To answer questions like, "Can I make what I want from what I have?" or, "How much will it cost to obtain something?", the following ideas may be employed to build resource theories.

Symmetric Monoidal Preorders

A preorder $(X, \leq)$ may be given extra structure in the following way:

- identify some $I \in X$ — monoidal unit
- define a function $\otimes : X \times X \longrightarrow X$ — monoidal product

such that

a) if $x_1 \leq y_1$ and $x_2 \leq y_2$ then $x_1 \otimes x_2 \leq y_1 \otimes y_2$ — monotonicity
b) $I \otimes x = x = x \otimes I$ — unitality
c) $x \otimes (y \otimes z) = (x \otimes y) \otimes z$ — associativity
d) $x \otimes y = y \otimes x$ — symmetry

Such a structure is called a symmetric monoidal preorder.

→ A monoid is a set M, a function $* : M \times M \to M$ and some $e \in M$ such that $*$ is unital w.r.t e and associative.

→ e.g: $\mathbf{Bool} = (\mathbb{B}, \leq, \text{true}, \wedge)$

↓ more useful

$\wedge$	F	T
F	F	F
T	F	T

- $(\mathbb{B}, \leq, \text{false}, \vee)$

$\vee$	F	T
F	F	T
T	T	T

- Cost $:= ([0,\infty], \geq, 0, +)$

→ Note: $(X, \leq)^{op} := (X, \geq)$

Monoidal Monotone Maps

- A map $f: (P, \leq_P) \longrightarrow (Q, \leq_Q)$ such that
  - $I_Q \leq_Q f(I_P)$
  - $f(p_1) \otimes_Q f(p_2) \leq_Q f(p_1 \otimes p_2)$

also called lax monoidal monotones

→ e.g.: For Bool $= (\mathbb{B}, \leq, \text{true}, \wedge)$, Cost $= ([0,\infty], \geq, 0, +)$ we have $g: \text{Bool} \longrightarrow \text{Cost}$ with $g(F) := \infty$, $g(T) := 0$.

Enrichment

2.3 Enrichment

In this section we will introduce $\mathcal{V}$-categories, where $\mathcal{V}$ is a symmetric monoidal preorder. We will see that **Bool**-categories are preorders, and that **Cost**-categories are a nice generalization of the notion of metric space.

2.3.1 $\mathcal{V}$-categories

While $\mathcal{V}$-categories can be defined even when $\mathcal{V}$ is not symmetric, i.e. just obeys conditions (a)–(c) of Definition 2.2, certain things don't work quite right. For example, we will see later in Exercise 2.75 that the symmetry condition is necessary in order for products of $\mathcal{V}$-categories to exist. Anyway, here's the definition.

Definition 2.46. Let $\mathcal{V} = (V, \leq, I, \otimes)$ be a symmetric monoidal preorder. A $\mathcal{V}$-*category* $\mathcal{X}$ consists of two constituents, satisfying two properties. To specify $\mathcal{X}$,

(i) one specifies a set $\text{Ob}(\mathcal{X})$, elements of which are called *objects*; ob

(ii) for every two objects x, y, one specifies an element $\mathcal{X}(x, y) \in V$, called the *hom-object*.[2] for every $x, y \in \text{Ob}(\mathcal{X})$, $\exists\ \mathcal{X}(x,y) \in V$ morphisms

The above constituents are required to satisfy two properties:

(a) for every object $x \in \text{Ob}(\mathcal{X})$ we have $I \leq \mathcal{X}(x, x)$, and

(b) for every three objects $x, y, z \in \text{Ob}(\mathcal{X})$, we have $\mathcal{X}(x, y) \otimes \mathcal{X}(y, z) \leq \mathcal{X}(x, z)$.

We call $\mathcal{V}$ the *base of the enrichment* for $\mathcal{X}$ or say that $\mathcal{X}$ is *enriched* in $\mathcal{V}$.

