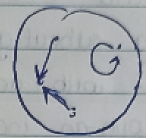
Structure, Subcategories & Insertion.

→ A category of richer structures: Endomaps of sets

- structure provides cohesion amongst the ingredients of category to make it richer.
- let S be a category ~~where~~ the objects ~~are maps~~ ~~between~~ & let T be structure (set of axioms) such that $T = \cdot\circlearrowleft$ (single endomap).

Now, T in S would be $S^{\circlearrowleft}$.

- where $S^{\circlearrowleft}$ is a category of where objects are sets-with-an-endomap.

let $X^{\circlearrowleft\alpha}$ (be object) = 

- Maps in category $S^{\circlearrowleft}$:

let the objects be $X^{\circlearrowleft\alpha}$ & $Y^{\circlearrowleft\beta}$ then $f: X \to Y$ is defined as

$f: X \to Y$ in S and $\boxed{f\circ\alpha = \beta\circ f}$ OR $X^{\circlearrowleft\alpha} \xrightarrow{f} Y^{\circlearrowleft\beta}$ in $S^{\circlearrowleft}$

- composition in category $S^{\circlearrowleft}$

let the objects be $X^{\circlearrowleft\alpha}$, $Y^{\circlearrowleft\beta}$ & $Z^{\circlearrowleft\gamma}$

$$X^{\circlearrowleft\alpha} \xrightarrow{f} Y^{\circlearrowleft\beta} \quad \& \quad Y^{\circlearrowleft\beta} \xrightarrow{g} Z^{\circlearrowleft\gamma}$$

To check:

$$X^{\circlearrowleft\alpha} \xrightarrow{f} Y^{\circlearrowleft\beta} \xrightarrow{g} Z^{\circlearrowleft\gamma}$$

$$(g\circ f)\circ\alpha = \gamma\circ(g\circ f)$$

L.H.S: $g\circ(f\circ\alpha) = g\circ(\beta\circ f)$

$= (g\circ\beta)\circ f = (\gamma\circ g)\circ f = \gamma\circ g\circ f$

→ $\boxed{gf\alpha = g\beta f = \gamma g f}$

- Identity:

$$X^{\circlearrowleft\alpha} \xrightarrow{1_X} X^{\circlearrowleft\alpha}$$

then: $1_X \circ \alpha = \alpha \circ 1_X \Rightarrow$ True.

associativity & Identity laws are also applicable in $S^{\circlearrowleft}$ as we have used them above

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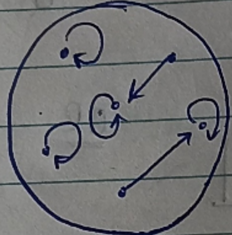
Isomorphism & more:

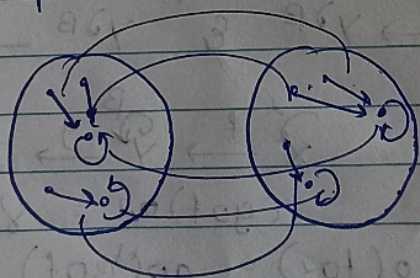
- In the category of sets-with-an-endomap, i.e, $S^{\circlearrowleft}$ isomorphism means that the structure of the two endomaps is same. In particular, the two endomaps must have same number of fixed points (Brouwer's theorem), the same number of cycles of respective lengths (i.e, 3 cycles of length 2 in each, 2 cycles of length 5 in each... etc)

- Idempotents ($e \circ e = e$)
let S^{c} be a subcategory (a part of category with more structure) with an endomap which is idempotent, i.e, a set-~~with-an~~-endomap $X^{\circlearrowleft\alpha}$ is an object in S^{c} if and only if $\alpha \circ \alpha = \alpha$.

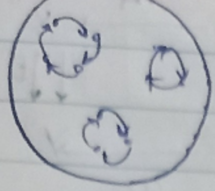


fixed point or reaches fixed point in one step.

