Category

Recipe of a Category:

Let C be a category

- Objects are $Ob(C)$
- Morphisms are $Hom(C)$
- Composition exists for morphisms

Need to follow two laws:

- Associativity in composition
- Identity morphisms must exist

Finsets: Category with objects: finite sets, morhpisms: functions

Identity Law:

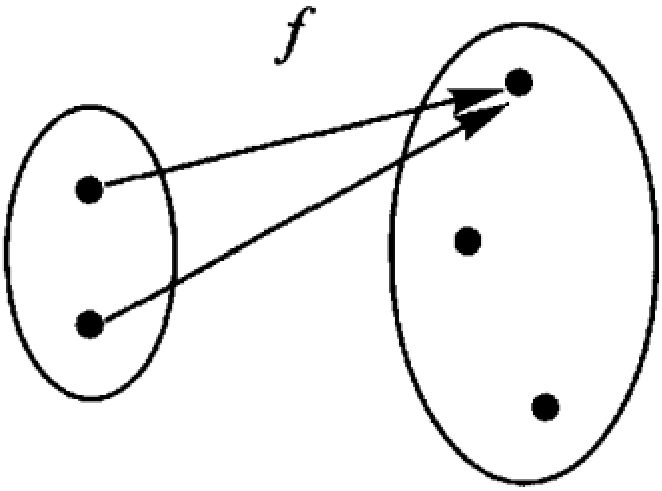
$$f : A \to B \implies f \circ 1_A = f \text{ and } 1_B \circ f = f$$

Associative Law:

$$h \circ (f \circ g) = (h \circ f) \circ g$$

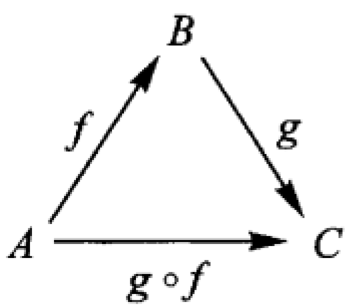
Diagrams

Internal:



You get to see the meat of what the map does to the internal components of the object; the focus is on the specific effect that a map has

External:



You don't need to know what goes on inside; you focus on what the maps do and how they are related

Some terms

1. **Point**: For a set X, a point is the map f where $f : 1 \rightarrow X$ (1 is a set having only 1 element)
2. **Isomorphism:** An invertible map (iso = same, morph = form) (has a unique inverse)

   $$f : A \rightarrow B \text{ is isomorphic if } \exists g : B \rightarrow A \text{ s.t. } f \circ g = 1_B \text{ and } g \circ f = 1_A$$

   Here A and B are isomorphic
   Reflexive: $A \cong A$
   Symmetric: If $A \cong B$, then $B \cong A$
   Transitive: $A \cong B$ and $B \cong C$ then $A \cong C$
3. **Automorphism**: An isomorphism whose domain is the same as the co-domain

4. **Retraction:** For a map $f : A \to B$ if there exists a map $g : B \to A$ such that $g \circ f = 1_A$ then g is a retraction for f

   Such an f is called a ***monomorphism*** (mono = one) (one-one mapping) (injective) which leads to a property in f

   $$f \circ x_1 = f \circ x_2 \implies x_1 = x_2 \forall x_1, x_2 : T \to A$$

   Pretty intuitive if you ask me, whenever there is a one-one mapping from A to B going back from B to A shouldn't be a problem, just follow the arrows back and for whichever item in B is not mapped to you can arbitrarily chose how to map it back without stopping the retract from existing. If multiple items in A map to the same item in B then you do not need the inputs to be necessarily equal.
5. **Section:** For a map $f : A \to B$ if there exists a map $g : B \to A$ such that $f \circ g = 1_B$ then g is a section for f

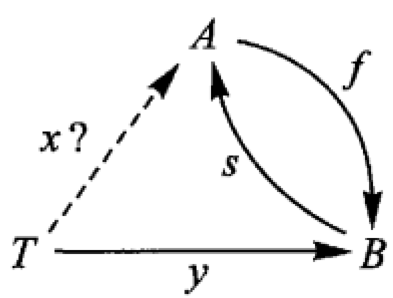
   Such an f is called an ***epimorhpism*** (epi = onto, after) (the map's coming after all of the items in B leaving no one alone I suppose) (surjective) which leads to a property in f

   $$x_1 \circ f = x_2 \circ f \implies x_1 = x_2 \forall x_1, x_2 : B \to T$$

   Pretty intuitive if you ask me, whenever there is a onto mapping from A to B then post-compositions are equal iff the maps composing are equal themselves because every element from B would be mapped further. Composition doesn't care about the intermediate stops you make in the flow of mapping.
6. **Idempotent Map:** A map $e : X \to X$ such that $e \circ e = e$

   Often an idempotent is of the form $r \circ s$ where $s : X \to A, r : A \to X$ s.t. $s \circ r = 1_A$

Some results

1. If a map $f : A \to B$ has a section, then $\exists s \text{ s.t. } f \circ s = 1_B$. Now pre-compose with y to get $f \circ s \circ y = y \implies f \circ x = y$
   There exists such an $x : T \to A$ for any map $y : T \to B$
   

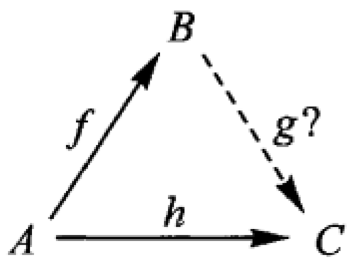
2. If a map $f : A \to B$ has a retraction, then $\exists r \text{ s.t. } r \circ f = 1_A$. Now post-compose with y to get $y \circ r \circ f = y \implies x \circ f = y$

There exists such an $x : B \to T$ for any map $y : A \to T$

3. "The order of going back is always in reverse" - words to live by?
   Sections and retracts show transitivity whenever the objects involved possess sections and retracts.

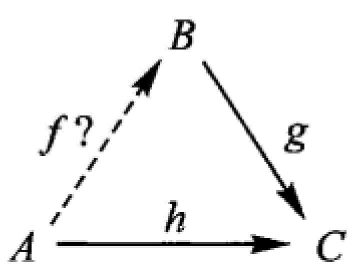
Determination and Choice Problems

Determination (Falling)



- g is uniquely determined by f where $g \circ f = h$
- The retraction problem is a determination problem in g as the retraction where $h = 1_A$ and $C = A$

Choice (Lifting)



- f is a choice for which $g \circ f = h$
- The retraction problem is a determination problem in g as the retraction where $h = 1_A$ and $C = A$

Misc Fun Stuff

Pick's formula: Area = #(Interior Points) + #(Boundary Points)/2 - 1

Sorting, Stacking, Combining - What maps are often used for if you think about it and determining / choosing boils down to going back