1	0	120	88		
2		0	48	104	
3			0	84	104
4				0	56
5					0

$s[i,j]$

i \ j	1	2	3	4	5
1		1	1		
2			2	3	
3				3	3
4					4
5					

- $m[1,3] = \min \begin{cases} m[1,1] + m[2,3] + 5 * 4 * 2, \\ m[1,2] + m[3,3] + 5 * 6 * 2 \end{cases} = 88$ $A_1 A_{2..3}$
- $m[2,4] = \min \begin{cases} m[2,2] + m[3,4] + 4 * 6 * 7, \\ m[2,3] + m[4,4] + 4 * 2 * 7 \end{cases} = 104$ $A_{2..3} A_4$
- $m[3,5] = \min \begin{cases} m[3,3] + m[4,5] + 6 * 2 * 4, \\ m[3,4] + m[5,5] + 6 * 7 * 4 \end{cases} = 104$ $A_3 A_{4..5}$

$l = 4$

i	0	1	2	3	4	5
p_i	5	4	6	2	7	4

$$m[i,i] = 0$$
$$m[i,j] = \min_{i \le k < j} [i,k] + m[k+1,j] + p_{i-1} p_k p_j$$

$m[i,j]$

i \ j	1	2	3	4	5
1	0	120	88	158	
2		0	48	104	136
3			0	84	104
4				0	56
5					0

$$m[1,4] = \min \begin{Bmatrix} \mathbf{m[1,3] + m[4,4] + 5 * 2 * 7}, \\ m[1,2] + m[3,4] + 5 * 6 * 7 \\ m[1,1] + m[2,4] + 5 * 4 * 7 \end{Bmatrix} = 158 \qquad A_{1..3}A_4$$

$s[i,j]$

i \ j	1	2	3	4	5
1		1	1	3	
2			2	3	3
3				3	3
4					4
5					

$$m[2,5] = \min \begin{Bmatrix} m[2,4] + m[5,5] + 4 * 7 * 4, \\ \mathbf{m[2,3] + m[4,5] + 4 * 2 * 4} \\ m[2,2] + m[3,5] + 4 * 6 * 4 \end{Bmatrix} = 136 \qquad A_{2..3}A_{4..5}$$

$l = 5$

i	0	1	2	3	4	5
p_i	5	4	6	2	7	4

$$m[i,i] = 0$$
$$m[i,j] = \min_{i \le k < j} [i,k] + m[k+1,j] + p_{i-1} p_k p_j$$

$\boldsymbol{m[i,j]}$

i \ j	1	2	3	4	5
1	0	120	88	158	184
2		0	48	104	136
3			0	84	104
4				0	56
5					0

$\boldsymbol{s[i,j]}$

i \ j	1	2	3	4	5
1		1	1	3	3
2			2	3	3
3				3	3
4					4
5					

$$m[1,5] = \min \begin{cases} m[1,4] + m[5,5] + 5 * 7 * 4, \\ \boldsymbol{m[1,3] + m[4,5] + 5 * 2 * 4}, \\ m[1,2] + m[3,5] + 5 * 6 * 4, \\ m[1,1] + m[2,5] + 5 * 4 * 4 \end{cases} = 184$$

$$\boldsymbol{A_{1..3} A_{4..5}}$$

$l = 5$

i	0	1	2	3	4	5
p_i	5	4	6	2	7	4

$$m[i,i] = 0$$
$$m[i,j] = \min_{i \le k < j} [i,k] + m[k+1,j] + p_{i-1} p_k p_j$$

$m[i,j]$

i \ j	1	2	3	4	5
1	0	120	88	158	184
2		0	48	104	136
3			0	84	104
4				0	56
5					0

$s[i,j]$

i \ j	1	2	3	4	5
1		1	1	3	3
2			2	3	3
3				3	3
4					4
5					

Optimal solution is

$$(A_{1..s[1,5]} A_{s[1,5]+1..5}) = (A_{1..3} A_{4..5})$$
$$= (A_{1..s[1,3]} A_{s[1,3]+1..3}) A_{4..5} \quad = ((A_{1..1} A_{2..3}) A_{4..5})$$
$$= ((A_1 (A_{2..s[2,3]} A_{s[2,3]+1..3})) A_{4..5})$$
$$= ((A_1 (A_{2..2} A_{3..3})) A_{4..5})$$
$$= ((A_1 (A_2 A_3)) A_{4..5})$$
$$= ((A_1 (A_2 A_3))(A_{4..s[4,5]} A_{s[4,5]+1,5}))$$
$$= ((A_1 (A_2 A_3))(A_4 A_5))$$