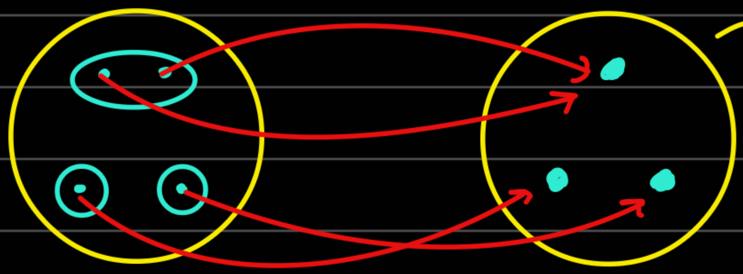
Partitions

→ A partition of a set is a surjection onto another set.

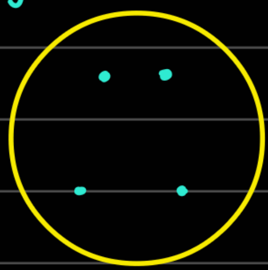


→ The "parts"

(or any other function isomorphic to this)

So we may ditch the blobs entirely and think in terms of this function.

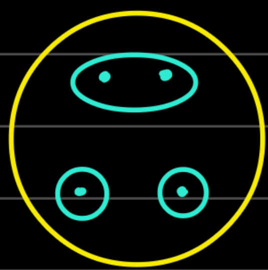
→ We can order partitions. Let $A :=$



We say $P_1 \le P_2$ if $\exists$ a function $P_1 \longrightarrow P_2$ that makes the following diagram commute:

$$\begin{tikzcd} A \arrow[r] \arrow[dr] & P_1 \arrow[d] \\ & P_2 \end{tikzcd}$$

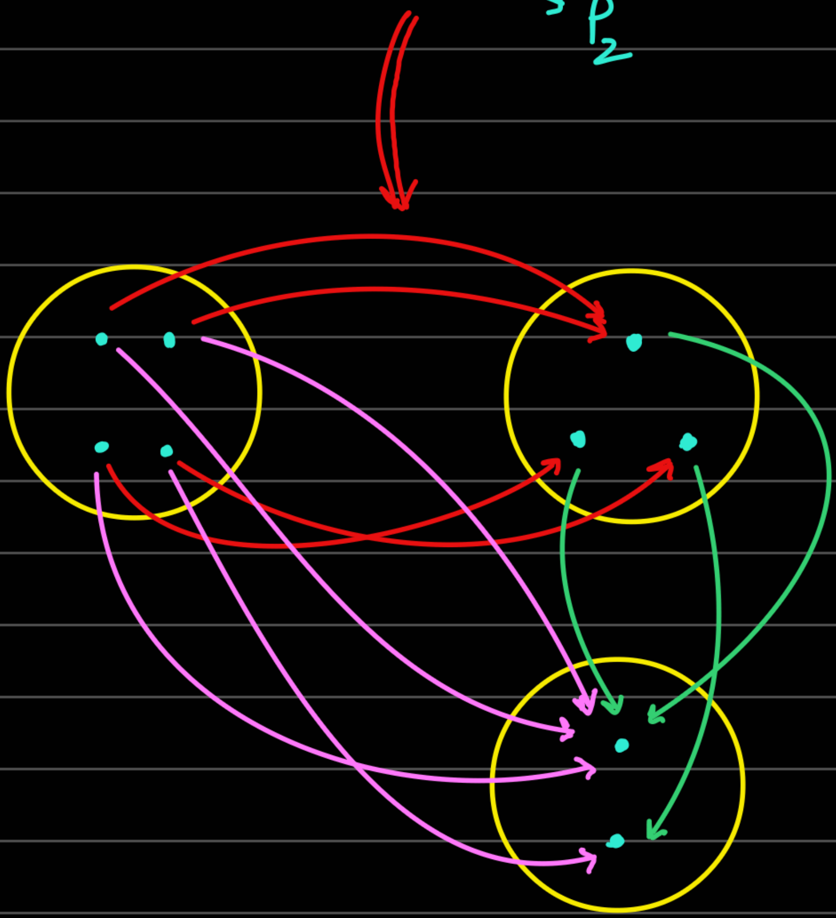
e.g: $P_1 =$



(3-element set)

$P_2 =$

(2-element set)

