NERCCS 2022

The low-dimension hypothesis implies higher-order interactions in complex systems

Vincent Thibeault, Antoine Allard, Patrick Desrosiers

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Dynamica

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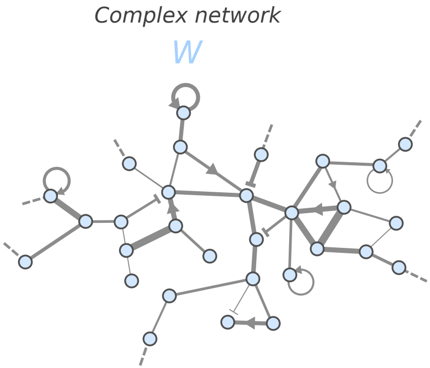
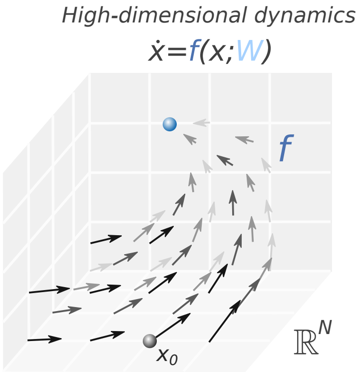
Complex systems : high dimension and emergent phenomena





A low-dimensional description of a high-dimensional complex system ? Paradox ?

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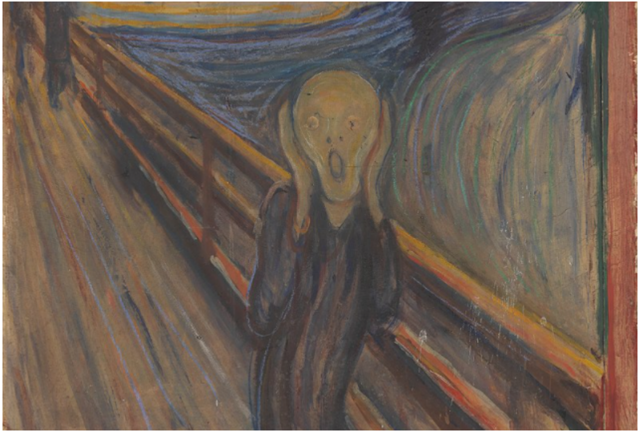


"The Scream of Dimensionality"

Nature Vol. 261 June 10 1976 459

review article

Simple mathematical models with very complicated dynamics

Robert M. May*

E.g. : Logistic equations

First-order difference equations arise in many contexts in the biological, economic and social sciences. Such equations, even though simple and deterministic, can exhibit a surprising array of dynamical behaviour, from stable points, to a bifurcating hierarchy of stable cycles, to apparently random fluctuations. There are consequently many fascinating problems, some concerned with delicate mathematical aspects of the fine structure of the trajectories, and some concerned with the practical implications and applications. This is an interpretive review of them.

Statistical physics and biology

GIORGIO PARISI

The relationship between biology and physics has often been close and, at times, uneasy. During this century many physicists have moved to work in biology. Amongst the most famous are Francis

have a satisfactory formulation of the laws.

However, a knowledge of the laws that govern the behaviour of the constituent elements of the system does not necessarily imply an understanding of the

About spin-glasses:

enormous richness and complexity of such an apparently simple system. A more detailed description would take us

THE GENERAL AND LOGICAL THEORY OF AUTOMATA

JOHN VON NEUMANN

The Institute for Advanced Study

mind. The natural systems are of enormous complexity, and it is clearly necessary to subdivide the problem that they represent into several parts.

Neural Networks, Vol. 6, pp. 801–806, 1993

0893-6080/93 $6.0(

Approximation of Dynamical Systems by Continuous Time Recurrent Neural Networks

KEN-ICHI FUNAHASHI AND YUICHI NAKAMURA

Toyohashi University of Technology

(*Received* 16 *March* 1992; *revised and accepted* 10 *November* 1992)

Abstract—*In this paper, we prove that any finite time trajectory of a given n-dimensional dynamical system can be approximately realized by the internal state of the output units of a continuous time recurrent neural network with n output units, some hidden units, and an appropriate initial condition. The essential idea of the proof is to embed*

Many random network models have a random matrix representation of the form

$$W \approx A + \underbrace{X}_{\text{noise}}$$

where A is some ***low-rank**** matrix :

