Notes

14 December 2024 12:17

Consider the following three questions you might ask yourself:

- Given what I have, is it *possible* to get what I want? → bool
- Given what I have, what is the *minimum cost* to get what I want? → cost
- Given what I have, what is the *set of ways* to get what I want? → Power

We begin with a formal definition of symmetric monoidal preorders.

Definition 2.2. A *symmetric monoidal structure* on a preorder $(X, \leq)$ consists of two constituents:

(i) an element $I \in X$, called the *monoidal unit*, and

(ii) a function $\otimes \colon X \times X \to X$, called the *monoidal product*.

These constituents must satisfy the following properties, where we write $\otimes(x_1, x_2) = x_1 \otimes x_2$:

(a) for all $x_1, x_2, y_1, y_2 \in X$, if $x_1 \leq y_1$ and $x_2 \leq y_2$, then $x_1 \otimes x_2 \leq y_1 \otimes y_2$,

(b) for all $x \in X$, the equations $I \otimes x = x$ and $x \otimes I = x$ hold,

(c) for all $x, y, z \in X$, the equation $(x \otimes y) \otimes z = x \otimes (y \otimes z)$ holds, and

(d) for all $x, y \in X$, the equation $x \otimes y = y \otimes x$ holds.

We call these conditions *monotonicity*, *unitality*, *associativity*, and *symmetry* respectively. A preorder equipped with a symmetric monoidal structure, $(X, \leq, I, \otimes)$, is called a *symmetric monoidal preorder*.

monoid ⇒ way of combining

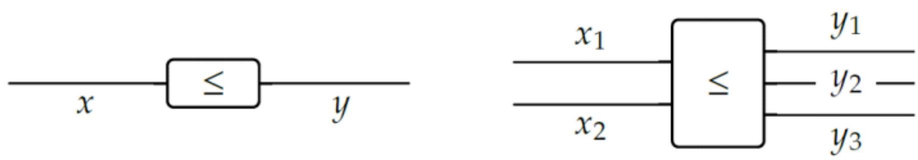
Wires = elements
Boxes = relationships
Parallelism = combination

We consider wires in parallel to represent the monoidal product of their labels, so we consider both cases above to represent the element $x \otimes y$. Note also that a wire labeled

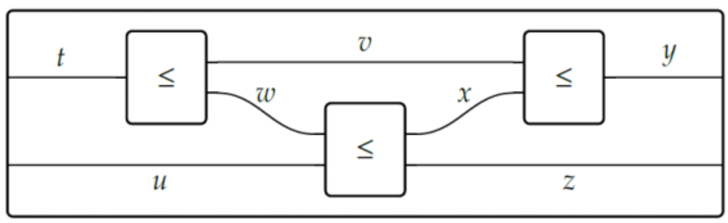
I or an absence of wires:

both represent the monoidal unit I; another way of thinking of this is that the unit is the empty monoidal product.

A wiring diagram runs between a set of parallel wires on the left and a set of parallel wires on the right. We say that a wiring diagram is *valid* if the monoidal product of the elements on the left is less than the monoidal product of those on the right. For example, if we have the inequality $x \leq y$, the the diagram that is a box with a wire labeled x on the left and a wire labeled y on the right is valid; see the first box below:



Finally, the symmetry condition (d), that $x \otimes y = y \otimes x$, says that a diagram is valid even if its wires cross:



(2.15)

The inner boxes in Eq. (2.15) translate into the assertions:

$$t \leq v + w \qquad w + u \leq x + z \qquad v + x \leq y \tag{2.16}$$

and the outer box translates into the assertion:

$$t + u \leq y + z. \tag{2.17}$$

You can add more axioms to symmetric monoidal preorders and get -> discard, copy, etc

The Book Preorder

Example 2.27 (Booleans with AND). We can define a monoidal structure on $\mathbb{B}$ by letting the monoidal unit be `true` and the monoidal product be $\wedge$ (AND). If one thinks of `false` $= 0$ and `true` $= 1$, then $\wedge$ corresponds to the usual multiplication operation $*$. That is, with this correspondence, the two tables below match up:

$\wedge$	false	true
false	false	false
true	false	true

$*$	0	1
0	0	0
1	0	1

(2.28)

One can check that all the properties in Definition 2.2 hold, so we have a monoidal preorder which we denote **Bool** $:= (\mathbb{B}, \leq, \texttt{true}, \wedge)$.