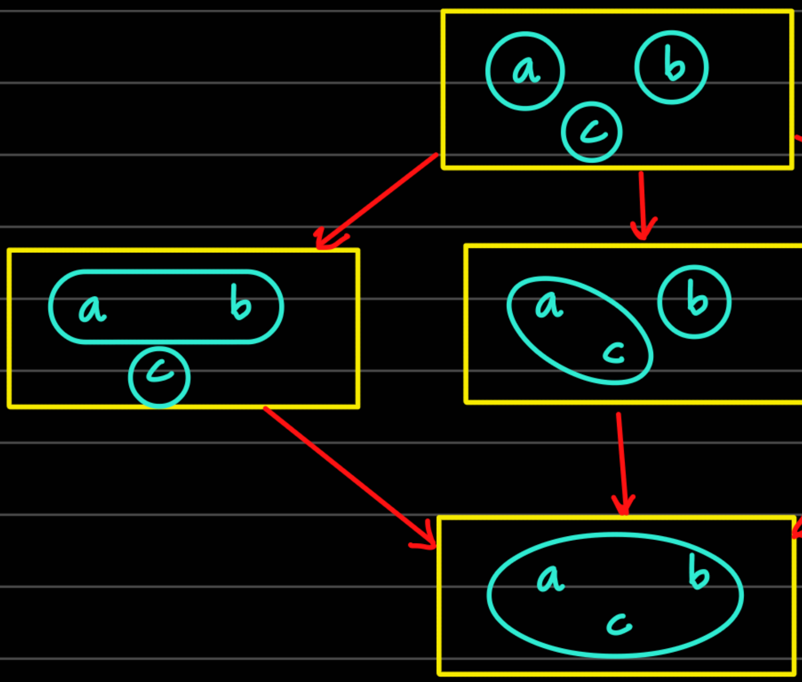


So $P_1 \le P_2$

more refined ⟶ less refined

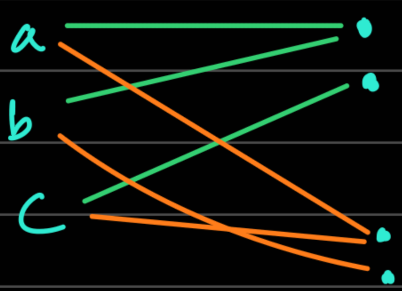
less blobs ⟶ more blobs

So a sense of 'order' may be observed.



$\leq$: Reflexive and transitive.

Note :



Cannot define a function from one of the two blobs to the other such that the triangle commutes, so no ordering within the two-blobs.

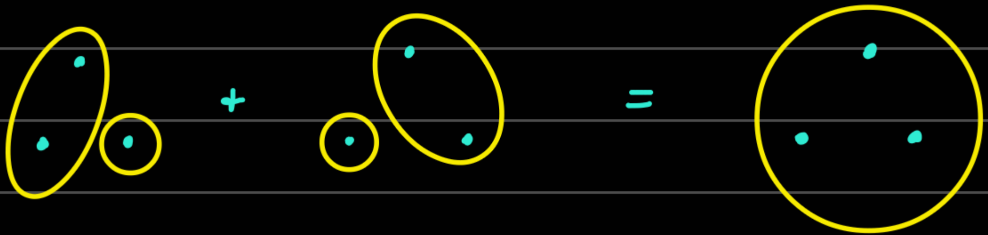
Pre-order

→ It is a set S equipped with a relation "$\leq$" $\in S \times S$, such that

a) $\forall s \in S, \quad s \leq s$.
b) $\forall s_i \in S, \quad$ if $s_1 \leq s_2$ and $s_2 \leq s_3$ then $s_1 \leq s_3$.

(Reflexivity and transitivity)

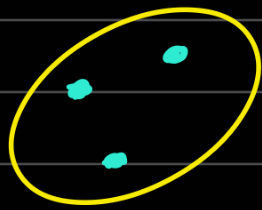
→ "Order creates join" : Join of any two partitions is the smallest partition $\geq$ both.



(More on this later.)

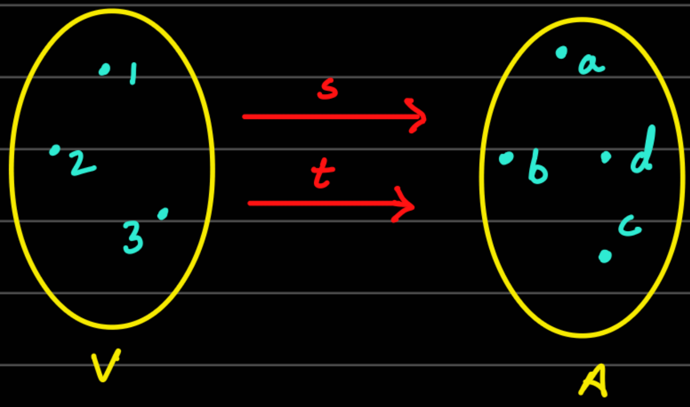
→ <u>Discrete preorders</u> : only order relation is '=', i.e. $x \leq x$ and if $x \neq y$ then neither $x \leq y$ or $y \leq x$ hold.

• So every set is an example.



