Week Twelve

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25^{th} December

- **Co-design diagrams** are similar to a *UWD*, each *boxes* represent **feasibility relations** (*design constraints* in the below figure), each *wire* represents a **preorder of resources** ($x \leqslant y$ represents *availability of x given y*): the wire on the left represent a **team's output** (which should be greater than or equal to the usage, hence, represented by '$\leqslant$'), the wire on the right represents the **team's input** requirements to generate output.

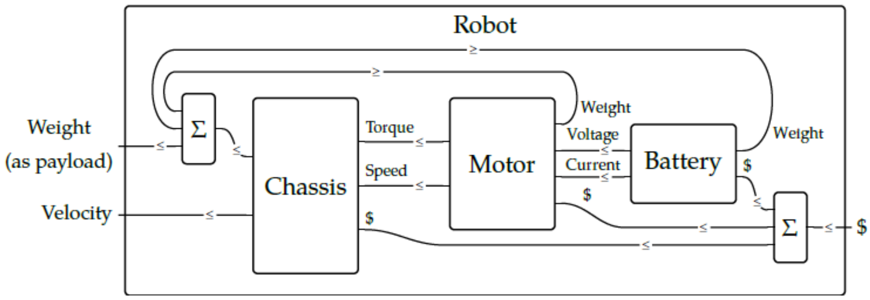

Figure 1: Example for a co-design diagram (Eq 4.1)

A *feasibility relation* matches resource production with requirements. $\forall (p, r) \in P \times R$, where P and R are the preorders of resources to be **produced** and **required** respectively, the box says **true** or **false** for that pair.

Hence, feasibility relations define a function $\Phi : P \times R \longrightarrow \mathbf{Bool}$ as:

(a) $(\Phi(p, r) \;\;\&\;\; p' \leqslant p) \implies \Phi(p', r)$, ie, if p amount of produce can be made given r, you can also produce less $p' \leqslant p$ with the same resources r.

(b) $(\Phi(p, r) \;\;\&\;\; r \leqslant r') \implies \Phi(p', r)$, ie, if p amount of produce can be made given r, with $r' \geqslant r$ resources, you can produce p.

- Let $\mathcal{X} = (X, \leqslant_X)$ and $\mathcal{Y} = (Y, \leqslant_Y)$ be preoders. A **feasibility relation** for $\mathcal{X}$ given $\mathcal{Y}$ is a monotone map:

$$\Phi : \mathcal{X}^{op} \times \mathcal{Y} \longrightarrow \mathbf{Bool}$$

We denote this by $\Phi : \mathcal{X} \nrightarrow \mathcal{Y}$. Given $x \in X$ and $y \in Y$, if $\Phi(x, y)$, we say *x can be obtained given y*.

This map is said to be monotone because by definition:

$$x' \leqslant_X x \;\;\&\;\; y \leqslant_Y y' \implies \Phi(x, y) \leqslant_{\mathbf{Bool}} \Phi(x', y').$$

26^{th} December

- **$\mathcal{V}$-profunctor**: Let $\mathcal{V} = (V, \leqslant, I, \otimes)$ be a **quantale** (a closed symmetric monoid with all joins existing), and let $\mathcal{X}$ and $\mathcal{Y}$ be $\mathcal{V}$-categories. A **$\mathcal{V}$-profunctor** $\Phi : \mathcal{X} \nrightarrow \mathcal{Y}$ is a $\mathcal{V}$-functor:

$$\Phi : \mathcal{X}^{op} \times \mathcal{Y} \longrightarrow \mathcal{V}$$

- **Bool**-profunctors and **Cost**-profunctors can be interpreted as bridges. See ex 4.11, 4.13. Also see **feasibility matrix** (ex 4.12).

  Profunctor can be optained via **matrix multiplication**. (See remark 4.16)

- The category **Feas** has objects as *preorders* and morphisms as *feasibility relations* (**Bool**-profunctor) and their composition is given by using $\wedge$ in place of $\otimes$ in the composite equation given in the below point.

