DYNAMIX

* Compositionality: Cases where systems/relationships can be combined to form new systems/relationships.

Observation is inherently lossy. (To extract info, drop details)

Central theme of category: Study of structures and structure-preserving maps.

A map $f: X \to Y$ is a kind of observation of X "via" a specified relationship it has with another object Y.

* Which aspects of X one wants to preserve under f becomes the qn (what category are you working in)

Eg: Order preserving: $x \leq y \Rightarrow f(x) \leq f(y)$

The structure "order" is preserved.

Metric - preserving : $|x-y| \leq |f(x)-f(y)|$

* In category theory, we will keep control over which aspects of our systems are being preserved under mapping.

(Less structures getting preserved) $\Rightarrow$ (More suprises with the operation)

Generative effects.

We would need system (Cat), observation (f), system-level operation (NOT preserved by obs.)

A simple system: Consider 3 points. $(\bullet, \circ, *)$

system

Connections are symmetric $(a \to b) \Rightarrow (b \to a)$

Transitive: $(a \to b), (b \to c) \Rightarrow (a \to c)$

Operation "Joing": Combine their connections

$A \vee B$ has connection btw x & y if $\exists$ $z_1, \dots, z_n$ (x is conn to $z_1 \to z_i \to z_{i+1} \to z_n \to y$)

(a) (b) (c)

The fn ϕ (sys) is a boolean that returns if (•) and (✱) are connected.

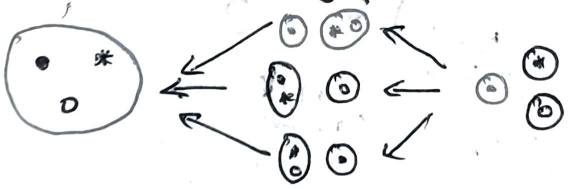
$\phi(a) = F$ $\phi(b) = F$ $\phi(a \vee b) = T$ ϕ does not preserve the structure.

∴ We conclude that it should also take o as input ⇒ (More info is required)

Ordering systems: (Joining is not preserved by ϕ) but order will get ?

* Systems are ordered in some hierarchy.

$A \leq B \Rightarrow$ (If x is connected to y in A) then x "must" be connected to y

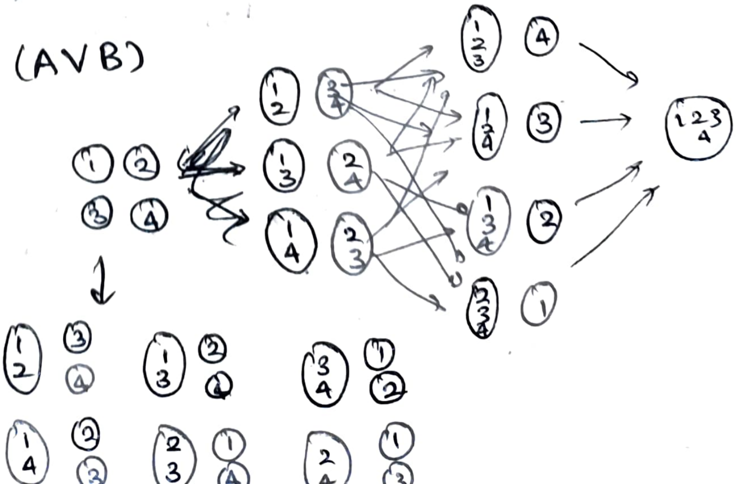


Arrow from A to B means $A \leq B$

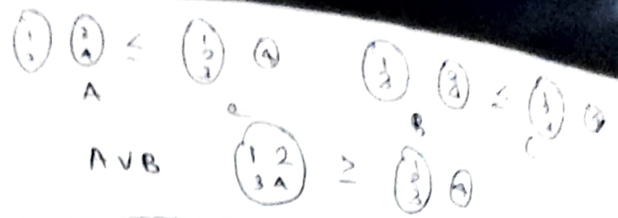
* This diagram is called Hasse diagrams

$A \leq (A \vee B)$ $B \leq (A \vee B)$

ex:



$(A \vee B) \leq C$ is false



6) $\mathbb{B} = \{$true, false$\}$ (false → true)

This satisfies $\neg(A \leq B)$ iff (A is True and B is false)

logical op is $A \Rightarrow B$

- $A \vee B$ (least element that is greater than both A & B)

∴ $F \vee F = F$ else (T)

Coming back to $(\circ, 0, *)$

$$\boxed{\underset{(F)}{\phi(A)} \vee \underset{(F)}{\phi(B)} \leq \underset{(T)}{\phi(A \vee B)}}$$ (It is not $\phi(A) \vee \phi(B) = \phi(A \vee B)$)

Generative effect.