1. $\mathcal{G}(N,p)$ *model* : rank 1, homogeneous ;
2. *Chung-Lu model* : rank 1, inhomogeneous ;
3. *Stochastic block model* with q communities : rank q ;
4. *Soft configuration model* : rank ~ 1, inhomogeneous ;
5. ...
6. *Many maximum-entropy random graph models* (e.g. ERGM)

* Rank : number of linearly independent rows/columns of a matrix

PHYSICAL REVIEW X **9,** 031050 (2019)

Exact Rank Reduction of Network Models

Eugenio Valdano[1] and Alex Arenas[2]

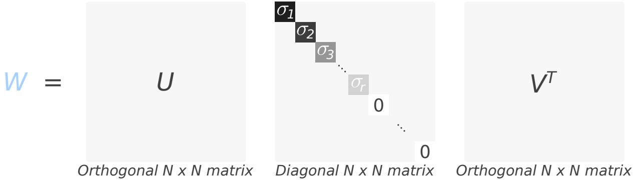
[1]*Center for Biomedical Modeling, The Semel Institute for Neuroscience and Human Behavior, David Geffen School of Medicine, 760 Westwood Plaza, University of California Los Angeles, Los Angeles, California 90024, USA*

[2]*Departament d'Enginyeria Informàtica i Matemàtiques, Universitat Rovira i Virgili, 43007 Tarragona, Spain*

However, we argue that r must be much smaller than the size of the system ($r \ll n$), as useful physical models are usually designed to depend on few—fundamental—parameters, compared to the size and complexity of the system under study. For this reason, we use this decomposition to classify and easily solve large network models using their low-rank linear algebraic structure.

Can we verify this widespread assumption for real complex networks?

Singular Value Decomposition (SVD) and effective rank



Singular Value Decomposition (SVD) and effective rank

$$W = U \begin{pmatrix} \sigma_1 & & & & & & \\ & \sigma_2 & & & & & \\ & & \sigma_3 & & & & \\ & & & \ddots & & & \\ & & & & \sigma_r & & \\ & & & & & 0 & \\ & & & & & & \ddots \\ & & & & & & & 0 \end{pmatrix} V^T \simeq U_n \begin{pmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \end{pmatrix} V_n^T$$

Orthogonal N x N matrix — *Diagonal N x N matrix* — *Orthogonal N x N matrix* — *N x n* — *n x n* — *n x N*

Optimal low-rank approximation
(Eckart-Young theorem)
(Exact for n = r)

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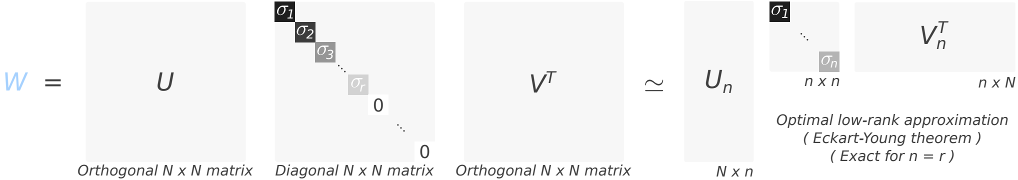
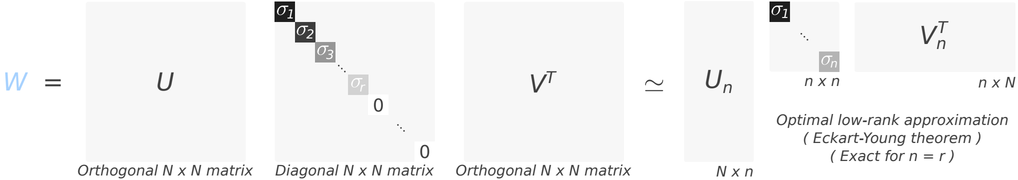


***Effective rank**: a number that indicate how many **singular values** are **significant**.*

Singular Value Decomposition (SVD) and effective rank



Effective rank: *a number that indicate how many* ***singular values*** *are* ***significant***.