- **Composition of $\mathcal{V}$-profunctors**: Let $\mathcal{V}$ be a quantale and $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ be $\mathcal{V}$-categories, and let $\Phi : \mathcal{X} \nrightarrow \mathcal{Y}$ and $\Psi : \mathcal{Y} \nrightarrow \mathcal{Z}$ be $\mathcal{V}$-profunctors. Their **composite** $\Psi \circ \Phi : \mathcal{X} \nrightarrow \mathcal{Z}$ is given by:

$$(\Psi \circ \Phi)(p, r) = \bigvee_{q \in Q} (\Phi(p, q) \otimes \Psi(q, r))$$

  Composition of profunctors is associative. (Page 129)

- For any **skeletal quantale** $\mathcal{V}$, the category $\mathbf{Prof}_{\mathcal{V}}$ has objects as $\mathcal{V}$-categories $\mathcal{X}$, whose morphisms are $\mathcal{V}$-profunctors $\mathcal{X} \nrightarrow \mathcal{Y}$, and with composite defined in the above point.

  Hence, **Feas**:= $\mathbf{Prof}_{\mathbf{Bool}}$.

  The identity morphism is given by the *unit-profunctor* $U_{\mathcal{X}} : \mathcal{X} \nrightarrow \mathcal{X}$,

$$U_{\mathcal{X}}(x, y) := \mathcal{X}(x, y)$$

$$\forall \Phi : \mathcal{P} \nrightarrow \mathcal{Q} \qquad \Phi \circ U_{\mathcal{P}} = \Phi = U_{\mathcal{Q}} \circ \Phi$$

  Proof for the above identity is in page 128.

- A monoidal category is a *categorified* monoidal preorder.

- Let $F : \mathcal{P} \longrightarrow \mathcal{Q}$ be a $\mathcal{V}$-functor. The **companion** of F ($\widehat{F} : \mathcal{P} \nrightarrow \mathcal{Q}$) and the **conjoint** of F ($\check{F} : \mathcal{Q} \nrightarrow \mathcal{P}$) are defined as:

$$\widehat{F}(p, q) := Q(F(p), q) \;\; \& \;\; \check{F}(q, p) := Q(q, F(p))$$

  The **companion** profunctor represents a bridge from $\mathcal{P}$ to $\mathcal{Q}$. Reversing the arrows result in the **conjoint** profunctor representing bridge from $\mathcal{Q}$ to $\mathcal{P}$.

- **$\mathcal{V}$-enriched adjunction** is a pair of $\mathcal{V}$-functors $F : \mathcal{P} \to \mathcal{Q}$ and $G : \mathcal{Q} \to \mathcal{P}$ such that:

$$\mathcal{P}(p, G(q)) \cong \mathcal{Q}(F(p), q)$$

  In this figure, $\forall p \in \mathcal{P} \;\; \& \;\; q \in \mathcal{Q}$, the above condition holds true except for the pair $(1, c)$, hence F and G **do not** form an *enriched* adjunction pair.

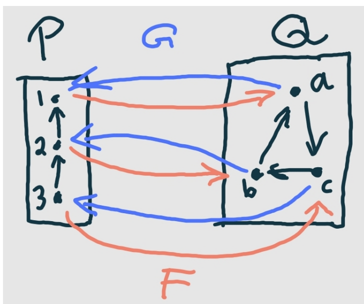

Figure 2: Example

- If $\mathcal{P}$ and $\mathcal{Q}$ are enriched in skeletal quantale $\mathcal{V}$ The companion of the adjoint F is equal to the conjoint of the adjoint G. (see ex 4.41)

  This can be used to prove that: $\widehat{\mathrm{id}} = \check{\mathrm{id}}$.

