Week Six

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Section 19: Terminal Objects

- From any set (even for **null set**), there is **only one map** to a *singleton* set.
- T is an object in a category $\mathcal{C}$, which is said to be **terminal** only if for any object X in $\mathcal{C}$:
  - ⋆ at least one map exists from X to T.
  - ⋆ that map should be the only map from X to T.

  Using these two conditions we can say that: (See page.229)

  "*There exists multiple terminal objects which are isomorphic to each other.*"

- In the category of *endomaps*, we can say that the **singleton set** equipped with an endomap from the **point to itself**, is a terminal '*set-with-endomap*'.

  The mapping from an endomap X to this terminal object also follows the 'structure preserving rule'.

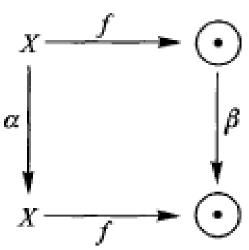


Figure 1: Map from X to T

Section 20: Points

- In the start of this section, we see an example which shows how we can use a *terminal object* (defined in the category) to **select an item** from an object (of the same category). Hence, we can define:

  "*A point of an object X is the map $T \longrightarrow X$*"

  where, T is the terminal object of the category.

- In different categories, the meaning of the word 'point' is different from what we think of. For example, in the category of endomaps, the term 'point' refers to **fixed point** (See page.232). So, if an endomap does **not** have a fixed point, we say it doesnt have '*points*' (which doesn't mean it doesnt have elements!) (See page.233)

Category	Terminal object	'Points of X' means...
$\mathcal{C}$	T	map $T \longrightarrow X$
$\mathcal{S}$	$\boxed{\cdot}$	element of X
$\mathcal{S}^{\circlearrowleft}$ endomaps of sets	$\boxed{\cdot\circlearrowleft}$	fixed point or equilibrium state

20th August

Product

- A product of A and B (in category $\mathcal{C}$, also called **factors of P**) is:
  - ⋆ an object P in $\mathcal{C}$.
  - ⋆ a pair of maps: $P \xrightarrow{p_1} A$ and $P \xrightarrow{p_2} B$ such that,
    for every other object X in $\mathcal{C}$, with pair of maps $X \xrightarrow{q_1} A$ and $X \xrightarrow{q_2} B$, there exist **exactly one map** $X \xrightarrow{q} P$: $q_1 = p_1 \circ q$ and $q_2 = p_2 \circ q$.

  From this definition, we can say that if there exists two products sharing the same **factors**, the products must be **isomorphic**.
- '3D-Space' can be considered as the **product** of three *linearly independent* axes.
- For Products in 'categories of endomaps',

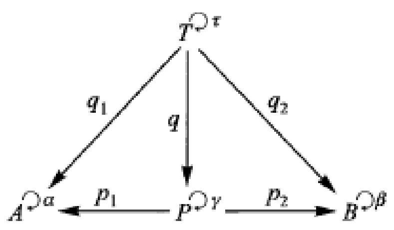
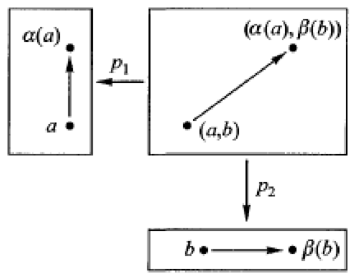


Figure 2: Product in category $S^{\circlearrowleft}$



Internal diagram of $A^{\circlearrowleft} \longrightarrow P \longleftarrow B^{\circlearrowleft}$

We get another condition from the '*structure preserving rule*':

$$\gamma(a,b) = (\alpha(a), \beta(b))$$

Proof:
As we know, P contains elements of the type (a,b) where $a \in A$ and $b \in B$.

$$p_1(a,b) = a \quad p_2(a,b) = b$$
$$\Rightarrow (p_1 + p_2)(a,b) = a + b$$
$$(Or) \quad (a,b) = (p_1+p_2)^{-1}[a+b] \tag{1}$$

