# Week 1

Siva Sundar, EE23B151

June 2024

## 19th June

I started reading the book "**Conceptual Mathematics**". I have read the first two sections today. Key points for each session are listed below:

**Section 1:**

- Firstly, we see examples for Categories:
  1. Galileo's bird's flight puzzle which talks about a relationship between the objects 'time' and 'space' where the bird travels.
  2. The 'space' talked above can again be split into two objects: the 'plane' where the shadow of the bird lie, and the level of flight of bird which is a 'line'.
  3. Other examples like the a category of two dishes (a relation with 1st and 2nd course dishes).
- In the next part, we relate many topics in set theory with category theory like functions as morphisms etc. A **category of finite sets** contains:
  1. Data for the Category:
     (1) Objects: the sets $\mathbf{A}, \mathbf{B}, \mathbf{C}, ...$
     (2) Maps: functions like $f, g, ....$
  2. Rules:
     (1) Identity law: if $\mathbf{A} \xrightarrow{f} \mathbf{B}$, then, $I_B \circ f = f \;\;\&\;\; f \circ I_A = f$.
     (2) Associative law: $h \circ (g \circ f) = (h \circ g) \circ f$.

**Section 2:**

- Some definitions: Consider the category $\mathbf{A} \xrightarrow{f} \mathbf{B}$,
  1. The set **A** is called the Domain of map '$f$'.
  2. The set **B** is called the Co-Domain of map '$f$'.
  3. A **rule** for map '$f$', is that each element in **A** must be mapped to only one element in **B**.
- **Test for equality** of two maps:
  A **point** of a set **A** is a map from a **singleton set 1** to **A**. Using this, we can say that " two maps $f$ and $g$ with domain **A** and co-domain **B** are said to be equal iff for all points $\mathbf{1} \xrightarrow{a} A$, $f \circ a = g \circ a$, then $f = g$. "
- Internal and External Diagrams:
  - Internal: uses the arrow diagrams where the elements of the set are shown.
  - External: shows mapping with arrows between sets as a whole without explicitly showing the elements in them.

**Section 3:**

- Total number of maps from set **A** to set **B** is given by: $n(B)^{n(A)}$, where $n(X)$ represents the number of elements in set **X**
- Isomorphisms: a map $\mathbf{A} \xrightarrow{f} \mathbf{B}$ is called Isomorphic or invertible, if there exist another map $\mathbf{B} \xrightarrow{g} \mathbf{A}$ such that, $f \circ g = I_B$ and $g \circ f = I_A$. This map '$g$' is called the inverse map of '$f$'. (If both domain and co-domain are equal, then this isomorphism is called **Automorphism**.)
- Isomorphs are **Reflexive, Symmetric**, and **Transitive**.
- From Ex.1(T), we can see that for two isomorphs $f$ and $g$, the inverse of the composition $f \circ g$ is $g^{-1} \circ f^{-1}$
- **Determination and Choice Problems**:

$$\begin{tikzcd} & B \arrow[dr, dashed, "g?"] & \\ A \arrow[ur, "f"] \arrow[rr, "h"'] & & C \end{tikzcd}$$

**Determination Problems** requires us to find the map $g$ (which we call the **determination** of $h$ by $f$) if both $f$ and $h$ are given, such that, $h = g \circ f$.
If such $g$ exist, then we say $h$ can be **determined** by $f$.

$$\begin{tikzcd} & B \arrow[dr, "g"] & \\ A \arrow[ur, dashed, "f?"] \arrow[rr, "h"'] & & C \end{tikzcd}$$

**Choice Problems** requires us to find the map $f$ if both $f$ and $h$ are given, such that, $h = g \circ f$. When the map $f$ is fixed, we get a lot of "*choices*" for the map $g$.

**Note:** When the set **B** is a "singleton set", then the maps $f$ and $h$ are constant maps.

- **Retractions, Sections and Idempotents:**
  1. Retractions are the solution maps $r$ for the determination problem: $r \circ f = I_A$.
  2. Sections are the solution maps $s$ for the choice problem: $f \circ s = I_B$.
  3. Idempotents are the maps $e$ such that: $e \circ e = e$. (eg: $e = f \circ r$)

**Theorems**:

$$\begin{tikzcd} & A \arrow[dr, bend left, "f"] & \\ T \arrow[ur, dashed, "x"] \arrow[rr, "y"'] & & B \arrow[ul, bend left, "s"] \end{tikzcd}$$

- If a map $\mathbf{A} \xrightarrow{f} \mathbf{B}$ has a **section**, then for any **T** and any map $\mathbf{T} \xrightarrow{y} \mathbf{B}$, there exist a map $\mathbf{T} \xrightarrow{x} \mathbf{A}$, such that $f \circ x = y$.
- If a map $\mathbf{A} \xrightarrow{x} \mathbf{B}$ has a **retraction**, then for any **T** and any map $\mathbf{A} \xrightarrow{y} \mathbf{T}$, there exist a map $\mathbf{B} \xrightarrow{f} \mathbf{T}$, such that $f \circ x = y$.
  (**Note:** Maps with retractions are **monomorphic** and maps with sections are **epimorphic**.)
- **Uniqueness of Inverses**: if a map $f$ has many retractions $r_1, r_2, \ldots$ and sections $s_1, s_2, \ldots$, then:
  1. All of the sections are equal to each other, same is true for retractions.
  2. Both section and retraction are equal. ($f$ is an **Isomorphism**)