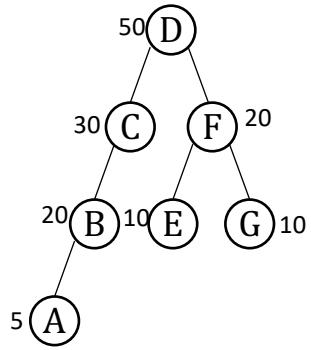
Worked example of the Optimal BST Problem

input

i	1	2	3	4	5	6	7
a_i	A	B	C	D	E	F	G
$f(a_i)$	5	20	30	50	10	20	10



Optimal (min-cost) given input
Its cost is

5*4 + 20*3 + 30*2 + 50*1 + 10*3 + 20*2 + 10*3 = 290

Worked example of the Optimal BST Problem

input

i	1	2	3	4	5	6	7
a_i	A	B	C	D	E	F	G
$f(a_i)$	5	20	30	50	10	20	10

Set: $w[i,j] = f(a_i) + \cdots + f(a_j)$

Define: $e[i,j] =$ the minimum cost of any BST on $a_i, \ldots, a_j$

Recurrence: $e[i,j] = \min_{i \le k \le j} \{e[i,k-1] + e[k+1,j] + w[i,j]\}$

Initial Conditions: for $i > j$ $e[i,j] = 0$

and $e[i,i] = w[i,i] = f(a_i)$

$e[i,j]$

i \ j	1	2	3	4	5	6	7
1	5						
2		20					
3			30				
4				50			
5					10		
6						20	
7							10

- $e[1,1] = f(a_1) = 5$
- $e[2,2] = f(a_2) = 20$
- $e[3,3] = f(a_3) = 30$
- $e[4,4] = f(a_4) = 50$
- $e[5,5] = f(a_5) = 10$
- $e[6,6] = f(a_6) = 20$
- $e[7\ 7] = f(a_7) = 10$

$root[i,j]$

i \ j	1	2	3	4	5	6	7
1	1						
2		2					
3			3				
4				4			
5					5		
6						6	
7							7

- $root[1,1] = 1$
- $root[2,2] = 2$
- $root[3,3] = 3$
- $root[4,4] = 4$
- $root[5,5] = 5$
- $root[6,6] = 6$
- $root[7,7] = 7$

Ⓐ Ⓑ Ⓒ Ⓓ Ⓔ Ⓕ Ⓖ

i	1	2	3	4	5	6	7
a_i	A	B	C	D	E	F	G
$f(a_i)$	5	20	30	50	10	20	10

$e[i,j]$

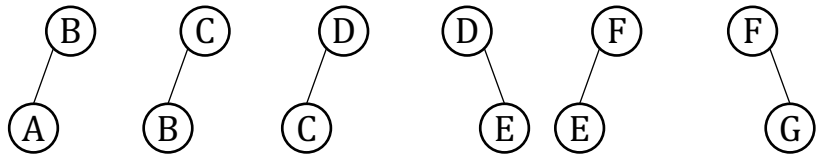
i \ j	1	2	3	4	5	6	7
1	5	30					
2		20	70				
3			30	110			
4				50	70		
5					10	40	
6						20	40
7							10

- $e[1,2] = \min\limits_{1\le k\le 2}\{e[1,k-1] + e[k+1,2] + w[1,2]\} = \min\{45, 30\} = 30$
- $e[2,3] = \min\limits_{2\le k\le 3}\{e[2,k-1] + e[k+1,3] + w[2,3]\} = \min\{80, 70\} = 70$
- $e[3,4] = \min\limits_{2\le k\le 3}\{e[3,k-1] + e[k+1,4] + w[3,4]\} = \min\{130, 110\} = 110$
- $e[4,5] = \min\limits_{4\le k\le 5}\{e[4,k-1] + e[k+1,5] + w[4,5]\} = \min\{70, 110\} = 70$
- $e[5,6] = \min\limits_{5\le k\le 6}\{e[5,k-1] + e[k+1,6] + w[5,6]\} = \min\{50, 40\} = 40$
- $e[6,7] = \min\limits_{6\le k\le 7}\{e[6,k-1] + e[k+1,7] + w[6,7]\} = \min\{40, 50\} = 40$

