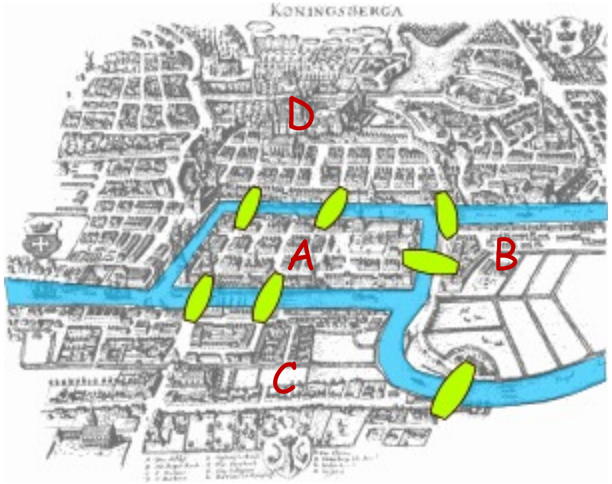
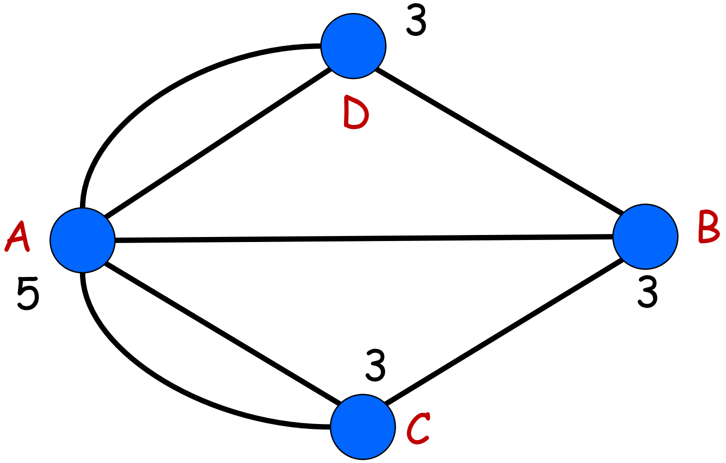
Introduction to Graphs

The Seven Bridges of Königsberg

Q: Can you find a path to cross all seven bridges, each exactly once?





Q: (Reformulated as a graph problem) Can you find a path in the graph that includes every edge exactly once?

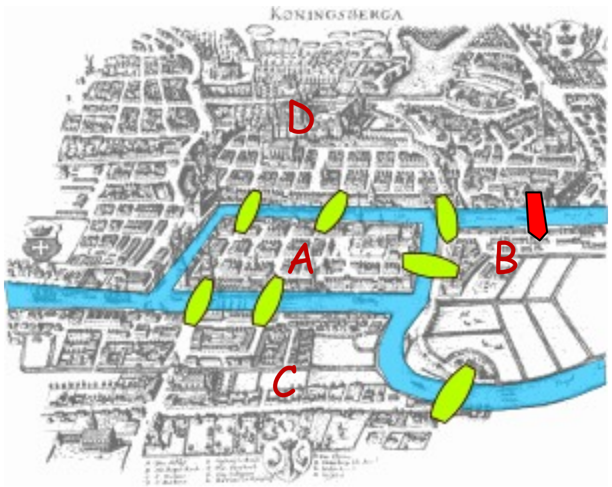
A: Not possible.

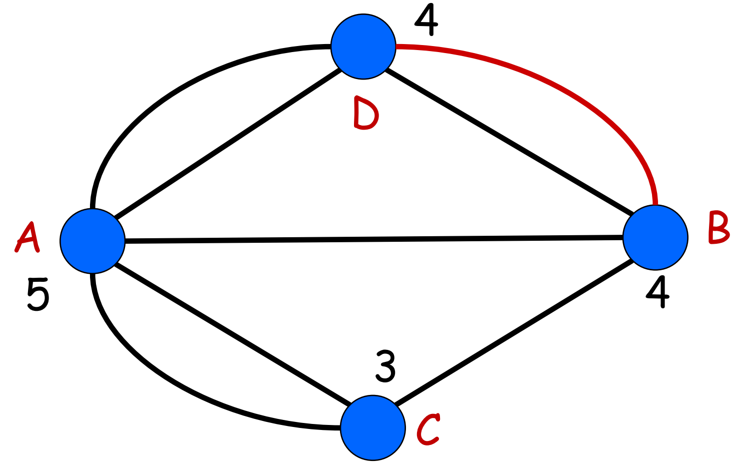
Theorem: A (multi)graph has such a path (known as an Euler path) iff it contains exactly 0 or 2 vertices with an odd degree.

Q: Can a graph have exactly one vertex with an odd degree?

Seven Bridges of Königsberg: Solution

Solution: Build one more bridge to remove 2 odd-degree vertices.

