Week One

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19th June

I started reading the book "**Conceptual Mathematics**". I have read the first two sections today. Key points for each session are listed below:

Section 1:

- Firstly, we see examples for Categories:
  1. Galileo's bird's flight puzzle which talks about a relationship between the objects 'time' and 'space' where the bird travels.
  2. The 'space' talked above can again be split into two objects: the 'plane' where the shadow of the bird lie, and the level of flight of bird which is a 'line'.
  3. Other examples like the a category of two dishes (a relation with 1st and 2nd course dishes).
- In the next part, we relate many topics in set theory with category theory like functions as morphisms etc. A **category of finite sets** contains:
  1. Data for the Category:
     (1) Objects: the sets $\mathbf{A}, \mathbf{B}, \mathbf{C}, ...$
     (2) Maps: functions like $f, g, ...$
  2. Rules:
     (1) Identity law: if $\mathbf{A} \xrightarrow{f} \mathbf{B}$, then, $I_B \circ f = f \;\; \& \;\; f \circ I_A = f$.
     (2) Associative law: $h \circ (g \circ f) = (h \circ g) \circ f$.

Section 2:

- Some definitions: Consider the category $\mathbf{A} \xrightarrow{f} \mathbf{B}$,
  1. The set **A** is called the Domain of map 'f'.
  2. The set **B** is called the Co-Domain of map 'f'.
  3. A **rule** for map 'f', is that each element in **A** must be mapped to only one element in **B**.
- **Test for equality** of two maps:
  A **point** of a set **A** is a map from a **singleton set 1** to **A**. Using this, we can say that " two maps f and g with domain **A** and co-domain **B** are said to be equal iff for all points $\mathbf{1} \xrightarrow{a} A$, $f \circ a = g \circ a$, then $f = g$. "
- Internal and External Diagrams:
  - Internal: uses the arrow diagrams where the elements of the set are shown.
  - External: shows mapping with arrows between sets as a whole without explicitly showing the elements in them.

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Section 3:

- Total number of maps from set **A** to set **B** is given by: $n(B)^{n(A)}$, where $n(X)$ represents the number of elements in set **X**
- Isomorphisms: a map $\mathbf{A} \xrightarrow{f} \mathbf{B}$ is called Isomorphic or invertible, if there exist another map $\mathbf{B} \xrightarrow{g} \mathbf{A}$ such that, $f \circ g = I_B$ and $g \circ f = I_A$. This map 'g' is called the inverse map of 'f'. (If both domain and co-domain are equal, then this isomorphism is called **Automorphism**.)
- Isomorphs are **Reflexive, Symmetric**, and **Transitive**.
- From Ex.1(T), we can see that for two isomorphs f and g, the inverse of the composition $f \circ g$ is $g^{-1} \circ f^{-1}$
- **Determination and Choice Problems**:

$$\begin{tikzcd} & B \arrow[dr, dashed, "g?"] & \\ A \arrow[ur, "f"] \arrow[rr, "h"'] & & C \end{tikzcd}$$

Determination Problems requires us to find the map g (which we call the **determination** of h by f) if both f and h are given, such that, $h = g \circ f$.
If such g exist, then we say h can be **determined** by f.

$$\begin{tikzcd} & B \arrow[dr, "g"] & \\ A \arrow[ur, dashed, "f?"] \arrow[rr, "h"'] & & C \end{tikzcd}$$

Choice Problems requires us to find the map f if both f and h are given, such that, $h = g \circ f$. When the map f is fixed, we get a lot of "*choices*" for the map g.

Note: When the set **B** is a "singleton set", then the maps f and h are constant maps.

- **Retractions, Sections and Idempotents:**
  1. Retractions are the solution maps r for the determination problem: $r \circ f = I_A$.
  2. Sections are the solution maps s for the choice problem: $f \circ s = I_B$.
  3. Idempotents are the maps e such that: $e \circ e = e$. (eg: $e = f \circ r$)

  Theorems:

  – If a map $\mathbf{A} \xrightarrow{f} \mathbf{B}$ has a **section**, then for any **T** and any map $\mathbf{T} \xrightarrow{y} \mathbf{B}$, there exist a map $\mathbf{T} \xrightarrow{x} \mathbf{A}$, such that $f \circ x = y$.

$$\begin{tikzcd} & A \arrow[dr, bend left, "f"] & \\ T \arrow[ur, dashed, "x"] \arrow[rr, "y"'] & & B \arrow[ul, bend left, "s"] \end{tikzcd}$$

  – If a map $\mathbf{A} \xrightarrow{x} \mathbf{B}$ has a **retraction**, then for any **T** and any map $\mathbf{A} \xrightarrow{y} \mathbf{T}$, there exist a map $\mathbf{B} \xrightarrow{f} \mathbf{T}$, such that $f \circ x = y$.
  (**Note:** Maps with retractions are **monomorphic** and maps with sections are **epimorphic**.)