The Cost Preorder

Example 2.37 (Lawvere's monoidal preorder, **Cost**). Let $[0, \infty]$ denote the set of non-negative real numbers—such as 0, 1, $15.33\overline{3}$, and 2π—together with ∞. Consider the preorder $([0, \infty], \geq)$, with the usual notion of $\geq$, where of course $\infty \geq x$ for all $x \in [0, \infty]$.

There is a monoidal structure on this preorder, where the monoidal unit is 0 and the monoidal product is $+$. In particular, $x + \infty = \infty$ for any $x \in [0, \infty]$. Let's call this

monoidal preorder

$$\textbf{Cost} := ([0, \infty], \geq, 0, +),$$

because we can think of the elements of $[0, \infty]$ as costs. In terms of structuring "getting from here to there," **Cost** seems to say "getting from a to b is a question of cost." The monoidal unit being 0 will translate into saying that you can always get from a to a at no cost. The monoidal product being $+$ will translate into saying that the cost of getting from a to c is at most the cost of getting from a to b *plus* the cost of getting from b to c. Finally, the "at most" in the previous sentence is coming from the $\geq$.

Opposite Categories

Proposition 2.38. Suppose $\mathcal{X} = (X, \leq)$ is a preorder and $\mathcal{X}^{\text{op}} = (X, \geq)$ is its opposite. If $(X, \leq, I, \otimes)$ is a symmetric monoidal preorder then so is its opposite, $(X, \geq, I, \otimes)$.

Monoidal Monotones

Definition 2.41. Let $\mathcal{P} = (P, \leq_P, I_P, \otimes_P)$ and $\mathcal{Q} = (Q, \leq_Q, I_Q, \otimes_Q)$ be monoidal preorders. A *monoidal monotone* from $\mathcal{P}$ to $\mathcal{Q}$ is a monotone map $f : (P, \leq_P) \to (Q, \leq_Q)$, satisfying two conditions:

(a) $I_Q \leq_Q f(I_P)$, and
(b) $f(p_1) \otimes_Q f(p_2) \leq_Q f(p_1 \otimes_P p_2)$

for all $p_1, p_2 \in P$.

There are strengthenings of these conditions that are also important. If f satisfies the following conditions, it is called a *strong monoidal monotone*:

(a′) $I_Q \cong f(I_P)$, and
(b′) $f(p_1) \otimes_Q f(p_2) \cong f(p_1 \otimes_P p_2)$;
and if it satisfies the following conditions it is called a *strict monoidal monotone*:
(a″) $I_Q = f(I_P)$, and
(b″) $f(p_1) \otimes_Q f(p_2) = f(p_1 \otimes_P p_2)$.

Enriched Categories

Definition 2.46. Let $\mathcal{V} = (V, \leq, I, \otimes)$ be a symmetric monoidal preorder. A $\mathcal{V}$-*category* $\mathcal{X}$ consists of two constituents, satisfying two properties. To specify $\mathcal{X}$,
(i) one specifies a set $\mathrm{Ob}(\mathcal{X})$, elements of which are called *objects*;
(ii) for every two objects x, y, one specifies an element $\mathcal{X}(x, y) \in V$, called the *hom-object*.[2]
The above constituents are required to satisfy two properties:
(a) for every object $x \in \mathrm{Ob}(\mathcal{X})$ we have $I \leq \mathcal{X}(x, x)$, and
(b) for every three objects $x, y, z \in \mathrm{Ob}(\mathcal{X})$, we have $\mathcal{X}(x, y) \otimes \mathcal{X}(y, z) \leq \mathcal{X}(x, z)$.
We call $\mathcal{V}$ the *base of the enrichment* for $\mathcal{X}$ or say that $\mathcal{X}$ is *enriched* in $\mathcal{V}$.

Theorem 2.49. There is a one-to-one correspondence between preorders and **Bool**-categories.

Definition 2.53. A *Lawvere metric space* is a **Cost**-category.