<u>Note</u> : If $x \cong y$ ($x \leq y$ and $y \leq x$) $\Rightarrow x = y$ then the preorder is called a partial/skeletal preorder (poset: partially ordered set).

<u>Graphs</u>



A graph G (V, A, s, t) consists of vertices (in V) connected to each other through arrows (in A) with source and target vertices specified (through $s: V \to A$ and $t: V \to A$ respectively).

$$
\begin{tikzcd}
1 \arrow[rr, "a"] \arrow[dr, "c"'] & & 2 \arrow[dl, "b"] \arrow[loop above, "d"] \\
& 3 &
\end{tikzcd}
$$

→ <u>Fun fact</u> : The Hasse diagram for the power set of a finite set is a cube of dimension n, when ordering is as per inclusion.

So if $X = \{0,1\}$, $P(X)$ is depicted as

$$
\begin{tikzcd}
& \{0,1\} & \\
\{0\} \arrow[ur] & & \{1\} \arrow[ul] \\
& \phi \arrow[ul] \arrow[ur] &
\end{tikzcd}
$$

Monotone Map

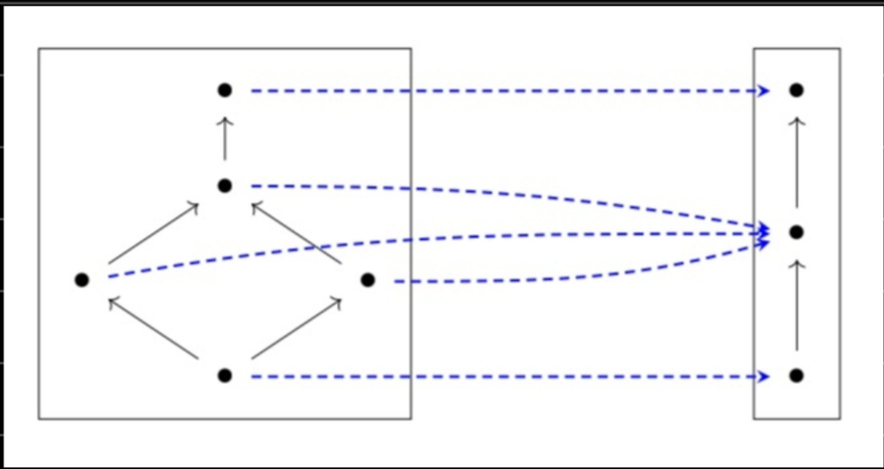
$x \leq_A y$ $\quad A, \leq_A$

$\downarrow f$

$f(x) \leq_B f(y)$ $\quad B, \leq_B$

Such a function $f : A \to B$ is called monotonic.

e.g:



Meets and Joins

Definition 1.81. Let $(P, \leq)$ be a preorder, and let $A \subseteq P$ be a subset. We say that an element $p \in P$ is a *meet* of A if

(a) for all $a \in A$, we have $p \leq a$, and

(b) for all q such that $q \leq a$ for all $a \in A$, we have that $q \leq p$.

GLB

We write $p = \bigwedge A$, $p = \bigwedge_{a \in A} a$, or, if the dummy variable a is clear from context, just $p = \bigwedge_A a$. If A just consists of two elements, say $A = \{a, b\}$, we can denote $\bigwedge A$ simply by $a \wedge b$.

Similarly, we say that p is a *join* of A if

(a) for all $a \in A$ we have $a \leq p$, and

(b) for all q such that $a \leq q$ for all $a \in A$, we have that $p \leq q$.

LUB

We write $p = \bigvee A$ or $p = \bigvee_{a \in A} a$, or when $A = \{a, b\}$ we may simply write $p = a \vee b$.

discrete preorder

→ Meets/Joins may not exist : If $P \equiv P(P, =)$ and $A = \{p, q\}$

then $\bigvee A$ or $\bigwedge A$ does not exist because comparison is not possible.

→ **Meets / Joins not unique** :

$$\begin{tikzcd} a & d \\ b \arrow[u] \arrow[ur] \arrow[r, bend right] & c \arrow[u] \arrow[ul] \arrow[l, bend right] \end{tikzcd}$$

let $A = \{a, d\}$

$\bigwedge A = b, c$

$b \leq c, \quad c \leq b \Rightarrow b \cong c$

→ In a power set, $A \wedge B = A \cap B$

$A \vee B = A \cup B$

→ In $\mathbb{B}$, meet ⟷ AND

join ⟷ OR

Generative Effects

→ A monotone map $f : P \to Q$ preserves meets if $f(a \wedge b) \cong f(a) \wedge f(b)$

" " " joins if $f(a \vee b) \cong f(a) \vee f(b)$

It is said to have a generative effect if $\exists$ elements $\in P$ s.t. the above conditions are not satisfied.