$x : \text{Ob}(\mathcal{X})$
$+ \mathcal{X}(x,y)$
$x, y \in \text{Ob}(\mathcal{X})$
$I \leq \mathcal{X}(x,x)$

Example 2.47. As we shall see in the next subsection, from every preorder we can

Metric Space

2.3.3 Lawvere metric spaces

Metric spaces offer a precise way to describe spaces of points, each pair of which is separated by some distance. Here is the usual definition:

Definition 2.51. A *metric space* (X, d) consists of:

(i) a set X, elements of which are called *points*, and

(ii) a function $d\colon X \times X \to \mathbb{R}_{\geq 0}$, where $d(x, y)$ is called the *distance between x and y*.

These constituents must satisfy four properties:

(a) for every $x \in X$, we have $d(x, x) = 0$,

(b) for every $x, y \in X$, if $d(x, y) = 0$ then $x = y$,

(c) for every $x, y \in X$, we have $d(x, y) = d(y, x)$, and

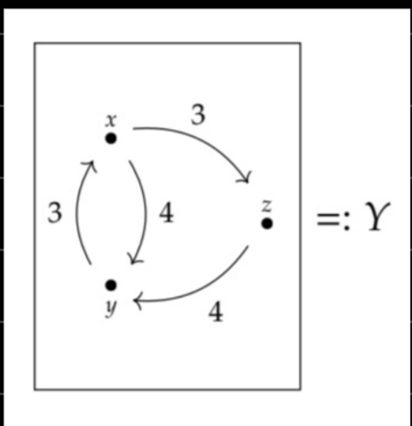
(d) for every $x, y, z \in X$, we have $d(x, y) + d(y, z) \geq d(x, z)$.

The fourth property is called the *triangle inequality*.

If we ask instead in (ii) for a function $d\colon X \times X \to [0, \infty] = \mathbb{R}_{\geq 0} \cup \{\infty\}$, we call (X, d) an *extended* metric space.

→ It is a Cost-category.

→ Metric spaces with weighted graphs :



$d(\nearrow)$	x	y	z
x	0	4	3
y	3	0	6
z	7	4	0

2.4.1 Changing the base of enrichment

Any monoidal monotone $\mathcal{V} \to \mathcal{W}$ between symmetric monoidal preorders lets us convert $\mathcal{V}$-categories into $\mathcal{W}$-categories

Construction 2.64. Let $f\colon \mathcal{V} \to \mathcal{W}$ be a monoidal monotone. Given a $\mathcal{V}$-category $\mathcal{C}$, one forms the associated $\mathcal{W}$-category, say $\mathcal{C}_f$ as follows.

(i) We take the same objects: $\mathrm{Ob}(\mathcal{C}_f) := \mathrm{Ob}(\mathcal{C})$.

(ii) For any $c, d \in \mathrm{Ob}(\mathcal{C})$, put $\mathcal{C}_f(c,d) := f(\mathcal{C}(c,d))$.

Example 2.65. As an example, consider the function $f\colon [0,\infty] \to \{\texttt{true}, \texttt{false}\}$ given by

$I_V = 0 \quad I_W = \text{true}$

$$f(x) := \begin{cases} \texttt{true} & \text{if } x = 0 \\ \texttt{false} & \text{if } x > 0 \end{cases} \tag{2.66}$$

$f(I_V) = f(0) = \text{true}$ so $I_W \leq f(I_V)$

It is easy to check that f is monotonic and that f preserves the monoidal product and monoidal unit; that is, it's easy to show that f is a monoidal monotone. (Recall Exercise 2.44.)

Thus f lets us convert Lawvere metric spaces into preorders.

$f(x) \wedge f(y) = T$ if $x, y = 0$
$= F$ else

$f(x+y) = T$ if $x, y = 0$
F else

Exercise 2.67. Recall the "regions of the world" Lawvere metric space from Exercise 2.57 and the text above it. We just learned that, using the monoidal monotone f in Eq. (2.66), we can convert it to a preorder. Draw the Hasse diagram for the preorder corresponding to the regions: US, Spain, and Boston. How could you interpret this preorder relation? ◊

2.4.2 Enriched functors

The notion of functor provides the most important type of relationship between categories.