(Here isomorphism means: correspondance between fixed points & correspondance between ~~fix~~ branches at fixed points)



- Automorphism
let $S^{\circlearrowleft}$ be a subcategory with an endomap that is invertible. $X^{\circlearrowleft\alpha}$ is an object in $S^{\circlearrowleft}$ if & only if the endomap α has an inverse, i.e. a map β such that $\alpha \circ \beta = 1_X$ & $\beta \circ \alpha = 1_X$.
Essentially there should be no branching in internal diagram (i.e, ~~due to endomap it is injective but~~ cycles make it injective & surjective)



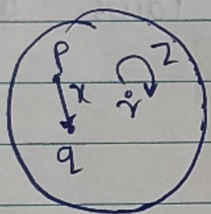
→ category of graphs

let $S^{\downarrow\downarrow}$ be category, where the object is a pair of maps with ~~same domain~~ same domain & same co-domain, so we can say that $S^{\downarrow}$ is a subcategory of $S^{\downarrow\downarrow}$

object: let X, Y be two sets, & s & t be two maps (~~structural~~ (s for source & t for target).

$$\begin{array}{c} X \\ s\downarrow\downarrow t \\ Y \end{array}$$

is the ~~obj~~ graph
∴ rather 'irreflexive directed multigraphs".



$p, q, r \in Y$ &, x & $z \in X$

$s(x) = p$ $\quad t(x) = q$

$s(z) = r$ $\quad t(z) = r$

maps: let $\begin{array}{c} X \\ s\downarrow\downarrow t \\ Y \end{array}$ $\quad \begin{array}{c} X' \\ s'\downarrow\downarrow t' \\ Y' \end{array}$ be objects

then the map in this category should 'preserve the structure'

Def: A map in $S^{\downarrow\downarrow}$ from $\begin{array}{c} X \\ s\downarrow\downarrow t \\ Y \end{array}$ to $\begin{array}{c} X' \\ s'\downarrow\downarrow t' \\ Y' \end{array}$ is a pair of map of sets

$X \xrightarrow{f_A} X'$, $Y \xrightarrow{f_D} Y'$ (A: Arrows
D: Dots)

such that $f_D \circ s = s' \circ f_A$ & $f_D \circ t = t' \circ f_A$

$$\begin{tikzcd} X \arrow[r, "f_A"] \arrow[d, shift right, "s"'] \arrow[d, shift left, "t"] & X' \arrow[d, shift right, "s'"'] \arrow[d, shift left, "t'"] \\ Y \arrow[r, "f_D"'] & Y' \end{tikzcd}$$

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Composition:

$$\begin{tikzcd} X \arrow[r, "f_A"] \arrow[d, shift right, "s"'] \arrow[d, shift left, "t"] & X' \arrow[r, "g_A"] \arrow[d, shift right, "s'"'] \arrow[d, shift left, "t'"] & X'' \arrow[d, shift right, "s''"'] \arrow[d, shift left, "t''"] \\ Y \arrow[r, "f_D"'] & Y' \arrow[r, "g_D"'] & Y'' \end{tikzcd}$$

Now, $g \circ f$:

$$\begin{tikzcd} X \arrow[r, "g_A \circ f_A"] \arrow[d, shift right, "s"'] \arrow[d, shift left, "t"] & X'' \arrow[d, shift right, "s''"'] \arrow[d, shift left, "t''"] \\ Y \arrow[r, "g_D \circ f_D"'] & Y'' \end{tikzcd}$$

to preserve the structure

$$(g_D \circ f_D) \circ s = s'' \circ (g_A \circ f_A) \text{ \& }$$
$$(g_D \circ f_D) \circ t = t' \circ (g_A \circ f_A)$$

Identity:

$$\begin{tikzcd} X \arrow[r, "1_X"] \arrow[d, shift right, "s"'] \arrow[d, shift left, "t"] & X \arrow[d, shift right, "s"'] \arrow[d, shift left, "t"] \\ Y \arrow[r, "1_Y"'] & Y \end{tikzcd}$$

I_X to $1_X = 1_Y \circ t = t$

$s \circ 1_X = 1_Y \circ s = s$

Associativity & identity laws are applicable