- A $\mathcal{V}$-profunctor $\Phi : \mathcal{X} \nrightarrow \mathcal{Y}$ can be thought of as a $\mathcal{V}$-category with $\mathcal{X}$ on the left and $\mathcal{Y}$ on the right. This construction is called **Collage of the Profunctor**. (denoted as $\mathbf{Col}(\Phi)$, see definition in page 131)

28^{th} Decemeber

- **Categorification** is the idea of generalising a 'thing we know' by adding **structure** to it, such that, what were formerly *properties* become structures. Removing this new structure from the '*categorified* thing' allows us to get the 'thing we knew' earlier.

  Categorified Preorders are equivalent to Categories.

  There exists categories with infinitely many structures, called **∞-categories**.

- A **symmetric monoidal structure** on a category $\mathcal{C}$ consists of:

  ⋆ $I \in \mathrm{Ob}(\mathcal{C})$ called the **monoidal unit**.

  ⋆ Functor called **monoidal product**: $\otimes : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$.

  which are subject to well-behaved, natural isomorphisms:

  (a) $\lambda_c : I \otimes c \cong c \forall c \in \mathrm{Ob}(\mathcal{C})$
  (b) $_c : c \otimes I \cong c \forall c \in \mathrm{Ob}(\mathcal{C})$
  (c) $\alpha_{c,d,e} : (c \otimes d) \otimes e \cong c \otimes (d \otimes e) \forall c, d, e \in \mathrm{Ob}(\mathcal{C})$
  (d) $\alpha_{c,d} : c \otimes d \cong d \otimes c \forall c, d \in \mathrm{Ob}(\mathcal{C})$ called the **swap map** such that $\sigma \circ \sigma = \mathrm{id}$.

  An example would be the monoidal category $(\mathbf{Set}, \{1\}, \times)$. (See ex 4.49)

- A category **enriched** in $\mathcal{V}$, or a **$\mathcal{V}$-category**, call it $\mathcal{X}$, has the following:

  (1) A collection $\mathrm{Ob}(\mathcal{X})$, elements of which are called **objects**.
  (2) $\forall x, y \in \mathrm{Ob}(\mathcal{X})$, we define the **hom-object** for x, y as $\mathcal{X}(x, y) \in \mathcal{V}$.
  (3) $\forall x \in \mathrm{Ob}(\mathcal{X})$, we define the **identity element** $\mathrm{id}_x : I \longrightarrow \mathcal{X}(x, x)$
  (4) $\forall x, y, z \in \mathrm{Ob}(\mathcal{X})$, we define the **composite** $\mathcal{X}(x, y) \otimes \mathcal{X}(y, z) \to \mathcal{X}(x, z)$.

- **Compact closed categories**: Let $(\mathcal{C}, I, \otimes)$ be a symmetric monoidal category, and $c \in \mathrm{Ob}(\mathcal{C})$ an object. A **dual for** c ($\in \mathrm{Ob}(\mathcal{C})$) consists of:

  ⋆ a morphism $\eta_c : I \to c^* \otimes c$, called the **unit of** c.

  ⋆ a morphism $\epsilon_c : c \otimes c^* \to I$, called the **counit of** c.

  If $\forall c \in \mathrm{Ob}(\mathcal{C})$ there exists a dual for c, then we say $(\mathcal{C}, I, \otimes)$ is **compact closed**.

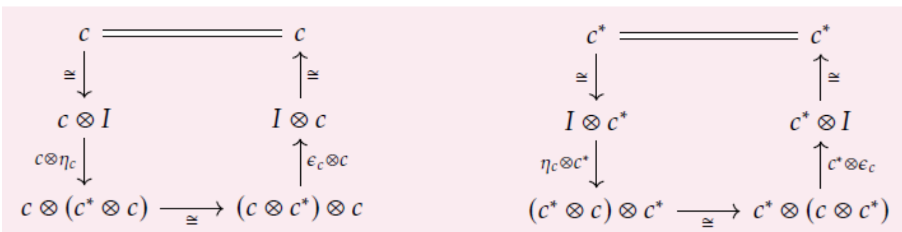

Figure 3: Commutative diagram representing the conditions, also called **snake equations** (page 141)

- If $\mathcal{C}$ is a compact closed category, then:

  1. It is monoidal closed. and $\forall c \in \mathrm{Ob}(\mathcal{C})$:
  2. If c^* and c' are both duals to c, then they are isomorphic.
  3. c and its double-dual are isomorphic: $c \cong c^{**}$.