$root[i,j]$

i \ j	1	2	3	4	5	6	7
1	1	2					
2		2	3				
3			3	4			
4				4	4		
5					5	6	
6						6	6
7							7

- $root[1,2] = 2$
- $root[2,3] = 3$
- $root[3,4] = 4$
- $root[4,5] = 4$
- $root[5,6] = 6$
- $root[6,7] = 6$



i	1	2	3	4	5	6	7
a_i	A	B	C	D	E	F	G
$f(a_i)$	5	20	30	50	10	20	10

$e[i,j]$

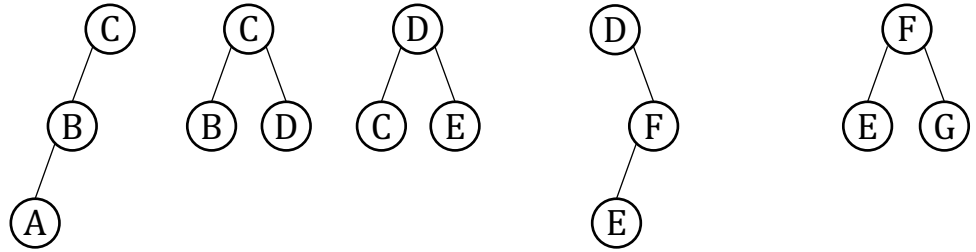
i \ j	1	2	3	4	5	6	7
1	5	30	85				
2		20	70	170			
3			30	110	130		
4				50	70	120	
5					10	40	60
6						20	40
7							10

- $e[1,3] = \min_{1 \le k \le 3}\{e[1,k-1] + e[k+1,3] + w[1,3]\} = \min\{125, 90, 85\} = 85$
- $e[2,4] = \min_{2 \le k \le 4}\{210, 170, 170\} = 170$
- $e[3,5] = \min_{3 \le k \le 5}\{160, 130, 200\} = 130$
- $e[4,6] = \min_{4 \le k \le 6}\{120, 150, 150\} = 120$
- $e[5,7] = \min_{5 \le k \le 7}\{80, 60, 80\} = 60$

$root[i,j]$

i \ j	1	2	3	4	5	6	7
1	1	2	3				
2		2	3	3			
3			3	4	4		
4				4	4	4	
5					5	6	6
6						6	6
7							7

- $root[1,3] = 3$
- $root[2,4] = 3$
- $root[3,5] = 4$
- $root[4,6] = 4$
- $root[5,7] = 6$



i	1	2	3	4	5	6	7
a_i	A	B	C	D	E	F	G
$f(a_i)$	5	20	30	50	10	20	10

$e[i,j]$

i \ j	1	2	3	4	5	6	7
1	5	30	85				
2		20	70	170			
3			30	110	130		
4				50	70	120	
5					10	40	60
6						20	40
7							10

$root[i,j]$

i \ j	1	2	3	4	5	6	7
1	1	2	3				
2		2	3	3			
3			3	4	4		
4				4	4	4	
5					5	6	6
6						6	6
7							7

$$e[1,4] = \min_{1\le k\le 4}\{e[1,k-1] + e[k+1,4] + w[1,4]\}$$

$$= \min_{1\le k\le 4}\begin{Bmatrix} e[1,0] + e[2,4] + 105, & e[1,1] + e[3,4] + 105, \\ e[1,2] + e[4,4] + 105, & e[1,3] + e[5,4] + 105 \end{Bmatrix}$$

$$= \min_{1\le k\le 4}\begin{Bmatrix} 0 + 170 + 105, & 5 + 110 + 105, \\ 30 + 50 + 105, & 85 + 0 + 105 \end{Bmatrix}$$

$$= \min\{275, 220, 185, 190\}$$

$$= 185$$

i	1	2	3	4	5	6	7
a_i	A	B	C	D	E	F	G
$f(a_i)$	5	20	30	50	10	20	10

$e[i,j]$

i \ j	1	2	3	4	5	6	7
1	5	30	85	185			
2		20	70	170			
3			30	110	130		
4				50	70	120	
5					10	40	60
6						20	40
7							10

