Week Ten

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2^{nd} December

- **Graphs**: consists of the following:
  - Set V which contains **vertices** and set A with **arrows**,
  - s and t are the **source** and **target** functions respectively.

  Note: From every graph we can get a *preorder*. **Hasse Diagram** is a graph that gives a *presentation* of a preorder $(P, \leqslant)$. (See page.14)
- *Total order*: They are *posets* (partially ordered sets), with an additional condition: "*for all* x, y, either $x \leqslant y$ or $y \leqslant x$". (They should be *comparable*)
- **Partitions** can be made from preorders. (See page.16)
- Preorder of **upper sets** ($U(X)$ contains q, if $p, q \in X$ and $p \leqslant q$) on a *discrete preorder* on set X is same as power set $P(X)$.
- **Product Preorder:** Given $(P, \leqslant)$ and $(Q, \leqslant)$, we define $(P \times Q, \leqslant)$ such that:

$$(p, q) \leqslant (p', q') \iff p \leqslant p' \;\;\&\;\; q \leqslant q'$$

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- **Monotone map** is a *structure preserving* function $f : A \to B$, such that:

$$\forall x, y \in A, \text{ if } x \leqslant_A y \text{ then } f(x) \leqslant_B f(y).$$

  Cardinality is a function which maps a set to a natural number (which is the number of elements in the set). This function is a monotone map, as:

$$\text{if } X \subseteq Y, \text{ then } n(X) \leqslant n(Y).$$

  If a map $f : X \to Y$ exists, then there exists a monotone map $g; \mathrm{Prt}(Y) \to \mathrm{Prt}(X)$. ($\mathrm{Prt}(X)$ gives the set of all partitions on X).

  If f and g are monotones, then $f \circ g$ is also monotone.

  Let P be a preorder. Monotone maps $P \to \mathcal{B}$ are in one-to-one correspondence with upper sets of P. (See page.22).
- *Yoneda Lemma:* to know an element is the same as knowing its upper set (the relationships it has with other elements). (see page.20).
- **Pullback map**: Let P and Q be preorders, and $f : P \to Q$ be a monotone map. Then we can define a montone map $g : U(Q) \to U(P)$ which is called the *pullback along f*. (U(X) is the set of all uppersets of X).

7^{th} December

- For a preorder $(P, \leqslant)$, and $A \subseteq P$ be a subset, we say $p \in P$ is a **meet** of A if
  - ⋆ $\forall a \in A$, we have $p \leqslant a$.
  - ⋆ $\forall q$, $q \leqslant a$ $\forall a \in A$, we have $q \leqslant p$.

  We denote meet 'p' as: $p \cong \bigwedge A$ or $p \cong \bigwedge_{a \in A} a$. This represents the *greatest lower bound* of the subset A. As the **GLB** is the "greatest among ***all*** lower bounds", we can say this is a **Universal property**.

- Similarly, for the preoreder discussed above, we say p is a **join** of A if:
  - ⋆ $\forall a \in A$, we have $a \leqslant p$.
  - ⋆ $\forall q$, $a \leqslant q$ $\forall a \in A$, we have $p \leqslant q$.

  We denote join p as: $p \cong \bigvee A$ or $p \cong \bigvee_{a \in A} a$. This represents the *lowest upper bound* of subset A. This is also a universal property.

- Any two things defined by the **same** universal property are automatically **equivalent** in a way known as '*unique up to unique* ***isomorphism***'. For example, we can see that if there exists two meets p and q for a preorder, they will be isomorphic to each other by definition.

- In a *discrete preorder*, there exist **no meets nor joins**.

- In any partial order (where $\cong$ and $=$ are the same), $p \vee p = p \wedge p = p$. (See page 25)

- In a power set $\mathrm{P}(X)$, for subsets, say $A, B \in X$, the meet is their intersection, ie, $A \wedge B = A \cap B$ and their join is their union, $A \vee B = A \cup B$.

- For a preorder P, $A \subseteq B \subseteq P$, then we say
  - ⋆ if meets of A and B exist, then $\bigwedge B \leqslant \bigwedge A$
  - ⋆ if joins of A and B exist, then $\bigvee A \leqslant \bigvee B$

- A monotone map $f : P \to Q$ has a **generative effect** if there exist elements $a, b \in P$ such that:

  $$f(a) \vee f(b) \ncong f(a \vee b)$$

  If the monotone map dosen't have a generative effect, then it will preserve the meets.

- A **Galois connection** between two preorders P and Q is a **pair of monotone maps** $f : P \to Q$ and $g : Q \to P$ such that:

  $$f(p) \leqslant q \iff p \leqslant g(q)$$

  We say f is the *left adjoint* and g is the *right adjoint* of the Galois connection.

- If P and Q are **total orders** and $f : P \to Q$ and $g : Q \to P$ are drawn with **arrows bending counterclockwise**, then f is **left adjoint** to g *iff* the arrows **do not cross**. (See page 28)

- Galois connections are a kind of relaxed version of isomorphisms. (Page 30)

- Right adjoints **preserve meets**, and Left adjoints **preserve joins** (See *Adjoint Functor Theorem*). Hence, left afjoints will not have generative effects.