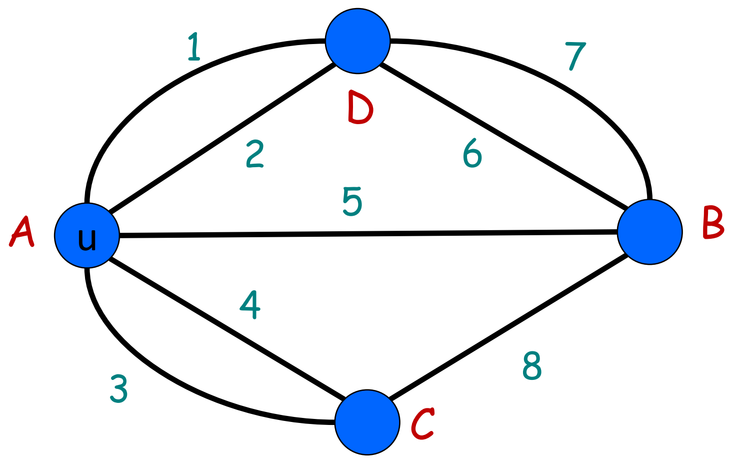




Algorithm:

```
u ← any odd-degree vertex
if no such vertex exists
     u ← any vertex
while u has an edge not taken yet
     take that edge (u,v)
     u ← v
```

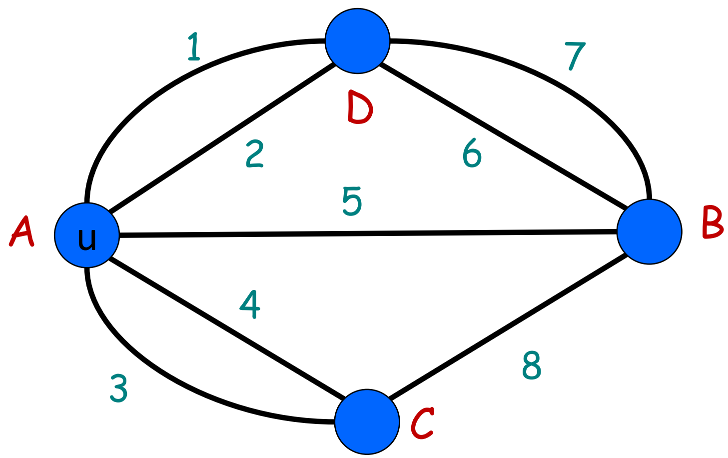
Seven Bridges of Königsberg: Initial Algorithm

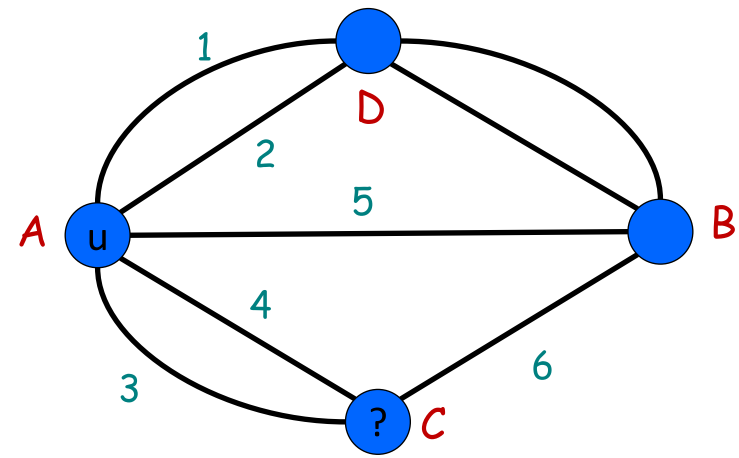


Algorithm:

$u \leftarrow$ **any odd-degree vertex**
if no such vertex exists
    $u \leftarrow$ **any vertex**
while u **has an edge not taken yet**
    take that edge (u, v)
    $u \leftarrow v$

Seven Bridges of Königsberg: Initial Algorithm



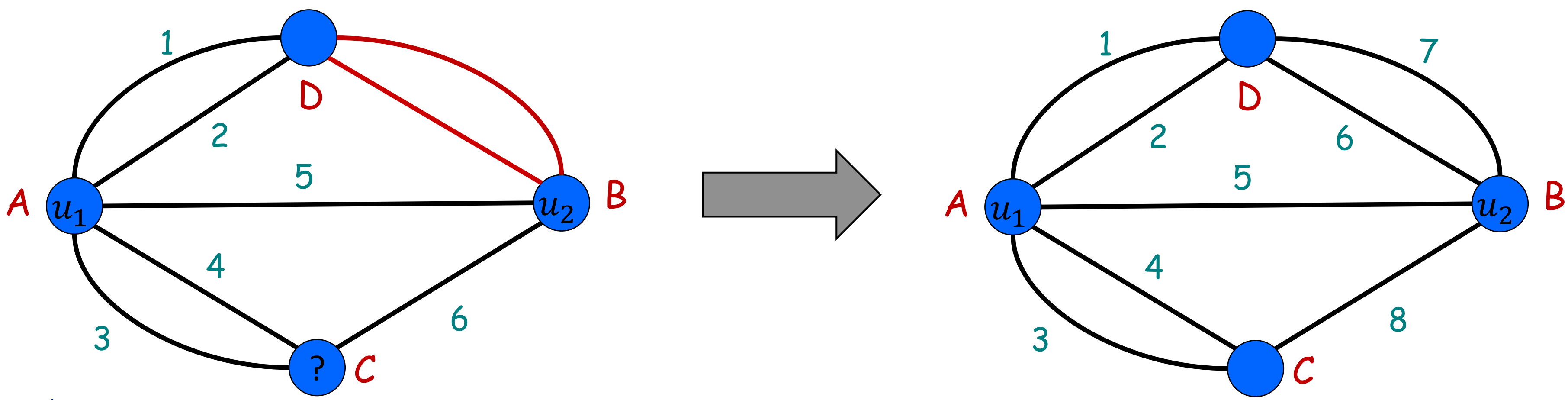


Algorithm:

```
u ← any odd-degree vertex
if no such vertex exists
     u ← any vertex
while u has an edge not taken yet
     take that edge (u,v)
     u ← v
```

But, this "algorithm" may get stuck...