$root[i,j]$

i \ j	1	2	3	4	5	6	7
1	1	2	3	3			
2		2	3	3			
3			3	4	4		
4				4	4	4	
5					5	6	6
6						6	6
7							7

$$e[1,4] = \min_{1\le k\le 4}\{e[1,k-1] + e[k+1,4] + w[1,4]\}$$

$$= \min_{1\le k\le 4}\begin{Bmatrix} e[1,0] + e[2,4] + 105, & e[1,1] + e[3,4] + 105, \\ e[1,2] + e[4,4] + 105, & e[1,3] + e[5,4] + 105 \end{Bmatrix}$$

$$= \min_{1\le k\le 4}\begin{Bmatrix} 0 + 170 + 105, & 5 + 110 + 105, \\ 30 + 50 + 105, & 85 + 0 + 105 \end{Bmatrix}$$

$$= \min\{275, 220, 185, 190\}$$

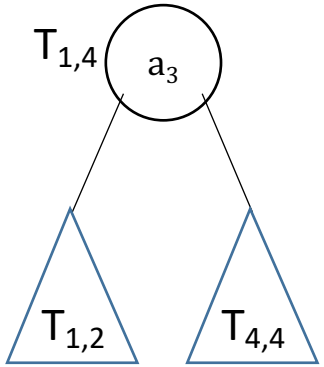
$$= 185$$

i	1	2	3	4	5	6	7
a_i	A	B	C	D	E	F	G
$f(a_i)$	5	20	30	50	10	20	10

$e[i,j]$

i \ j	1	2	3	4	5	6	7
1	5	30	85	185			
2		20	70	170			
3			30	110	130		
4				50	70	120	
5					10	40	60
6						20	40
7							10

- $e[1,4] = \min_{1 \le k \le 4} \{e[1,k-1] + e[k+1,4] + w[1,4]\} = \min\{275, 220, 185, 190\} = 185$



$root[i,j]$

i \ j	1	2	3	4	5	6	7
1	1	2	3	3			
2		2	3	3			
3			3	4	4		
4				4	4	4	
5					5	6	6
6						6	6
7							7

- $root[1,4] = 3$

i	1	2	3	4	5	6	7
a_i	A	B	C	D	E	F	G
$f(a_i)$	5	20	30	50	10	20	10

$e[i,j]$

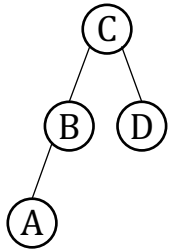
i \ j	1	2	3	4	5	6	7
1	5	30	85	185			
2		20	70	170	190		
3			30	110	130	180	
4				50	70	120	150
5					10	40	60
6						20	40
7							10

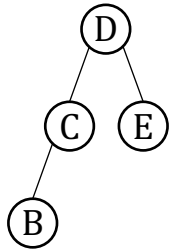
- $e[1,4] = \min_{1 \le k \le 4}\{e[1,k-1] + e[k+1,4] + w[1,4]\} = \min\{275, 220, 185, 190\} = 185$
- $e[2,5] = \min\{240, 200, 190, 280\} = 190$
- $e[3,6] = \min\{230, 180, 240, 240\} = 180$
- $e[4,7] = \min\{150, 180, 170, 210\} = 150$

