Week Six

Siva Sundar, EE23B151

August 2024

19th August

Section 19: Terminal Objects

- From any set (even for **null set**), there is **only one map** to a *singleton* set.
- T is an object in a category $\mathcal{C}$, which is said to be **terminal** only if for any object X in $\mathcal{C}$:
  - ⋆ at least one map exists from X to T.
  - ⋆ that map should be the only map from X to T.

  Using these two conditions we can say that: (See page.229)

  "*There exists multiple terminal objects which are isomorphic to each other.*"

- In the category of *endomaps*, we can say that the **singleton set** equipped with an endomap from the **point to itself**, is a terminal '*set-with-endomap*'.

  The mapping from an endomap X to this terminal object also follows the 'structure preserving rule'.

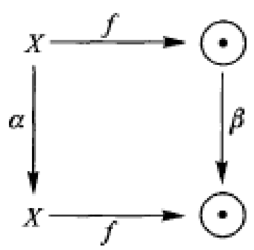


Figure 1: Map from X to T

Section 20: Points

- In the start of this section, we see an example which shows how we can use a *terminal object* (defined in the category) to **select an item** from an object (of the same category). Hence, we can define:

  $$\text{“}\textit{A point of an object } X \textit{ is the map } T \longrightarrow X\text{”}$$

  where, T is the terminal object of the category.

- In different categories, the meaning of the word 'point' is different from what we think of. For example, in the category of endomaps, the term 'point' refers to **fixed point** (See page.232). So, if an endomap does **not** have a fixed point, we say it doesnt have '*points*' (which doesn't mean it doesnt have elements!) (See page.233)

Category	Terminal object	'Points of X' means...
$\mathcal{C}$	T	map $T \longrightarrow X$
$\mathcal{S}$	$\boxed{\cdot}$	element of X
$\mathcal{S}^{\circlearrowleft}$ endomaps of sets	$\boxed{\cdot\circlearrowleft}$	fixed point or equilibrium state

20th August

Product

- A product of A and B (in category $\mathcal{C}$, also called **factors of P**) is:
  - ⋆ an object P in $\mathcal{C}$.
  - ⋆ a pair of maps: $P \xrightarrow{p_1} A$ and $P \xrightarrow{p_2} B$ such that,
    for every other object X in $\mathcal{C}$, with pair of maps $X \xrightarrow{q_1} A$ and $X \xrightarrow{q_2} B$, there exist **exactly one map** $X \xrightarrow{q} P$: $q_1 = p_1 \circ q$ and $q_2 = p_2 \circ q$.

  From this definition, we can say that if there exists two products sharing the same **factors**, the products must be **isomorphic**.
- '3D-Space' can be considered as the **product** of three *linearly independent* axes.
- For Products in 'categories of endomaps',

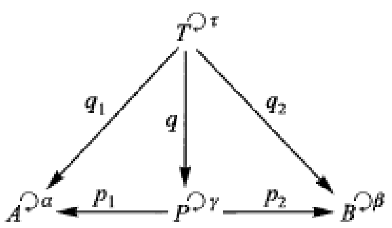
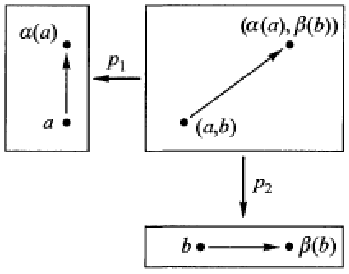


Figure 2: Product in category $S^{\circlearrowleft}$



Internal diagram of $A^{\circlearrowleft} \longrightarrow P \longleftarrow B^{\circlearrowleft}$

We get another condition from the '*structure preserving rule*':

$$\gamma(a,b) = (\alpha(a), \beta(b))$$

Proof:
As we know, P contains elements of the type (a,b) where $a \in A$ and $b \in B$.

$$p_1(a,b) = a \quad p_2(a,b) = b$$

$$\Rightarrow (p_1 + p_2)(a,b) = a + b$$

$$(Or) \quad (a,b) = (p_1 + p_2)^{-1}[a + b] \tag{1}$$

By structure preserving conditions:

$$p_1\gamma = \alpha p_1 \quad p_2\gamma = \beta p_2$$

$$\Rightarrow (p_1 + p_2)\gamma = \alpha p_1 + \beta p_2$$

$$(Or) \quad \gamma = (p_1 + p_2)^{-1}[\alpha p_1 + \beta p_2]$$

Now, applying γ on the element (a,b) yields:

$$\gamma(a,b) = (p_1 + p_2)^{-1}[\alpha p_1 + \beta p_2](a,b)$$

$$\Rightarrow (p_1 + p_2)^{-1}[\alpha p_1(a,b) + \beta p_2(a,b)]$$

$$\Rightarrow (p_1 + p_2)^{-1}[\alpha(a) + \beta(b)]$$

$$\boxed{\gamma(a,b) = (\alpha(a), \beta(b))} \qquad \text{(used (1))}$$

Petri-nets

It is a 4-tuple[1] $N = (P, T, F, m_0)$, where:

- P: set of all **Places**. (p, q, r, s in fig.3)
- T: set of all **Transitions**. (t, u, v)
- P and T are **disjoint**.
- F: **Flow relation** that defines the *arcs*[2].
  $F \subseteq (P \times T) \cup (T \times P)$
  $(F = \{(p,t),(r,t),(v,p),(t,q)...(From, To)\})$
- m_0 is the initial marking, assigns **tokens**[3] to their initial places.
  $m_0 : P \longrightarrow \mathbb{N}$ ($\mathbb{N}$ is the set of *natural numbers*)
  (Here, $m_0 = [p, r^2]$, which reads, p has 1 element, r has two elements.)