Seven Bridges of Königsberg: Algorithm



Algorithm:

Hierholzer's algorithm (1873)

$u \leftarrow$ **any odd-degree vertex**
Find-Path(u)

while there are still edges not yet taken
    $u \leftarrow$ **any previously seen vertex that is endpoint of untaken edge**
    $p \leftarrow$ **Find-Path**(u)
    insert p **into existing path at** u

<u>Find-Path(u):</u>
while u **has an edge not taken yet**
    take that edge (u, v)
    $u \leftarrow v$

1. Algorithm chooses $u = u_1 = \mathrm{A}$.
Finds path $p_1 = ADACABC$.
Get stuck and stops

2. It then chooses $u = u_2 = B$.
Finds cycle $p_2 = BDB$.
Gets stuck.

2. It then inserts p_2 into 1 to create final solution.

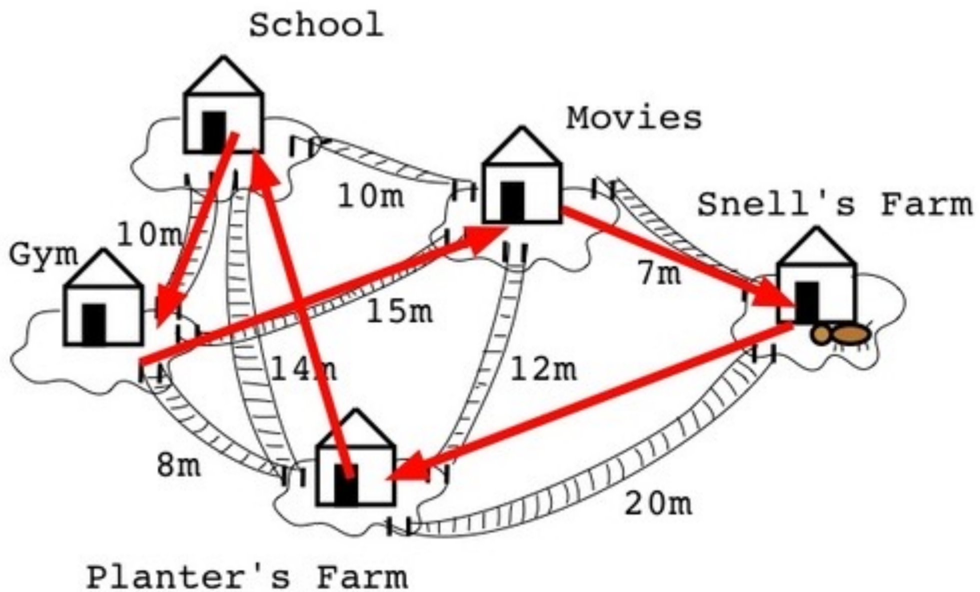
Graph Applications

Graph	Nodes	Edges
transportation	street intersections	highways
communication	computers	fiber optic cables
World Wide Web	web pages	hyperlinks
social	people	relationships
food web	species	predator-prey
software systems	functions	function calls
scheduling	tasks	precedence constraints
circuits	gates	wires

- Because they model ``relationships", graphs are ubiquitous.
- Instead of solving problems in one application, we focus on designing algorithms to solve problems in *abstract graphs*.
- These can then be used in many different application areas!

Traveling Salesman Problem

Q: How to visit all places, and then return to starting point, travelling the shortest possible distance.



