Symmetry-driven embedding of networks in hyperbolic space

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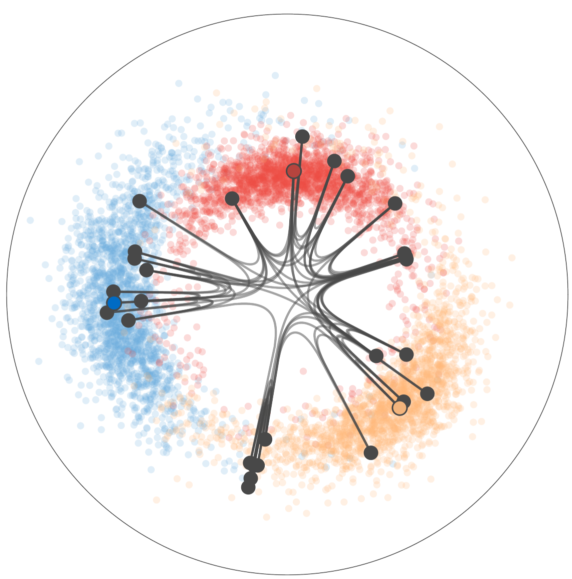
Network geometry is a versatile yet simple framework that captures several observed properties of empirical networks, such as non-vanishing clustering, sparsity, and power-law degree distribution [1]. This accurate description is achieved by positioning vertices in a metric space (usually hyperbolic) and connecting them according to their proximity. To use this framework inferentially, one must find vertex coordinates that best reproduce the observed topology. Unfortunately, the problem is non-convex, which makes the inference of the coordinates of the vertices a challenging task. Common embedding techniques circumvent this difficulty by using a combination of simplifying heuristics, greedy algorithms, and machine learning techniques [2–5]. These approaches yield a pointwise estimate, which ignores the estimation's uncertainty and the possibility of having many adequate embeddings of the same graph.

We address both issues simultaneously using a Bayesian approach [6] in which the posterior distribution indicates the plausibility of each embedding (see left panel of the Figure). By sampling the posterior, we can estimate error bars for the coordinates, but also for the graphs and geometric properties that ensue from the embedding. Furthermore, we find a method to synthetically generate graphs that have two plausible embeddings, inducing a multimodal posterior distribution.

Correctly sampling the posterior is a technically challenging problem because of the large number of local maxima. Indeed, we find that naive Hamiltonian Monte Carlo—the state-of-the-art method for sampling continuous random variables [7]—fails to adequately explore the parameter space. We side-step this issue using cluster-based transformations, leveraging natural symmetries of the hyperbolic space (see right panel of the Figure).

[1] Nat. Rev. Phys. **3**, 114–135 (2021).
[2] New J. Phys. **21**, 123033 (2019).
[3] Phys. Rev. E **104**, 044315 (2021).
[4] Nat. Commun. **8**, 1615 (2017).
[5] IEEE/ACM Trans. Netw. **26**, 920–933 (2018).
[6] arXiv:2406.10711 (2024).
[7] J. Stat. Softw. **76**, 1–32 (2017).



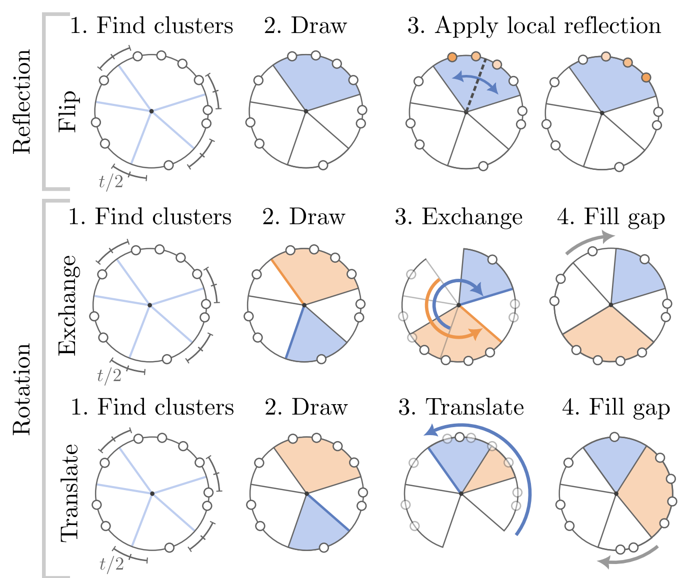


Figure: (left) Probabilistic hyperbolic embedding of a synthetic graph of 30 vertices. Black points and dark-colored points are the median coordinates of each vertex. Light-colored points are sampled positions for the three highlighted vertices. Lines are edges drawn using hyperbolic geodesics. The sample contains 2,000 points. (right) Cluster-based transformations used in the sampling algorithm. The *flip* transformation targets the reflection symmetry while the *exchange* and *translate* target the rotation symmetries. The radial coordinate of the vertices is not shown in the schematics.