

Pullbacks (Fibred Product)

In the category *Set* a 'pullback' is a subset of the cartesian product of two sets.

Say X and Y are your two sets and then P is a subset of $(X \times Y)$ such that the square commutes.

$$\begin{array}{ccc} P & \xrightarrow{p_2} & Y \\ p_1 \downarrow & & \downarrow g \\ X & \xrightarrow{f} & Z \end{array}$$

Its akin to finding a "common part" between the two sets X and Y which are related by some Z (Subset of the cartesian product)

Suppose:

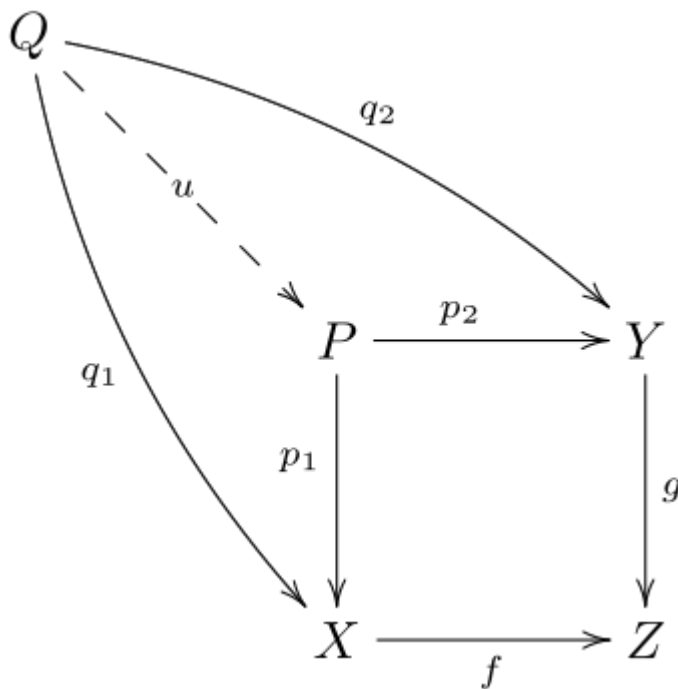
- $Z = \{1, 2, 3\}$
- $X = \{a, b\}$ with $f(a) = 1, f(b) = 2$
- $Y = \{x, y\}$ with $g(x) = 1, g(y) = 1$

Then:

- $P = \{(a, x), (a, y)\}$

Given f, g the pullback is P with p_1, p_2

To be universal, P must be equal upto a morphism to every Q



Pushouts (Fibred Sum)

In the category *Set* a 'pushout' is a quotient of the disjoint union of two sets.

Say X and Y are your two sets and then P is a subset of $(X \times Y)$ such that the square commutes.

Suppose:

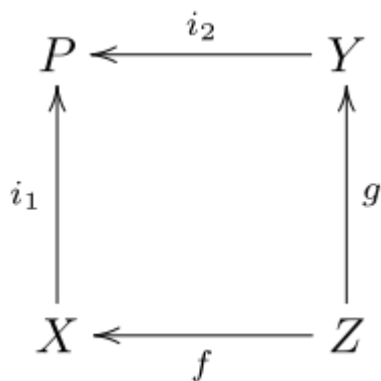
- $Z = \{1, 2\}$
- $X = \{a, b, c\}$ with $f(1) = a$, $f(2) = b$
- $Y = \{x, y, z\}$ with $f(1) = x$, $f(2) = y$

Then:

