Taylor Series

A Taylor Series is an expansion of some function into an **infinite sum of terms**, where each term has a larger exponent like x, x^2, x^3, etc.

Example: The Taylor Series for e^x

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \dots$$

says that the function: e^x

is equal to the infinite sum of terms: $1 + x + x^2/2! + x^3/3! + \dots$ etc

(Note: ! is the Factorial Function.)

Does it really work? Let's try it:

Example: e^x for x=2

Well, we already know the answer is $\mathbf{e^2} = 2.71828... \times 2.71828... =$ **7.389056...**

But let's try more and more terms of our infinte series:

Terms	Result
1+2	**3**

$1+2+\frac{2^2}{2!}$	**5**
$1+2+\frac{2^2}{2!}+\frac{2^3}{3!}$	**6.3333...**
$1+2+\frac{2^2}{2!}+\frac{2^3}{3!}+\frac{2^4}{4!}$	**7**
$1+2+\frac{2^2}{2!}+\frac{2^3}{3!}+\frac{2^4}{4!}+\frac{2^5}{5!}$	**7.2666...**
$1+2+\frac{2^2}{2!}+\frac{2^3}{3!}+\frac{2^4}{4!}+\frac{2^5}{5!}+\frac{2^6}{6!}$	**7.3555...**
$1+2+\frac{2^2}{2!}+\frac{2^3}{3!}+\frac{2^4}{4!}+\frac{2^5}{5!}+\frac{2^6}{6!}+\frac{2^7}{7!}$	**7.3809...**
$1+2+\frac{2^2}{2!}+\frac{2^3}{3!}+\frac{2^4}{4!}+\frac{2^5}{5!}+\frac{2^6}{6!}+\frac{2^7}{7!}+\frac{2^8}{8!}$	**7.3873...**

It starts out really badly, but it then gets better and better!

Try using "**2^n/fact(n)**" and **n=0** to 20 in the Sigma Calculator and see what you get.

Here are some common Taylor Series:

Taylor Series expansion	**As Sigma Notation**
$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$	$\sum_{n=0}^{\infty} \frac{x^n}{n!}$
$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$	$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$

cos x $= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots$ $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots \quad \text{for } |x| < 1 \quad \sum_{n=0}^{\infty} x^n$$

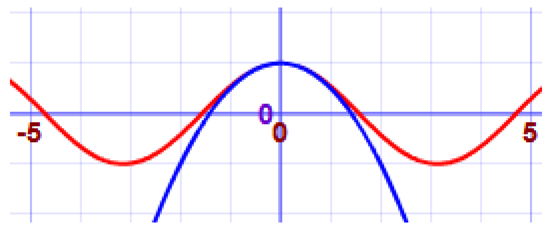
(There are many more.)

Approximations

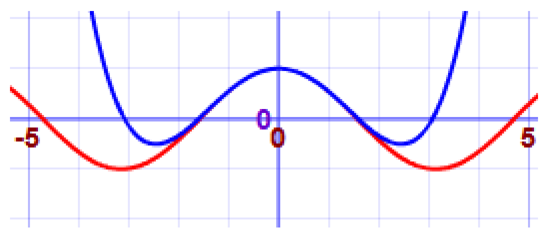
We can use the first few terms of a Taylor Series to get an approximate value for a function.

Here we show better and better approximations for **cos(x)**. The red line is **cos(x)**, the blue is the approximation (try plotting it yourself) :

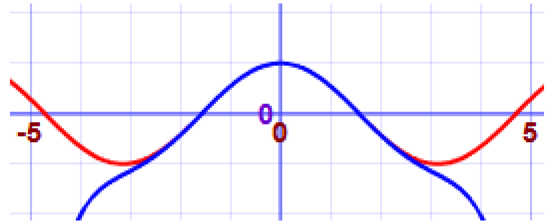
$1 - x^2/2!$



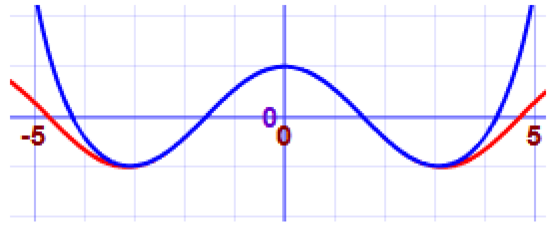
$1 - x^2/2! + x^4/4!$



$1 - x^2/2! + x^4/4! - x^6/6!$



$1 - x^2/2! + x^4/4! - x^6/6! + x^8/8!$



You can also see the Taylor Series in action at Euler's Formula for Complex Numbers.

