Differential Equations

⇒ Linear D.E.

* The derivatives of independant variables should be of power 1.
* independant variable in transdental series should not be present
* i.v. shouldn't be multiplied with any of its derivatives

Solutions of D.E. :

* First order ODE : $\frac{dy}{dx} = f(x,y)$

(OR) $F\left(x, y, \frac{dy}{dx}\right) = 0$

* $y(x)$ is a sol^n to this ODE if

it's 1^{st} order derivative exists and also satisfies the eq.

⇒ Geometrical intrepretation of 1^{st} order ODE :-

$$\frac{dy}{dx} = f(x,y) \longrightarrow (1)$$

$$y(x_0) = y_0 \text{ (const.)} \rightarrow (2)$$

* $f(x,y)$ gives the slope of $y = f(x,c)$ curve at every points and passes through (x_0, y_0)

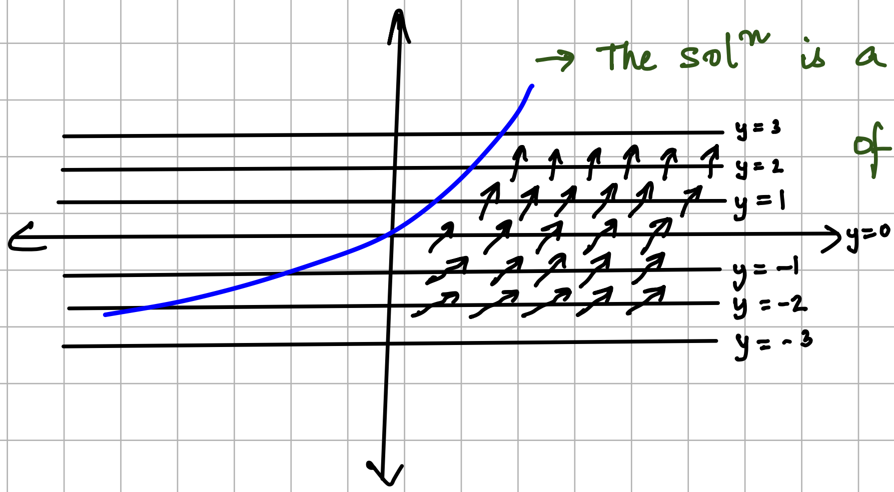
Directional feild :- $\frac{dy}{dx} = y+2 = f(x,y)$

$y(0) = 0$

Interpreting this,

like, slope of every sol^n along $y=0 \Rightarrow f(x,y) = 2$ and so on.

$y=$	slope
$y=3$	5
2	4
1	3
0	2
-1	1
-2	0
-3	-1



→ The sol^n is a curve in dir^n of dir^n feilds

⇒ First order ODE :

① Variable sep. form :

Eg :

$$\frac{\cos y}{\sin x}\frac{dy}{dx} = \frac{-x}{x^2+1}$$

$$\int \frac{\cos y}{\sin y}\,dy = \int -\frac{x}{x^2+1}\,dx$$

$$\ln|\sin y| = -\int \frac{1}{2}\frac{1}{t+1}\,dt$$

$$\ln|\sin y| = -\frac{1}{2}\ln(1+t)$$

$$\ln|\sin y| = \ln\frac{1}{\sqrt{1+x^2}} + c$$

$$|\sin y| = \frac{c}{\sqrt{1+x^2}}$$

$$\boxed{y = \sin^{-1}\left(\frac{c}{\sqrt{1+x^2}}\right)}$$

★ Reduction to V.S. form :

* Converting all y as $\frac{y}{x}$ by app. division or vice versa for x and do,

$$y = vx \quad (\text{or vice versa})$$

FORM (1) :

$\Rightarrow$ f is a homogeneous of degree 'n'

$$(\text{if } f(\lambda x, \lambda y) = \lambda^n f(x,y))$$

after $y = vx$,

Q. $2xy\frac{dy}{dx} = 3y^2 - x^2$

$y = vx$

$\frac{dy}{dx} = x\frac{dv}{dx} + v$

$$2v\frac{dy}{dx} = 3v^2 - 1$$

$$2v\left(x\frac{dv}{dx} + v\right) = 3v^2 - 1$$

$$2vx\frac{dv}{dx} = v^2 - 1$$

$$\frac{2v}{v^2-1}\,dv = \frac{1}{x}\,dx$$

$$\ln(v^2-1) = \ln x + c$$

$$v^2 - 1 = cx$$

$$\frac{y^2}{x^2} - 1 = cx$$

$y^2 = (cx+1)x^2$; $\boxed{y = x\sqrt{cx+1}}$

FORM 2 :

$$\frac{dy}{dx} = \frac{ax+by+c}{lx+my+n}$$

case 1 : $am = bl$,
$lx + my = v(x)$

case 2: $am \neq bl$,

first $x = X + h$, $y = Y + k$
$dx = dX$, $dy = dY$

so, $\frac{dy}{dx} = \frac{dY}{dX}$

and the eq. reduces to,

$$\frac{dY}{dX} = \frac{a(X+h) + b(Y+k) + c}{l(X+h) + m(Y+k) + n}$$

and

choose h, k, s.t.
$ah + bk + c = 0$ and $lh + mk + n = 0$

so,

$$\frac{dY}{dX} = \frac{aX+bY}{lX+mY}$$ $\rightarrow$ This is in homogeneous form so can be converted to V.S. form.