Definition 2.51. A *metric space* (X, d) consists of:
(i) a set X, elements of which are called *points*, and
(ii) a function $d\colon X \times X \to \mathbb{R}_{\geq 0}$, where $d(x, y)$ is called the *distance between x and y*.

A lawvere metric space drops criteria (b), (c) from the distance function
(a) for every $x \in X$, we have $d(x, x) = 0$,
(b) for every $x, y \in X$, if $d(x, y) = 0$ then $x = y$,
(c) for every $x, y \in X$, we have $d(x, y) = d(y, x)$, and
(d) for every $x, y, z \in X$, we have $d(x, y) + d(y, z) \geq d(x, z)$.

Changing enrichment

2.4.1 Changing the base of enrichment

Any monoidal monotone $\mathcal{V} \to \mathcal{W}$ between symmetric monoidal preorders lets us convert $\mathcal{V}$-categories into $\mathcal{W}$-categories.

Construction 2.64. Let $f\colon \mathcal{V} \to \mathcal{W}$ be a monoidal monotone. Given a $\mathcal{V}$-category $\mathcal{C}$, one forms the associated $\mathcal{W}$-category, say $\mathcal{C}_f$ as follows.
(i) We take the same objects: $\mathrm{Ob}(\mathcal{C}_f) := \mathrm{Ob}(\mathcal{C})$.
(ii) For any $c, d \in \mathrm{Ob}(\mathcal{C})$, put $\mathcal{C}_f(c, d) := f(\mathcal{C}(c, d))$.

Enriched Functors

Definition 2.69. Let $\mathcal{X}$ and $\mathcal{Y}$ be $\mathcal{V}$-categories. A $\mathcal{V}$-*functor from $\mathcal{X}$ to $\mathcal{Y}$*, denoted $F\colon \mathcal{X} \to \mathcal{Y}$, consists of one constituent:
(i) a function $F\colon \mathrm{Ob}(\mathcal{X}) \to \mathrm{Ob}(\mathcal{Y})$
subject to one constraint
(a) for all $x_1, x_2 \in \mathrm{Ob}(\mathcal{X})$, one has $\mathcal{X}(x_1, x_2) \leq \mathcal{Y}(F(x_1), F(x_2))$.

Product Category

Definition 2.74. Let $\mathcal{X}$ and $\mathcal{Y}$ be $\mathcal{V}$-categories. Define their $\mathcal{V}$-*product*, or simply *product*, to be the $\mathcal{V}$-category $\mathcal{X} \times \mathcal{Y}$ with
(i) $\mathrm{Ob}(\mathcal{X} \times \mathcal{Y}) := \mathrm{Ob}(\mathcal{X}) \times \mathrm{Ob}(\mathcal{Y})$,
(ii) $(\mathcal{X} \times \mathcal{Y})((x, y), (x', y')) := \mathcal{X}(x, x') \otimes \mathcal{Y}(y, y')$,
for two objects (x, y) and (x', y') in $\mathrm{Ob}(\mathcal{X} \times \mathcal{Y})$.

Example 2.76. Let $\mathcal{X}$ and $\mathcal{Y}$ be the Lawvere metric spaces (i.e. **Cost**-categories) defined by the following weighted graphs:

$$\mathcal{X} := \boxed{\begin{tikzcd} \overset{A}{\bullet} \arrow[r, "2"] & \overset{B}{\bullet} \arrow[r, "3"] & \overset{C}{\bullet} \end{tikzcd}} \qquad \boxed{\begin{tikzcd} \overset{p}{\bullet} \arrow[d, bend right, "5"'] \\ \underset{q}{\bullet} \arrow[u, bend right, "8"'] \end{tikzcd}} =: \mathcal{Y} \tag{2.77}$$

Their product is defined by taking the product of their sets of objects, so there are six objects in $\mathcal{X} \times \mathcal{Y}$. And the distance $d_{X\times Y}((x, y), (x', y'))$ between any two points is given by the sum $d_X(x, x') + d_Y(y, y')$.