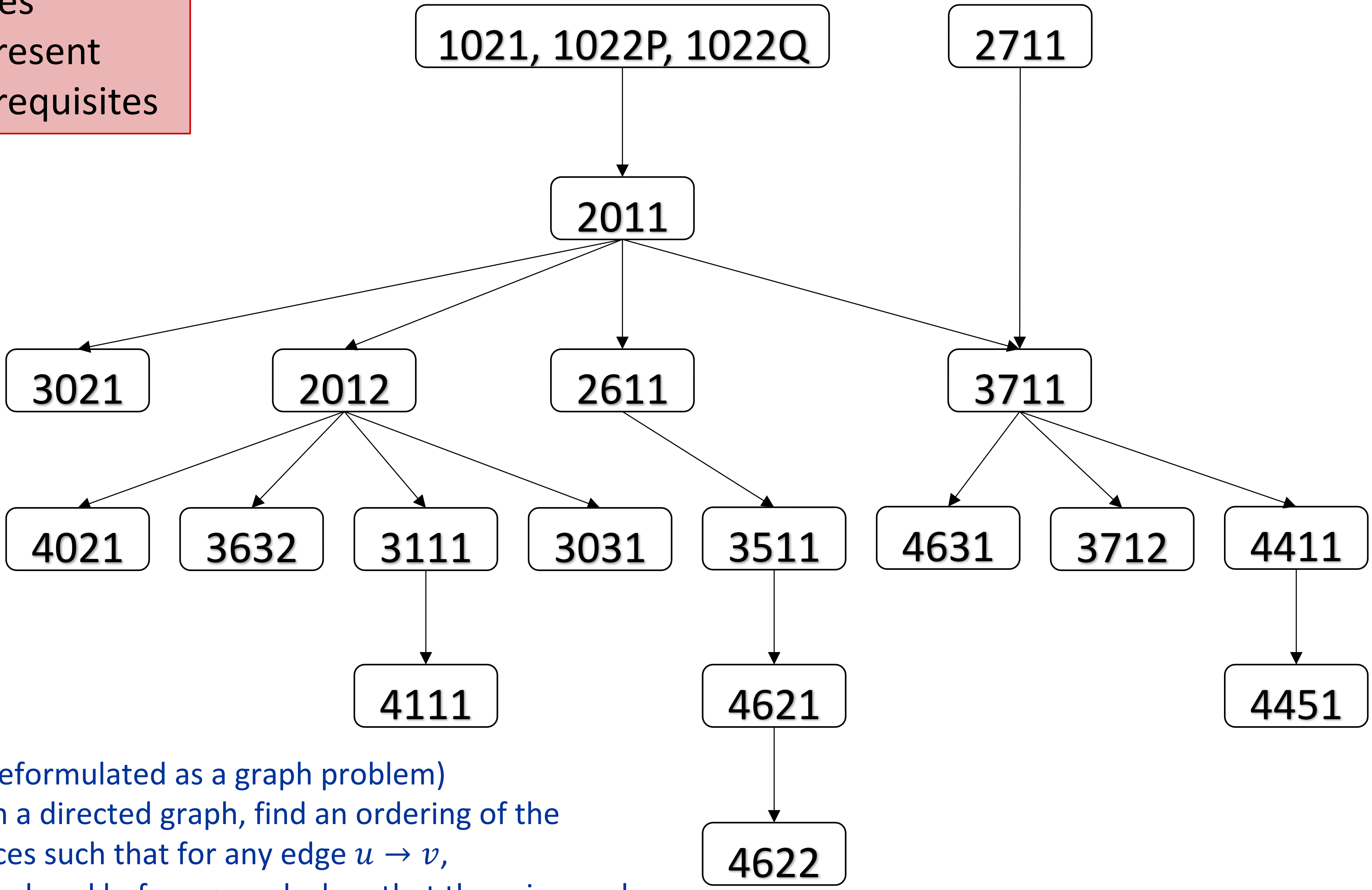
Q: (Reformulated as a graph problem)
Given a graph in which edges have weights (lengths), how to find a cycle with minimum total weight that includes all vertices?

A: Don't know.

- Don't have an algorithm that runs in polynomial time.
  (Conjecture is that such an algorithm doesn't exist.)
- This is actually equivalent to the P = NP problem (still open).

Partial COMP Course Dependency Chart

Edges represent prerequisites

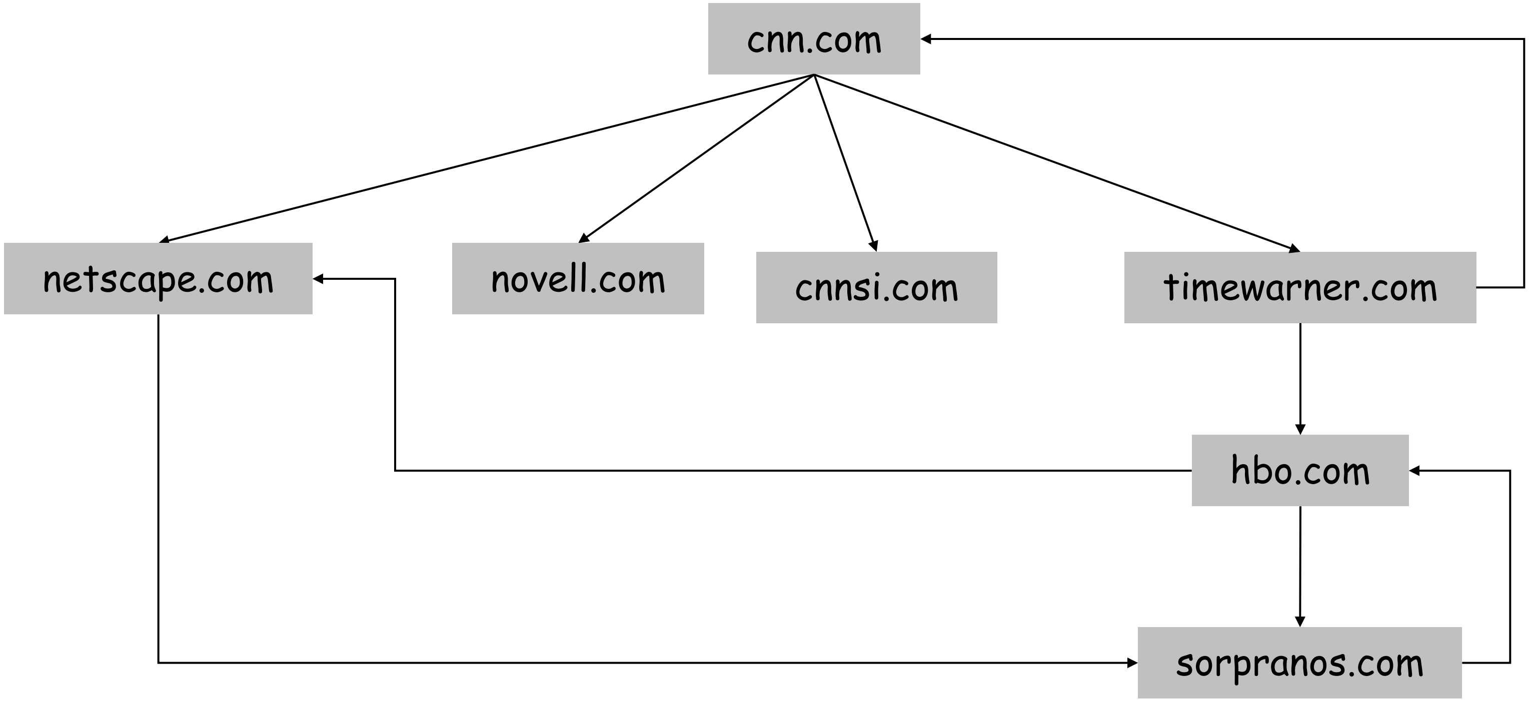


Q: (Reformulated as a graph problem)
Given a directed graph, find an ordering of the vertices such that for any edge $u \rightarrow v$, u is ordered before v, or declare that there is a cycle in the graph.

