The low-dimension hypothesis implies higher-order interactions in complex systems

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Complex systems are often modeled with high-dimensional nonlinear dynamics on complex networks [1] [Fig. 1 (a)]. To get insights on their emergent phenomena, it is typically assumed [2] without a precise statement that these dynamics can be reduced to a few number of equations involving a low-rank matrix that describes the network —we call it the low-dimension hypothesis. We verify the hypothesis for real complex networks of different origins by showing that their effective rank is significantly lower than their actual rank [Fig. 1 (b)]. We then introduce a dimension reduction for general dynamical systems on networks that gives an optimal low-dimensional dynamics of observables [Fig. 1 (c)]. We demonstrate that higher-order interactions [3] between the observables naturally emerge from the dimension reduction. Spectral upper-bounds on the errors of the low-dimensional dynamics and numerical simulations for dynamics on real networks finally provide conditions for exact dimension reduction and intuitions that support the low-dimension hypothesis of complex systems [Fig. 1 (d)].

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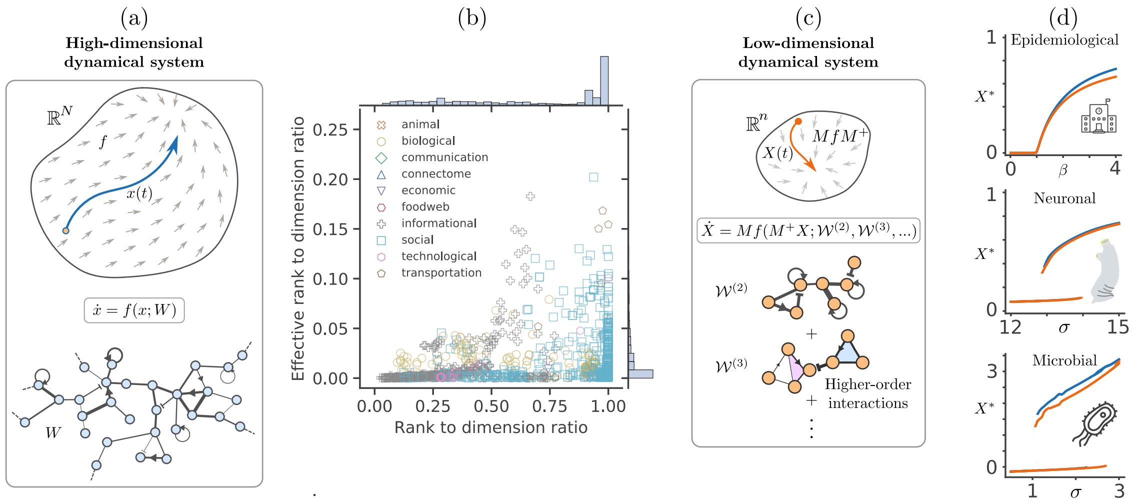

FIG. 1. (a) A N-dimensional dynamics of vector field f depending on a network W and some trajectory $x(t)$. (b) The effective (stable) rank $\|W\|_F^2/\|W\|_2^2$ divided by N against the rank divided by N of real networks of different origins. (c) A n-dimensional dynamics ($n < N$) of vector field MfM^+ where M is a $n \times N$ reduction (lumping) matrix that defines the linear observable $X = Mx$. Depending on the nonlinear terms of f, the optimal vector field depends on different order r tensors denoted $\mathcal{W}^{(r)}$. (d) The stable bifurcation branches of three different dynamics where the blue curves are the equilibrium points of the high-dimensional dynamics while the orange curves are the ones of the reduced dynamics. Top: Quenched-mean field Susceptible-Infected-Susceptible dynamics on a undirected network of high school contacts. The complete dynamics is of dimension $N = 327$ and the reduced dynamics has dimension $n = 1$. Center: Excitatory Wilson-Cowan dynamics on the weighted and directed connectome of *Ciona intestinalis*. $N = 213$, $n = 20$. Bottom: Microbial dynamics on a signed, weighted, and directed gut microbiome network [4]. $N = 838$, $n = 80$.