- So we have seen ϕ preserves order but not join operation and we saw an inequality with it also.

<u>Order:</u>

Preview of sets, fns, relations: Set: collection of things known as elements

$\mathbb{B} \Rightarrow$ set of booleans. $\underline{3} = \{1, 2, 3\}$

Union: $\phi \cup A = A$ Intersection: $\phi \cap A = \phi$ $A \times B$: (x, y) $x \in A$ $y \in B$

$A \sqcup B$: $(x, 1), (y, 2)$ $x \in X$, $y \in Y$ $\{1, h\} \cup \{1, 2, 3\}$
(Disjoint union) $= \{h, 1, 2, 3, 1\}$

- Relation btw X and Y is a subset of $X \times Y$.

Partitions: If A is a set, a partition of A consists of a set P and for each $p \in P$, a nonempty subset $A_p \subseteq A$ such that $A = \bigcup_{p \in P} A_p$ and if $p \neq q$ $A_p \cap A_q = \phi$ ☺

Binary relation: relation btw X and X.

Equivalence relation: It is a binary relation on A if
(i) $a \sim a$ $\forall a \in A$
(ii) $a \sim b$ iff $b \sim a$ $\forall a, b \in A$
(iii) if $a \sim b$ and $b \sim c$ then $a \sim c$

* Let A be a set. There is a one-one correspondence between partition(A) and equivalence relations on A.

Proof: $(\sim)$ is an equivalente relation.

Closed set: $\forall x \in X$ and $x' \sim x$ we have $x' \in X$

Connected: $x \sim y$ $\forall x, y \in X$ $p \in$ (closed, connected sets)

(i) A_p is non-empty. As $a \in A$ and $a \sim a$

A_p can have that subsets

(iii) $p \neq q$ $A_p \cap A_q \neq \phi$

Say $A_p \cap A_q \neq \phi = \{c\}$
$\exists x_p \in A_p$ s.t $x_p \sim c$
$\exists x_q \in A_q$ s.t $c \sim x_q$
(contradiction)

* Given an equivalence relation $(\sim)$ on A, we say quotient $A/\sim$ of A under $\sim$ is the set of parts of the corresponding partition.

Functions: Most frequently used relations.

Let S and T be sets. fn from S to T is a subset $F \in (S \times T)$. $\forall s \in S$ $\exists$ unique $t \in T$, s.t $(s,t) \in F$.

Surjective fn, $\forall t \in T$ $\exists$ a ~~unique~~ $s \in S$ with $F(s) = t$.

Injective fn: If $\forall$ $t \in T$ and $s_1, s_2 \in S$ s.t $F(s_1) = F(s_2) = t$

$\Rightarrow s_1 = s_2$

Arbitrary fn: $\underline{3} \to \underline{3}$; Surjective fn: $\underline{3} \twoheadrightarrow \underline{2}$

Injective fn: $\underline{2} \rightarrowtail \underline{3}$; Bijective fn: $\underline{3} \xrightarrow{\cong} \underline{3}$

Non-surjectiveness & Non-injectiveness can be anywhere

$id_{X}(x) = x$ (Identity fn)

* A partition on set A can be understood in terms of surjective fn. $f: A \twoheadrightarrow P$, where P is any other set, $f^{-1}(p) \subseteq A$, one for each element $p \in P$.

$f(11) = a$ $f(12) = a$ $f(13) = b$

$f(21) = c$ $f(22) = d$ $f(23) = d$

$F: X \to Y$ and $G: Y \to Z$ $\quad X \to Z$ is $G(F(x))$ $\forall x \in X$

→ A preorder relation on a set X is a binary relation on X, here donated with ($\leq$)

(a) $x \leq x$; and b) if $x \leq y$ and $y \leq z$ then $x \leq z$

(This satisfies the cdt of equivalence except symmetry)

• We call a pair $(X, \leq)$ consisting of a set equipped with a pre-order relation as preorder

Discrete preorders : Order relations on X are of the form $x \leq x$; if $x \neq y$ neither $x \leq y$ nor $y \leq x$ holds true

Co-discrete preorders : $(X, \leq)$ Binary relation on $X \otimes X$ $\forall x, y$ in X we have $x \leq y$

Booleans: $\mathbb{B} = \{F, T\}$ is a pre-order $F \leq T$

• A preorder is a partial order if we have $(x \approx y \Rightarrow x = y) \rightarrow$ This condition is called skeletality

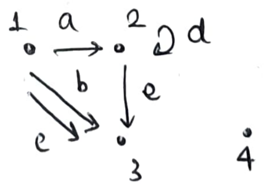
→ Discrete ~~partial~~ pre order is a partial order

Codiscrete preorder is a partial order for 1 element set.

