Category Theory:

- Ground breaking paper of Eilenberg & Mac lane : " A general theory of natural consequence"
- Multiplication often appears in the guise of independent choices.
- Logical relation of and & multiplication (They are different manifestation of same idea)
- Galileo & flight of bird:

He reduced the motion of an bird in space to two ~~sp~~ simple/ special motions in vertical line & horizontal plane

$$\begin{tikzcd}
\boxed{\text{Space}} \arrow[r, "\text{motion of}", "\text{level of flight of bird}"'] \arrow[d, "\text{shadow of flight of bird}"'] & \boxed{\text{Line}} \\
\boxed{\text{Plane}} &
\end{tikzcd}$$

→ The category of sets: [Understanding categories via sets]
A map of sets is a process from one set to another. We investigate the composition of maps (following one process by a second process), & find that the algebra of composition of maps resembles the algebra of multiplication of numbers, but its interpretation is much richer

Date: ____

~~Categ~~

→ To Understand category, we can consider a familiar example sets (one of the categories).

[A category is usually defined by the following ingredients:

1) Objects
2) Morphism
3) Composition
4) Identity

and ~~these~~ governed by

- Identity laws
- Associativity laws.]

- In sets:

1) objects are nothing but finite sets
2) Morphisms are maps (also called transformation, function)
3) Composition is nothing but composition of maps itself
4) A special kind of map/morphism

Illustration: let A, B, C & D be objects

$f : A \to B$ can be written as $A \xrightarrow{f} B$

$g : B \to C$

$h : C \to D$, be maps

now $g \circ f = g(f)$, i.e., there exists a map from $A \to C$ ~~such that~~ k, such that $g \circ f = g(f) = h$

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[rr, bend right, "h"'] & B \arrow[r, "g"] & C \end{tikzcd}$$

and identity $1_A : A \to A$ such that $\forall a_i \in A \; [i \in \mathbb{N}]$

$1_A(a_i) = a_i$

Rules for category

i) The identity laws:

(a) If $A \xrightarrow{1_A} A \xrightarrow{g} B$ ~~(b)~~ then $A \xrightarrow{g \circ 1_A = g} B$

(b) If $A \xrightarrow{f} B \xrightarrow{1_B} B$

$A \xrightarrow{1_B \circ f = f} B$

ii) The Associative Laws:

If $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$

[A point of set X is a map $1 \longrightarrow X$]

~~The algebra of composition:~~

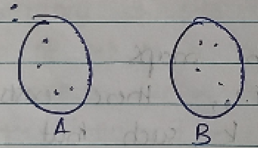
then $A \xrightarrow{h\circ(g\circ f) = (h\circ g)\circ f} D$.

i.e, $h\circ(g\circ f) = (h\circ g)\circ f$

[order matters, i.e, $g\circ f \neq f\circ g$ (generally)]

- A map f of set involves three things
I a set (A), called the domain of the map f
II a set (B), called the codomain of the map f
III a rule (or process) for f, assigning to each element of the domain A exactly one element of the codomain B
(* ~~too~~ two functions/maps fn & n are equal if their domains & codomains are equal, i.e, $A \xrightarrow{m} B$ & $A \xrightarrow{g} B$ & for each point $1 \xrightarrow{a} A$
$f\circ a = g\circ a$ $\forall a \in A$)

- No. of maps from A to B:
n(A) represents cardinality of A
~~due to the~~ from the concept of independent choices:



total no. of maps from A to B = $[n(B)]^{n(A)}$

- The algebra of composition
(understanding some special ~~different~~ kinds of maps/morphisms)

→ composition of morphisms is analogous to multiplication of numbers
(except that in composition order matters & commutative law isn't applicable)
→ To understand the analogy of division of numbers in set category we need to understand two kind of morphisms

Ex 2:

~~proposition: suppose the map $A \xrightarrow{f} B$ has~~

II choice problem: (or lifting problem)
given g & h as shown, to find $\cancel{g}f$ is called lifting problem

$$\begin{tikzcd} & B \arrow[dr, "g"] & \\ A \arrow[ur, "f?"] \arrow[rr, "h"'] & & C \end{tikzcd}$$

a particular solution f can be called as choice

<u>special case of section</u>:
if $h = 1_A$, $A = C$ then $f = s$
$g \circ s = h = 1_A$
s is called section

Propositions:
1) ~~If the~~ suppose a map $A \xrightarrow{f} B$ has a retraction. Then for any set T & for any pair of maps $T \xrightarrow{x_1} A$ & $T \xrightarrow{x_2} A$ from any set T to A if $f \circ x_1 = f \circ x_2 \Longleftrightarrow x_1 = x_2$
& f is called injective function or <u>monomorphism</u>

$$\begin{tikzcd} T \arrow[r, shift left, "x_1"] \arrow[r, shift right, "x_2"'] & A \arrow[r, "f"] \arrow[rr, bend left, "1_A"] & B \arrow[r, "r"] & A \end{tikzcd}$$

2) suppose a map $A \xrightarrow{f} B$ has a section.
Then for any set T & any pair $B \xrightarrow{t_1} T$, $B \xrightarrow{t_2} T$ of maps from B to T if $t_1 \circ f = t_2 \circ f$.
$\Leftrightarrow t_1 = t_2$
& f is called surjective function or epimorphism

$$\begin{tikzcd} & B \arrow[dr, "f"] & \\ T \arrow[ur, "t_1"] \arrow[rr, shift left, "t_1 \circ f"] \arrow[rr, shift right, "t_2 \circ f"'] \arrow[dr, "t_2"'] & & A \\ & B \arrow[ur, "f"'] & \end{tikzcd}$$

Date: ___

- If the single determination problem has a solution for f (a.k.a retraction for f) then every determination problem with the same f has a solution.
- If the single choice problem has a solution for f (a.k.a section for f) then every choice problem involving this same f has a solution.

The above statements are interesting, as they are analogous to numbers saying that, $1/5$ is inverse of 5, & hence for $x \times 5 = 3$ ~~can b~~
$x = 1/5 \times 3$, where instead of a reciprocal the inverse & multiplication are used

Idempotent: An endomap (whose domain & codomain ~~are~~ same) ~~is~~ e is called idempotent if $e \circ e = e$

Uniqueness of an isomorphism: ~~For a morphism $f: A \to B$,~~ ~~ist~~ For an isomorphism $f: A \to B$ there exists only one inverse f^{-1}.

~~Def~~ Isomorphism & automorphism

A map f is called an isomorphism if there exists another map f^{-1} which is both a retraction & a section for f.

$$A \underset{f^{-1}}{\overset{f}{\rightleftarrows}} B \qquad f \circ f^{-1} = 1_B \qquad f^{-1} \circ f = 1_A$$

A map which is both an endomap and at the same time an isomorphism is usually called by the one word automorphism.