

# This IS an Algorithm: Adding Two Numbers

**Input:** Two numbers  $x$  and  $y$  (potentially very long), each consisting of  $n$  digits:  $x = \overline{x_n x_{n-1} \dots x_1}, y = \overline{y_n y_{n-1} \dots y_1}$

**Output:** A number  $z = \overline{z_{n+1} z_n \dots z_1}$ , such that  $z = x + y$ .

```
c ← 0
for i ← 1 to n
    zi ← xi + yi + c
    if zi ≥ 10 then c ← 1, zi ← zi - 10
    else c ← 0
zn+1 ← c
```

$$\begin{array}{r} 529501233 \\ +612345678 \\ \hline 1141846911 \end{array}$$

You've been running algorithms all your life!

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 $c \leftarrow 0$ 
for  $i \leftarrow 1$  to  $n$ 
     $z_i \leftarrow x_i + y_i + c$ 
    if  $z_i \geq 10$  then  $c \leftarrow 1$ ,  $z_i \leftarrow z_i - 10$ 
    else  $c \leftarrow 0$ 
 $z_{n+1} \leftarrow c$ 
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 $c_1 \leftarrow 0$ 
for  $i \leftarrow 1$  to  $n$ 
     $u_i \leftarrow x_i + y_i + c_i$ 
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Slightly revised  
(but equivalent)

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$c_i$											0
$u_i$											11
$z_i$											

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