• A graph consists of $(G = (V, A, s, t))$ a set V whose elements are vertices, A : arrow set and two fns $s, t : A \to V$ (source, target fns)

Given $a \in A$; $s(a) = v$ and $t(a) = w$ $(v \to w)$

Eg: $G =$

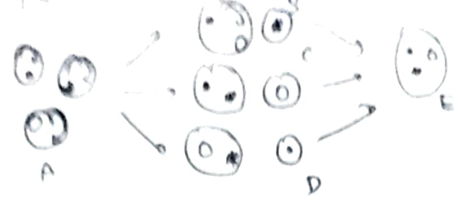


$s(a) = 1$ $t(a) = 2$
$s(b) = 1$ $t(b) = 3$
$s(c) = 1$ $t(c) = 3$
$s(d) = 2$ $t(d) = 2$
$s(e) = 2$ $t(e) = 3$

* From every graph we can get a pre-order. (Hasse diagram itself is a graph). presentation of a preorder $(P, \leq)$. Elements of P are vertices V in G and $\leq$ is given by $v \leq w$ iff $\exists$ a path $v \to w$. (

1.41) Yes it is a preorder

1.42)



$A \le B$ $A \le C$ $A \le D$ $A \le$
$B \le E$ $C \le E$ $D \le E$

1.43) Total order: $\forall x, y$ either $x \le y$ or $y \le x$

x, y of a preorder are comparable $x \le y$ or $y \le x$.

* Total order is preorder where every two elements are comparable.

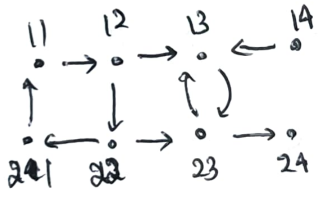
1.44) No two discrete elements are comparable.

1.45) $\mathbb{N} = \{0, 1, 2, \dots\}$ are a pre-order $0 \le 1$ and $5 \le 100$.

$0 \to 1 \to 2 \to 3 \dots$

1.48) Usual ordering is indeed a total order.

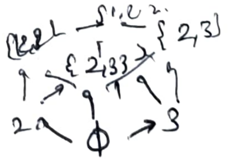
Partition from pre-order:



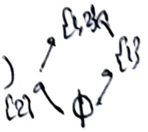
$(11 \le 21)$ & $(21 \le 11)$ $\therefore$ equivalence

Power set: $P(X)$ is the power set (set of subsets of X)

Ordering all elements of $P(X)$ will form a cube in n-dim. Eg: $X = \{1, 2, 3\}$

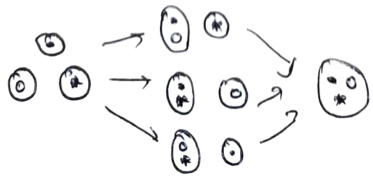


$P(\phi) = \phi$ (That's all) $P\{1\} = \phi \to 1$ $P(\{1, 2\})$



* The set $Prt(A)$: set of all partitions of A.

For $\{\cdot, *, o\}$ it would have 5.



→ We can order partitions by fineness.
A partition P is finer than a partition Q,
$\forall\, p \in P, \exists\, q \in Q$ s.t $A_p \subseteq A_q$ interms of
surjective fns: $f: A \twoheadrightarrow P$ $g: A \twoheadrightarrow Q$ $\exists\, h: P \to Q$ s.t
$f \circ h = g$ (P is finer than Q) or (Q is coarser than P

Obs. identity obv.

P has more elem. than Q

1.53] Coarsest partition (group everything to 1)
Finest partition (individual groupping)

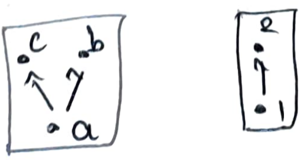
1.54] Preorder $(P, \le)$ An upper set in P is
a subset $\mathcal{U}$ of P satisfying if $p \in \mathcal{U}$ and
$p \le q$ then $q \in \mathcal{U}$. "If p is an element then
so is the anything bigger".