World Wide Web

Web graph.

- Node: web page.
- Edge: hyperlink from one page to another (directed).



Social Networks

Social network graph.

- Nodes: people.
- Edges: relationship between two people
  - Can be directed or undirected

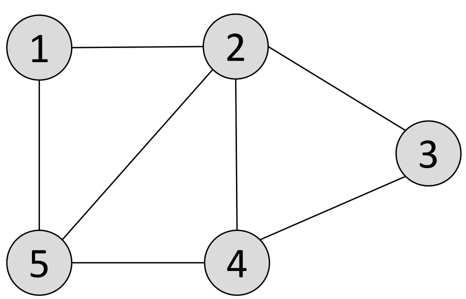
Reference: Valdis Krebs, http://www.firstmonday.org/issues/issue7_4/krebs

Undirected and Directed Graphs

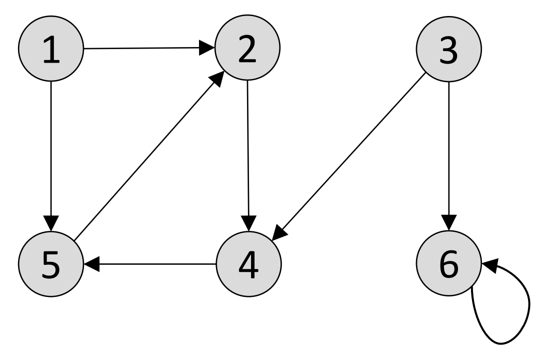
Graph. $G = (V, E)$

- V: set of nodes (vertices).
- E: set of edges between pairs of nodes.
- There are two different types of graphs: Undirected and Directed
- In most cases we assume simple (as opposed to multi) graphs, where there is at most 1 (2) edges between the same pair of nodes in undirected (directed) graphs.

Undirected graph



Directed graph



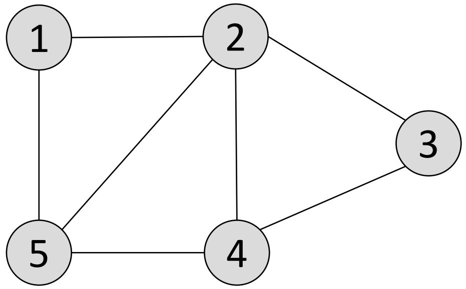
Undirected Graphs

Graph. $G = (V, E)$

- V: set of nodes (vertices).
- E: set of edges between pairs of nodes.
- Abusing notation, we also use V and E to denote the number of nodes and edges. We sometimes also use $n = |V|$, $m = |E|$.

Undirected graphs

- Edges have no specified "direction"
- Degree: $\deg(v) = \#$ edges touching v
- $\sum_{v \in V} \deg(v) = 2|E|$



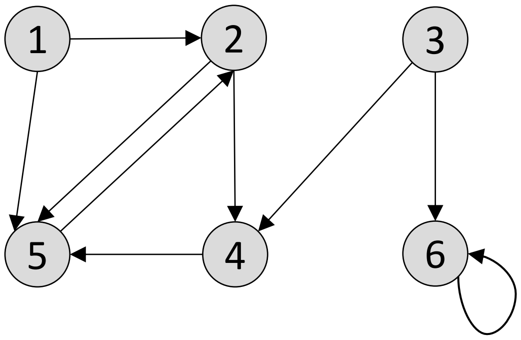
Directed Graphs

Graph. $G = (V, E)$

- V: set of nodes (vertices).
- E: set of edges between pairs of nodes.
- Abusing notation, we also use V and E to denote the number of nodes and edges. We sometimes also use $n = |V|$, $m = |E|$.

Directed graphs.

- Edges have directions
- If an edge exists in both directions, we will represent it using 2 edges in opposite directions
- Outdegree: $\deg^{out}(v)$ = # edges leaving v;
  Indegree: $\deg^{in}(v)$ = # edges entering v.
- $\sum_{v \in V} \deg^{out}(v) = \sum_{v \in V} \deg^{in}(v) = |E|$



Exercise on Number of Edges

What is the maximum number of edges in a **directed** graph G with $|V|$ nodes?

- Each node connects to $|V| - 1$ other nodes. Therefore the maximum number of edges is $|V|(|V| - 1)$.

What is the maximum number of edges in a **undirected** graph G with $|V|$ nodes?