E.g. : Stable rank $\quad srank(W) = \sum_{n=1}^{r} \sigma_n^2 / \sigma_1^2$

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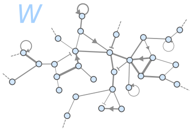
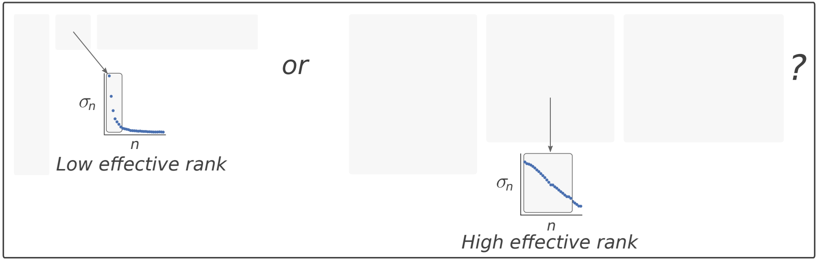
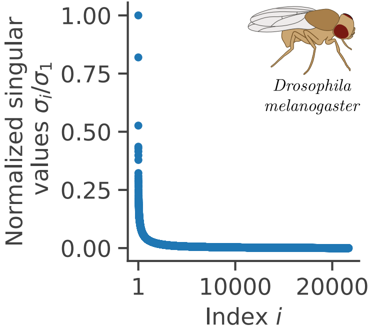
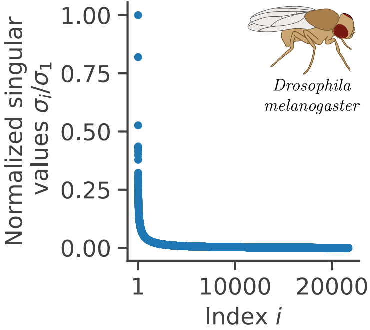
Complex network

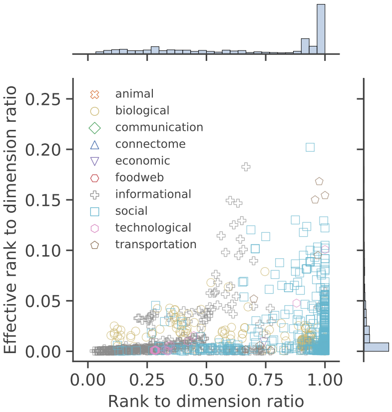




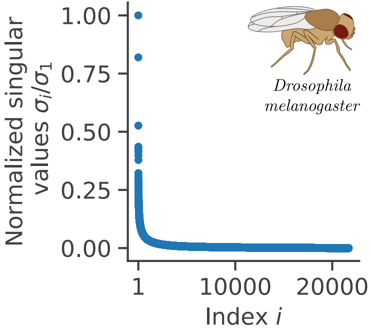
Experimental result

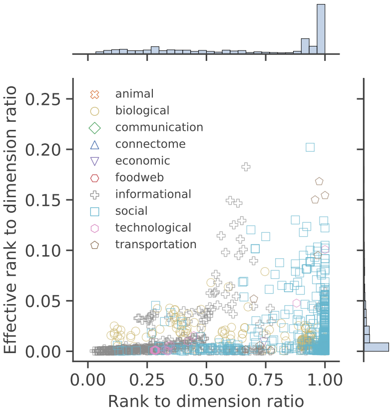



Normalized singular values σ_i/σ_1
1.00
0.75
0.50
0.25
0.00
1
10000
20000
Index i
Drosophila melanogaster

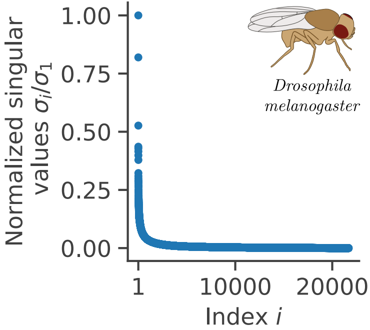

Effective rank to dimension ratio
0.25
0.20
0.15
0.10
0.05
0.00
0.00
0.25
0.50
0.75
1.00
Rank to dimension ratio
animal
biological
communication
connectome
economic
foodweb
informational
social
technological
transportation

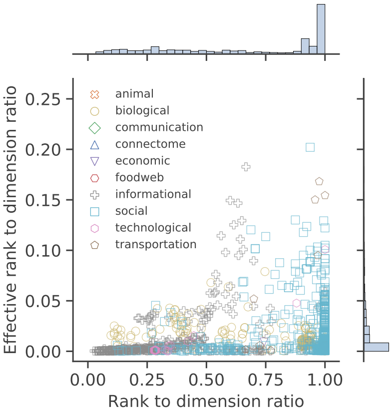
Experimental result





Suggest that the low-rank hypothesis is justified for real complex networks!