- The below picture shows how an **undirected** wiring diagram equipped with the morphisms —η_c, ϵ_c— can represent a **directed** wiring diagram.

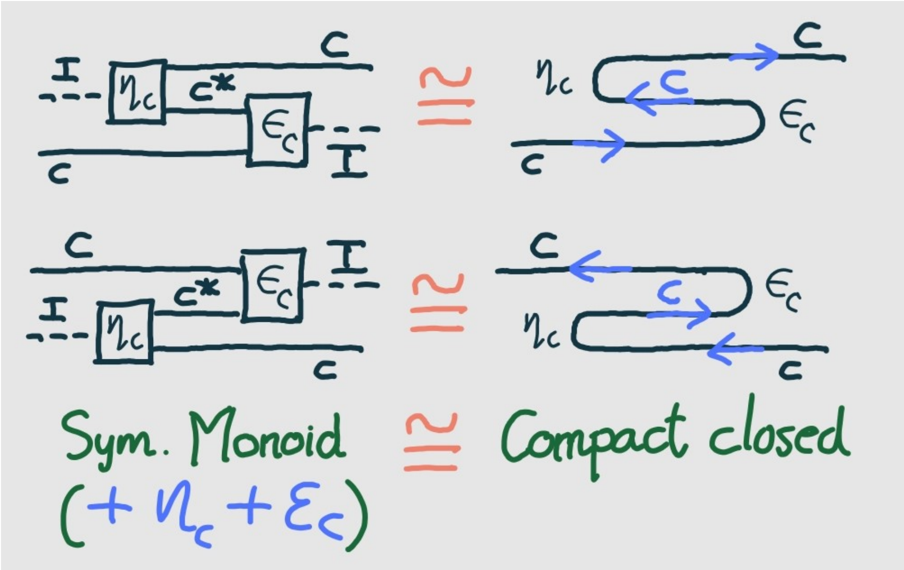

Figure 4: Snake equations in terms of *Symmetric Monoidal categories*.

- The category **Corel** contains *objects* as *finite sets* and *morphisms* as **corelations**. This category can be equipped with a **symmetric monoidal structure** $(\varnothing, \bigsqcup)$ which is also **compact closed** with *dual of each set as itself*.
- The category **Prof**$_\mathcal{V}$ (where $\mathcal{V}$ is *skeletal quantale*) can be given the structure of a compact closed category, with **monoidal product** given by product of $\mathcal{V}$-categories.

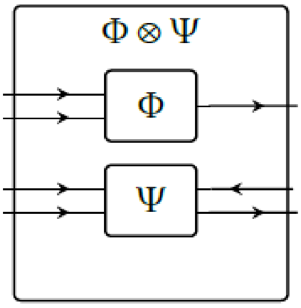

Figure 5: $(\Phi \times \Psi)((x_1, y_1), (x_2, y_2)) := \Phi(x_1, x_2) \otimes \Psi(y_1, y_2)$

The **monoidal unit** in **Prof**$_\mathcal{V}$ is $\mathbf{1}$, which contains only one object.

Duals in **Prof**$_\mathcal{V}$ are **opposite categories**. The *unit* and *counit* are $\mathcal{V}$-profunctors defined as:

- ⋆ $\eta_\mathcal{X} : \mathbf{1} \times \mathcal{X}^{\text{op}} \times \mathcal{X} \to \mathcal{V}$, with $\eta_\mathcal{X}(1, x, x') := \mathcal{X}(x, x')$
- ⋆ $\epsilon_\mathcal{X} : \mathcal{X}^{\text{op}} \times \mathcal{X} \nrightarrow \mathbf{1}$, with $\epsilon_\mathcal{X}(x, x', 1) := \mathcal{X}(x, x')$