- Based on the previous answer, $|V|(|V| - 1)/2$ because we replace 2 directed edges with an undirected one

Exercises on Node Degree

1. Show that if all nodes in a graph have degree 3, then the number of nodes $|V|$ must be even

- We have that the sum of all degrees must be equal to twice the number of edges $|E|$. Therefore, $3|V| = 2|E|$. For $3|V|$ to be even, $|V|$ must be even

2. Assume a graph, where all nodes have an odd degree. Show that $|V|$ must be even

- Similar to the previous, but now we do not know the degree of each node. Instead lets say that the degree of node v_i is: $d(v_i) = 2k_i + 1$, where k_i is an integer (e.g., if $d(v_i) = 1$, then $k_i = 0$, if $\mathrm{d}(v_i) = 3$, then $k_i = 1$, etc). Then

$$\sum_{i=1}^{|V|} d(v_i) = \sum_{i=1}^{|V|} (2k_i + 1) = |V| + 2\sum_{i=1}^{|V|} k_i = 2|E| \implies |V| = 2|E| - 2\sum_{i=1}^{|V|} k_i$$

- Since $|V|$ equals the difference of 2 even numbers, it must be also even

Exercise on Hand Shaking

Show that at a party, the number of guests who shake hands an odd number of times is even

- We construct the graph $G = (V, E)$ where each guest becomes a node. There is an edge between two vertices if and only if the corresponding guests shake hands at the party
- Equivalently, we want to prove that "in every graph, the number of nodes with odd degree is even".
- Let V_{odd} be the set of vertices with odd degree, and V_{even} the set of vertices with even degree: $V_{odd} \cup V_{even} = V$ and $V_{odd} \cap V_{even} = \emptyset$. We have

$$\sum_{i=1}^{|V|} d(v_i) = \sum_{i=1}^{|V_{odd}|} d(v_i) + \sum_{i=1}^{|V_{even}|} d(v_i) = 2|E| \implies \sum_{i=1}^{|V_{odd}|} d(v_i) = 2|E| - \sum_{i=1}^{|V_{even}|} d(v_i)$$

- The sum of even degrees must be even. Since the sum of odd degrees equals the difference of 2 even numbers, it must be also even. Using the same reasoning as the previous exercise, we have:

$$\sum_{i=1}^{|V_{odd}|} (2k_i + 1) = |V_{odd}| + 2 \sum_{i=1}^{|V_{odd}|} k_i \text{ is even} \implies |V_{odd}| \text{ is even}$$

More Exercises on Node Degrees

Q: In a group of 8 people, some of them shake hands. Is it possible that everyone shaked hands with a different number of people?

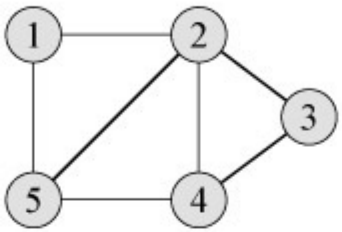
Solution: Everyone had between 0 and 7 handshakes. It is not possible that someone shook hands with everyone and someone else with no one. So, there must be at least 2 people with the same number of handshakes.

Q: In a simple, connected graph on 6 vertices, the degrees of 5 vertices are 1, 2, 3, 4, 5 respectively. What is the degree of the 6 th vertex?

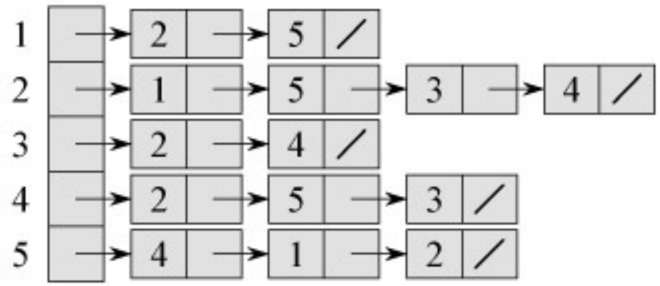
Solution 1: Let us call the 5 vertices with known degrees v_1, v_2, v_3, v_4, v_5, where $d(v_i) = i$. The degree of v_6 is unknown. Node v_5 is connected by an edge to every node, so the only neighbor of v_1 is v_5. Node v_4 is connected to every node except v_1. Therefore the two neighbors of v_2 are v_4 and v_5. For v_6, we know that it is connected by an edge to v_4 and v_5, and not connected to v_1 and v_2. The same is true for v_3. Since the degree of v_3 is 3, v_3 is connected by an edge to v_6, therefore the degree of v_6 is 3.

Solution 2: Since the sum of the degrees is even, the missing number has to be odd: 1, 3, or 5. Use the first half of the first solution. For v_6, we know that it is connected by an edge to v_4 and v_5 and not connected to v_1 and v_2, this rules out the degree being 5 or 1.

Graph Representation: Adjacency List and Adjacency Matrix



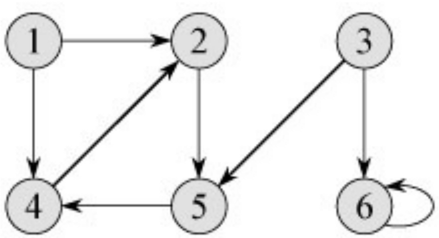
(a)



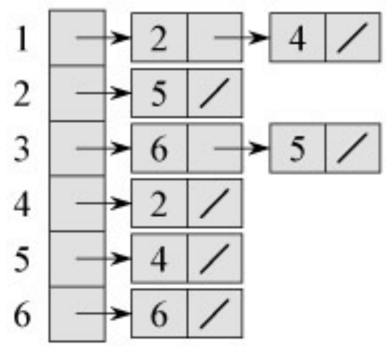
(b)

	1	2	3	4	5
1	0	1	0	0	1
2	1	0	1	1	1
3	0	1	0	1	0
4	0	1	1	0	1
5	1	1	0	1	0

(c)



(a)



(b)

	1	2	3	4	5	6
1	0	1	0	1	0	0
2	0	0	0	0	1	0
3	0	0	0	0	1	1
4	0	1	0	0	0	0
5	0	0	0	1	0	0
6	0	0	0	0	0	1

(c)

Graph Representation 2

Adjacency list.