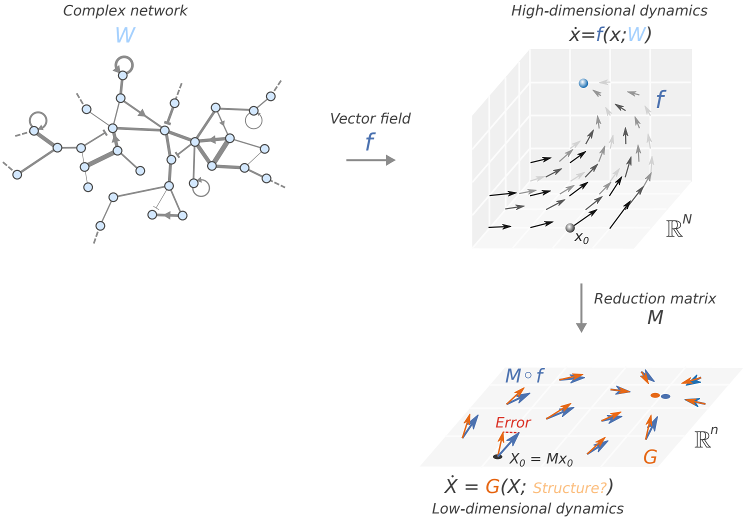
Suggest that the low-rank hypothesis is justified for real complex networks!

What's the consequence for *dynamics on complex networks*?

Low-dimension hypothesis

Complex systems can be reduced to a **few number of equations** that capture their **large-scale behavior**.

Dimension reduction of dynamical systems is about aligning vector fields.



Choose the vector field G : Least-squares

Complete dynamics : $\dot{x} = f(x)$

Reduced dynamics : $\dot{X} = G(X)$ where $X = Mx$

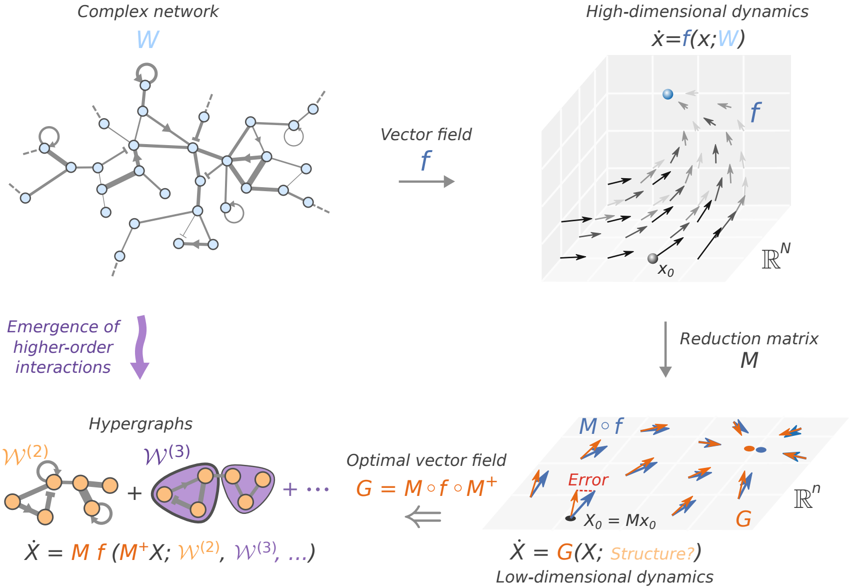
THEOREM (SIMPLIFIED)

The vector field G^* that minimizes the quadratic error between the projected dynamics $\dot{p} = f(p)$ with $p = M^+Mx$ and the reduced dynamics in $\mathbb{R}^N$ [$M^+G(X)$] is

$$G^*(X) = Mf(M^+X).$$

Proof : Just use least-squares.

A surprise : Higher-order interactions



Examples

$$\text{QMF SIS}: \quad \dot{x}_i = -\alpha x_i + \beta(1 - x_i)\sum_{j=1}^{N} W_{ij} x_j, \quad i \in \{1, ..., N\}.$$

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Reduced QMF SIS :

$$\dot{X}_\mu = -\alpha X_\mu + \beta \sum_{\nu=1}^{n} \mathcal{W}_{\mu\nu}^{(2)} X_\nu - \beta \sum_{\nu,\tau=1}^{n} \mathcal{W}_{\mu\nu\tau}^{(3)} X_\nu X_\tau, \quad \mu \in \{1, ..., n\}$$