- **Closure operator** $j : P \to P$ on a preorder P is a **monotone** map with:
  - ⋆ $p \leqslant j(p)$
  - ⋆ $j(j(p)) \cong j(p)$

  They can be made by composing left adjoint f with its right adjoint g. The other composite map $g \circ f$ (*interior map*) satisfies: $(g \circ f)(p) \leqslant p$.

10^{th} December

- A $\mathcal{V}$-category is a set of objects where $\mathcal{V}$ provides the structure for assessing "getting from point a" to "point b". Examples of such categories are:
  - ⋆ A **Bool**-category, where the answer for "getting from a to b" is **true/false**.
  - ⋆ A **Cost**-category, where the answer is a **cost**, $d \in [0, 1]$.
  - ⋆ A **Set**-category where the question of getting from point a to point b has a set of answers (elements of which might be called **methods**).
- Preorders are denoted as $(P, \leqslant)$, where we have two structures: X being a set and $\leqslant$ being the **relation** which is **transitive** and **reflexive**.
- A *symmetric monoidal* structure on $(X, \leqslant)$ consists of:
  - ⋆ an element $I \in X$ (monoidal unit).
  - ⋆ a function $\otimes : X \times X \longrightarrow X$ (monoidal product), which satisfies *monotonicity, unitality, symmetry,* and *associativity* (MUSA). (Page 42)

  A preorder equipped with a symmetric monoidal structure, $(X, \leqslant, I, \otimes)$, is called a **symmetric monoidal preorder**. Replacing all the conditions in page 42 with $\cong$ in place of $=$, makes it a **weak monoidal structure**.
- **Wiring diagrams**: the **wires** represent *elements*, the **boxes** represent *relationships*, and the wiring diagrams themselves show how relationships can be **combined**. We call boxes and wires **icons**.

  Wires **in parallel** to represent the **monoidal product** of their labels. (Page 44-45)

  No line represents the monoidal unit I.

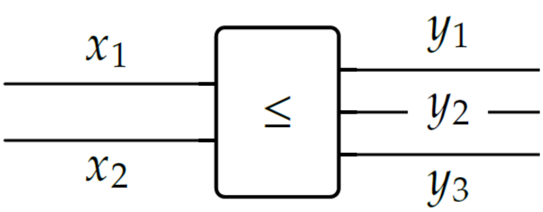

Figure 1: Valid only if $x_1 \otimes x_2 \leqslant y_1 \otimes y_2 \otimes y_3$

  Reflexivity says that $x \leqslant x$, this means the diagram just consisting of a wire is always valid. **Transitivity** allows us to connect diagrams together. See Page 46.
- These monoidal structures can be used to analyse real-life cases such as Chemical reactions or Manufacturing. In the latter, we add a new axiom called the **discard axiom**: $x \leqslant I \ \forall x \in X$. In another case, like informatics, we have a **copy axiom**: $x \leqslant x + x \ \forall x \in X$.

13^{th} December

- Let $\mathcal{P} = (P, \leqslant_P, I_P, \bigotimes_P)$ and $\mathcal{Q} = (Q, \leqslant_Q, I_Q, \bigotimes_Q)$ be monoidal preorders. A X-**monoindal monotone** from $\mathcal{P}$ to $\mathcal{Q}$ is a monotone map $f : (P, \leqslant_P) \longrightarrow (Q, \leqslant_Q)$, satisfying: ($X$ and α are placeholders)

  ⋆ $I_Q \ \alpha \ f(I_P)$

  ⋆ $f(p_1) \otimes_Q f(p_2) \ \alpha \ f(p_1 \otimes_P p_2), \quad \forall \, p_1, p_2 \in P$

  where, X = lax (α is $\leqslant$), oplax (α is $\geqslant$), strong (α is $\cong$), strict (α is $=$)

- $\mathcal{V}$ − *categories* **MUST** follow **M** and **S** in **MUSA**. Let $\mathcal{V} = (V, \leqslant, I, \bigotimes)$ be a symmetric monoidal preorder. A $\mathcal{V}$ − *category* $\mathcal{X}$ consists of two consituents:

  ⋆ a set $\mathrm{Ob}(\mathcal{X})$, elements of which are called **objects** ($\notin V$).

  ⋆ $\forall x, y \in \mathrm{Ob}(\mathcal{X})$, one specifies an element $\mathcal{X}(x, y) \in V$, called the **hom-object**.

  These constituents follow the below properties:

  ⋆ $\forall x \in \mathrm{Ob}(\mathcal{X})$ we have $I \leqslant \mathcal{X}(x, x)$.

  ⋆ $\forall x, y, z \in \mathrm{Ob}(\mathcal{X})$ we have $\mathcal{X}(x, y) \otimes \mathcal{X}(y, z) \leqslant \mathcal{X}(x, z)$.

  We say **V** the **base of the enrichment** for **X** or **X** is **enriched** in **V**.

- There is a *one-to-one* correspondence between preorders and **Bool**-categories. (See ex-2.47 and page 59)

- A **metric space** (X, d) consists of:

  ⋆ a set X, elements are *points*.