- $P = \{ \{(a, 0), (x, 1)\}, \{(b, 0), (y, 1)\}, \{(c, 0)\}, \{(z, 1)\} \}$

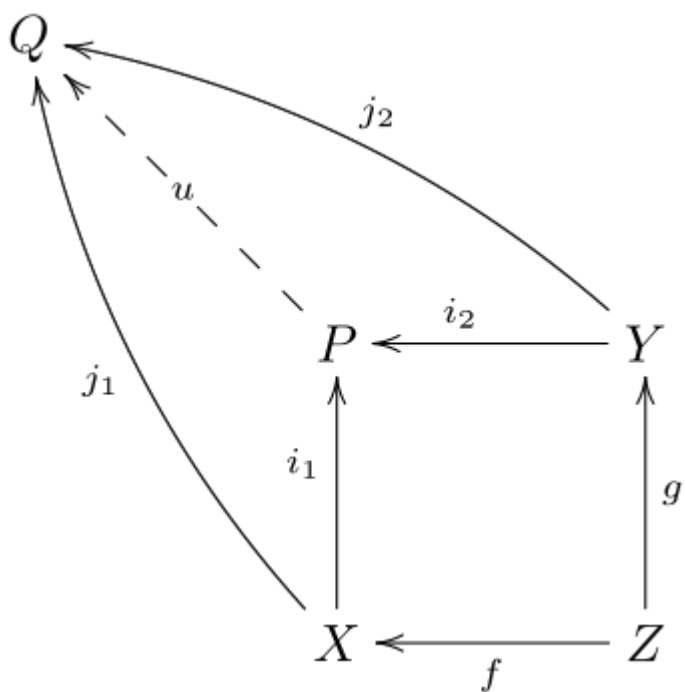
It's akin to "gluing together" two sets X and Y which are related by some Z

(Take disjoint union such that elements related by Z are identified together in a subset)



Given f, g the pushout is P with p_1, p_2

To be universal, P must be equal upto a morphism to every Q



Spans

Any diagram of this form is called a span: $Y \leftarrow X \rightarrow Z$

A span is nothing but a diagram relating two objects in a category via some other object

The colimit of a span is a pushout

Cospans

A cospan is a span in the opposite category of whatever category the span was defined in.

Opposite category is denoted by C^{op} which means all morphisms have their dom, codom reversed.

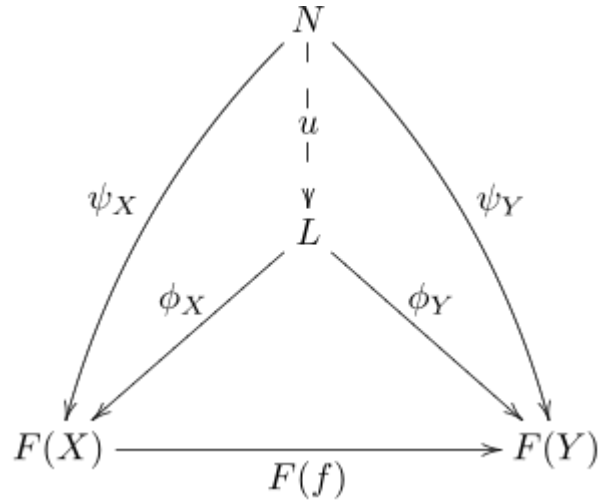
The limit of a cospan is a pullback

I hate the co reversal in this mess!

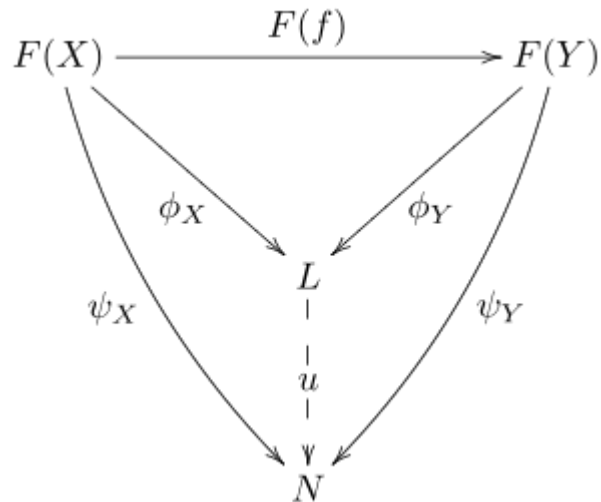
Limits and Colimits

Generalizations of these to universal properties.

Limits restrict unique morphisms in such cones



Co-limits restrict unique morphisms in upside-down cones



Think of limits as the terminal objects in the category of cones and co-limits as the initial objects in that category.