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Kuramoto-Sakaguchi : $\dot{z}_j = i\omega_j z_j + \sum_{k=1}^{N} W_{jk}[z_k e^{-i\alpha} - z_j^2 \bar{z}_k e^{i\alpha}]$

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$$\mathcal{W}_{\mu\alpha\beta\gamma}^{(4)} = \sum_{j,k=1}^{N} M_{\mu j} M_{j\alpha}^{+} M_{j\beta}^{+} W_{jk} M_{k\gamma}^{+}.$$

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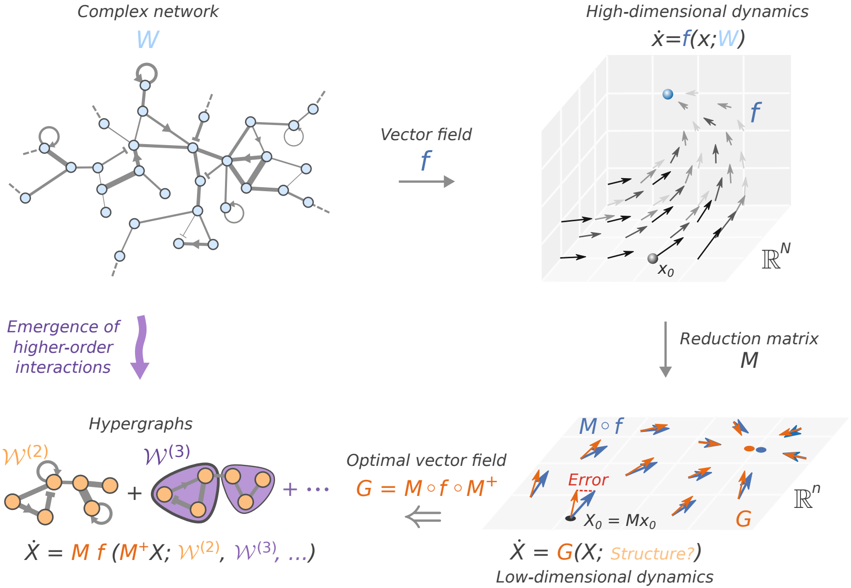
How does the error behave according to the dimension n of the reduced system ?

Alignment error $\mathcal{E}_f(x)$



Upper bound on the alignment error $\mathcal{E}_f(x)$

Theorem (simplified)

The alignment error $\mathcal{E}_f(x)$ for some $x \in \mathbb{R}^N$ is upper-bounded by

$$\mathcal{E}_f(x) \leq \frac{1}{\sqrt{n}}\left[\|V_n^\top J_x(x', y')(I - V_n V_n^\top)x\| + \frac{\sigma_{n+1}}{\sigma_1}\|V_n^\top J_y(x', y')\|_2\|x\|\right].$$

σ_i : *i-th singular values of W*

V_n : *n-truncated right singular vector matrix*

J_x, J_y : *Jacobian matrices evaluated at some point x', y'*

n : *dimension of the reduced system*

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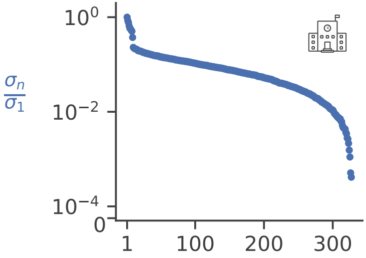
n : *dimension of the reduced system*

$J_x(x', y') = aI$ and $n \geq \mathrm{rank}(W)$ $\Rightarrow$ *Exact* dimension reduction

Alignment error for dynamics on real complex networks



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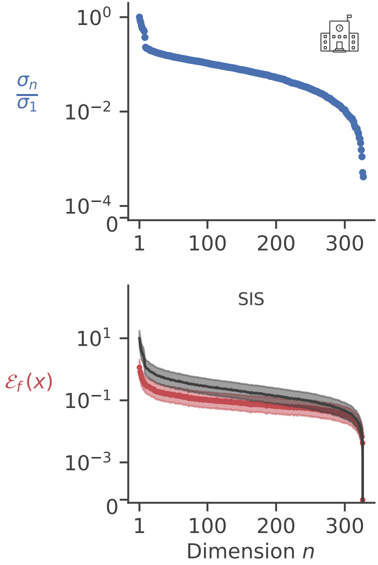