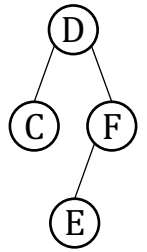
$root[i,j]$

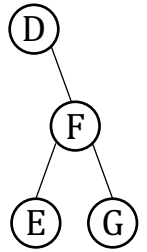
i \ j	1	2	3	4	5	6	7
1	1	2	3	3			
2		2	3	3	4		
3			3	4	4	4	
4				4	4	4	4
5					5	6	6
6						6	6
7							7

- $root[1,4] = 3$
- $root[2,5] = 4$
- $root[3,6] = 4$
- $root[4,7] = 4$









i	1	2	3	4	5	6	7
a_i	A	B	C	D	E	F	G
$f(a_i)$	5	20	30	50	10	20	10

$e[i,j]$

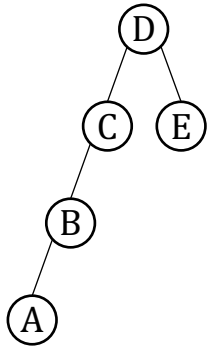
i \ j	1	2	3	4	5	6	7
1	5	30	85	185	210		
2		20	70	170	190	240	
3			30	110	130	180	210
4				50	70	120	150
5					10	40	60
6						20	40
7							10

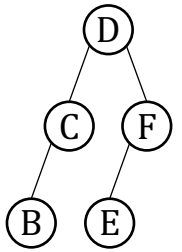
- $e[1,5] = \min_{1 \le k \le 5}\{e[1,k-1] + e[k+1,5] + w[1,5]\} = \min\{305, 250, 215, 210, 300\} = 210$
- $e[2,6] = \min\{310, 270, 240, 320, 320\} = 240$
- $e[3,7] = \min\{270, 210, 270, 260, 300\} = 210$

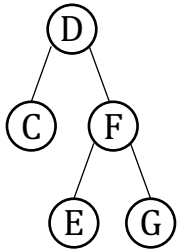
$root[i,j]$

i \ j	1	2	3	4	5	6	7
1	1	2	3	3	4		
2		2	3	3	4	4	
3			3	4	4	4	4
4				4	4	4	4
5					5	6	6
6						6	6
7							7

- $root[1,5] = 4$
- $root[2,6] = 4$
- $root[3,7] = 4$







i	1	2	3	4	5	6	7
a_i	A	B	C	D	E	F	G
$f(a_i)$	5	20	30	50	10	20	10

$e[i,j]$

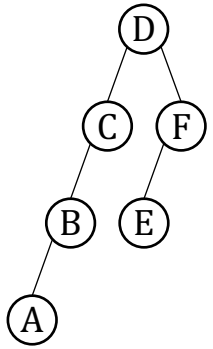
i \ j	1	2	3	4	5	6	7
1	5	30	85	185	210	260	
2		20	70	170	190	240	270
3			30	110	130	180	210
4				50	70	120	150
5					10	40	60
6						20	40
7							10

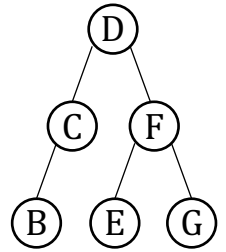
- $e[1,6] = \min\limits_{1\le k\le 6}\{e[1,k-1] + e[k+1,6] + w[1,6]\} = \min\{375, 320, 285, 260, 340, 345\} = 260$
- $e[2,7] = \min\ \{350, 310, 270, 350, 340, 380\} = 270$

$root[i,j]$

i \ j	1	2	3	4	5	6	7
1	1	2	3	3	4	4	
2		2	3	3	4	4	4
3			3	4	4	4	4
4				4	4	4	4
5					5	6	6
6						6	6
7							7

