

# Universal Properties

Describes the nature of an object in terms of how its related to everything else (the universe)

## Things to keep in mind

"Our goal is to understand everything in terms of maps and their composition, so we should ask ourselves: what special property do singleton sets have? We want the answer to involve maps."



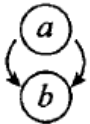
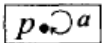
"to say that Chad is 'universally admired' means:  
For every person  $X$  in the world,  $X$  admires Chad."

## Terminal Objects

$S$  is a terminal object of  $C$  if for each object  $X$  in a category  $C$ , there exists only one  $C$ -map from  $X$  to  $S$

If  $S_1$  and  $S_2$  are terminal objects, then they are isomorphic in  $C$  and there exists only one unique isomorphism between them

Since there is only one isomorphism between them, it is better to call the terminal object 1

Category	Terminal object	'Points of $X$ ' means...
$\mathcal{C}$	$T$	$\text{map } T \longrightarrow X$
$\mathcal{S}$		element of $X$
$\mathcal{S}^{\circlearrowright}$ endomaps of sets		fixed point or equilibrium state
$\mathcal{S}^{\downarrow}$ irreflexive graphs	 or 	?

# Points

A map from a terminal object 1 to any other object  $X$

Points can be used to probe things inside an object

Points are often useful in "separating" maps (showing they are distinct)

A "point" in the category of dynamical systems ends up being a "fixed point"

## Initial (Coterminal) Objects

$S$  is an initial object of  $C$  if for each object  $X$  in a category  $C$ , there exists only one  $C$ -map from  $S$  to  $X$

In the category of sets, the null set is the initial object and the map that exists to every other set is the empty function (mapping from nothingness)

If  $S_1$  and  $S_2$  are initial objects, then they are isomorphic in  $C$  and there exists only one unique isomorphism between them

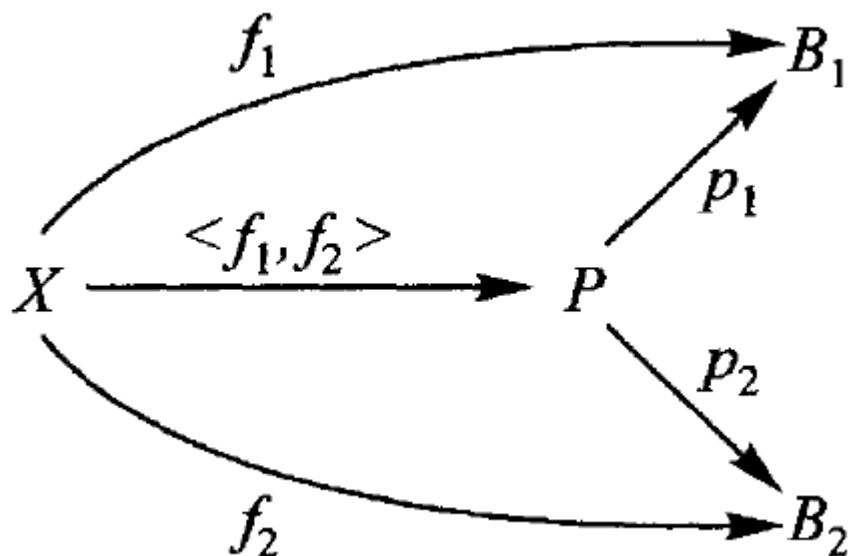
Since there is only one isomorphism between them, it is better to call the terminal object 0

## Null object

If an object is both initial and terminal, it is called the null object or zero object for that category

## Products

$P$  is a product of  $B_1, B_2$  if for every  $X$  in the category and a pair of maps  $f_1, f_2$  there exists a unique map  $\langle f_1, f_2 \rangle$  (determined uniquely using  $f_1, f_2$ ) from  $X$  to  $P$  such that the below diagram commutes



$p_1, p_2$  are projection maps

**Definition:** Suppose that  $A$  and  $B$  are objects in a category  $\mathcal{C}$ . A **product** of  $A$  and  $B$  (in  $\mathcal{C}$ ) is

1. an object  $P$  in  $\mathcal{C}$ , and
2. a pair of maps,  $P \xrightarrow{p_1} A$ ,  $P \xrightarrow{p_2} B$ , in  $\mathcal{C}$  satisfying:

for every object  $T$  and every pair of maps  $T \xrightarrow{q_1} A$ ,  $T \xrightarrow{q_2} B$ , there is exactly one map  $T \xrightarrow{q} P$  for which  $q_1 = p_1 \circ q$  and  $q_2 = p_2 \circ q$ .