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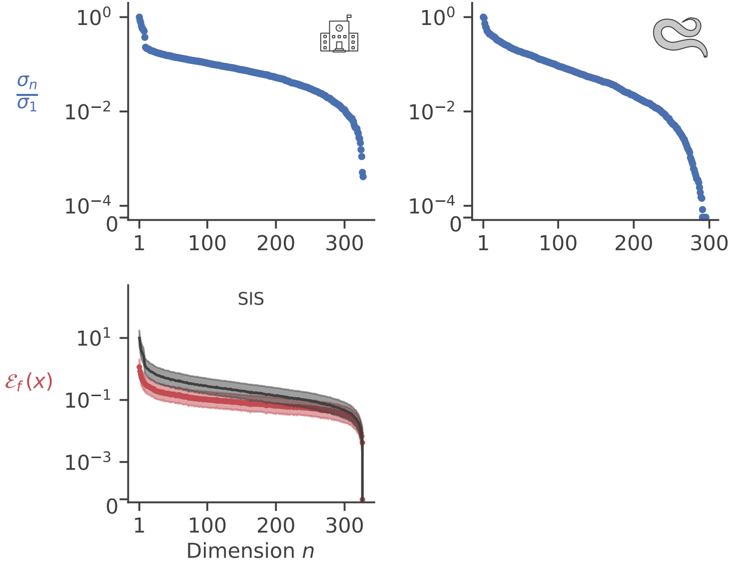

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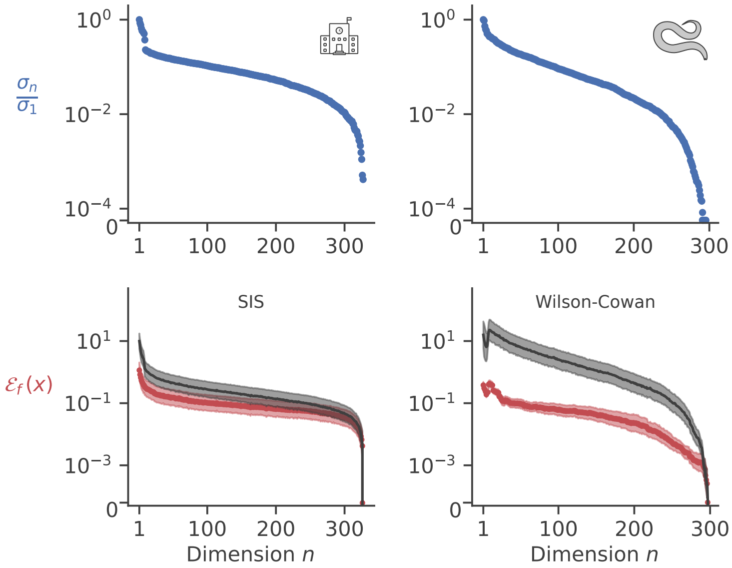

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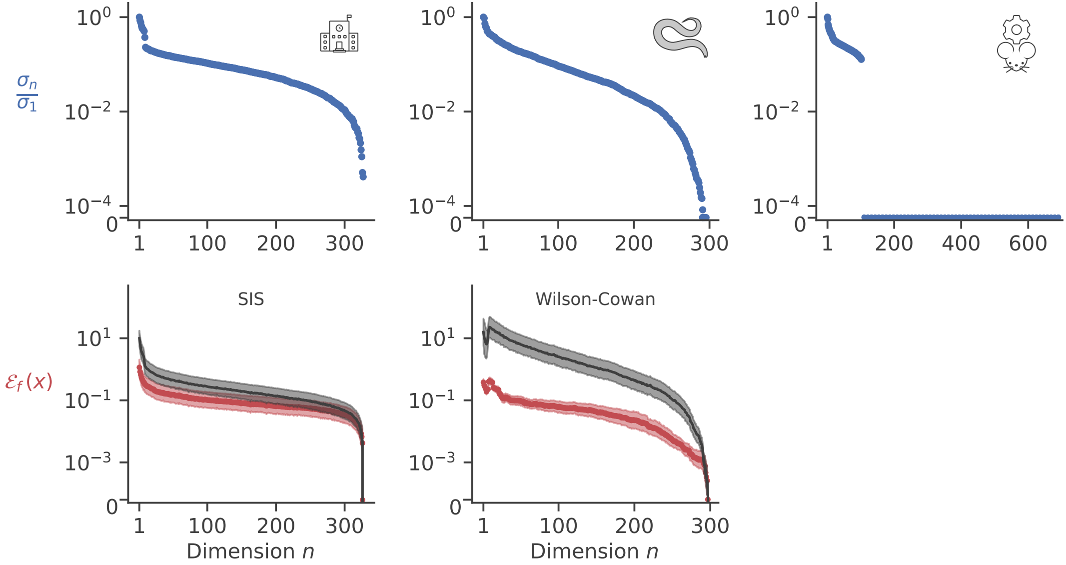