For $(\mathbb{B}, \le)$ $\phi \to \{\text{true}\} \to \{\text{False}, \text{true}\}$

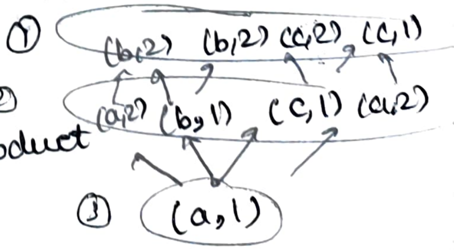
1.55) Upperset of discrete preorder will
be single have just all elements as they
are not related

1.56] Product preorder: Given pre-orders
$(P, \le)$ and $(Q, \le)$ we define $(P \times Q, \le)$
by the cdt $(p, q) \le (p', q')$ iff $p \le p'$ and $q \le q'$

1.57]



Product



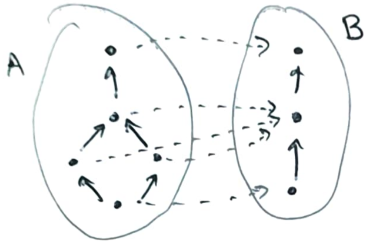
$\phi \to ① \to ② \to ③$

Opposite pre-order: $(P, \leq)$ and pre-order $(P, \leq^{op})$
$p \leq^{op} q$ iff $q \leq p$

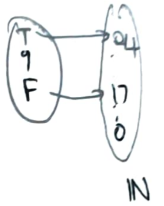
Monotone maps:

* The most important sort of relationship between preorders is called a monotone map, considered the right notion of "structure-preserving map" for preorders.

Defn: $(A, \leq_A)$ and $(B, \leq_B)$ is a fn $f: A \to B$ s.t $\forall\, x, y \in A$, if $x \leq_A y$ then $f(x) \leq_B f(y)$



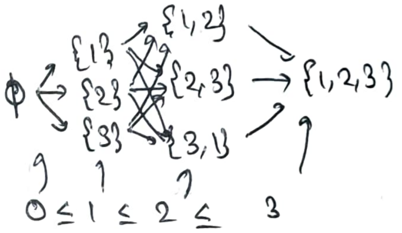
eg:



(Monotone map preserves order)

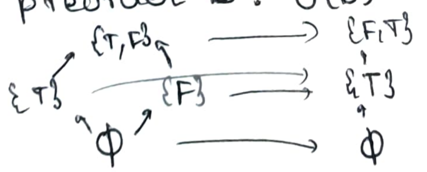
1.62) The map $|\cdot|: P(X) \to \mathbb{N}$ sending each subset S to its no. of elements $|S|$.

1.63) $X = \{0, 1, 2\}$



1.64] $(P, \leq)$, the map $U(P) \to P(P)$ sending each upper set of $(P, \leq)$ to itself. It is a monotone map. ↳ (If nothing is related (discrete) it is id)

1.65] preorder $\mathbb{B}$. $U(\mathbb{B})$



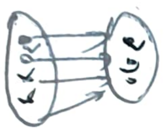
1.66] Yoneda lemma:

$\uparrow p := \{p' \in P \mid p \leq p'\}$ is an upper set as $p_o \in \uparrow P \Rightarrow p_f \in \uparrow P \;\; \forall\, p_f \geq p_o$

and $p \leq p'$ iff $\uparrow(p') \in \uparrow(p)$

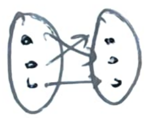
1.67] $f : (P, \leq_P) \to (Q, \leq_Q)$ satisfies "monotonicity property. If P is a ~~mono~~ discrete preorder then every fn $P \to Q$ satisfies the monotonicity

1.68) $S : X \twoheadrightarrow P$ for some set P. A surjective fn $f : X \twoheadrightarrow Y$ induces a monotone map $f^* : Prt(Y) \to Prt(X)$ going backwards.



(Monotone, # goes from low to high is must)

1.69)



1.70) Identity fn is monotone

1.72) $id_P : P \to P$ is a monotone map $(P, \leq) \to (P, \leq^{op})$ iff $\forall p, q \in P$ we have $p \leq q$, $q \leq p$

when this is true we call it a dagger preorder.

∴ It is also a equivalence relation

∴ skeletal dagger preorder is discrete preorder

Defn. Let $(P, \leq_P)$ and $(Q, \leq_Q)$ be preorders. A monotone fn $f : P \to Q$ is called isomorphism if $\exists g : Q \to P$ monotone s.t $f \circ g = id_P$ $g \circ f = id_Q$

"Isomorphism is just relabeling of the elements"

1.18) $P \to B$ are in one-to-one correspondence with upper sets of P.

