

Terminal Objects

→ Let C be a category, then a terminal object $T \in C$ is s.t.
 $\forall X \in C \exists ! \text{ morphism } f: X \rightarrow T.$

→ e.g: For $C = \text{Set}$, T is any one element set.

$C = \text{Top}$, T is a one-point space.

$C = \text{Poset}$, T is the maximal element in the partially ordered set.

→ "Uniqueness" \Rightarrow unique upto an isomorphism, not always equal elements (as is the case here)

→ Proof of uniqueness: Let T and T' be terminal objects.

$\Rightarrow f: T \rightarrow T'$ is a unique morphism
and $g: T' \rightarrow T$ is a unique morphism

$$T \xrightarrow{f} T' \xrightarrow{g} T$$

$$T' \xrightarrow{g} T \xrightarrow{f} T'$$

$$\Rightarrow gf = 1_T, fg = 1_{T'}$$

$\Rightarrow T$ and T' are isomorphic.

→ Non-examples:
(1) $\bullet \rightrightarrows \bullet$ (not unique)
(2) $\bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \dots$

Initial Objects (dual to terminal)

→ Let C be a category, then an initial object $I \in C$ is s.t.

$$\forall X \in C, \exists ! \text{ morphism } f: I \rightarrow X$$

→ An initial object in C is a terminal object in C^{op} .

→ e.g: For $C = \text{Set}$, \emptyset is initial.

The empty category is initial.

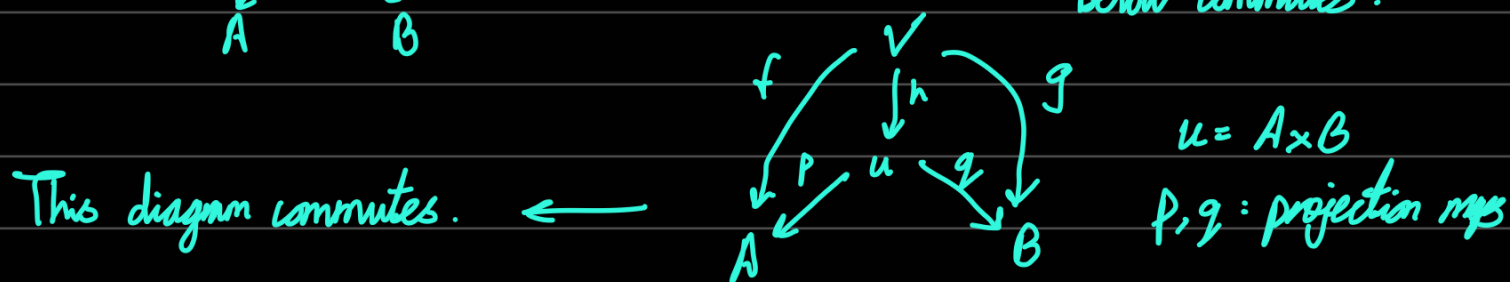
The category with one object, one morphism is terminal.

(how is empty category valid?)

Products

→ Given objects A and B in C , a product $A \times B$ is given by

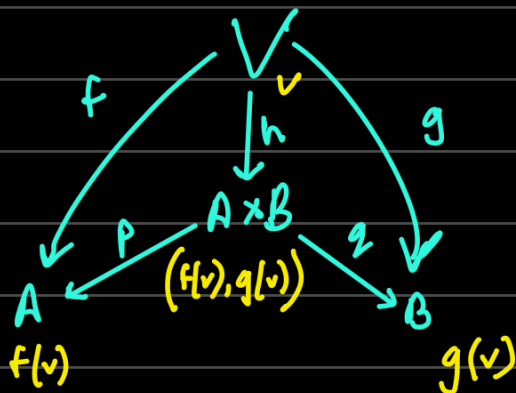
s.t. \forall $f: V \rightarrow A$ $g: V \rightarrow B$ $\exists!$ a factorisation h such that the diagram below commutes.



→ e.g: If A and B are sets then u is the Cartesian product



So,



→ In $C = \text{Vect}$, the product is the direct sum.

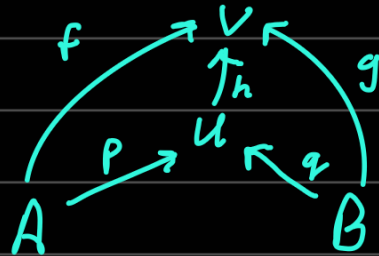
→ non e.g: tensor products are not categorical products

Coproducts

→ Given objects A and B , the coproduct u is given by

s.t. \forall V $\begin{array}{c} f \nearrow \\ A \end{array} \begin{array}{c} V \\ \nwarrow g \\ B \end{array}$, $\exists! h$ such that the diagram below commutes.

p, q : injection maps



→ In $C = \text{Set}$, u is the disjoint union of A and B .