Week Four

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Section 11

- "*A Set has the potentiality to carry all sorts of structure with the help of maps.*" Sets lack **structure** and can be mapped to any other set. However, introducing a map between two sets creates a notion of structure. In category theory, this "structure" can be **preserved** or **destroyed** by mappings between categories, highlighting a **key difference** between categories (sets+maps) and sets.
- An automorphism of a finite set is also known as a **permutation** of the set.
- Suppose $A^{\circlearrowright^{\alpha}}$ and $B^{\circlearrowright^{\beta}}$ have A isomorphic to B as sets, we **cannot** conclude that $A^{\circlearrowright^{\alpha}}$ is isomorphic to $B^{\circlearrowright^{\beta}}$. (Page 159, Ex.3 and Ex.4)

Section 12

- ⋆ The category $S^{\circlearrowright}$ has **practical uses:** *Dynamical systems/Automata.* We have the set X (in $S^{\circlearrowright}$) of all the different **possible states** of the system, and the endomap α of X which takes each state x to the state in which the system will be one unit of time later.
- In a **finite** dynamical system, every state eventually **settles** into a cycle.
- "**Family Trees**" are categories of sets with **two endomaps**, namely, 'mother' and 'father'.

Section 13: Monoids

- A category with exactly **one object** is a **monoid**.
- **'Structure-preserving'** interpretation of one category into another is a **functor**.
- A **discrete-time dynamical system** is just a **functor** from a 'monoid' (whose mappings are natural numbers) to the 'category of sets'. For **continuous-time**, use real numbers for mappings in the monoid. (Page 168,169)

19th July

Section 14:

- "*Although certain important properties are **'preserved'** by f, they are **not necessarily 'reflected'**.*" For example, if $X^{\circlearrowright^{\alpha}}$ contains a point x which is 'not a **fixed point**' of α, $y = f(x)$ (in $Y^{\circlearrowright^{\beta}}$) 'can be a fixed point' of β. (Page: 171, Ex.4)

- A functor f **preserves** the property of being in a small cycle, but the 'size of the cycle' **may decrease**. (Page 171, Ex.5)

- **Accessibility:** the *(positive)* property states that any point x in $X^{\circlearrowright^{\alpha}}$, has a point $\bar{x}$ such that $x = \alpha(\bar{x})$. This property is preserved by the map $S^{\circlearrowright}$, ie, if x is a **value** of α, then $f(x)$ is a value of β (in $Y^{\circlearrowright^{\beta}}$).

- A *negative property* of x is **not** being a fixed point. **Negative properties** tend **not** to be 'preserved', but instead they tend to be **reflected**. (Page 171 Ex.4)

  A map $X^{\circlearrowright^{\alpha}} \xrightarrow{f} Y^{\circlearrowright^{\beta}}$ in $S^{\circlearrowright}$ *'reflects'* a property means that if the value of f at x has the property, then x itself has the property.

- Naming the elements that have a given period by maps using an object called "**the cycle of length n (denoted by C_n)**", whose elements are in the set $\{0, 1, 2, \ldots, n-1\}$, which is a '*successor endomap*' where successor to '$n-1$' is 0.

- The maps from this cycle C_n to any object $Y^{\circlearrowright^{\beta}}$ '**name**' exactly the elements of period n in $Y^{\circlearrowright^{\beta}}$.

- Another object we can you to study dynamical systems besides C_n is $N = \mathbb{N}^{\circlearrowright^{\sigma}}$ ($\sigma(n) = n + 1$), where any mapping f from this object to another object $Y^{\circlearrowright^{\beta}}$ always follows $f(n) = \beta^n(y)$,
  This allows us to avoid using the map β on Y and instead, *precompose* the map σ on the mapping from N to Y for getting the **next state**.

- Presentations of Dynamical systems:
  - ⋆ In an endomap $X^{\circlearrowright^{\alpha}}$, the points which start a loop are called **generators**. (Page 183)
  - ⋆ Using these generators and the mapping, we can 'name' all the other points in the endomap. This creates a **list of labels (L)** for each point in the endomap.
  - ⋆ While finding these labels, we come across equations **(R)** which **relates** the 'generator points'.
  - ⋆ Applying $f \circ \alpha = \beta \circ f$ on equation set **(R)**, we see that the other enodmap $Y^{\circlearrowright^{\beta}}$ must contain points which follow the equation set **(R)** where α is replaced by β, so that there exist a '**structure-preserving**' map f.
  - ⋆ Both **(L)** and **(R)** are collectively called **presentation** of $X^{\circlearrowright^{\alpha}}$, and this can be used to **find the number of maps** (structure-preserving) between the two objects. (Page 183,184)
  - ⋆ Even **infinite** dynamical systems may have **finite presentations**. For example, $N^{\circlearrowright^{\sigma}}$ is presented by one *generator*, 0, and **no equations!**