2. Monoidal Preorders & Enrichment

* $\mathcal{V}$ - category is a set of objects, where $\mathcal{V}$.

"Structures the question" of getting from one object to another (also the methods of getting)

Eg: Bool - category → the question of getting from point a to b is either true/false.

Cost - category → the question of getting from point a to b is a cost d $\in [0, \infty]$

Set - category → the question has set of answers (whose elements are called methods)

SYMMETRIC MONOIDAL PREORDER: is a preorder $(X, \leq)$ consists of

(i) an element $I \in X$ called monoidal unit

(ii) a function $\otimes : X \times X \longrightarrow X$ called monoidal product

&

Must satisfy, where $\otimes(x_1, x_2) = x_1 \otimes x_2$

monotonicity

(a) If $x_1 \leq y_1$ & $x_2 \leq y_2$ then $x_1 \otimes x_2 \leq y_1 \otimes y_2$

unitality

(b) $\forall x \in X$ $\quad I \otimes x = x \otimes I = x$

(c) $x \otimes (y \otimes z) = (x \otimes y) \otimes z$ & (d) $x \otimes y = y \otimes x$

rep : $(X, \leq, I, \otimes)$ a symmetric monoidal preorder

Example : ① $(\mathbb{R}, \leq, 0, +)$

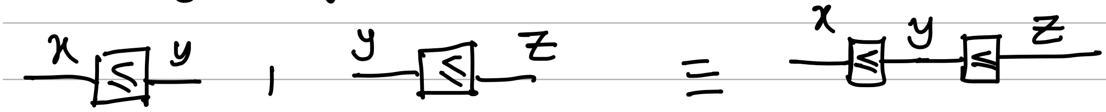
② $(\text{Disc}_M, =, e, *)$ if $(M, e, *)$ is a commutative monoid

Disc_M → Discrete preorder M

e → identity element of M

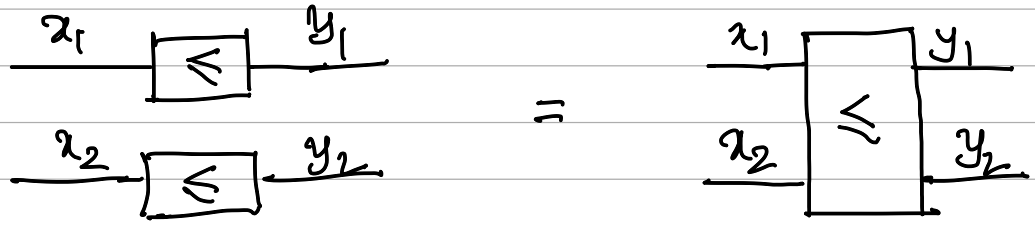
$*$ → monoid multiplication

* Wiring Diagram :-



x, y (two wires) $\Rightarrow$ $x \otimes y$ (Two wires in parallel)

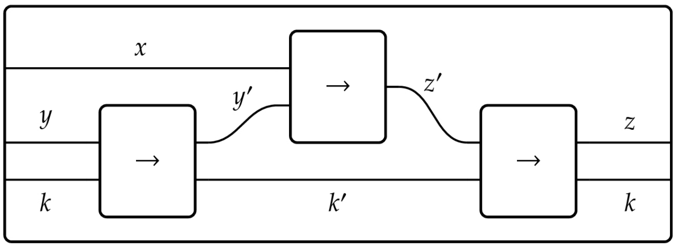
I or "nothing" $\Rightarrow$ Monoidal unit I



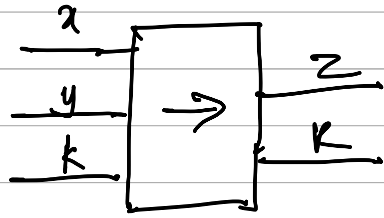
Example in che : $(\text{Mat}, \rightarrow, 0, +)$

$$\underline{2H_2O + 2Na} \underline{\rightarrow} 2NaOH + H_2$$

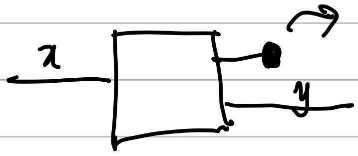
$H_2O, Na, NaOH, H_2 \in \text{Mat}$ (0 is for sake)



