V Cohesion :

We ~~are~~ started this chapter to study generative effects and how can we "remove them".

• What are generative effects in monotone map f?

We say that a monotone map $f: P \to Q$ preserves meets if $f(a \wedge b) \cong f(a) \wedge f(b)$ for all $a, b \in P$.
We say f preserves joins if $f(a \vee b) \cong f(a) \vee f(b)$ for all $a, b \in P$

We say that a monotone map $f: P \to Q$ has a generative effect if there exists elements $a, b \in P$ such that
$f(a) \vee f(b) \ncong f(a \vee b)$

Proof of generative effect of $f: P \to Q$
Given : $a \in P$ and $b \in P$ then $f(a), f(b) \in Q$
To prove: $f(a) \vee f(b) \ncong f(a \vee b)$
Proof: $a \vee b$ is the least upperbound of $\{a, b\} \subseteq P$ (1)
$f(a) \vee f(b)$ is the least upper bound (2) of $\{f(a), f(b)\} \subseteq Q$
$a \leq a \vee b \rightarrow f(a) \leq f(a \vee b)$ $\because f$ is Monotone
$b \leq a \vee b \rightarrow f(b) \leq f(a \vee b)$

$f(a) \vee f(b) \leq f(a \vee b)$ $[\because f(a) \vee f(b)$ is the least upper bound $]$

- How are generative effects removed?

<u>Galois connections</u>.
a Galois connections are pair of maps ~~indoto~~, ~~indowed~~ with left adjoint and right adjoint, a conditions to preserve joins and meets.
The generalized idea of Galois connections (in Category Theory) is adjunctions.

Def: A ~~getor~~ Galois connection between preorders P and Q is a pair of monotone maps $f: P \to Q$ and $g: Q \to P$ such that

$$f(p) \leq q \quad \text{iff} \quad p \leq g(q)$$

f: left adjoint of g (of the GC)
g: right adjoint of f (of the GC)

- How Galois connections preserve meets and joins?

i) Suppose that $f: P \to Q$ and $g: Q \to P$ are monotone maps. The following are equivalent:
a) f and g form a Galois connection where f is left adjoint to g,
b) for every $p \in P$ and $q \in Q$ we have

$$p \leq g(f(p)) \quad \text{and} \quad f(g(q)) \leq q$$

Proof of $f(a \vee b) \cong f(a) \vee f(b)$.

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Preposition (To prove):

i) (Right adjoint preserves meets), i.e., let $f: P \to Q$ be left adjoint to $g: Q \to P$. Suppose $A \subseteq Q$ any subset and let $g(A) = \{g(a) \mid a \in A\}$ be it its image. Then if has a meet $\wedge A \in Q$ then $g(A)$ has a meet $\wedge g(A) \in P$ and we have:

$$g(\cancel{\&} \wedge A) = \wedge g(A)$$

ii) (left adjoint preserves joins), i.e, if $A \subseteq P$ is any subset that has a join $\vee A \in P$, then $f(A)$ has a join $\vee f(A)$ in Q and we have:

$$f(\vee A) = A \vee f(A)$$

Given:

i) $f: P \to Q$, $g: Q \to P$, $A \subseteq Q$, $m = \wedge A$, g & f are monotone maps

ii) $\cancel{f: P \to Q}$, $A \subseteq P$ and $n = \vee A$ (n is join of A)

Proof:

i) $m = \wedge A$

for $\forall a \in A$ $g(m) \leq g(a)$ ($\because$ g is monotone map)

. So ~~g is a~~ $g(m)$ is <u>a</u> <u>lower bound</u> of the set $g(A)$

consider b to be any ~~another~~ lower bound of $g(A)$. So $\forall a \in A$ we have $b \leq g(a)$. By def^n of GC $f(b) \leq a$ $\forall a$. So $f(b)$ is <u>a lower bound</u> of the set A. W.K.T m is <u>meet of A</u> so, <u>$f(b) \leq m$</u>.

Now again using GC, $\boxed{b \leq g(m)}$ for any lower bound b. Hence, we say that $g(m)$ is greatest lower bound

$$g(\wedge A) = \wedge g(A) \Rightarrow \wedge g(A) = g(m) \quad [m = \wedge A]$$

ii) $n = \vee A$

for all $a \in A$ $a \leq n$.

$f(a) \leq f(n)$ [$\because f$ is monotonic map]. so $f(n)$ is an upper bound of the set $f(A)$.

Consider c to be any other upper bound of $f(A)$

so, $f(a) \leq c$ $\forall a \in A \subseteq P$. ~~so f(a)~~ ~~c is a lower~~ ~~bound of the set~~ A. $a \leq g(c)$ ($\because$ $g(c)$), so $g(c)$ is a lower bound of the set w.r.t $\vee A = n$

so, $n \leq g(c)$ again $\boxed{f(n) \leq c}$ ($\because$ $g(c)$)

for any upper bound of c.

Hence, $f(n)$ is least upper bound or join of $f(A)$

$f(\vee A) = \vee f(A)$

- What conditions on f allow us to remove generative effects?

<u>Adjoint</u> <u>functor</u> <u>theorem</u>: <u>for</u> <u>pre-orders</u>.

Suppose Q is a preorder that has all meets if and only if is a right is a right adjoint (has left adjoint)

Similarly, if P has all joins and Q is any preorder, a monotone map $f: P \to Q$ preserves joins if and only if it is a left adjoint.