A greatest lower bound - meet   least upper bound - join

Defn: $(P, \leq)$ be a preorder and $A \subseteq P$ be subset

$p \in P$ is a meet of A if

(i) $\forall a \in A$ we have $p \leq a$ and

(ii) $\forall q$ s.t $q \leq a$, $\forall a \in A$, we have $q \leq p$

We write $p = \bigwedge(A)$, $p = \bigwedge_{a \in A} a$

III^{ly} for join $\forall a \in A$ we have $a \leq p$ and

$\forall q$ s.t $a \leq q$ $\forall a \in A$, we have $p \leq q$

$p = \bigvee(A)$

In a power set $P(X)$. Let $A, B \subseteq X$ $A \wedge B = A \cap B$ and $A \vee B = A \cup B$.

We say that the map $f: P \to Q$ has a generative effect $\exists$ $a, b \in P$ s.t $f(a) \vee f(b) \not\cong f(a \vee b)$

Galois connections

* Relaxed version of isomorphism.
- It is a pair of monotone maps $f: P \to Q$ and $g: Q \to P$ s.t $f(p) \leq q$ iff $p \leq g(q)$

f: Left adjoint g: Right adjoint

• Monotone maps are just observations.
$\phi : P \to Q$ is a phenomenon of P observed by Q
generative effect of ϕ is failure to preserve joins.

1.92) $f : P \to Q$ preserves meets if $f(a \wedge b) = f(a) \wedge f(b)$ $\forall\, a, b \in P$.

• There is more stuff in $f(a \vee b)$ than in $f(a) \vee f(b)$

$$f(a) \vee f(b) \leq f(a \vee b)$$

Preserving meets $\Rightarrow$ restricting to sub-system

Galois connections :

Given two pre-orders, it is a pair of maps back and forth $\left(\begin{smallmatrix} P \to Q \\ Q \to P \end{smallmatrix}\right)$. (relaxed version of isomorphism)

Defn: Galois conn. btw P and Q ~~are~~ is a pair of monotone maps $f : P \to Q$ and $g : Q \to P$ s.t

$$f(p) \leq q \text{ iff } p \leq g(q)$$

$f \to$ left adjoint
$g \to$ right adjoint.

eg: ~~f(x)~~ consider $3x$. $\left\lceil \frac{x}{3} \right\rceil \leq y$ iff $x \leq 3y$
left adjoint

Right adj : $3x \leq y$ iff $x \leq \left\lfloor \frac{y}{3} \right\rfloor$

1.99) 

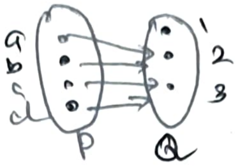
(f, g) are Galois conn.

$f(2) \leq 1'$ but $2 \nleq g(1')$

1.79) $f: P \to Q$ we can define a monotone map $f^*: U(Q) \to U(P)$?

$f: P \to Q$ has a one-one correspondence with $U(Q)$ (making $U: Q \to \mathbb{B}$ $u(q) = \text{true}$ ~~if q is an image of f~~) if $q \in Q$ some upper set

Define $f \circ U: P \to \mathbb{B}$ sends $p \to$ true iff $f(p) \in U$



$U(Q) = \{1, 2\}$

$U(P) = \{a, b\}$

OK so, $f: P \to Q$ monotone defines a monotone $U: Q \to \mathbb{B}$ For any upper set.
From this define $V: P \to \mathbb{B}$ which would ask if $f(p) \in U$ (chosen upper set). If so, this will correspond to another upper set

$U = \{1, 2, 3\}$
$V = \{a, b, c\}$

$f \circ U: P \to \mathbb{B}$ (Monotonic)

Peace!!!

Example for Galois conn.:

$\mathbb{R}$ $\mathbb{Z}$
$f: P \to Q$
$g: Q \to P$

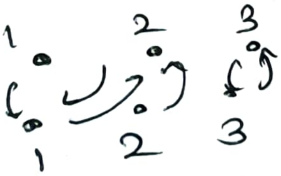
$\lceil \frac{x}{3} \rceil \leq y$ iff $x \leq 3y$
L.A R.A

$x \leq 3y$ $\frac{x}{3} \leq y$
$\lceil \frac{x}{3} \rceil \leq y$ $\lceil \frac{x}{3} \rceil \leq y$

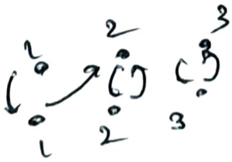
Similarly $3x \leq y$ iff $\lfloor \frac{y}{3} \rfloor \geq x$ ($\lfloor \frac{y}{3} \rfloor \geq x$, $\frac{y}{3} \geq x$

$x \leq \lfloor \frac{y}{3} \rfloor$

Remember $f(x) \leq y$ iff $x \leq g(y)$

1.99) 

f is L.A of g
$f(1) \leq 2 \Rightarrow 1 \leq g(2)$
$f(2) \leq 1 \Rightarrow 2 \leq g(1)$

