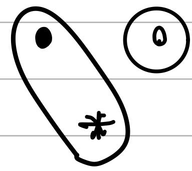
1. Generative Effects

In cat. theory there are these structures that are preserved in a category, and less the structure is preserved more suprises occurs when we observe its operations

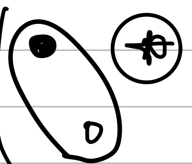
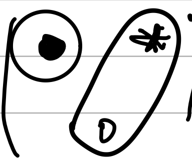
Take a system of 3 points which are connected in some 'way' → There are total 5 ways of making a system from these 3 points

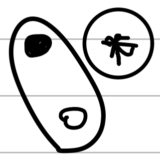
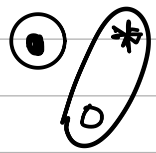
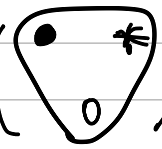


* Suppose ϕ is an observation - 'whether • is connected to * or not' which results in true in 2 cases & false in remaining 3.

Now an operation 'JOIN' is defined. ($\vee$)

* Over lap 2 systems (in mind)
* Ensure TRANSITIVITY

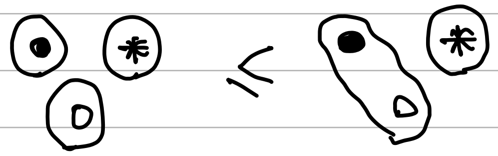
Now ϕ () = ϕ () = false

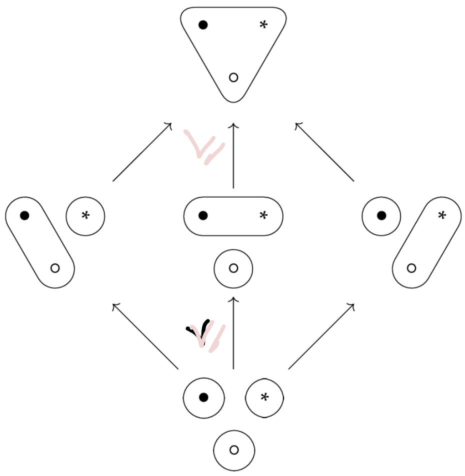
BUT.. ϕ ( $\vee$ ) = ϕ () = true

↳ This is an example of generative effect.

'JOIN'ing is not preserved by the observation ϕ

ORDER : Given systems A & B we say that $A \leq B$ if
whenever x is connected to y in A, then x is connected to y in B (not necessarily in the other way around)

eg. 



→ order hierarchy known as 'Hasse Diagrams'

Note: for any A & B in this diagram ($\vee \rightarrow$ join)

$A \leq A \vee B$ or $B \leq A \vee B$
& If $A \leq C$ & $B \leq C$;
$A \vee B \leq C$

set $\mathbb{B} = \{ \text{true}, \text{false} \}$ has an order false $\leq$ true

use the logic if $A \leq B$ then $A \rightarrow B$ here

false 'can' $\rightarrow$ true, false 'can' $\rightarrow$ false,
true $\rightarrow$ true but true $\not\rightarrow$ false

so, $f \leq t$, $f \leq f$, $t \leq t$ but $t \not\leq f$

Exercise 1.7.

1. $t \vee f$: $t \leq t \vee f$ & $f \leq t \vee f$ so $t \vee f = t$
2. $f \vee t = t$, 3. $t \vee t = t$,
4. $t \vee f = f$

In our previous ϕ observation,

ϕ preserves the $\leq$ order

{ If $A \leq B$ if $\bullet$ & $*$ is connected in A then so is in B. }

but, $\phi(A) \vee \phi(B) \leq \phi(A \cup B)$

(exercise 1.7. captures this)

but this was not preserved by the JOIN operation

$(f \vee f = t)$

Disjoint union of 2 sets : $A = \{1, 2\}$

$B = \{1, 6, 7\}$

$A \sqcup B = \{(1,1), (2,1), (1,2), (6,2), (7,2)\}$

contains elements of form $(x,1)$ & $(y,2)$ where $x \in A$, $y \in B$

PARTITION :

Definition 1.14. If A is a set, a *partition* of A consists of a set P and, for each $p \in P$, a nonempty subset $A_p \subseteq A$, such that

$$A = \bigcup_{p \in P} A_p \qquad \text{and} \qquad \text{if } p \neq q \text{ then } A_p \cap A_q = \varnothing. \tag{1.15}$$

Book def :

We may denote the partition by $\{A_p\}_{p \in P}$. We refer to P as the set of *part labels* and if $p \in P$ is a part label, we refer to A_p as the p^{th} *part*. The condition (1.15) says that each element $a \in A$ is in exactly one part.