  ⋆ a function $d : X \times X \longrightarrow \mathcal{R}_{\geqslant 0}$, where $d(x, y)$ is called the **distance between points** x **and** y.

  These constituents must satisfy:

  ⋆ $\forall x \in X, \ d(x, x) = 0$

  ⋆ $\forall x \in X, \ d(x, y) = 0 \Leftrightarrow x = y$

  ⋆ $\forall x \in X, \ d(x, y) = d(y, x)$

  ⋆ $\forall x \in X, \ d(x, y) + d(y, z) \geqslant d(x, z) \quad$ (Triangle inequality).

  If we change $\mathcal{R}_{\geqslant\prime}$ to $[0, \infty]$, we call (X, d) an **extended** metric space.

- A *Lawvere metric space* is a **Cost**-category. (Page 61)

- **Cost**-weighted graphs are similar to Hasse diagrams but the edges are labelled with numbers $w \geqslant 0$. It represents a Lawvere metric space.

  Such graphs G can be converted to matrices M_G whose row and column represents **points** in the graph and the entries represent the **edge weight** ($w \in [0, \infty]$) between the row point and column point.

14^{th} December

- Changeing the *base of enrichment*: Let $f : \mathcal{V} \longrightarrow \mathcal{W}$ be a monoidal monotone. Given a $\mathcal{V}$-category $\mathcal{C}$, one forms the associated $\mathcal{W}$-category $\mathcal{C}_f$ as follows: (page 64)
  - ⋆ $\text{Ob}(\mathcal{C}_f) := \text{Ob}(\mathcal{C})$
  - ⋆ $\forall c, d \in \text{Ob}(\mathcal{C}),\ \mathcal{C}_f(c, d) := f(\mathcal{C}(c, d))$
- **Enriched Functors:** Let $\mathcal{X}$ and $\mathcal{Y}$ be $\mathcal{V}$-categories. A $\mathcal{V}$-functor $\mathcal{F} : \mathcal{X} \longrightarrow \mathcal{Y}$ consists of:
  - $\mathcal{F} : \text{Ob}(\mathcal{X}) \longrightarrow \text{Ob}(\mathcal{Y})$

  subject to one constraint:
  - $\forall x_1, x_2 \in \text{Ob}(\mathcal{X}),\ \mathcal{X}(x_1, x_2) \leqslant \mathcal{Y}(\mathcal{F}(x_1), \mathcal{F}(x_2))$
- **Product $\mathcal{V}$-categories**: Let $\mathcal{X}$ and $\mathcal{Y}$ be $\mathcal{V}$-categories. $\mathcal{V}$-product is defined as the $\mathcal{V}$-category $\mathcal{X} \times \mathcal{Y}$ (see ex. 2.75) with:
  - ⋆ $\text{Ob}(\mathcal{X} \times \mathcal{Y}) := \text{Ob}(\mathcal{X}) \times \text{Ob}(\mathcal{Y})$
  - ⋆ $(\mathcal{X} \times \mathcal{Y})((x, y), (x', y')) := \mathcal{X}(x, x') \times \mathcal{Y}(y, y')$
- **Symmetric monoidal closed preorders**: A symmetric monoidal preorder $\mathcal{V} = (V, \leqslant, I, \otimes)$ is called **closed** (see remark 2.81), if $\forall v, w \in V$, there is an element $v \multimap w$ in $\mathcal{V}$, called the **hom-element**, with the property:

$$(a \otimes v) \leqslant w \iff a \leqslant (v \multimap w) \qquad \forall a, v, w \in V$$

- For $v \in V$ and $A \subseteq V$, we have $v \otimes \bigvee_{a \in A} a \cong \bigvee_{a \in A} (v \otimes a)$. (Page 70)
- **Unital Commutative Quantale** is a symmetric monoidal closed preorder $\mathcal{V} = (V, \leqslant, I, \otimes, \multimap)$ that has all joins: $\bigvee A$ exists $\forall A \subseteq V$. Empty join $\bigvee \varnothing$ is denoted as '0'.
- If $\mathcal{X}$ is a $\mathcal{V}$-category $(X, \leqslant_X, I_X, \otimes_X)$ with objects $U, V \in X$ Generalized **Hausdorff distance** is given by:

$$\mathcal{X}(U, V) = \bigwedge_{u \in U} \bigvee_{v \in V} \mathcal{X}(u, v)$$

- **Generalised Matrix Multiplication:** Let $\mathcal{V} = (V, \leqslant, \otimes, I)$ be a quantale. Given X and Y, a $\mathcal{V}$-matrix is a function $M : X \times Y \longrightarrow V$. $\forall x \in X \& y \in Y$, we say $M(x, y)$ is the $(x, y)^{th}$ entry.

  Say we have two such $\mathcal{V}$-matrices $M : X \times Y \longrightarrow V$ and $M : Y \times Z \longrightarrow V$. Their product $(M * N) : X \times Z \longrightarrow V$, whose entries are given by:

$$(M * N)(x, z) := \bigvee_{y \in Y} M(x, y) \otimes N(y, z)$$