- A node-indexed array of lists.
- Given node u, retrieving all neighbors in $\Theta(\deg(u))$ time
- Given u, v, checking if (u, v) is an edge takes $\Theta(\deg(u))$ time.
- Space: $\Theta(|V| + |E|)$.

Adjacency matrix.

- A $|V| \times |V|$ matrix.
- Given node u, retrieving all neighbors in $\Theta(|V|)$ time
- Given u, v, checking if (u, v) is an edge takes $O(1)$ time.
- Space: $\Theta(|V|^2)$.

Note:

- Adjacency lists are more commonly used, since most graphs are *sparse*.
- Usually, assume no self-loops and duplicated edges.
  - Thus, for undirected graphs, $0 \leq |E| \leq |V|(|V| - 1)/2$
  - For directed graphs, $0 \leq |E| \leq |V|(|V| - 1)$
- Can convert from one to the other in $\Theta(|V|^2)$ time.

Paths and Connectivity

Def. A path in a (directed or undirected) graph $G = (V, E)$ is a sequence P of nodes $v_1, v_2, \ldots, v_{k-1}, v_k$ such that (v_i, v_{i+1}) is an edge. The length of the path is $k-1$ (i.e., # edges in the path).

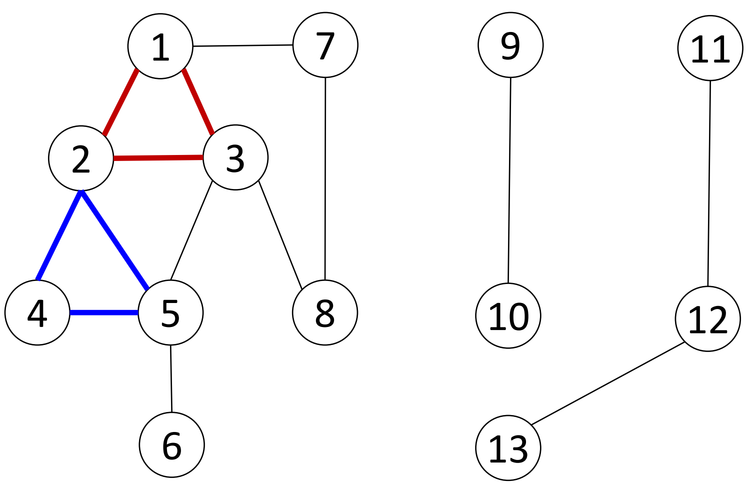
Def. A path is simple if all nodes are distinct.

Def. An undirected graph is connected if for every pair of nodes u and v, there is a path between u and v.

Theorem: For a connected graph, $|E| \geq |V| - 1$.

Def. A cycle is a path $v_1, v_2, \ldots, v_{k-1}, v_k$ in which $v_1 = v_k$, $k > 2$,

Def. A cycle is simple if the first $k-1$ nodes are all distinct.



2132 is a simple cycle.
2132542 is a nonsimple cycle.

Exercise on Network Connectivity

Q: Suppose in a wireless network of n mobile devices, each device is within communication range with at least $n/2$ other devices (assuming n is an even number). Show that all devices are connected.

Reformulate as a graph problem: Let $G = (V, E)$ be an undirected graph in which each node has degree $\geq n/2$. Show that G is connected.

Pf: Consider any two nodes $u, v \in V$. There are two cases:

- If edge $(u, v) \in E$, then u and v are connected.
- If edge $(u, v) \notin E$ then u, v must have a common neighbor, say w, because
  - V contains $n - 2$ nodes other than u and v.
  - u and v each have $\geq n/2$ neighbors among these $n - 2$ nodes.
- Thus, u and v have at least a common neighbor.
- The above argument holds for any two nodes u, v, so G is connected.

Q: If the threshold $n/2$ is changed to $n/2 - 1$, does the claim still hold?

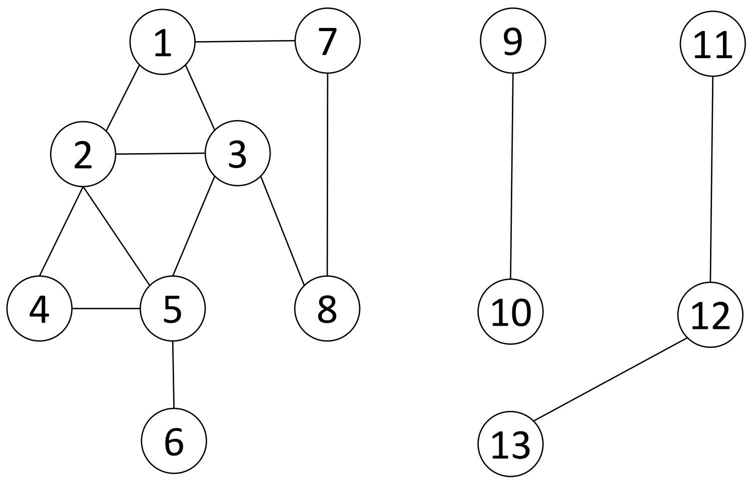
Connectivity and Shortest Path

s-t connectivity problem.
Given two nodes s and t, is there a path from s to t?

s-t shortest path problem.
Given two node s and t, what is the shortest path from s to t?

Def: The length of the path (in terms of number of edges) is the distance from s to t.

