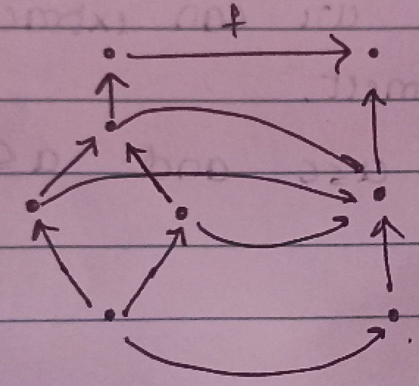
III Monotone Maps:

Def: A monotone map between preorders $(A, \leq_A)$ and $(B, \leq_B)$ is a function $f: A \rightarrow B$ such that, for all elements $x, y \in A$ if $x \leq_A y \leftrightarrow$ then $f(x) \leq_B f(y)$

Ex:

A =  = B

i) this map is preorder (structure) preserving.

ii) For any pre-order $(P, \leq_P)$, the identity function is monotone.

If $(Q, \leq_Q)$ and $(R, \leq_R)$ are preorders and if $f: P \rightarrow Q$ and $g: Q \rightarrow R$ are monotone, then $(f ; g): P \rightarrow R$ is also monotone.

iii) For a set X, we define $Prt(X)$ as set of partitions on X [that is number of surjection from $X \rightarrow P$ ~~such tha~~ $n(X) \geq n(P)$]

Any surjective function $f : X \twoheadrightarrow Y$ induces a monotone map $f^* : Prt(Y) \to Prt(X)$, going backwards.
It is defined by sending a partition (an element of $Prt(X)$) $s : Y \twoheadrightarrow P$ to the composite $f ; s : X \twoheadrightarrow P$
(mention of dagger preorder in page no. 21)

- Isomorphism of monotone maps:

Def let $(P, \leq_P)$ and $(Q, \leq_Q)$ be pre-orders. A monotone function $f : P \to Q$ is called an isomorphism if there exists a monotone map $g : Q \to P$ such that $f ; g = id_P$ and $g ; f = id_Q$. For and $p \in P$ and $q \in Q$, we have

$$p = g(f(p)) \text{ and } q = g(f(q))$$

- Pullback map:

Def: let P and Q be preorders, and $f : P \to Q$ be a monotone ~~mon~~ monotone map map. Then we can define a monotone map $f^* : U(Q) \to U(P)$ sending an upper set $U \subseteq Q$ to the upper set $f^{-1}(U) \subseteq P$. We call this <u>pullback along</u> f