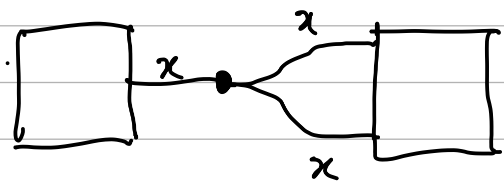
→ corresponds to



↳ k acts as 'catalyst' in this reaction



→ This means 'discard'



$x \leq x + x \quad \forall x \in X$

Examples:

<u>Bool</u>

$(\mathbb{B}, \leq, \text{true}, \wedge)$

↳ and. (also works for OR ($\vee$))

<u>Cost</u>

$([0, \infty], \geq, 0, +)$

prop: If $(X, \leq, I, \otimes)$ is a symmetric monoidal preorder, then the opposite preorder, $(X, \geq, I, \otimes)$ is also a SMP.

$\{ \mathcal{X} = (X, \leq) \; ; \; \mathcal{X}^{op} = (X, \geq) \}$

⇒ Monoidal monotone maps :

$$\mathcal{P} = (P, \leq_P, I_P, \otimes_P)$$

$$\mathcal{Q} = (Q, \leq_Q, I_Q, \otimes_Q)$$

a monoidal monotone from $\mathcal{P}$ to $\mathcal{Q}$ is a monotone map f : satisfies,

(a) $I_Q \leq_Q f(I_P)$ &

(b) $f(P_1) \otimes_Q f(P_2) \leq_Q f(P_1 \otimes_P P_2)$

for all $P_1, P_2 \in P$

* It is a strong Monoidal Monotone if the $\leq_Q$ in (a), (b) is '$\cong$'

* A strict Monoidal Monotone is more rigid, if the $\leq_Q$ in (a), (b) is '$=$'

also called 'lax monoidal monotones' or 'monoidal functors' & the dual is 'oplax monoidal monotones'

Eg : $i : (\mathbb{N}, \leq, 0, +) \to (\mathbb{R}, \leq, 0, +)$ monotone

$i(n) = n$, if $m \leq n$ then $i(m) \leq i(n)$ ↗

$0 \leq 0$

$i(m) + i(n) \leq i(m+n)$ } precisely it is $=$ instead of $\leq$

so, strict monoidal monotone.

②

Bool $\qquad\qquad$ Cost

$g : (\mathbb{B}, \leq, \text{true}, \wedge) \to ([0, \infty], \geq, 0, +)$

$g(\text{false}) = \infty \qquad g(\text{true}) = 0$

NOTE

The operator $\otimes$ for bool is 'AND' $(\wedge)$

(i) $I_Q \leq g(I_P) \Rightarrow 0 \leq 0 \Rightarrow$ TRUE

(ii) $g(\text{true}) + g(\text{false}) \geq g(\text{true} \wedge \text{false})$
$= g(\text{false}) = \infty$

$= \quad 0 + \infty \geq \infty$ ✓

$0 + 0 \geq 0$ ✓

$\infty + 0 \geq \infty$ ✓

$\infty + \infty \geq \infty$ ✓

& all are $= \infty$, strict monoidal monotone

* Enrichment :

Def :

Definition 2.46. Let $\mathcal{V} = (V, \leq, I, \otimes)$ be a symmetric monoidal preorder. A $\mathcal{V}$-*category* $\mathcal{X}$ consists of two constituents, satisfying two properties. To specify $\mathcal{X}$,

(i) one specifies a set $\mathrm{Ob}(\mathcal{X})$, elements of which are called *objects*;

(ii) for every two objects x, y, one specifies an element $\mathcal{X}(x, y) \in V$, called the *hom-object*.[2]

The above constituents are required to satisfy two properties:

(a) for every object $x \in \mathrm{Ob}(\mathcal{X})$ we have $I \leq \mathcal{X}(x, x)$, and

(b) for every three objects $x, y, z \in \mathrm{Ob}(\mathcal{X})$, we have $\mathcal{X}(x, y) \otimes \mathcal{X}(y, z) \leq \mathcal{X}(x, z)$.

We call $\mathcal{V}$ the *base of the enrichment* for $\mathcal{X}$ or say that $\mathcal{X}$ is *enriched* in $\mathcal{V}$.

* from any preorder we can construct a 'Bool' category.