The problem can be defined on either an undirected or directed graph. If edges have weights, the distance is the sum of edge weights in the path.



- 6 & 1 are connected
- 6 & 9 are not connected
- The shortest path from 6 to 8 (6538) has length 3. Note that longer paths exist

Trees

Def. An undirected graph is a tree if it is connected and does not contain a cycle.

Def. An undirected graph is a forest if it does not contain a cycle (i.e., a collection of trees).

Theorem (simpler version of Theorem B.4 in textbook):

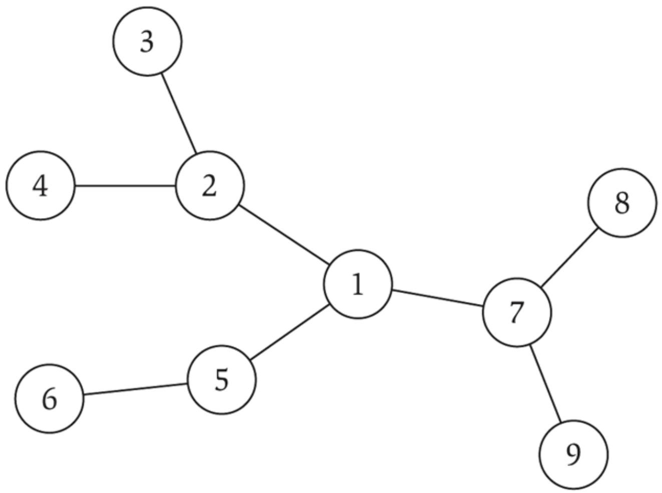
Let G be an undirected graph. Any two of the following statements imply the third (hence G is a tree).

(1) G is connected.

(2) G does not contain a cycle.

(3) $|E| = |V| - 1$.

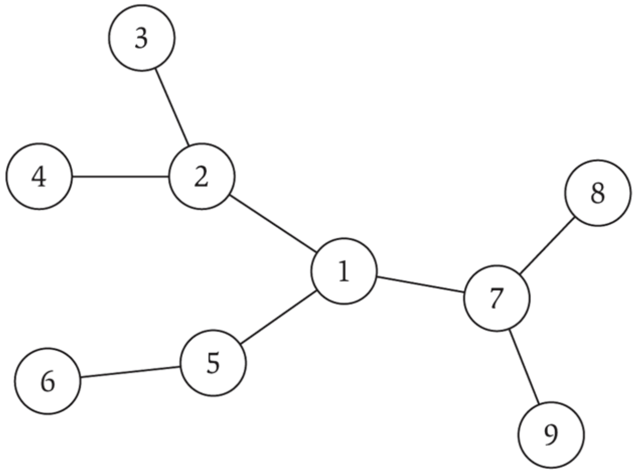
Proof: (Omitted)



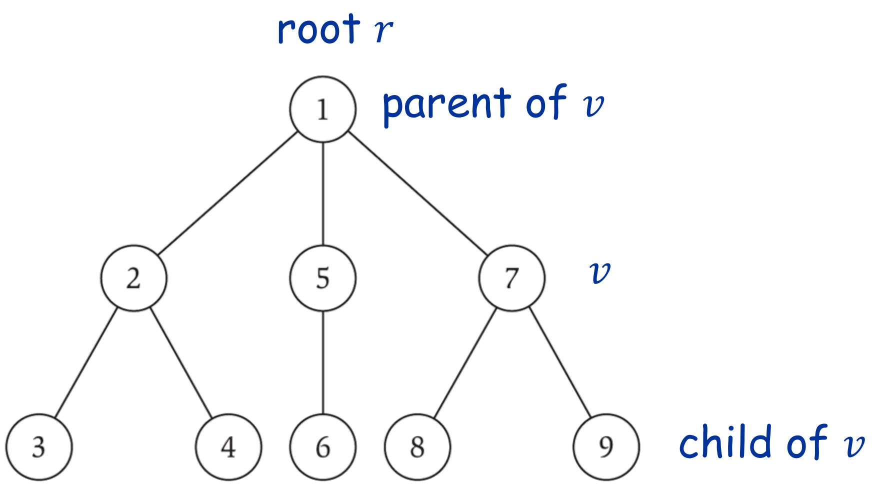
$|E| = 8, |V| = 9$

Rooted Trees

Rooted tree. Given a tree T, choose a root node r and orient each edge away from r.



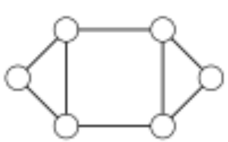
a tree



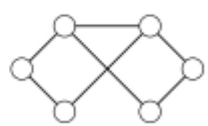
the same tree, rooted at 1

Exercise on Graph Properties (optional)

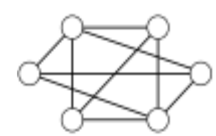
- Graph coloring is the procedure of assignment of colors to each vertex of a graph G such that no adjacent vertices get same color. The smallest number of colors required to color a graph G is called its chromatic number of that graph.
- A bipartite graph is a graph whose vertices can be divided into two disjoint sets U and V, so that each edge connects a vertex in U to one in V.
  - Equivalently, a bipartite graph is a graph that does not contain any odd-length cycles.
  - What is the chromatic number of bipartite graphs?
- A planar graph is a graph that can be drawn on the plane in such a way that its edges intersect only at their endpoints (i.e., they do not cross).



Non-Bipartite
Planar



Bipartite
Planar



Non-Bipartite
Planar