Examine the following graph, and make sure you understand how easy it is to derive from the weighted graphs for $\mathcal{X}$ and $\mathcal{Y}$ in Eq. (2.77):

$$\mathcal{X} \times \mathcal{Y} = \boxed{\begin{tikzcd} \overset{(A,p)}{\bullet} \arrow[r, "2"] \arrow[d, bend right, "5"'] & \overset{(B,p)}{\bullet} \arrow[r, "3"] \arrow[d, bend right, "5"'] & \overset{(C,p)}{\bullet} \arrow[d, bend right, "5"'] \\ \underset{(A,q)}{\bullet} \arrow[r, "2"] \arrow[u, bend right, "8"'] & \underset{(B,q)}{\bullet} \arrow[r, "3"] \arrow[u, bend right, "8"'] & \underset{(C,q)}{\bullet} \arrow[u, bend right, "8"'] \end{tikzcd}}$$

Monoidal Closed Preorders

Definition 2.79. A symmetric monoidal preorder $\mathcal{V} = (V, \leq, I, \otimes)$ is called *symmetric monoidal closed* (or just *closed*) if, for every two elements $v, w \in V$, there is an element $v \multimap w$ in $\mathcal{V}$, called the *hom-element*, with the property

$$(a \otimes v) \leq w \quad \text{iff} \quad a \leq (v \multimap w). \tag{2.80}$$

for all $a, v, w \in V$.

Proposition 2.87. Suppose $\mathcal{V} = (V, \leq, I, \otimes, \multimap)$ is a symmetric monoidal preorder that is closed. Then

(a) For every $v \in V$, the monotone map $- \otimes v \colon (V, \leq) \to (V, \leq)$ is left adjoint to $v \multimap - \colon (V, \leq) \to (V, \leq)$.

(b) For any element $v \in V$ and set of elements $A \subseteq V$, if the join $\bigvee_{a \in A} a$ exists then so does $\bigvee_{a\in A} v \otimes a$ and we have

$$\left(v \otimes \bigvee_{a \in A} a\right) \cong \bigvee_{a \in A} (v \otimes a). \tag{2.88}$$

(c) For any $v, w \in V$, we have $v \otimes (v \multimap w) \leq w$.

(d) For any $v \in V$, we have $v \cong (I \multimap v)$.

(e) For any $u, v, w \in V$, we have $(u \multimap v) \otimes (v \multimap w) \leq (u \multimap w)$.

Quantales

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Definition 2.90. A *unital commutative quantale* is a symmetric monoidal closed preorder $\mathcal{V} = (V, \leq, I, \otimes, \multimap)$ that has all joins: $\bigvee A$ exists for every $A \subseteq V$. In particular, we often denote the empty join by $0 := \bigvee \varnothing$.

Proposition 2.96. Let $\mathcal{P} = (P, \leq)$ be a preorder. It has all joins iff it has all meets.

Remark 2.97. The notion of Hausdorff distance can be generalized, allowing the role of **Cost** to be taken by any quantale $\mathcal{V}$. If $\mathcal{X}$ is a $\mathcal{V}$-category with objects X, and $U \subseteq X$ and $V \subseteq X$, we can generalize the usual Hausdorff distance, on the left below, to the formula on the right:

$$d(U, V) := \sup_{u \in U} \inf_{v \in V} d(u, v) \qquad \mathcal{X}(U, V) := \bigwedge_{u \in U} \bigvee_{v \in V} \mathcal{X}(u, v).$$

For example, if $\mathcal{V} = \textbf{Bool}$, the Hausdorff distance between sub-preorders U and V answers the question "can I get into V from every $u \in U$," i.e. $\forall_{u \in U}.\, \exists_{v \in V}.\, u \leq v$. Or for another example, use $\mathcal{V} = \mathsf{P}(M)$ with its interpretation as modes of transportation, as in Exercise 2.62. Then the Hausdorff distance $d(U, V) \in \mathsf{P}(M)$ tells us those modes of transportation that will get us into V from every point in U.

Proposition 2.98. Suppose $\mathcal{V} = (V, \leq, I, \otimes)$ is any symmetric monoidal preorder that has all joins. Then $\mathcal{V}$ is closed—i.e. it has a $\multimap$ operation and hence is a quantale—if and only if $\otimes$ distributes over joins; i.e. if Eq. (2.88) holds for all $v \in V$ and $A \subseteq V$.