But 

Not $f(1) \leq 1$
$1 \leq g(2)$ ✓, $2 \leq g(1)$
$f(2) \not\leq 1$

\# If one is identity another (~~LA~~ RA) must be identity

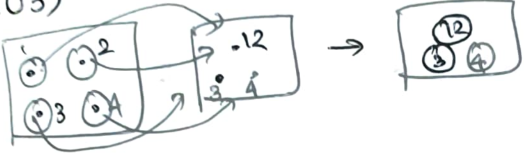
Back to partitions:

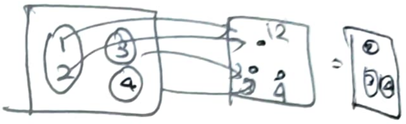
$g: S \to T$. This will induce a Galois conn.

$g_!: Prt(S) \leftrightarrows Prt(T)$

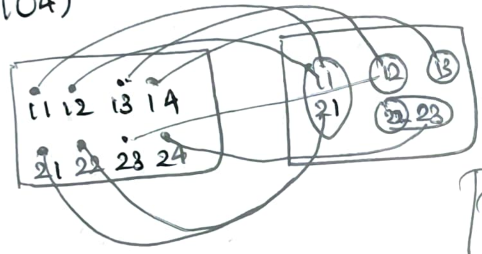
$g: S \to T$. $g_!: Prt(S) \to Pr(T)$. $\sim_S$ (A partition)
obtain a partition $\sim_T$. $t_1, t_2 \in T$ are same part
$t_1 \sim_T t_2$ $\exists\, s_1, s_2 \in S$. (This might not satisfy transitivity. So we must ensure this manually.

1.103)



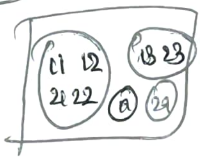


1.104)

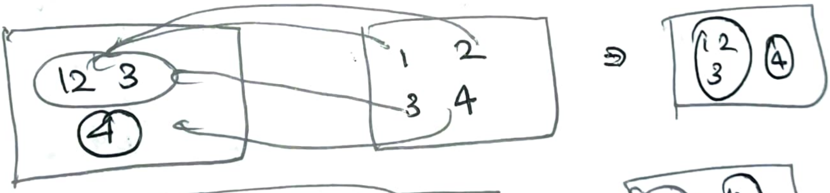


$g_!$ (Pulls it back)

11,21 are conn. $\Rightarrow$ (11,12,21,22)
12 is conn $\Rightarrow$ (13,23)
13 is conn $\Rightarrow$ (14)
22,23 are conn. $\Rightarrow$ (24)

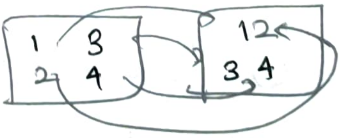


1.105)



1,2 are always connected.

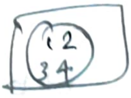
1.106) $g: S \to T$



① partition $c: S \twoheadrightarrow P$

$g_!(c)$

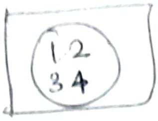
② Coarser partition. $d: T \twoheadrightarrow P'$

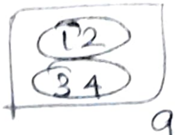


d

③ Non-coarser partition.

e

$g^*(d)$: 

$g^*(e) =$ 

$g^* \to RA$
$g_\downarrow \to LA$
(G.L law)

(5) $g_\downarrow(c) \le d$ and $g_\downarrow(c) \not\le e$

$\Downarrow$ $\Downarrow$

$c \le g^*(d)$ (Yes) $c \not\le g^*(e)$ (Same level)

Basic theory of Galois connections:

$f: P \to Q$ and $g: Q \to P$ are monotone maps.

(a) f & g are Galois conn. (f is left adj. to g)

(b) $\forall\, p \in P, q \in Q$; $p \le \cancel{f}\, g(f(p))$ and $f(g(q)) \le q$