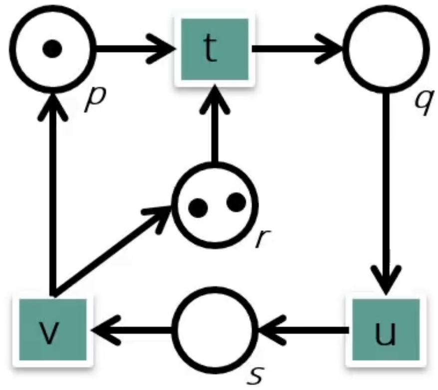


Figure 3: Petri-net example

Arcs and Transitions:

To check whether an **arc** should get **activated** or not, we use the **weight function**, defined as $w : F \to \mathbb{N}^0$ ($\mathbb{N}^0$ is the set of Natural numbers **with zero**),

$$w(p,t) = \begin{cases} 1 & \text{if } (p,t) \in F \\ 0 & \text{otherwise} \end{cases}$$

To **fire** a transition, there must be **at least one token** in **all the input places** of a transition, as a transition uses one token from all input places to create tokens in its output places. Mathematically,

$$\forall p \in P, \quad w(p,t) \leqslant m(p)$$

After firing a transition, it **creates tokens** in its **output** places, hence, if m' represent next state:

$$\forall p \in P, \quad m'(p) = m(p) - w(p,t) + w(t,p)$$

Example (Fig.3)

Given, $m_0 = [p, r^2]$, the transition t uses one element from its inputs (p,r) and we see the next state:

$$[p, r^2] \xrightarrow{t} [p, r^2] - [p, r] + [q] = [r, q] = m_1$$

Now, as p is empty, t cant fire, but now u can fire as its inputs have elements:

$$[r, q] \xrightarrow{u} [r, q] - [q] + [s] = [r, s] = m_2$$

Now, u cant fire, but v can:

$$[r, s] \xrightarrow{v} [r, s] - [s] + [r, p] = [p, r^2] = m_0$$

Now we can use these states and draw the **Labeled Transition System**, which, in this case, is a cycle of three states.

[1]An **ordered** set with 4 elements.

[2]Mappings between places and transitions. The places from which an arc runs to a transition are called the **input** places of the transition; the places to which arcs run from a transition are called the **output** places of the transition.

[3]Elements in each place, denoted by dots.

22nd August

Section 22: Universal mapping properties and Incidence relations

- In the definition of terminal objects, we mentioned 'for all X', 'for each X' or 'for every X'. Such properties can be called **Universal** properties.
- In a category $\mathcal{C}$, we can find a small class $\mathcal{A}$ of objects in $\mathcal{C}$ which can be used to understand more complex objects X by means of maps $A \xrightarrow{x} X$ from objects in $\mathcal{A}$. This map x is called figure of shape A in X.
- In sets, the points of X are in a sense all there is to X, so that we often use the words 'point' and 'element' interchangeably, whereas in dynamical systems points are fixed states, and in graphs they are loops.
- In category of sets, *'if two maps agree on all points, they are the same map'*.

  Such a property is not possible in category of endo-maps and graphs 'using points'.
- *Given, any pair of maps:* $X^{\circlearrowright^{\alpha}} \underset{g}{\overset{f}{\rightrightarrows}} Y^{\circlearrowright^{\beta}}$ *, if for all figures* $\mathbb{N}^{\circlearrowright^{\sigma}} \xrightarrow{x} X^{\circlearrowright^{\alpha}}$ *of shape* $\mathbb{N}^{\circlearrowright^{\sigma}}$ *it is true that* $fx = gx$*, then* $f = g$. (See page.246-248)
- **Incidence relation**:

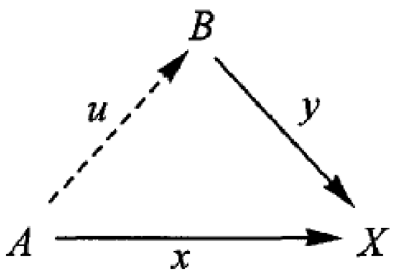


Figure 4: Case 1

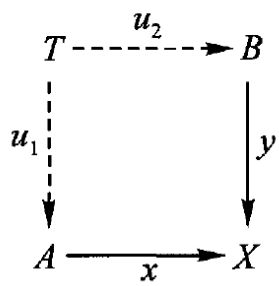


Figure 5: Case 2

  Consider we have figures x from A to X and figure y from B to X, then we can check the extend of overlap (or incidence) of these two figures:

  If we have a map u from A to B, such that $yu = x$ (fig.4), then we say x *is incident to* y.

  If we have another object T which gives maps u_1 and u_2 as shown in fig.5 such that $yu_2 = xu_1$, we can say the same. In this case however, we are introducing another figure from T *to* X which splits the square to two triangles which satisfy the first case.
- In the category of graphs, an object with one dot (say D) and an object with two dots linked by an arrow (say A), constitute the basic figure-type.
  We can say that:
  "Given any two maps f,g from X to Y, if $fx = gx$ *for all figures* x *from D to X of shape D and A to X of shape A, then* $f = g$*"*

References

[1] Wikipedia: Petri net

[2] Youtube: A formal introduction to Petri nets