

Week Five

Siva Sundar, EE23B151

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6th August

SEIR Model

- An extension to “**SIR model**”.
- The full-form is as follows:
 - ★ ‘**S**’ stands for “*Susceptible*” to disease. Entities in this set are **vulnerable** to diseases.
 - ★ ‘**E**’ stands for “*Exposed*”. Entities in this set **are infected**, but they are **not infectious** to other Entities in **S**.
 - ★ ‘**I**’ stands for “*Infected*”. Entities in this set can **infect** others as well.
 - ★ ‘**R**’ stands for “*Recovered*” from infection.
- **Assumptions:**
 - ★ For the entities in ‘E’ to infect others, it takes some time (ie, for moving from the set ‘E’ to ‘I’). Let’s assume this follows “**exponential distribution**” with parameter as “*a*” (hence, time average is a^{-1}).
 - ★ Birth rate Λ **equals** death rate $N\mu$, making N (population size, $S + E + I + R = N$) a constant. The term ‘ μ ’ is the *per capita death rate*.

7th August

- Set of **Differential Equations** for the model:

$$\frac{dS}{dt} = \mu N - \mu S - \frac{\beta IS}{N}$$

$$\frac{dE}{dt} = \frac{\beta IS}{N} - (\mu + a)E$$

$$\frac{dI}{dt} = aE - (\gamma + \mu)I$$

$$\frac{dR}{dt} = \gamma I - \mu R$$

Notice that as $S + E + I + R = N$:

$$\frac{dS}{dt} + \frac{dE}{dt} + \frac{dI}{dt} + \frac{dR}{dt} = 0$$

- **Reproduction Number:**

The *basic reproduction number* is the **expected number** of cases **directly generated by one case** in a population where **all individuals are susceptible to infection**. (**Assuming**, individuals are **not** infected nor immunized (naturally or through vaccination))

For this model,

- ★ Exposed individuals become infectious with *probability* ‘**a**’.
- ★ *Total cases* for an exposed individual: $\mu + a$. (be infected or die)
- ★ Duration of infection: $(\gamma + \mu)^{-1}$.
- ★ Disease spreads from Infectious individuals with *probability* β .

$$\therefore R_0 = (\text{Prob. transmission per contact}) \times (\text{Duration of infection}) \\ \times (\text{Probability of surviving ‘Exposed’ stage})$$

$$\Rightarrow R_0 = \beta \times \frac{1}{\mu + \gamma} \times \frac{a}{a + \gamma} \quad (\text{See Ref.[2]})$$

• States of Equilibrium:

- ★ *Disease-Free-Equilibrium (DFE)*: disease is **not** present in the population. Mathematically, $(S, E, I, R) = (N, 0, 0, 0)$.
- ★ *Endemic Equilibrium (EE)*: disease **persists** in the population at a **constant level** over time. It represents a *steady-state* situation where the disease is constantly present in the population but at a *stable* level.

For a biologically meaningful initial condition as follows:

$$(S(0), E(0), I(0), R(0)) \in \{(S, E, I, R) \in [0, N]^4 : S, E, I, R \geq 0, S + E + I + R = N\}$$

it holds that:

$$R_0 \leq 1 \Rightarrow \lim_{t \rightarrow \infty} (S(t), E(t), I(t), R(t)) = DFE = (N, 0, 0, 0)$$

$$R_0 > 1, I(0) > 0 \Rightarrow \lim_{t \rightarrow \infty} (S(t), E(t), I(t), R(t)) = EE$$

8th August

CLICK HERE to see the **Julia** code which **solves** the **Differential Equations** seen above. Below are some output figures for different values of a, β, μ and γ .

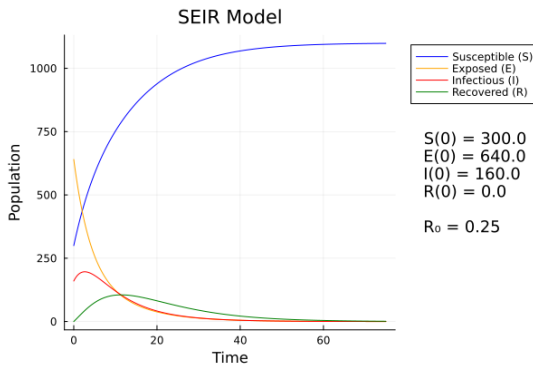


Figure 1: $a = \beta = \mu = \gamma = 0.1$ (DFE)

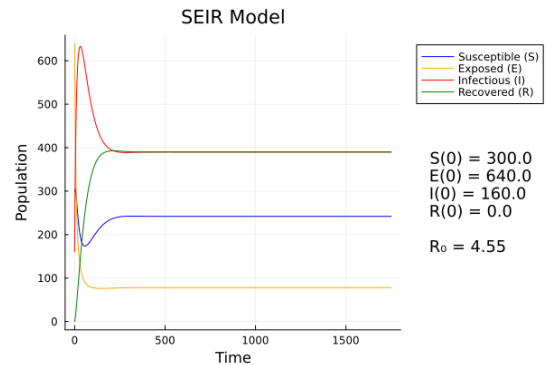


Figure 2: $a = \beta = 0.1$ & $\mu = \gamma = 0.01$ (EE)

10th August

How did we arrive at the Differential Equation set?

- People migrating from 'S' to 'E' can be thought of as a first order kinetic equation as its *similar* to **multiplication of bacteria** in a culture. Say, the rate constant be λ .

$$\frac{dS}{dt} = -\lambda S + \dots$$

$$\frac{dE}{dt} = +\lambda S + \dots$$

- As we know, the people in the set 'S' are **exposed** to infection due to the people in 'I'. Hence, the exposure rate λ depends on 'I':

$$\text{Rate of S to E } (\lambda) = (\text{Rate of Transmission } (\beta) \times \text{P(infection | contact occurred)})$$

$$\Rightarrow \lambda = \frac{\beta I}{N}$$

- By similar reasoning given in first point, we can also say transitions 'E to I' and 'I to R' follow first order kinetics as well, with rate constants σ (Incubation rate) and γ (Recovery rate) respectively:

$$\frac{dE}{dt} = +\lambda S - \sigma E + \dots$$

$$\frac{dI}{dt} = +\sigma E - \gamma I + \dots$$

$$\frac{dR}{dt} = \gamma I + \dots$$

Up to this derivation, we used [3] as reference. In it, they then remove the dots (...) stating that there are *no other influence* to the system. But in reality, there is **birth** (Λ) and **death rate** (μN) that affects the total system!

- Birth rate increases only susceptible individuals while death rate decreases all the groups. Hence:

$$\frac{dS}{dt} = \Lambda - \mu S - \frac{\beta IS}{N}$$

$$\frac{dE}{dt} = \frac{\beta IS}{N} - (\mu + a)E$$

$$\frac{dI}{dt} = aE - (\gamma + \mu)I$$

$$\frac{dR}{dt} = \gamma I - \mu R$$

By, taking the sum of all the equations and taking $S + E + I + R = N$, the total population, we get:

$$\frac{dN}{dt} = \Lambda - \mu N$$

- So if we take a system with constant population N , then: $\Lambda = \mu N$. Replacing this in the **DE set** yields the required model!

References:

- [1] Wikipedia: Compartmental models in epidemiology
- [2] Introduction to SEIR Models by Swiss TPH
- [3] The SEIR model of infectious diseases