What is this Magic?

How can we turn a function into a series of power terms like this?

Well, it isn't really magic. First we say we **want** to have this expansion:

$$f(x) = c_0 + c_1(x\text{-}a) + c_2(x\text{-}a)^2 + c_3(x\text{-}a)^3 + \ldots$$

Then we choose a value "a", and work out the values c_0 , c_1 , c_2 , ... etc

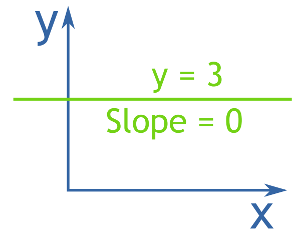
And it is done using **derivatives** (so we must know the derivative of our function)

Quick review: a derivative gives us the slope of a function at any point.

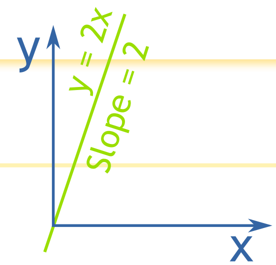
These basic derivative rules can help us:

- The derivative of a constant is **0**

- The derivative of **ax** is **a** (example: the derivative of **2x** is **2**)
- The derivative of $\mathbf{x^n}$ is $\mathbf{nx^{n-1}}$ (example: the derivative of $\mathbf{x^3}$ is $\mathbf{3x^2}$)



We will use the little mark ′ to mean "derivative of".



OK, let's start:

To get c_0, choose x=a so all the (x-a) terms become zero, leaving us with:

$$f(a) = c_0$$

So $\mathbf{c_0 = f(a)}$

To get c_1, take the derivative of f(x):

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \ldots$$

With x=a all the (x-a) terms become zero:

$$f'(a) = c_1$$

So $\mathbf{c_1 = f'(a)}$

To get c_2, do the derivative again:

$$f''(x) = 2c_2 + 3\times2\times c_3(x\text{-}a) + \ldots$$

With x=a all the (x-a) terms become zero:

$$f''(a) = 2c_2$$

So $\mathbf{c_2 = f''(a)/2}$

In fact, a pattern is emerging. Each term is

- the next higher derivative ...
- ... divided by all the exponents so far multiplied together (for which we can use factorial notation, for example 3! = 3×2×1)

And we get:

$$f(x) = f(a) + \frac{f'(a)}{1!}(x\text{-}a) + \frac{f''(a)}{2!}(x\text{-}a)^2 + \frac{f'''(a)}{3!}(x\text{-}a)^3 + \ldots$$

Now we have a way of finding our own Taylor Series:

For each term: take the next derivative, divide by n!, multiply by $(x\text{-}a)^n$

Example: Taylor Series for cos(x)

Start with:

$$f(x) = f(a) + \frac{f'(a)}{1!}(x\text{-}a) + \frac{f''(a)}{2!}(x\text{-}a)^2 + \frac{f'''(a)}{3!}(x\text{-}a)^3 + \ldots$$

The derivative of **cos** is **−sin**, and the derivative of **sin** is **cos**, so:

- f(x) = cos(x)
- f'(x) = −sin(x)
- f''(x) = −cos(x)
- f'''(x) = sin(x)
- etc...

And we get:

$$\cos(x) = \cos(a) - \frac{\sin(a)}{1!}(x\text{-}a) - \frac{\cos(a)}{2!}(x\text{-}a)^2 + \frac{\sin(a)}{3!}(x\text{-}a)^3 + \ldots$$

Now put **a=0**, which is nice because **cos(0)=1** and **sin(0)=0**:

$$\cos(x) = 1 - \frac{0}{1!}(x\text{-}0) - \frac{1}{2!}(x\text{-}0)^2 + \frac{0}{3!}(x\text{-}0)^3 + \frac{1}{4!}(x\text{-}0)^4 + \ldots$$

Simplify:

$$\cos(x) = 1 - x^2/2! + x^4/4! - \ldots$$

Try that for sin(x) yourself, it will help you to learn.

Or try it on another function of your choice.

The key thing is to know the derivatives of your function f(x).

Note: A **Maclaurin Series** is a Taylor Series where **a=0**, so all the examples we have been using so far can **also** be called Maclaurin Series.

Question 1 Question 2 Question 3 Question 4 Question 5
Question 6 Question 7 Question 8 Question 9 Question 10