Proof: $f(p) \le q$ iff $p \le g(q)$

$g(f(p)) \le g(q)$ As $g(q)$

Let $q := f(p)$ $\therefore f(p) \le q \Rightarrow p \le g(q)$

(1.109) $\Rightarrow$ $\boxed{p \le g(f(p))}$ |||ly $p := g(q)$

$\boxed{f(g(q)) \le q}$ $p \le g(q) \Rightarrow f(p) \le q$

$\therefore$ If

(1.110) If $f: P \to Q$ is right adjoint g, then for any other right adjoint g',

it means $g(q) \cong g'(q)$ $\forall\, q \in Q$

$\forall\, p \in P$ $p \le g(f(p))$ and $g'(f(p)) \ge p$

$\forall\, q \in Q$ $f(g(q)) \le q$ $f(g'(q)) \le q$

Let $g(q) \le \underbrace{g'(q)}_{p'} \Rightarrow f(g(q)) \le q$

$q \le f(p)$

Let $g(q) \leq g'(q)$ $(f(g(q)) \leq q)$ Also $f(g'(q)) \leq q$
$(p \leq g'(q)) \Rightarrow (f(p) \leq q)$ $\Rightarrow g'(q) \leq g(q)$

Right adjoint preserves meets (lower bound)

Let $f: P \to Q$ be L.A to $g: Q \to P$ $\left(f(g(q)) \leq q \quad g(f(p)) \geq p \right)$

Suppose $A \subseteq Q$ and $g(A) ::= \{g(a) \mid a \in A\} \subseteq P$

If A has a meet $\wedge A \in Q$ then $g(A)$ has a meet $\wedge g(A) \in P$ s.t $g(\wedge A) \cong \wedge g(A)$

Proof:

Let $m ::= \wedge A$ be its meet $g(m) \leq g(a) \; \forall a \in A$

To show: If $p \leq a \; \forall a \in A$ then $m \geq p$.

$g(p) \leq g(a) \; \forall a \in A$ then $f(g(p)) \leq a$

Suppose $\forall a \in A$ $b \leq g(a)$ To prove: $b \leq g(m)$

$f(b) \leq a \Rightarrow f(b) \leq m \Rightarrow b \leq g(m)$ ☺

III^{ly} L.A preserves join. (g is LA of f)

Let $j ::= \vee A$ be its meet $g(j) \geq g(a) \; \forall a \in A$

Let $p \in Q$ s.t $p \geq a$ then $p \geq j$

$p \geq a \Rightarrow g(p) \geq g(a) \Rightarrow f(g(p)) \geq a \Rightarrow f(g(p)) \geq$

$\Rightarrow g(p) \geq g(j)$ Hence proved.

Eg: RA need not preserve joins.

Adjoint for preorders: $(Q, \le)$ has all meets and P be any preorder. If $g: Q \to P$ preserves meets iff its a right adjoint.

III^{ly} If P has all joins and $f: P \to Q$ preserves joins iff its a left adjoint.

Proof: If RA $\Rightarrow$ preservation is proved

If ~~LA~~ preservation ? We define

$f: P \to Q$ as $f(p) := \bigwedge \{q \in Q \mid p \le g(q)\}$

because if $f(p) \le q \Rightarrow p \le g(q)$ $\quad f(p) \ge f(p')$

Monotonicity - 2 $\quad$ If $p' \le p$ $\{q \in Q \mid p \le g(q)\}$

$\subseteq \{q \in Q \mid p' \le g(q)\}$ $\left(\begin{array}{c}\text{ஞாபகம் உண்டா?} \\ A \subseteq B \Rightarrow \bigwedge A \ge \bigwedge B\end{array}\right)$

To prove: $p \le f(g(q))$ $\forall\, p, q \in P, Q$ $\quad g(f(p)) \ge p$

We know that $p \le \bigwedge (g(q))$ $\left(\begin{array}{c}\forall\, g(q) \ge p \\ \text{Rule of } \wedge\end{array}\right)$

$\therefore\ p \le g(\wedge q) \le g(f(p))$ $\quad g(\wedge \{q \mid g(q) \ge p\})$

$\wedge\, g(q)$ s.t. $g(q) \ge p$

~~$g(q) \ge p$~~

~~$q \in Q$ $\quad f(g(q)) = \bigwedge (q'$ s.t. $g(q') \ge g(q))$~~

~~$q' \ge q$~~

~~$f(g(q)) \ge g(q)$ ... q~~

$f(g(q)) \le q$

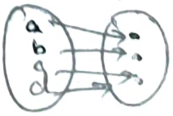
So preservation of meet $\Rightarrow$ Existence of a right adjoint

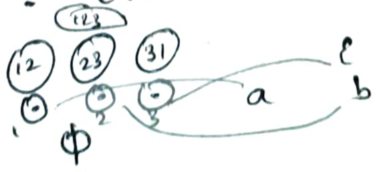
(1.117) $f: A \to B$ $\quad$ (A: set of apples) (B = set of buckets)

$f: A \to B$ (putting apple in a bucket)

$f^*: P(B) \to P(A)$ [monotone map]

* This map takes $B' \subseteq B$ to its preimage $f^{-1}(B') \subseteq A$. (It is monotonic)