We consider two different partitions $\{A_p\}_{p \in P}$ and $\{A'_{p'}\}_{p' \in P'}$ of A to be the same if for each $p \in P$ there exists a $p' \in P'$ with $A_p = A'_{p'}$. In other words, if two ways to divide A into parts are exactly the same—the only change is in the labels—then we don't make a distinction between them.

It is basically dividing the elements into groups - no element is common in any 2 groups & together they form the set

* All the diagrams above are partitions of the set $\{\circ, \bullet, *\}$

SYMBOLS :

$\rightarrow$ arbitrary func, $\twoheadrightarrow$ surjective, $\rightarrowtail$ injective

$\cong$ bijective

$F ; G$ means $G(F(x))$ {first F then G}

<u>EQUIVALENCE RELATION</u> : on A (A is a set)

It is a binary relation & follows reflexivity, symmetry & transitivity.

Prop : " There is a one-to-one correspondence b/w ways to partition A & equivalence relations on A."

$\downarrow$

proof : If a, b are in same partition they are related, by doing so we see it follows requirements of equivalence relation we can do the other way around too.

Def:

Given an equivalence relation $(\sim)$ in A, the quotient $A/\sim$ of A under $\sim$ is " set of parts of corresponding partition "

* Partition on a set A as surjective functions out of A :

" Given a surj. func. from $A \twoheadrightarrow P$, the preimages of each element of P form a partition."

* Preorders :

A equivalence relation where the symmetry condition is loosened.

Infix notation : $\leq$

We call $(X, \leq)$ → a set X equipped with a preorder.

- Discrete preorder ⇒ simply a collection of points of form $x \leq x \ \forall x$ & if $x \neq y$ neither of $x \leq y$ or $y \leq x$ holds
  $(X, =)$

  elements can't be "ordered" → automatically follows preorder def.

for any set

- Codiscrete preorder ⇒ simply the total binary relation $X \times X$ ⇒ $x \leq y$ & $y \leq x \ \forall x, y \in X$

$\mathbb{B}$ → {false, true} , Preorder is false $\leq$ true
(use false = 0 & true = 1)

* A preorder is a partial order (poset) if we have

$$\underbrace{x \cong y \text{ implies } x = y}$$

→ This prop. is called "skeletality"

⇒ Graph :

Definition 1.36. A *graph* $G = (V, A, s, t)$ consists of a set V whose elements are called *vertices*, a set A whose elements are called *arrows*, and two functions $s, t \colon A \to V$ known as the *source* and *target* functions respectively. Given $a \in A$ with $s(a) = v$ and $t(a) = w$, we say that a is an arrow from v to w.

By a *path* in G we mean any sequence of arrows such that the target of one arrow is the source of the next. This includes sequences of length 1, which are just arrows $a \in A$ in G, and sequences of length 0, which just start and end at the same vertex v, without traversing any arrows.

* total order : If $\underline{\forall}\ x, y$ either $x \leq y$ or $y \leq x$

(A codiscrete preorder has 'and' instead of 'or')

Eg : $(\mathbb{N}, \leq)$

⇒ Partition from preorder : all the complete loops in a preorder form a part & incomplete ones are parts with single elements

Eg :

*Preorder of Power sets : $(P(X), \subseteq)$

⇒ Ordering of Partitions by 'fineness' :

the coarsest partition is, one part (all in one)

finest partition is, every thing is in its own partition formally, A partition P is finer than Q if for every part $p \in P$ there is a part $q \in Q$ such that $A_p \subset A_q$ (recall what A_s are)

Upper sets :

Example 1.54 (Upper sets). Given a preorder $(P, \le)$, an *upper set* in P is a subset U of P satisfying the condition that if $p \in U$ and $p \le q$, then $q \in U$. "If p is an element then so is anything bigger." Write $\mathrm{U}(P)$ for the set of upper sets in P. We can give the set U an order by letting $U \le V$ if U is contained in V.

