Week Ten

Siva Sundar, EE23B151

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2^{nd} December

- **Graphs**: consists of the following:
  - Set V which contains **vertices** and set A with **arrows**,
  - s and t are the **source** and **target** functions respectively.

  Note: From every graph we can get a *preorder*. **Hasse Diagram** is a graph that gives a *presentation* of a preorder $(P, \leqslant)$. (See page.14)
- *Total order*: They are *posets* (partially ordered sets), with an additional condition: "*for all* x, y, either $x \leqslant y$ or $y \leqslant x$". (They should be *comparable*)
- **Partitions** can be made from preorders. (See page.16)
- Preorder of **upper sets** ($U(X)$ contains q, if $p, q \in X$ and $p \leqslant q$) on a *discrete preorder* on set X is same as power set $P(X)$.
- **Product Preorder:** Given $(P, \leqslant)$ and $(Q, \leqslant)$, we define $(P \times Q, \leqslant)$ such that:

$$(p, q) \leqslant (p', q') \iff p \leqslant p' \;\;\&\;\; q \leqslant q'$$

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- **Monotone map** is a *structure preserving* function $f : A \to B$, such that:

$$\forall x, y \in A, \text{ if } x \leqslant_A y \text{ then } f(x) \leqslant_B f(y).$$

  Cardinality is a function which maps a set to a natural number (which is the number of elements in the set). This function is a monotone map, as:

$$\text{if } X \subseteq Y, \text{ then } n(X) \leqslant n(Y).$$

  If a map $f : X \to Y$ exists, then there exists a monotone map $g; \text{Prt}(Y) \to \text{Prt}(X)$. ($\text{Prt}(X)$ gives the set of all partitions on X).

  If f and g are monotones, then $f \circ g$ is also monotone.

  Let P be a preorder. Monotone maps $P \to \mathcal{B}$ are in one-to-one correspondence with upper sets of P. (See page.22).
- *Yoneda Lemma:* to know an element is the same as knowing its upper set (the relationships it has with other elements). (see page.20).
- **Pullback map**: Let P and Q be preorders, and $f : P \to Q$ be a monotone map. Then we can define a montone map $g : U(Q) \to U(P)$ which is called the *pullback along f*. (U(X) is the set of all uppersets of X).

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- For a preorder $(P, \leqslant)$, and $A \subseteq P$ be a subset, we say $p \in P$ is a **meet** of A if
  - ⋆ $\forall a \in A$, we have $p \leqslant a$.
  - ⋆ $\forall q$, $q \leqslant a$ $\forall a \in A$, we have $q \leqslant p$.

  We denote meet 'p' as: $p \cong \bigwedge A$ or $p \cong \bigwedge_{a \in A} a$. This represents the *greatest lower bound* of the subset A. As the **GLB** is the "greatest among ***all*** lower bounds", we can say this is a **Universal property**.
- Similarly, for the preoreder discussed above, we say p is a **join** of A if:
  - ⋆ $\forall a \in A$, we have $a \leqslant p$.
  - ⋆ $\forall q$, $a \leqslant q$ $\forall a \in A$, we have $p \leqslant q$.

  We denote join p as: $p \cong \bigvee A$ or $p \cong \bigvee_{a \in A} a$. This represents the *lowest upper bound* of subset A. This is also a universal property.
- Any two things defined by the **same** universal property are automatically **equivalent** in a way known as '*unique up to unique **isomorphism***'. For example, we can see that if there exists two meets p and q for a preorder, they will be isomorphic to each other by definition.
- In a *discrete preorder*, there exist **no meets nor joins**.
- In any partial order (where $\cong$ and $=$ are the same), $p \vee p = p \wedge p = p$. (See page 25)
- In a power set $\mathrm{P}(X)$, for subsets, say $A, B \in X$, the meet is their intersection, ie, $A \wedge B = A \cap B$ and their join is their union, $A \vee B = A \cup B$.
- For a preorder P, $A \subseteq B \subseteq P$, then we say
  - ⋆ if meets of A and B exist, then $\bigwedge B \leqslant \bigwedge A$
  - ⋆ if joins of A and B exist, then $\bigvee A \leqslant \bigvee B$
- A monotone map $f : P \to Q$ has a **generative effect** if there exist elements $a, b \in P$ such that:
  $$f(a) \vee f(b) \ncong f(a \vee b)$$
  If the monotone map dosen't have a generative effect, then it will preserve the meets.
- A **Galois connection** between two preorders P and Q is a **pair of monotone maps** $f : P \to Q$ and $g : Q \to P$ such that:
  $$f(p) \leqslant q \iff p \leqslant g(q)$$
  We say f is the *left adjoint* and g is the *right adjoint* of the Galois connection.
- If P and Q are **total orders** and $f : P \to Q$ and $g : Q \to P$ are drawn with **arrows bending counterclockwise**, then f is **left adjoint** to g *iff* the arrows **do not cross**. (See page 28)
- Galois connections are a kind of relaxed version of isomorphisms. (Page 30)
- Right adjoints **preserve meets**, and Left adjoints **preserve joins** (See *Adjoint Functor Theorem*). Hence, left afjoints will not have generative effects.

- **Closure operator** $j : P \to P$ on a preorder P is a **monotone** map with:
  - ⋆ $p \leqslant j(p)$
  - ⋆ $j(j(p)) \cong j(p)$

  They can be made by composing left adjoint f with its right adjoint g. The other composite map $g \circ f$ (*interior map*) satisfies: $(g \circ f)(p) \leqslant p$.