- $root[1,6] = 4$
- $root[2,7] = 4$



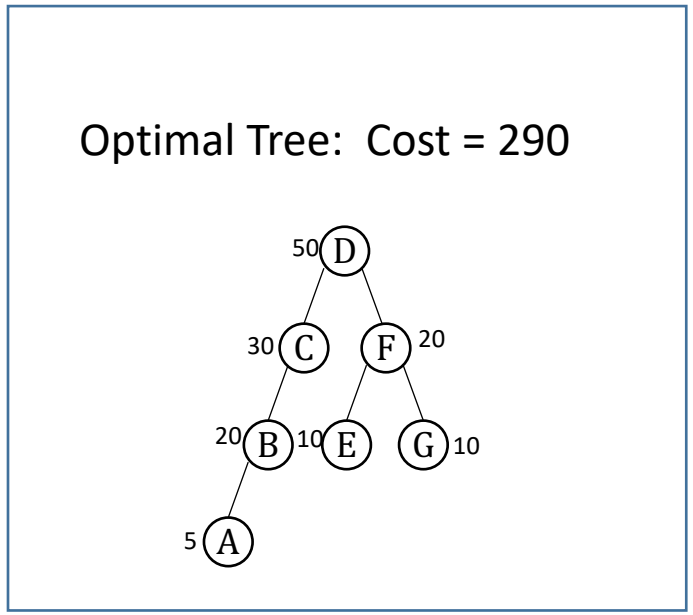


i	1	2	3	4	5	6	7
a_i	A	B	C	D	E	F	G
$f(a_i)$	5	20	30	50	10	20	10

$e[i,j]$

i \ j	1	2	3	4	5	6	7
1	5	30	85	185	210	260	290
2		20	70	170	190	240	270
3			30	110	130	180	210
4				50	70	120	150
5					10	40	60
6						20	40
7							10

- $e[1,7] = \min_{1 \le k \le 7}\{e[1,k-1] + e[k+1,7] + w[1,7]\} = \min\{415, 360, 325, 290, 370, 365, 405\} = 290$



$root[i,j]$

i \ j	1	2	3	4	5	6	7
1	1	2	3	3	4	4	4
2		2	3	3	4	4	4
3			3	4	4	4	4
4				4	4	4	4
5					5	6	6
6						6	6
7							7

- $root[1,7] = 4$

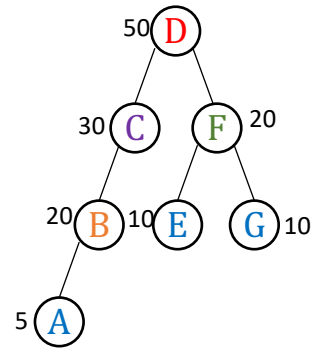
i	1	2	3	4	5	6	7
a_i	A	B	C	D	E	F	G
$f(a_i)$	5	20	30	50	10	20	10

$e[i,j]$

i \ j	1	2	3	4	5	6	7
1	5	30	85	185	210	260	290
2		20	70	170	190	240	270
3			30	110	130	180	210
4				50	70	120	150
5					10	40	60
6						20	40
7							10

- $e[1,7] = \min_{1 \le k \le 7}\{e[1,k-1] + e[k+1,7] + w[1,7]\} = \min\{415, 360, 325, 290, 370, 365, 405\} = 290$

Optimal Tree: Cost = 290



$root[i,j]$

i \ j	1	2	3	4	5	6	7
1	1	2	3	3	4	4	4
2		2	3	3	4	4	4
3			3	4	4	4	4
4				4	4	4	4
5					5	6	6
6						6	6
7							7

- $root[1,7] = 4$ (D)
- $root[1,3] = 3$ (C)
- $root[5,7] = 6$ (F)
- $root[1,2] = 2$ (B)
- $root[1,1] = 1$ (A)
- $root[5,5] = 5$ (E)
- $root[7,7] = 7$ (G)

i	1	2	3	4	5	6	7
a_i	A	B	C	D	E	F	G
$f(a_i)$	5	20	30	50	10	20	10