CrossEntropyLoss, KL Divergence, Entropy, and Logits

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1 Introduction

This document explores CrossEntropyLoss, Kullback-Leibler (KL) Divergence, entropy $H(P)$, and the role of logits in machine learning, with examples for two-class and three-class classification, and an application to LLM distillation including temperature effects.

2 CrossEntropyLoss and KL Divergence

CrossEntropyLoss measures the difference between a true distribution P and a predicted distribution Q:

$$H(P, Q) = -\sum_i P(i) \log Q(i),$$

while KL Divergence quantifies how Q diverges from P:

$$D_{KL}(P||Q) = \sum_i P(i) \log \left(\frac{P(i)}{Q(i)} \right).$$

2.1 Relationship: $H(P, Q) = H(P) + D_{KL}(P||Q)$

This is a mathematical identity:

$$D_{KL}(P||Q) = \sum_i P(i) (\log P(i) - \log Q(i)) = -H(P) + H(P, Q),$$

$$H(P, Q) = H(P) + D_{KL}(P||Q),$$

where $H(P)$ is the entropy of P, and D_{KL} is the additional cost of using Q.

2.2 Two-Class Example

True label: class 1, $P = [0, 1]$, predicted $Q = [0.3, 0.7]$.

2.2.1 CrossEntropyLoss

$$H(P,Q) = -[0 \cdot \log(0.3) + 1 \cdot \log(0.7)] = -\log(0.7) \approx 0.3567$$

2.2.2 KL Divergence

$$D_{KL}(P||Q) = 1 \cdot \log\left(\frac{1}{0.7}\right) = 0.3567$$

Since $H(P) = 0$, $H(P,Q) = D_{KL}(P||Q)$.

3 Entropy $H(P)$

Entropy measures uncertainty in P:

$$H(P) = -\sum_i P(i) \log P(i)$$

3.1 Two-Class Example

For $P = [0,1]$:

$$H(P) = -[0 \cdot \log(0) + 1 \cdot \log(1)] = 0$$

(Convention: $0 \log(0) = 0$).

For $P = [0.5, 0.5]$:

$$H(P) = -2 \cdot (0.5 \log(0.5)) \approx 0.6931$$

3.2 Three-Class Example

For $P = [0,1,0]$:

$$H(P) = -[0 \cdot \log(0) + 1 \cdot \log(1) + 0 \cdot \log(0)] = 0$$

For $P = [0.33, 0.33, 0.33]$:

$$H(P) = -3 \cdot (0.33 \log(0.33)) \approx 1.0974$$

4 Role of Logits

Logits are raw scores, converted to probabilities via softmax:

$$Q(i) = \frac{e^{z_i}}{\sum_j e^{z_j}}.$$

They are used directly in CrossEntropyLoss.

4.1 Two-Class Logit Example

Logits: $z = [1.0, 2.0]$, true label: class 1 ($P = [0, 1]$).

4.1.1 Softmax

$$e^{1.0} \approx 2.718, \quad e^{2.0} \approx 7.389$$

$$\text{Sum} = 10.107$$

$$Q = [0.269, 0.731]$$

4.1.2 CrossEntropyLoss

$$\text{Loss} = -\log(0.731) \approx 0.313$$

If $z = [-1.0, 1.0]$:

$$Q = [0.119, 0.881], \quad \text{Loss} = -\log(0.881) \approx 0.127$$

4.2 Three-Class Logit Example

Logits: $z = [2.0, 1.0, -1.0]$, true label: class 1 ($P = [0, 1, 0]$).

4.2.1 Softmax

$$e^{2.0} \approx 7.389, \quad e^{1.0} \approx 2.718, \quad e^{-1.0} \approx 0.368$$

$$\text{Sum} = 10.475$$

$$Q = [0.705, 0.259, 0.035]$$

4.2.2 CrossEntropyLoss

$$\text{Loss} = -\log(0.259) \approx 1.349$$

If $z = [0.5, 2.5, -0.5]$:

$$Q = [0.114, 0.844, 0.042], \quad \text{Loss} = -\log(0.844) \approx 0.169$$

5 CrossEntropyLoss with Non-Zero Entropy: LLM Distillation Example

In LLM distillation, P is a soft distribution (e.g., from a teacher model), so $H(P) \neq 0$. CrossEntropyLoss remains applicable.

5.1 Three-Class Distillation Example

Teacher's $P = [0.7, 0.2, 0.1]$, student's $Q = [0.6, 0.3, 0.1]$.

5.1.1 Entropy $H(P)$

$$\begin{aligned} H(P) &= -[0.7\log(0.7) + 0.2\log(0.2) + 0.1\log(0.1)] \\ &\approx -[0.7 \cdot (-0.3567) + 0.2 \cdot (-1.6094) + 0.1 \cdot (-2.3026)] \\ &\approx 0.8019 \end{aligned}$$

5.1.2 CrossEntropyLoss

$$\begin{aligned} H(P,Q) &= -[0.7\log(0.6) + 0.2\log(0.3) + 0.1\log(0.1)] \\ &\approx -[0.7 \cdot (-0.5108) + 0.2 \cdot (-1.2040) + 0.1 \cdot (-2.3026)] \\ &\approx 0.8287 \end{aligned}$$

5.1.3 KL Divergence

$$D_{KL}(P||Q) = H(P,Q) - H(P) \approx 0.8287 - 0.8019 = 0.0268$$

Minimizing $H(P,Q)$ reduces D_{KL}.

5.2 Temperature in Distillation

In distillation, a temperature T softens distributions by scaling logits before softmax:

$$P(i) = \frac{e^{z_i/T}}{\sum_j e^{z_j/T}}.$$

Higher T flattens probabilities, emphasizing softer targets.

5.2.1 Example with Temperature

Teacher logits: $z_P = [2.0, 1.0, 0.0]$, student logits: $z_Q = [1.5, 1.0, 0.5]$, $T = 2$.

- Teacher P:

$$e^{2.0/2} = e^1 \approx 2.718, \quad e^{1.0/2} \approx 1.649, \quad e^{0.0/2} = 1$$

$$\text{Sum} = 5.367$$

$$P = [0.506, 0.307, 0.186]$$

- Student Q:

$$e^{1.5/2} \approx 2.117, \quad e^{1.0/2} \approx 1.649, \quad e^{0.5/2} \approx 1.284$$

$$\text{Sum} = 5.050$$

$$Q = [0.419, 0.327, 0.254]$$

5.2.2 CrossEntropyLoss with Temperature

$$\begin{aligned}
H(P,Q) &= -[0.506\log(0.419) + 0.307\log(0.327) + 0.186\log(0.254)] \\
&\approx -[0.506\cdot(-0.870) + 0.307\cdot(-1.118) + 0.186\cdot(-1.371)] \\
&\approx 1.038
\end{aligned}$$

At $T = 1$, $P = [0.665, 0.245, 0.090]$, $Q = [0.558, 0.245, 0.197]$, $H(P,Q) \approx 0.842$. Higher T increases loss but smooths learning.

6 Conclusion

The identity $H(P,Q) = H(P) + D_{KL}(P||Q)$ holds universally. For one-hot P, $H(P) = 0$, so $H(P,Q) = D_{KL}(P||Q)$. In distillation, $H(P) \neq 0$, and temperature adjusts softness, yet CrossEntropyLoss remains effective.