Definition 2.69. Let $\mathcal{X}$ and $\mathcal{Y}$ be $\mathcal{V}$-categories. A *$\mathcal{V}$-functor from $\mathcal{X}$ to $\mathcal{Y}$*, denoted $F\colon \mathcal{X} \to \mathcal{Y}$, consists of one constituent:

(i) a function $F\colon \mathrm{Ob}(\mathcal{X}) \to \mathrm{Ob}(\mathcal{Y})$

subject to one constraint

(a) for all $x_1, x_2 \in \mathrm{Ob}(\mathcal{X})$, one has $\mathcal{X}(x_1, x_2) \leq \mathcal{Y}(F(x_1), F(x_2))$.

2.4.3 Product $\mathcal{V}$-categories

If $\mathcal{V} = (V, \leq, I, \otimes)$ is a symmetric monoidal preorder and $\mathcal{X}$ and $\mathcal{Y}$ are $\mathcal{V}$-categories, then we can define their $\mathcal{V}$-product, which is a new $\mathcal{V}$-category.

Definition 2.74. Let $\mathcal{X}$ and $\mathcal{Y}$ be $\mathcal{V}$-categories. Define their *$\mathcal{V}$-product*, or simply *product*, to be the $\mathcal{V}$-category $\mathcal{X} \times \mathcal{Y}$ with

(i) $\mathrm{Ob}(\mathcal{X} \times \mathcal{Y}) := \mathrm{Ob}(\mathcal{X}) \times \mathrm{Ob}(\mathcal{Y})$, → Set of sets

(ii) $(\mathcal{X} \times \mathcal{Y})\big((x, y), (x', y')\big) := \mathcal{X}(x, x') \otimes \mathcal{Y}(y, y')$,

for two objects (x, y) and (x', y') in $\mathrm{Ob}(\mathcal{X} \times \mathcal{Y})$.

$$\mathcal{X} := \boxed{\begin{tikzcd} \overset{A}{\bullet} \arrow[r, "2"] & \overset{B}{\bullet} \arrow[r, "3"] & \overset{C}{\bullet} \end{tikzcd}} \qquad \boxed{\begin{tikzcd} \overset{p}{\bullet} \arrow[d, bend right, "5"'] \\ \underset{q}{\bullet} \arrow[u, bend right, "8"'] \end{tikzcd}} =: \mathcal{Y}$$

$$\mathcal{X} \times \mathcal{Y} = \boxed{\begin{tikzcd} \overset{(A,p)}{\bullet} \arrow[r, "2"] \arrow[d, bend right, "5"'] & \overset{(B,p)}{\bullet} \arrow[r, "3"] \arrow[d, bend right, "5"'] & \overset{(C,p)}{\bullet} \arrow[d, bend right, "5"'] \\ \underset{(A,q)}{\bullet} \arrow[r, "2"] \arrow[u, bend right, "8"'] & \underset{(B,q)}{\bullet} \arrow[r, "3"] \arrow[u, bend right, "8"'] & \underset{(C,q)}{\bullet} \arrow[u, bend right, "8"'] \end{tikzcd}}$$

Symmetric Monoidal Closed Preorders

Definition 2.79. A symmetric monoidal preorder $\mathcal{V} = (V, \leq, I, \otimes)$ is called *symmetric monoidal closed* (or just *closed*) if, for every two elements $v, w \in V$, there is an element $v \multimap w$ in $\mathcal{V}$, called the *hom-element*, with the property

$$(a \otimes v) \leq w \quad \text{iff} \quad a \leq (v \multimap w). \tag{2.80}$$

for all $a, v, w \in V$.

Quantales

Definition 2.90. A *unital commutative quantale* is a symmetric monoidal closed preorder $\mathcal{V} = (V, \leq, I, \otimes, \multimap)$ that has all joins: $\bigvee A$ exists for every $A \subseteq V$. In particular, we often denote the empty join by $0 := \bigvee \varnothing$.

