Notes on Matrix Analysis

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December 25, 2023

Abstract

The note is based on [Tao11].

1 The Geometry of Eigenvalues

1.1 mini-max formula of eigenvalue

Theorem 1.1 (Courant-Fischer min-max theorem)**.** *Let A be an $n \times n$ Hermitian matrix. Then we have*

$$\lambda_i(A) = \sup_{\dim(V)=i} \bar{f}(V) \stackrel{\text{def}}{=} \sup_{\dim(V)=i} \inf_{v \in V:|v|=1} v^* A v \tag{1.1}$$

1.2 Partial Trace

Definition 1.2 (Partial Trace)**.** *Given an $n \times n$ Hermitian matrix $\mathbf{A}$ and an m-dimensional subspace V of $\mathbb{C}^n$, we difine the partial trace*

$$\operatorname{tr}(\mathbf{A} \downharpoonright_V) \stackrel{\text{def}}{=} \sum_{i=1}^{m} v_i^* \mathbf{A} v_i, \tag{1.2}$$

where $v_1, \ldots v_m$ is any orthonormal basis of V.

Theorem 1.3 (Extremal partial trace)**.** *Given an $n \times n$ Hermitian matrix $\mathbf{A}$. Then for any $1 \leq k \leq n$, one has*

$$\sum_{j=n-k+1}^{n} \lambda_j(\mathbf{A}) \leq \operatorname{tr}(\mathbf{A} \downharpoonright_V) \leq \sum_{j=1}^{k} \lambda_j(\mathbf{A}). \tag{1.3}$$

It has a direct corollary as follows, since $a_{i_1 i_1} + \cdots + a_{i_k i_k} = \operatorname{tr}(\mathbf{A} \downharpoonright_{\langle e_{i_1}, \ldots, e_{i_k} \rangle})$.

Corollary 1.4 (Schur-Horn inequalities)**.**

$$\sum_{j=n-k+1}^{n} \lambda_j(\mathbf{A}) \leq a_{i_1 i_1} + \cdots + a_{i_k i_k} \leq \sum_{j=1}^{k} \lambda_j(\mathbf{A}). \tag{1.4}$$

The result can also be derived by an application of the Birkhoff's theorem for doubly stochastic matrices, implying that $\operatorname{diag}(\mathbf{A})$ *lies in the permutahedron of $\lambda_1 \ldots \lambda_n$.*

Theorem 1.5 (Schur-Horn theorem, 1954)**.** *The orbit of* $\operatorname{diag}(\mathbf{UAU}^*)$ *is exactly the permutahedron of $\lambda_1 \ldots \lambda_n$.*

proof sketch. It suffices to consider the $n = 2$ case. ∎

Since we provide a geometric description for the sum of top-k eigenvalues, we can naturally prove the Ky Fan inequality.

Theorem 1.6 (Ky Fan inequality)**.**

$$\sum_{j=1}^{k} \lambda_j(\mathbf{A}+\mathbf{B}) \leq \sum_{j=1}^{k} \lambda_j(\mathbf{A}) + \sum_{j=1}^{k} \lambda_j(\mathbf{B}) \tag{1.5}$$

Proof. Since

$$\mathrm{tr}(\mathbf{A}+\mathbf{B} \downharpoonright_V) = \mathrm{tr}(\mathbf{A} \downharpoonright_V) + \mathrm{tr}(\mathbf{B} \downharpoonright_V) \leq \sum_{j=1}^{k} \lambda_j(\mathbf{A}) + \sum_{j=1}^{k} \lambda_j(\mathbf{B}) \tag{1.6}$$

holds for any subspace V. Let V maximize the left hand side give the desired result. □

The extremal partial trace provides a geometric description the sum of top-k eigenvalues. Such description can be generalized to the sum of any k eigenvalues

Theorem 1.7 (Wielandt minimax formula)**.** *Let $1 \leq i_1 < \ldots i_k \leq n$ be integers. Define a partial flag to be a nested collection $V_1 \subset \cdots \subset V_k$ of subspaces of $\mathbb{C}^n$ such that $\dim(V_j) = i_j$ for all $1 \leq j \leq k$. Define the associated* Schubert variety *$X(V_1, \ldots, V_k)$ to be the collection of all k-dimensional subspaces W such that* $\dim(W \cap V_j) \leq j$*. Then for any $n \times n$ Hermitian matrix $\mathbf{A}$,*

$$\lambda_{i_1}(\mathbf{A}) + \cdots + \lambda_{i_k}(\mathbf{A}) = \sup_{(V_1,\ldots,V_k)\in\mathcal{V}_{i_1,\ldots,i_k}} \inf_{W\in X(V_1,\ldots,V_k)} \mathrm{tr}(\mathbf{A} \downharpoonright_W). \tag{1.7}$$

Proof. **Part 1:** We prove that

$$\lambda_{i_1}(\mathbf{A}) + \cdots + \lambda_{i_k}(\mathbf{A}) \geq \inf_{W\in X(V_1,\ldots,V_k)} \mathrm{tr}(\mathbf{A} \downharpoonright_W) \tag{1.8}$$

for $\forall (V_1, \ldots, V_k) \in \mathcal{V}_{i_1,\ldots,i_k}$. First we simplify the case to $i_1 = 1$. Second we find $W' \in X(V_2, \ldots, V_k)$ that minimize $\mathrm{tr}(\mathbf{A} \downharpoonright_{W'})$ which is at most $\lambda_{i_2}(\mathbf{A}) + \cdots + \lambda_{i_k}(\mathbf{A})$(using induction on k). Thirdly, we check and find the minimal j such that $\dim(W' \cap V_j) < j$, then we add $v \in \mathcal{P}_{W'^{\perp}} V_j$ into W' to be W, which is the construction we require.

Part 2: Let $\mathbf{A}$ be diagonalized and $V_j = \langle e_1, \ldots, e_{i_j} \rangle$. Now we prove that

$$\mathrm{tr}(\mathbf{A} \downharpoonright_W) \geq \lambda_{i_1}(\mathbf{A}) + \cdots + \lambda_{i_k}(\mathbf{A}) \tag{1.9}$$

for $\forall W \in X(V_1, \ldots, V_k)$. All we have to do is construct an orthogonal basis of W such that $v_j \in W \cap V_j \cap \mathrm{span}(v_1, \ldots, v_{j-1})^{\perp}$, that yields 1.9. □