By structure preserving conditions:

$$p_1\gamma = \alpha p_1 \quad p_2\gamma = \beta p_2$$
$$\Rightarrow (p_1+p_2)\gamma = \alpha p_1 + \beta p_2$$
$$(Or) \quad \gamma = (p_1+p_2)^{-1}[\alpha p_1 + \beta p_2]$$

Now, applying γ on the element (a,b) yields:

$$\gamma(a,b) = (p_1+p_2)^{-1}[\alpha p_1 + \beta p_2](a,b)$$
$$\Rightarrow (p_1+p_2)^{-1}[\alpha p_1(a,b) + \beta p_2(a,b)]$$
$$\Rightarrow (p_1+p_2)^{-1}[\alpha(a) + \beta(b)]$$
$$\boxed{\gamma(a,b) = (\alpha(a), \beta(b))} \qquad \text{(used (1))}$$

Petri-nets

It is a 4-tuple[1] $N = (P, T, F, m_0)$, where:

- P: set of all **Places**. (p, q, r, s in fig.3)
- T: set of all **Transitions**. (t, u, v)
- P and T are **disjoint**.
- F: **Flow relation** that defines the *arcs*[2].
  $F \subseteq (P \times T) \cup (T \times P)$
  $(F = \{(p,t), (r,t), (v,p), (t,q)...(From, To)\})$
- m_0 is the initial marking, assigns **tokens**[3] to their initial places.
  $m_0 : P \longrightarrow \mathbb{N}$ ($\mathbb{N}$ is the set of *natural numbers*)
  (Here, $m_0 = [p, r^2]$, which reads, p has 1 element, r has two elements.)

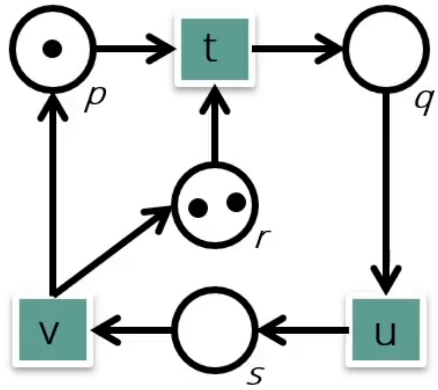


Figure 3: Petri-net example

Arcs and Transitions:

To check whether an **arc** should get **activated** or not, we use the **weight function**, defined as $w : F \to \mathbb{N}^0$ ($\mathbb{N}^0$ is the set of Natural numbers **with zero**),

$$w(p,t) = \begin{cases} 1 & \text{if } (p,t) \in F \\ 0 & \text{otherwise} \end{cases}$$

To **fire** a transition, there must be **at least one token** in **all the input places** of a transition, as a transition uses one token from all input places to create tokens in its output places. Mathematically,

$$\forall p \in P, \quad w(p,t) \leqslant m(p)$$

After firing a transition, it **creates tokens** in its **output** places, hence, if m' represent next state:

$$\forall p \in P, \quad m'(p) = m(p) - w(p,t) + w(t,p)$$

Example (Fig.3)

Given, $m_0 = [p, r^2]$, the transition t uses one element from its inputs (p,r) and we see the next state:

$$[p, r^2] \xrightarrow{t} [p, r^2] - [p, r] + [q] = [r, q] = m_1$$

Now, as p is empty, t cant fire, but now u can fire as its inputs have elements:

$$[r, q] \xrightarrow{u} [r, q] - [q] + [s] = [r, s] = m_2$$

Now, u cant fire, but v can:

$$[r, s] \xrightarrow{v} [r, s] - [s] + [r, p] = [p, r^2] = m_0$$

Now we can use these states and draw the **Labeled Transition System**, which, in this case, is a cycle of three states.

[1] An **ordered** set with 4 elements.

[2] Mappings between places and transitions. The places from which an arc runs to a transition are called the **input** places of the transition; the places to which arcs run from a transition are called the **output** places of the transition.

[3] Elements in each place, denoted by dots.

References

[1] Wikipedia: Petri net

[2] Youtube: A formal introduction to Petri nets