

# Week 1

Siva Sundar, EE23B151

June 2024

## 19th June

I started reading the book “**Conceptual Mathematics**”. I have read the first two sections today. Key points for each session are listed below:

### Section 1:

- Firstly, we see examples for Categories:
  1. Galileo’s bird’s flight puzzle which talks about a relationship between the objects ‘time’ and ‘space’ where the bird travels.
  2. The ‘space’ talked above can again be split into two objects: the ‘plane’ where the shadow of the bird lie, and the level of flight of bird which is a ‘line’.
  3. Other examples like the a category of two dishes (a relation with 1st and 2nd course dishes).
- In the next part, we relate many topics in set theory with category theory like functions as morphisms etc. A **category of finite sets** contains:
  1. Data for the Category:
    - (1) Objects: the sets **A**, **B**, **C**, ...
    - (2) Maps: functions like  $f, g, \dots$
  2. Rules:
    - (1) Identity law: if  $\mathbf{A} \xrightarrow{f} \mathbf{B}$ , then,  $I_B \circ f = f$  &  $f \circ I_A = f$ .
    - (2) Associative law:  $h \circ (g \circ f) = (h \circ g) \circ f$ .

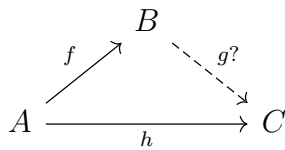
### Section 2:

- Some definitions: Consider the category  $\mathbf{A} \xrightarrow{f} \mathbf{B}$ ,
  1. The set **A** is called the Domain of map ‘ $f$ ’.
  2. The set **B** is called the Co-Domain of map ‘ $f$ ’.
  3. A **rule** for map ‘ $f$ ’, is that each element in **A** must be mapped to only one element in **B**.
- **Test for equality** of two maps:

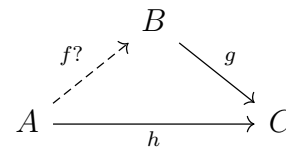
A **point** of a set **A** is a map from a **singleton set 1** to **A**. Using this, we can say that “ two maps  $f$  and  $g$  with domain **A** and co-domain **B** are said to be equal iff for all points  $\mathbf{1} \xrightarrow{a} \mathbf{A}$ ,  $f \circ a = g \circ a$ , then  $f = g$ . ”
- Internal and External Diagrams:
  - Internal: uses the arrow diagrams where the elements of the set are shown.
  - External: shows mapping with arrows between sets as a whole without explicitly showing the elements in them.

### Section 3:

- Total number of maps from set **A** to set **B** is given by:  $n(B)^{n(A)}$ , where  $n(X)$  represents the number of elements in set **X**
- Isomorphisms: a map  $\mathbf{A} \xrightarrow{f} \mathbf{B}$  is called Isomorphic or invertible, if there exist another map  $\mathbf{B} \xrightarrow{g} \mathbf{A}$  such that,  $f \circ g = I_B$  and  $g \circ f = I_A$ . This map 'g' is called the inverse map of 'f'. (If both domain and co-domain are equal, then this isomorphism is called **Automorphism**.)
- Isomorphs are **Reflexive**, **Symmetric**, and **Transitive**.
- From Ex.1(T), we can see that for two isomorphs  $f$  and  $g$ , the inverse of the composition  $f \circ g$  is  $g^{-1} \circ f^{-1}$
- **Determination and Choice Problems:**



**Determination Problems** requires us to find the map  $g$  (which we call the **determination** of  $h$  by  $f$ ) if both  $f$  and  $h$  are given, such that,  $h = g \circ f$ . If such  $g$  exist, then we say  $h$  can be **determined** by  $f$ .



**Choice Problems** requires us to find the map  $f$  if both  $f$  and  $h$  are given, such that,  $h = g \circ f$ . When the map  $f$  is fixed, we get a lot of “choices” for the map  $g$ .

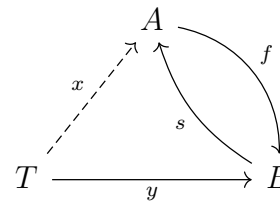
**Note:** When the set **B** is a “singleton set”, then the maps  $f$  and  $h$  are constant maps.

### • Retractions, Sections and Idempotents:

1. Retractions are the solution maps  $r$  for the determination problem:  $r \circ f = I_A$ .
2. Sections are the solution maps  $s$  for the choice problem:  $f \circ s = I_B$ .
3. Idempotents are the maps  $e$  such that:  $e \circ e = e$ . (eg:  $e = f \circ r$ )

### Theorems:

- If a map  $\mathbf{A} \xrightarrow{f} \mathbf{B}$  has a **section**, then for any  $\mathbf{T}$  and any map  $\mathbf{T} \xrightarrow{y} \mathbf{B}$ , there exist a map  $\mathbf{T} \xrightarrow{x} \mathbf{A}$ , such that  $f \circ x = y$ .



- If a map  $\mathbf{A} \xrightarrow{f} \mathbf{B}$  has a **retraction**, then for any  $\mathbf{T}$  and any map  $\mathbf{A} \xrightarrow{y} \mathbf{T}$ , there exist a map  $\mathbf{B} \xrightarrow{f} \mathbf{T}$ , such that  $f \circ x = y$ .

(**Note:** Maps with retractions are **monomorphic** and maps with sections are **epimorphic**.)

- **Uniqueness of Inverses:** if a map  $f$  has many retractions  $r_1, r_2, \dots$  and sections  $s_1, s_2, \dots$ , then:
  1. All of the sections are equal to each other, same is true for retractions.
  2. Both section and retraction are equal. ( $f$  is an **Isomorphism**)