For example, if $(\mathbb{B}, \le)$ is the booleans (Example 1.34), then its preorder of uppersets $\mathrm{U}(\mathbb{B})$ is

$$\boxed{\begin{array}{c}\{\mathtt{true}, \mathtt{false}\} \\ \uparrow \\ \{\mathtt{true}\} \\ \uparrow \\ \varnothing\end{array}}$$

The subset $\{\mathtt{false}\} \subseteq \mathbb{B}$ is not an upper set, because $\mathtt{false} \le \mathtt{true}$ and $\mathtt{true} \notin \{\mathtt{false}\}$.

⇒ Monotone map : $(A, \le_A) \to (B, \le_B)$

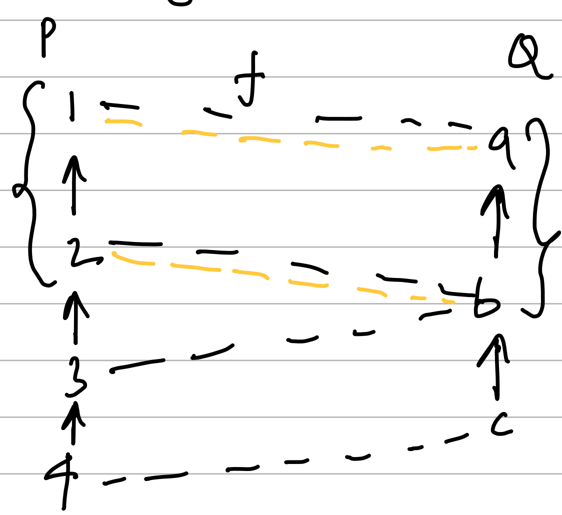
is a func. $f : A \to B$ s.t. $\forall\, x, y \in A$ if $x \le_A y$

then $f(x) \le_B f(y)$ (Order is preserved)

* Dagger preorder = equivalence $(x \le y$ & $y \le x)$

* An isomorphism in preorders is basically just a "relabeling of the elements"

Ex 1.79



pullback along f

⇒ MEET : Least Upper Bound } universal
⇒ JOIN : Greatest Lower Bound } properties

similar to 'AND', '$\cap$'

↳ similar to 'OR', '$\cup$'

$$* \quad p = \wedge A \quad \Rightarrow \quad p \text{ is a meet}$$
$$* \quad q = \vee A \quad \Rightarrow \quad q \text{ is a join}$$

* A preorder can have no meet or join (OR) multiple meets & joins

⇒ We say a monotone map f preserves meets if $f(a \wedge b) \cong f(a) \wedge f(b)$ & preserves joins if $f(a \vee b) \cong f(a) \vee f(b)$ i.e. if there are no generative effects

⇒ Galois connection : b/w 2 preorders P & Q is a pair of monotone maps $f: P \to Q$ & $g: Q \to P$ s.t.

$$f(p) \leq q \quad \text{iff} \quad p \leq g(q)$$

then f is left adjoint & g is right adjoint of the galois connection

Eg: the map $f(3x-)$ $\mathbb{Z} \to \mathbb{R}$ (sends $x \to 3x$)
$g: \mathbb{R} \to \mathbb{Z}$ be $\lceil -/3 \rceil \to$ greatest int above $-/3$

$$\lceil x/3 \rceil \leq y \quad \text{iff} \quad x < 3y$$

so, f, g form a galois connection

* prop : Left adjoints preserve joins & right adjoint preserve meets.

$$g(\wedge A) \cong \wedge g(A)$$

$$\& \quad f(\vee A) \cong \vee f(A)$$

⇒ Closure operator : is a monotone map $P \to P$ (P is a preorder)
s.t. $\forall p \in P$

(a) $p \leq j(p)$

(b) $j(j(p)) \cong j(p)$

⇒ Level shifting : We can have preorder of preorders which we called as level shift. (relation of relations & so on)

If $\mathrm{POS}(S)$ is the set of all preorder relations

if $\leq$ below $\leq'$, the we say $\leq \sqsubseteq \leq'$

if $a \leq b \Rightarrow a \leq' b \quad \forall a, b \in S$.