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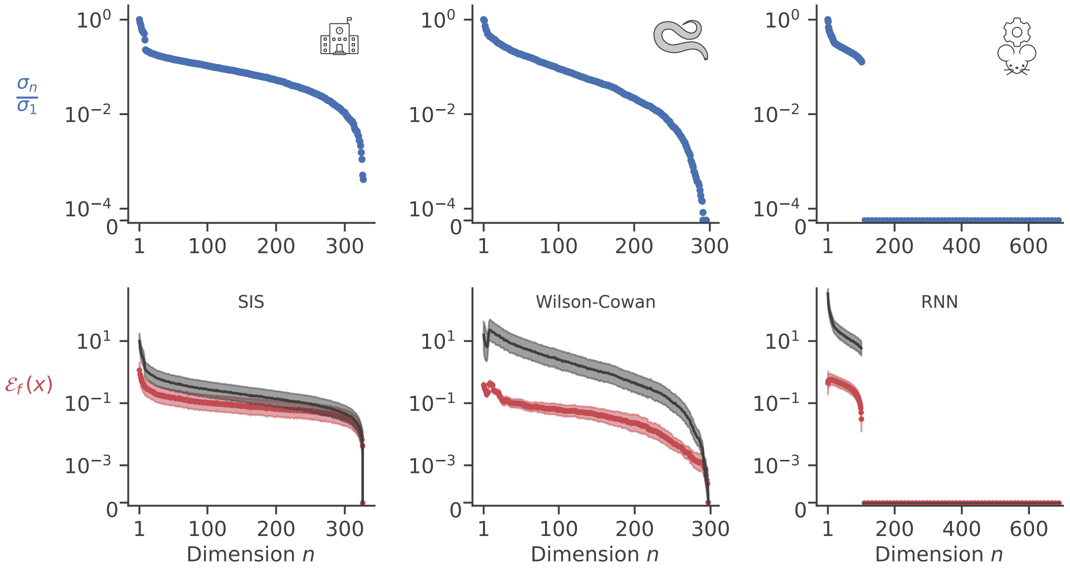


Take-home messages

1. Real networks have low *effective* ranks ;

2. Dimension reduction can lead to dynamics with *higher-order interactions* ;

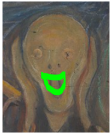
3. Alignment errors of reduced vector fields can rapidly decrease following the singular values of complex networks.

Take-home messages

1. Real networks have low *effective* ranks ;

2. Dimension reduction can lead to dynamics with *higher-order interactions* ;

3. Alignment errors of reduced vector fields can rapidly decrease following the singular values of complex networks. *Encouraging !*



Thank you for your attention!

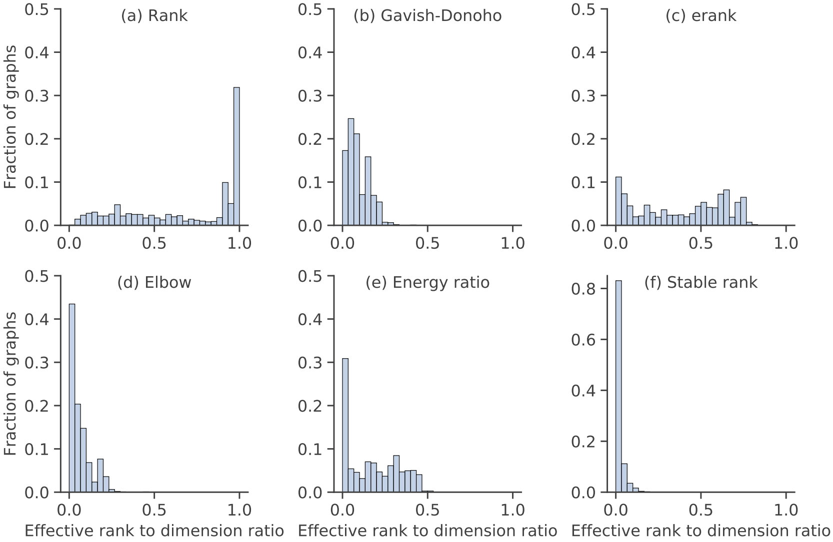
Thanks to the organizers!