$P(B) =$ 

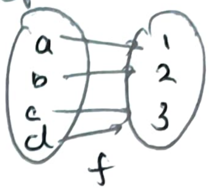
(It has both LA and RA)

$f_!(A') = \{ b \in B \mid \exists a \in A' \text{ s.t } f(a) = b \}$

$f_*(A') := \{ b \in B \mid \forall a \text{ s.t } f(a) = b \text{ we have } a \in A' \}$

Takes a set A' of apples and tells the bucket that contains atleast one of those apple

(Left Adjoint)



$A' = \{a, d\}$

$f_!(A') = \{1, 3\}$

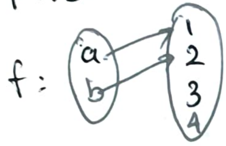
$f_*(A') = \emptyset$

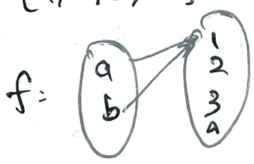
$f_!(f^*(B')) \leq B'$

take a set of apples A' and tells all buckets that are all-A'

$f_*(f^*(B')) \geq B'$

1.118) $X = \{a, b\}$ $Y = \{1, 2, 3, 4\}$





1) $B_1 = \{1, 3\}$ $B_2 = \{1, 2\}$

$f^*(B_1) = \{a\}$ $f^*(B_2) = \{a, b\}$

~~$f_!(A_2)$~~) $A_1 = \{a\}$ $A_2 = \{a, b\}$

$f_!(A_1) = \{1\}$ $f_!(A_2) = \{1, 2\}$

3. $f_*(A_1) = \{\cancel{1}\ 1, 3, 4\}$ $f_*(A_2) = \cancel{\emptyset}\ \{1, 2, 3, 4\}$

Closure operators:

Given a Galois connection with $f : P \to Q$ L.A to $g : Q \to P$ we can have $(g \circ f) : P \to P$ from preorder P to itself.

It will satisfy $p \leq (g \circ f)(p)$

Proof: Property

$\Rightarrow g(f(g(f(p)))) \cong g(f(p))$

(1) $(g \circ f \circ g \circ f)(p) \geq (g \circ f)(p)$

$(f \circ g \circ f)(p) \leq f(p)$

For $(f \circ g \circ f)(p) \in Q$

$f(g(q)) \leq q$

g is monotone

A closure operator $j: P \to P$ on a preorder P is a monotone map s.t $\forall p \in P$, we have

$p \le j(p)$; $j(j(p)) \cong j(p)$

eg: $7+2+3 = 12$ One expression written as another $\underset{x}{5+4} \le \underset{y}{9}$ (lesser)

j is reduction. $x \le j(x)$ and $j(j(x)) \cong j(x)$

1.22) Adjunctions $\Rightarrow$ Closure operators

lly closure $\Rightarrow$ Adjunctions.

P is a preorder. $j: P \to P$ a closure.

$fix_j ::= \{p \in P \mid j(p) \cong p\}$

L.A $j: P \to fix_j$ R.A $g: fix_j \to P$

$p \in P$ $q \in fix_j$ As $p \le j(p)$ $j(p) \le q \Rightarrow p \le q$ (eq)

<u>Level shifting</u>:

Given any set S, $\exists$ $Rel(S)$ of binary rel. on S.
$R \in Rel(S)$ is $\subset S \times S$. $\therefore$ $Rel(S)$ can be given an order via subset relation.

eg: $Rel(\{1\})$: $\phi \to \{(1,1)\}$

eg. $Rel(\{1,2\})$: $\phi \to$ $\{(1,1)\}$, $\{(1,2)\}$, $\{(2,1)\}$, $\{(2,2)\}$ $\to \{(1,1),(1,2)\}$ ...

lly $Pos(S)$ preorder relations on S ($\subseteq$
$\le \subseteq \le'$ means $a \le b \Rightarrow a \le' b$)

A preorder of preorder structures

- Every preorder relation is itself a relation
$\therefore$ inclusion map $Pos(S) \to Rel(S)$ is R.A
L.A is $Cl: Rel(S) \to Pos(S)$

(25) $S = \{1,2,3\}$

1) $(P, \leq)$:   $U(\leq) := \{(s_1, s_2) \mid s_1 \leq s_2\} \subseteq S \times S$

$U(\leq) = \{(1,3), (2,3)\}$

2) $Q \subseteq S \times S$ $\quad Q = \{ (1,3), \}$

$Q' = \{(1,3), (1,2), (3,3)\}$

3) $cl(Q) = \{(1,1), (2,2), (3,3)$