$\Rightarrow$ Exercise : (i) $(2x - 4y + 5)\frac{dy}{dx} + x - 2y + 3 = 0$

(2) $\frac{dy}{dx} = \frac{x+2y-3}{2x+y-3}$

1) $$\frac{dy}{dx} = \frac{-x+2y-3}{2x-4y+5}$$

$$v = 2x - 4y + 5$$

$$\frac{dv}{dx} = 2 - 4\frac{dy}{dx}$$

$$\frac{dv}{dx} - 2 = -4\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{4}\left(2 - \frac{dv}{dx}\right)$$

$$-\frac{v}{2} = -x + 2y - \frac{5}{2}$$

$$-\frac{v-1}{2} = -x + 2y - 3$$

$$2 - \frac{dv}{dx} = \cancel{4}^{2} \times \frac{(-v-1)}{\cancel{2}} \times \frac{1}{v}$$

$$2 - \frac{dv}{dx} = -2\frac{(v+1)}{v}$$

$$\frac{dv}{dx} = 2 + 2\frac{(v+1)}{v}$$

$$\frac{dv}{dx} = \frac{2(2v+1)}{v}$$

$$\frac{v\,dv}{2v+1} = 2dx \Rightarrow \left(1 - \frac{1}{2v+1}\right)dv = 4dx$$

$$\boxed{v - \ln(2v+1) = 4x}$$

⇒ Exact Differential Equation :

$$M(x,y)\,dx + N(x,y)\,dy = 0 \rightarrow (1)$$

This eq^n is exact diff. eq^n if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Def : An eq^n (1) is exact if $\exists\, f(x,y)$ having 1^{st} order continuous partial derivatives s.t.

$$\frac{\partial F}{\partial x} = M \quad ; \quad \frac{\partial F}{\partial y} = N$$

so that we can use this in eq (1)

$$\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = 0 \Rightarrow d(F) = 0$$

$\boxed{F = c}$ is a soln

Ex : $y^2 dx + 2xy\, dy = 0$

$d(xy^2) = 0 \ni F = xy^2$

solⁿ : $\boxed{xy^2 = C}$

$\Rightarrow$ Solution of exact diff. eqⁿ :

* if can't find F directly we use this

Q. $(3x^2 + 4xy)\,dx + (2x^2 + 2y)\,dy = 0$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = 4x$$

$\exists\, F(x,y) \ni \frac{\partial f}{\partial x} = 3x^2 + 4xy \rightarrow (1)$ $\quad \frac{\partial f}{\partial y} = 2x^2 + 2y \rightarrow (2)$

Integrate (1) partially w.r.t x,

$F = x^3 + 2x^2 y + k(y)$ $\rightarrow$ y is const. so it can be in const. of integration

put this in (2),

$$\cancel{2x^2} + \frac{\partial k}{\partial y} = 2y + \cancel{2x^2}$$

$$k = y^2 + c$$

$\therefore \quad F(x,y) = x^3 + 2x^2y + y^2 + c$

$sol^n = \quad F = \tilde{c}$

Take $\tilde{c} - c = c$

sol^n : $\boxed{x^3 + 2x^2y + y^2 = c}$

direct formula : $F = \int M dx + \int \left(N - \frac{\partial}{\partial y} \int M dx\right) dy + c$

$\Rightarrow$ Reduction to exact diff. eq^n :

$$P(x,y)\,dx + Q(x,y)\,dy = 0 \rightarrow (1)$$

If eq^n (1) is not exact,

we search for $\mu(x,y)$ s.t.

$$\underbrace{\mu P}_{M}\,dx + \underbrace{\mu Q}_{N}\,dy = 0$$

μ is called INTEGRATING FACTOR

It means, $\frac{\partial}{\partial y}(\mu P) = \frac{\partial}{\partial x}(\mu Q)$

$$\mu \frac{\partial P}{\partial y} + P\frac{\partial \mu}{\partial y} = \mu \frac{\partial Q}{\partial x} + Q \frac{\partial \mu}{\partial x}$$

$$P\frac{\partial \mu}{\partial y} - Q\frac{\partial \mu}{\partial x} = \mu\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \rightarrow (2)$$

CASE (1) : $\mu(x)$ so $\frac{\partial \mu}{\partial y} = 0$

So, eq^n 2, $\quad -Q\frac{d\mu}{dx} = \mu\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$

$$\frac{d\mu}{dx} = \mu \boxed{\left(\frac{1}{Q}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\right)} \to \text{If this is just } R(x)$$

we can use v.s. form

$$\frac{d\mu}{\mu} = R(x)\,dx$$

$$\boxed{\mu = c e^{\int R(x)\,dx}}$$

case 2: lly for $\mu(y)$.

Q. $-y\,dx + x\,dy = 0$

μ can be anything like

$$\mu(x) = \frac{1}{x^2} \text{ (or) } \mu(y) = \frac{1}{y^2} \text{ (or)}$$

$$\mu(x,y) = \frac{1}{x^2+y^2}$$

Ex: $2\sin(y^2)\,dx + xy\cos(y^2)\,dy = 0$

$$y(2) = \sqrt{\pi/2}$$