  – **Uniqueness of Inverses**: if a map f has many retractions $r_1, r_2, \ldots$ and sections $s_1, s_2, \ldots$, then:
    1. All of the sections are equal to each other, same is true for retractions.
    2. Both section and retraction are equal. (f is an **Isomorphism**)

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Section 4:

- **Algebraic Category:** is defined by a pair which contains a **Set of numbers** and the **rule or algebraic operation** between two numbers in the set which results in a new number which must be in the set.
  eg: $(\mathbb{R}, +)$: Real numbers with addition as the combining rule.
- Consider two such categories $(A, *)$ and $(A', *')$ and a map $A \xrightarrow{f} A'$. From the **commutative diagram** shown in the right, we can say that, there exists $f(a * b)$ such that:

  $$\boxed{f(a * b) = f(a) *' f(b)}$$

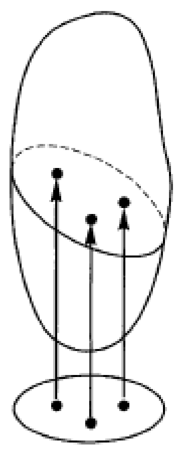
  $$\begin{tikzcd} A \arrow[r, "f"] & A' \\ A \arrow[u, "a*b"] \arrow[ur, "f(a*b)"] \arrow[r, "f"] & A' \arrow[u, "f(a)*'f(b)"'] \end{tikzcd}$$

  This is true for all algebraic categories and it is used as a **test** for checking whether the given category is an **acceptable algebraic category**.
- In geometry, we have "**Euclid's Category**" whose objects are **polygonal plane figures** and the map f has a **distance preserving property**, ie, if a and b are points of figure F, then the map f can only connect it to another figure F' whose points are $f(a)$ and $f(b)$ such that: $dist(a, b) = dist(f(a), f(b))$.

Section 5 and 6:

- A map that can be factored through **1** is called a **constant map**.
- Mapping can be visulized as **Stacking** or **Sorting**. A map $X \xrightarrow{f} B$ can be said to be a sorting of X into B "sorts". (We are talking about B as if it was some number!)
- Another view point is that of **sampling** the co-domain. For a map $A \xrightarrow{f} X$, we can say that f is a family of A-elements of X. In geometrical sense, we can say that we are superimposing an "A-shaped" figure on Figure X.

Section 8:



- The word '**section**' is short for '**cross section**'. Imagine holding a cucumber vertically over a table. The map projects each point in the cucumber perpendicularly onto its shadow on the table. If you slice through the cucumber as shown in the picture on the right, you get a section of that map. Generally, a section of the projection map can have any shape, not just a straight cut.
- To find the number of retractions from set **A** to set **B** we can use the formula: $n(A)^{n(B)-n(A)}$ which should be an natural number.
- For finding number of sections from set **A** to set **B** (given, a map already exists from **B** to **A**), take the product of the number of points in **B** with same image in A (due to the existing map).

Section 9:

- Isomorphism can be thought of as "same size" (for **finite** objects), ie, for $A \cong B$ (A is isomorphic to B), we can say that A and B have same number of elements.
- Let A and B be sets. The notation $A \trianglelefteq B$ means that **there exists at least one map** from A to B. It is **Reflexive** and **Transitive**.
- A is a **retract of** B means that there are maps $A \xrightarrow{s} B \xrightarrow{r} A$ with $rs = 1_A$. (We write this as $A \leq_R B$.)
- An **idempotent** $B \xrightarrow{e} B$ can be split into two maps, $A \underset{s}{\overset{r}{\rightleftharpoons}} B$, with the introduction of a new object A, and the two maps follow $r \circ s = 1_A$ and $s \circ r = e$.
- An **involution** is a map $A \xrightarrow{f} A$, such that, $f \circ f = 1_A$.