The *arXiv* will be out "soon".

For more info : vincent.thibeault.1@ulaval.ca

Questions?

Low-rank hypothesis for other effective ranks



Which error to minimize ?

Reduction matrix M Projector $P = M^+M$ Errors $\delta, \mathcal{E}, \Delta, \varepsilon$

$$
\begin{tikzcd}
x \arrow[rr, mapsto, "F"] \arrow[dr, mapsto, "P"] \arrow[ddr, mapsto, bend right, "M"'] & & F(x) \arrow[dr, dash, dashed, "\delta"'] \arrow[drr, dash, dashed, "\mathcal{E}"] & & \\
& Px \arrow[rr, mapsto, "F"] \arrow[d, mapsto, bend right, "M"'] & & F(Px) \arrow[r, dash, dashed, "\Delta"] \arrow[dr, mapsto, bend right, "M"] & M^+G(X) \\
& X \arrow[rr, mapsto, "G"'] \arrow[u, mapsto, bend right, "M^+"'] & & G(X) \arrow[r, dash, dashed, "\varepsilon"'] \arrow[ur, mapsto, bend right, "M^+"'] & MF(Px)
\end{tikzcd}
$$

It is described by N differential equations

$$\tau_x \, \dot{x}_i = -x_i + (1 - a x_i)/(1 + e^{-b(y_i - c)}), \quad \text{with} \quad y_i = \sum_{j=1}^{N} W_{ij} x_j$$

- x_i : Activation rate i
- τ_x : Time constant
- a : Refractory period
- b : Slope of the logistic function
- c : Activation threshold

Dynamics for the upper-bound

$$\dot{x} = f(x, Wx)$$

E.g. : SIS, Lotka-Volterra, microbial, Kuramoto-Sakaguchi, theta, Winfree, Wilson-Cowan, RNN, ...

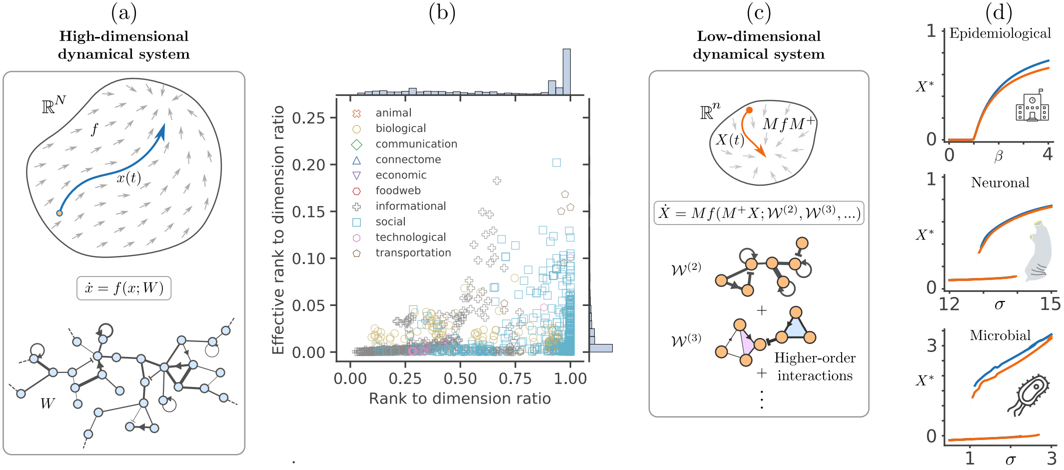


Figure: (a) ... (b) ... (c) ... (d) The stable bifurcation branches of three different dynamics where the blue curves are the equilibrium points of the high-dimensional dynamics while the orange curves are the ones of the reduced dynamics. Top : Quenched-mean field Susceptible-Infected-Susceptible dynamics on a undirected network of high school contacts. The complete dynamics is of dimension $N = 327$ and the reduced dynamics has dimension $n = 1$. Center : Excitatory Wilson-Cowan dynamics on the weighted and directed connectome of *Ciona intestinalis*. $N = 213$, $n = 20$. Bottom : Microbial dynamics on a signed, weighted, and directed gut microbiome network . $N = 838$, $n = 80$.