Example 2.91. In Example 2.83, we saw that **Cost** is monoidal closed. To check whether **Cost** is a quantale, we take an arbitrary set of elements $A \subseteq [0, \infty]$ and ask if it has a join $\bigvee A$. To be a join, it needs to satisfy two properties:

a. $a \geq \bigvee A$ for all $a \in A$, and

b. if $b \in [0, \infty]$ is any element such that $a \geq b$ for all $a \in A$, then $\bigvee A \geq b$.

In fact we can define such a join: it is typically called the *infimum*, or greatest lower bound, of A.[5] For example, if $A = \{2, 3\}$ then $\bigvee A = 2$. We have joins for infinite sets too: if $B = \{2.5, 2.05, 2.005, \ldots\}$, its infimum is 2. Finally, in order to say that $([0, \infty], \geq)$ has all joins, we need a join to exist for the empty set $A = \varnothing$ too. The first condition becomes vacuous—there are no a's in A—but the second condition says that for any $b \in [0, \infty]$ we have $\bigvee \varnothing \geq b$; this means $\bigvee \varnothing = \infty$.

Thus indeed $([0, \infty], \geq)$ has all joins, so **Cost** is a quantale.

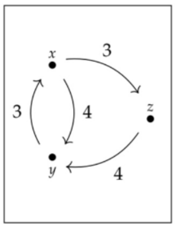
Matrix Multiplication

Definition 2.100. Let $\mathcal{V} = (V, \leq, \otimes, I)$ be a quantale. Given sets X and Y, a *matrix with entries in* $\mathcal{V}$, or simply a *$\mathcal{V}$-matrix*, is a function $M: X \times Y \to V$. For any $x \in X$ and $y \in Y$, we call $M(x, y)$ the *(x, y)-entry*.

Here is how you multiply $\mathcal{V}$-matrices $M: X \times Y \to V$ and $N: Y \times Z \to V$. Their product is defined to be the matrix $(M * N): X \times Z \to V$, whose entries are given by the formula

$$(M * N)(x, z) := \bigvee_{y \in Y} M(x, y) \otimes N(y, z). \tag{2.101}$$

→ We may find the distance matrix first by noting down M_y.

$Y :=$ 

M_Y	x	y	z
x	0	4	3
y	3	0	∞
z	∞	4	0

d_Y	x	y	z
x	0	4	3
y	3	0	6
z	7	4	0

The matrix M_Y can be thought of as recording the length of paths that traverse either 0 or 1 edges: the diagonals being 0 mean we can get from x to x without traversing any

$$\begin{matrix} 0 & 4 & 3 \\ 3 & 0 & \infty \\ \infty & 4 & 0 \end{matrix} \qquad \begin{matrix} 0 & 4 & 3 \\ 3 & 0 & \infty \\ \infty & 4 & 0 \end{matrix}$$

$$M^2(1,1) = \bigvee_j M(1,j) \otimes M(j,1) = 0$$

$$M^2(3,1) = \bigvee_j M(3,j) \otimes M(j,1) =$$

edges. When we can get from x to y in one edge we record its length in M_Y, otherwise we use ∞.

When we multiply M_Y by itself using the formula Eq. (2.101), the result M_Y^2 tells us the length of the shortest path traversing 2 edges or fewer. Similarly M_Y^3 tells us about the shortest path traversing 3 edges or fewer:

$M_Y^2 =$

↗	x	y	z
x	0	4	3
y	3	0	6
z	7	4	0

$M_Y^3 =$

↗	x	y	z
x	0	4	3
y	3	0	6
z	7	4	0

One sees that the powers stabilize: $M_Y^2 = M_Y^3$; as soon as that happens one has the matrix of distances, d_Y. Indeed M_Y^n records the lengths of the shortest path traverse n edges or fewer, and the powers will always stabilize if the set of vertices is finite, since the shortest path from one vertex to another will never visit a given vertex more than once.[7]