Eg:

$$\begin{tikzcd} & t & \\ & s \arrow[u] & \\ q \arrow[ur] & & r \arrow[ul] \\ & p \arrow[ul] \arrow[ur] & \end{tikzcd}$$

Let

$Ob(\mathcal{X}) = \{p, q, r, s, t\}$
↳ (i) condition

Let us take 'true' if $x \leq y$, so $\forall (x,y) \in Ob(\mathcal{X})$

we have $\mathcal{X}(x,y) = \text{true}$ if $x \leq y$ this satisfies the (ii) condition. The monoidal unit for bool is 'true'

we can verify, (a)

$$I = \text{true} \leq \mathcal{X}(x,x) = \text{true}$$

for (b),

$$\mathcal{X}(x,y) \otimes \mathcal{X}(y,z) \leq \mathcal{X}(x,z)$$

$\otimes = \wedge$ for bool category.

so, if $x \leq y$ & $y \leq z$, then,

$$\text{true} \wedge \text{true} \leq \mathcal{X}(x,z)$$

$$\text{true} \leq \text{true}$$

↳ (By preorder logic)

so, this preorder can be rep. as a Bool-category

The hom-object $\mathcal{X}(x,y)$ can be represented as a matrix :

$\leq$	p	q	r	s	t
p	true	true	true	true	true
q	false	true	false	true	true
r	false	false	true	true	true
s	false	false	false	true	true
t	false	false	false	false	true

⇒ Lawvere metric space:

Definition 2.51. A *metric space* (X, d) consists of:

(i) a set X, elements of which are called *points*, and

(ii) a function $d\colon X \times X \to \mathbb{R}_{\geq 0}$, where $d(x, y)$ is called the *distance between x and y*.

These constituents must satisfy four properties:

(a) for every $x \in X$, we have $d(x, x) = 0$,

(b) for every $x, y \in X$, if $d(x, y) = 0$ then $x = y$,

(c) for every $x, y \in X$, we have $d(x, y) = d(y, x)$, and

(d) for every $x, y, z \in X$, we have $d(x, y) + d(y, z) \geq d(x, z)$.

The fourth property is called the *triangle inequality*.

If we ask instead in (ii) for a function $d\colon X \times X \to [0, \infty] = \mathbb{R}_{\geq 0} \cup \{\infty\}$, we call (X, d) an *extended* metric space.

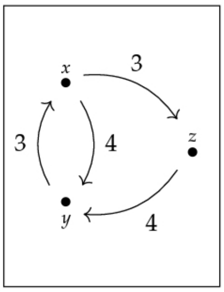
* It is basically a "Cost - Category" where $Ob(\mathcal{X})$ = set of points &

$$\mathcal{X}(x, y) = d(x, y)$$

& $0 \geq d(x, x)$ & second cond. of enrichment is seen as the triangle inequality

⇒ we can represent lawvere spaces as weighted graph

Metric Space:



$=: Y$

Matrix associated with Y,

$$M_Y := \begin{array}{c|ccc} \nearrow & x & y & z \\ \hline x & 0 & 4 & 3 \\ y & 3 & 0 & \infty \\ z & \infty & 4 & 0 \end{array}$$

Diagonal will be 0 & no edge is found ∞

* NOTE: $d(x,y)$ can $\neq d(y,x)$ as 'effort' is used more than distance.

⇒ Constructions of $\mathcal{V}$- categories :

① Changing base of enrichment :

from $\mathcal{V}$ category → $\mathcal{W}$ category by using a monoidal monotone.

Construction 2.64. Let $f : \mathcal{V} \to \mathcal{W}$ be a monoidal monotone. Given a $\mathcal{V}$-category $\mathcal{C}$, one forms the associated $\mathcal{W}$-category, say $\mathcal{C}_f$ as follows.

(i) We take the same objects: $\mathrm{Ob}(\mathcal{C}_f) := \mathrm{Ob}(\mathcal{C})$.

(ii) For any $c, d \in \mathrm{Ob}(\mathcal{C})$, put $\mathcal{C}_f(c, d) := f(\mathcal{C}(c, d))$.

This constr. is a $\mathcal{W}$-category & can be verified with the conditions for enrichment.

② Enriched functors :

Definition 2.69. Let $\mathcal{X}$ and $\mathcal{Y}$ be $\mathcal{V}$-categories. A *$\mathcal{V}$-functor from $\mathcal{X}$ to $\mathcal{Y}$*, denoted $F : \mathcal{X} \to \mathcal{Y}$, consists of one constituent:

(i) a function $F : \mathrm{Ob}(\mathcal{X}) \to \mathrm{Ob}(\mathcal{Y})$

subject to one constraint

(a) for all $x_1, x_2 \in \mathrm{Ob}(\mathcal{X})$, one has $\mathcal{X}(x_1, x_2) \leq \mathcal{Y}(F(x_1), F(x_2))$.

