Week Six

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19th August

Section 19: Terminal Objects

- From any set (even for **null set**), there is **only one map** to a *singleton* set.
- T is an object in a category $\mathcal{C}$, which is said to be **terminal** only if for any object X in $\mathcal{C}$:
  - ⋆ at least one map exists from X to T.
  - ⋆ that map should be the only map from X to T.

  Using these two conditions we can say that: (See page.229)

  "*There exists multiple terminal objects which are isomorphic to each other*."

- In the category of *endomaps*, we can say that the **singleton set** equipped with an endomap from the **point to itself**, is a terminal '*set-with-endomap*'.

  The mapping from an endomap X to this terminal object also follows the 'structure preserving rule'.

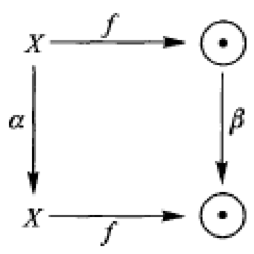


Figure 1: Map from X to T

Section 20: Points

- In the start of this section, we see an example which shows how we can use a *terminal object* (defined in the category) to **select an item** from an object (of the same category). Hence, we can define:

  "*A point of an object X is the map $T \longrightarrow X$*"

  where, T is the terminal object of the category.

- In different categories, the meaning of the word 'point' is different from what we think of. For example, in the category of endomaps, the term 'point' refers to **fixed point** (See page.232). So, if an endomap does **not** have a fixed point, we say it doesnt have '*points*' (which doesn't mean it doesnt have elements!) (See page.233)

Category	Terminal object	'Points of X' means...
$\mathcal{C}$	T	map $T \longrightarrow X$
$\mathcal{S}$	$\boxed{\bullet}$	element of X
$\mathcal{S}^{\circlearrowleft}$ endomaps of sets	$\boxed{\bullet\circlearrowleft}$	fixed point or equilibrium state

20th August

Product

- A product of A and B (in category $\mathcal{C}$, also called **factors of P**) is:
  - ⋆ an object P in $\mathcal{C}$.
  - ⋆ a pair of maps: $P \xrightarrow{p_1} A$ and $P \xrightarrow{p_2} B$ such that,
    for every other object X in $\mathcal{C}$, with pair of maps $X \xrightarrow{q_1} A$ and $X \xrightarrow{q_2} B$, there exist **exactly one map** $X \xrightarrow{q} P$: $q_1 = p_1 \circ q$ and $q_2 = p_2 \circ q$.

  From this definition, we can say that if there exists two products sharing the same **factors**, the products must be **isomorphic**.
- '3D-Space' can be considered as the **product** of three *linearly independent* axes.
- For Products in 'categories of endomaps',

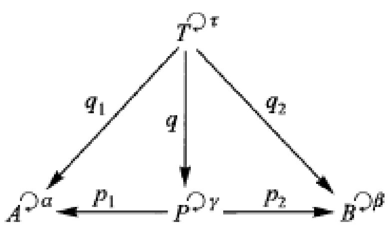
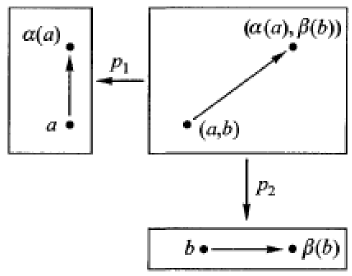


Figure 2: Product in category $S^{\circlearrowleft}$



Internal diagram of $A^{\circlearrowleft} \longrightarrow P \longleftarrow B^{\circlearrowleft}$

We get another condition from the '*structure preserving rule*':

$$\gamma(a, b) = (\alpha(a), \beta(b))$$

Proof:
As we know, P contains elements of the type (a, b) where $a \in A$ and $b \in B$.

$$p_1(a, b) = a \quad p_2(a, b) = b$$
$$\Rightarrow (p_1 + p_2)(a, b) = a + b$$
$$(Or) \quad (a, b) = (p_1 + p_2)^{-1}[a + b] \tag{1}$$

By structure preserving conditions:

$$p_1\gamma = \alpha p_1 \quad p_2\gamma = \beta p_2$$
$$\Rightarrow (p_1 + p_2)\gamma = \alpha p_1 + \beta p_2$$
$$(Or) \quad \gamma = (p_1 + p_2)^{-1}[\alpha p_1 + \beta p_2]$$

Now, applying γ on the element (a, b) yields:

$$\gamma(a, b) = (p_1 + p_2)^{-1}[\alpha p_1 + \beta p_2](a, b)$$
$$\Rightarrow (p_1 + p_2)^{-1}[\alpha p_1(a, b) + \beta p_2(a, b)]$$
$$\Rightarrow (p_1 + p_2)^{-1}[\alpha(a) + \beta(b)]$$
$$\boxed{\gamma(a, b) = (\alpha(a), \beta(b))} \qquad \text{(used (1))}$$