* The example for this is indirectly done when we say preorders & Bool categories have one-to-one correspondance.

* We used a functor there (which is the monotone map) to see preorders as Bool - cat. The other way can also be done

{ & Infact we proved equivalence b/w cat. of preorders & cat. of Bool cat. }

Opposite category $\mathcal{X}^{op}$ of $\mathcal{X}$:

(i) $\mathrm{Ob}(\mathcal{X}^{\mathrm{op}}) := \mathrm{Ob}(\mathcal{X})$, and

(ii) for all $x, y \in \mathcal{X}$, we have $\mathcal{X}^{\mathrm{op}}(x, y) := \mathcal{X}(y, x)$.

A $\mathcal{V}$-category $\mathcal{X}$ is a *dagger* $\mathcal{V}$-category if the identity function is a $\mathcal{V}$-functor $\dagger : \mathcal{X} \to \mathcal{X}^{\mathrm{op}}$. And a *skeletal* $\mathcal{V}$-category is one in which if $I \leq \mathcal{X}(x, y)$ and $I \leq \mathcal{X}(y, x)$, then $x = y$.

③ Product $\mathcal{V}$- categories :

Definition 2.74. Let $\mathcal{X}$ and $\mathcal{Y}$ be $\mathcal{V}$-categories. Define their $\mathcal{V}$-*product*, or simply *product*, to be the $\mathcal{V}$-category $\mathcal{X} \times \mathcal{Y}$ with

(i) $\mathrm{Ob}(\mathcal{X} \times \mathcal{Y}) := \mathrm{Ob}(\mathcal{X}) \times \mathrm{Ob}(\mathcal{Y})$,

(ii) $(\mathcal{X} \times \mathcal{Y})\big((x, y), (x', y')\big) := \mathcal{X}(x, x') \otimes \mathcal{Y}(y, y')$,

for two objects (x, y) and (x', y') in $\mathrm{Ob}(\mathcal{X} \times \mathcal{Y})$.

for preorders $(P, \leq_P)$ & $(Q, \leq_Q)$ the product

$(p_1, q_1) \leq (p_2, q_2)$ iff $p_1 \leq p_2$ <u>AND</u> $q_1 \leq q_2$

here $\otimes$ = AND $\Rightarrow$ This is a BOOL- product

Eg: b/w 2 Lawvere metric spaces :

$$\mathcal{X} := \begin{tikzcd} \overset{A}{\bullet} \arrow[r, "2"] & \overset{B}{\bullet} \arrow[r, "3"] & \overset{C}{\bullet} \end{tikzcd} \qquad \begin{tikzcd} \overset{p}{\bullet} \arrow[d, bend right, "5"'] \\ \underset{q}{\bullet} \arrow[u, bend right, "8"'] \end{tikzcd} =: \mathcal{Y}$$

$$\mathcal{X} \times \mathcal{Y} = \begin{tikzcd} \overset{(A,p)}{\bullet} \arrow[r, "2"] \arrow[d, bend right, "5"'] & \overset{(B,p)}{\bullet} \arrow[r, "3"] \arrow[d, bend right, "5"'] & \overset{(C,p)}{\bullet} \arrow[d, bend right, "5"'] \\ \underset{(A,q)}{\bullet} \arrow[r, "2"] \arrow[u, bend right, "8"'] & \underset{(B,q)}{\bullet} \arrow[r, "3"] \arrow[u, bend right, "8"'] & \underset{(C,q)}{\bullet} \arrow[u, bend right, "8"'] \end{tikzcd}$$

$$d_{X \times Y}\big((x,y), (x',y')\big) = d_X(x, x') + d_Y(y, y')$$

from def.

$\Rightarrow$ Monoidal closed preorders :

Definition 2.79. A symmetric monoidal preorder $\mathcal{V} = (V, \leq, I, \otimes)$ is called *symmetric monoidal closed* (or just *closed*) if, for every two elements $v, w \in V$, there is an element $v \multimap w$ in $\mathcal{V}$, called the *hom-element*, with the property

$$(a \otimes v) \leq w \quad \text{iff} \quad a \leq (v \multimap w). \tag{2.80}$$

for all $a, v, w \in V$.

Eg: Cost $= ([0,\infty], \geq, 0, +)$ is monoidal closed

define $x \multimap y$ as $\max(0, y-x)$

so, for any a, iff $a \geq y - x$

then, $x + a \geq y$

$\Leftrightarrow \max(0,a) \geq \max(0, y-x)$

$\Leftrightarrow a \geq \max(0, y-x)$

This is $\cong$ $a \geq x \multimap y$

prop:

This is the same cond. used in defining closed monoidal preorder

Proposition 2.87. Suppose $\mathcal{V} = (V, \leq, I, \otimes, \multimap)$ is a symmetric monoidal preorder that is closed. Then

(a) For every $v \in V$, the monotone map $- \otimes v\colon (V, \leq) \to (V, \leq)$ is left adjoint to $v \multimap -\colon (V, \leq) \to (V, \leq)$.

(b) For any element $v \in V$ and set of elements $A \subseteq V$, if the join $\bigvee_{a \in A} a$ exists then so does $\bigvee_{a \in A} v \otimes a$ and we have

$$\left(v \otimes \bigvee_{a \in A} a\right) \cong \bigvee_{a \in A} (v \otimes a). \tag{2.88}$$

$\Rightarrow$ Left adjoint preserves joins

(c) For any $v, w \in V$, we have $v \otimes (v \multimap w) \leq w$.

(d) For any $v \in V$, we have $v \cong (I \multimap v)$.

(e) For any $u, v, w \in V$, we have $(u \multimap v) \otimes (v \multimap w) \leq (u \multimap w)$.

$\Rightarrow$ Quantales:

Definition 2.90. A *unital commutative quantale* is a symmetric monoidal closed preorder $\mathcal{V} = (V, \leq, I, \otimes, \multimap)$ that has all joins: $\bigvee A$ exists for every $A \subseteq V$. In particular, we often denote the empty join by $0 := \bigvee \varnothing$.

Eg: Cost is a quantale, $\forall A \subseteq V$

we have join $\bigvee A$ which is min(A) & min(A) $\in$ Cost

for all the As. (or) infimum of A

prop: $\mathcal{P} = (P, \leq)$ It has all joins iff it has all meets.

> It says join & meet are dual.

⇒ Matrix Multiplication in a quantale:

Definition 2.100. Let $\mathcal{V} = (V, \leq, \otimes, I)$ be a quantale. Given sets X and Y, a *matrix with entries in* $\mathcal{V}$, or simply a $\mathcal{V}$*-matrix*, is a function $M\colon X \times Y \to V$. For any $x \in X$ and $y \in Y$, we call $M(x, y)$ the (x, y)*-entry*.

If $M : X \times Y \to V$ & $N : Y \times Z \to V$ are 2 $\mathcal{V}$-matrices, their product

$(M * N) : X \times Z \to V$ whose entries are

$$(M * N)(x, z) = \bigvee_{y \in Y} M(x, y) \otimes N(y, z)$$

Eg:

$X = \{1, 2, 3\}$ $\quad$ $Y = \{1, 2\}$ $\quad$ $Z = \{1, 2, 3\}$

$M : X \times Y \to \mathbb{B}$ $\quad$ $N : Y \times Z \to \mathbb{B}$

(true if $x \leq y$ else false)

Given,

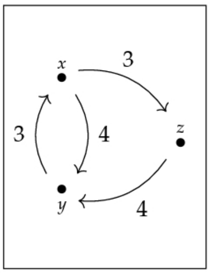
$$M = \begin{bmatrix} f & f \\ f & t \\ t & t \end{bmatrix} \qquad N = \begin{bmatrix} t & t & f \\ t & f & t \end{bmatrix}$$

$$M \times N(1,1) = \bigvee_{y \in Y} M(1, y) \wedge N(y, 1)$$

$$= (M(1,1) \wedge N(1,1)) \vee (M(1,2) \wedge M(2,1))$$

$$= (f \wedge t) \vee (f \wedge t) = f \vee f = f$$

Ily, for others.

for this graph : 

$=: Y$ $\quad M_Y = \begin{array}{c|ccc} & x & y & z \\ \hline x & 0 & 4 & 3 \\ y & 3 & 0 & \infty \\ z & \infty & 4 & 0 \end{array}$

$\infty \rightarrow$ if we cannot reach x to y in one step

We can obtain, M_Y^2 from the matrix multiplication of M_Y & M_Y

$\Rightarrow$ This gives the distance matrix to go from x to y in 2 steps or fewer (The ∞ helps here) and lly M_Y^3 gives the matrix for going from x to in 3 steps or fewer. For the above example $M_Y^2 = M_Y^3 \Rightarrow$ 2 steps is enough to go from any x to y.