Brouwer's Theorems:

Understanding different sides of Brouwer's theorem using the category of topological spaces and continuous maps.

- First side: Brouwer fixed point theorems:
    1. Let I be a line segment, including its endpoints (I for Interval) and suppose that f: I → 4 is a continuous endomap. Then this map must have a fixed point: a point x in I for which f(x) = x.)
    2. Let D be a closed disk (the plane figure consisting of all the points inside or on a circle), and f a continuous endomap of D. Then f has a fixed point. (*And in some special maps known as contraction maps, we can find the fixed point precisely using Banach's fixed point theorem for contraction maps. Illustrations for contraction maps: a map [cartography one] of an area crumpled and thrown on the actual are [that map represents] so that it also includes the crumpled map and so on…*)
    3. Any continuous endomap of a solid ball has a fixed point.

- Second side: Brouwer's retraction theorems:
  (*this is used because fixed point theorem cannot be proven and it also seems unintuitive*)
    4. Consider the inclusion map j: E → I of the two-point set E as boundary of the interval I. There is no continuous map which is a retraction for j.
       (*there will be some point x in E for which LHL ≠ RHL or the map is torn hence it is not continuous*)
       (*inclusion map j means E is a subset of I such that for all e ϵ E j(e)=e*)
    5. Consider the inclusion map j: C ---- D of the circle C as boundary of the disk D into the disk. There is no continuous map which is a retraction for j.
    6. Consider the inclusion j: S --4 B of the sphere S as boundary of the ball B into the ball. There is no continuous map which is a retraction for j.
- Proof Brouwer's theorem:
  (*contrapositive method: by saying not retraction theorem implies not fixed-point theorem*)
  To prove that the non-existence of a retraction implies that every continuous endomap has a fixed point, all we need to do is to assume that there is a continuous endomap of the disk which does not have any fixed point, and to build from it a continuous retraction for the inclusion of the circle into the disk. (proof in page no. 125 of Lawvere)
  (*Points to determine the relation between fixed point and retraction theorems:*
    1. *Let j: C→ D be, as before the inclusion of the circle into disk. Suppose that we have two continuous maps g and f: D → D the satisfies gj=j, then there must be x in D such that g(x)=f(x), comes from the fact that fixed point theorem is the special case of* $g=1_D$.
    2. *Suppose that A is a retract of X ↔ s: A → X and r: X → A with* $rs=1_A$. *Suppose also that X has fixed point property for*

maps from T (for every endomap f: X→ X there is map x: T →X for which fx=x) then A also has fixed point property for maps from T)

- Now we can deduce retraction theorem from fixed point theorem and above points:
  - Given: fixed point theorem and antipodal map (mapping diametrically opposite points) has no fixed point.
  - To prove: Consider and endomap f: D → D, D is the disk as above, then there doesn't exist retraction r.
    (where j: C → D and r: D → C and $rj=1_C$).
  - Proof: let C be the retract of D.
    We know that D has fixed point property implying C must also have fixed point property.
    But we know that C has an antipodal map for which there is no fixed point that means our assumption of r existing is wrong.
    Hence, retraction doesn't exist for fixed point property and C is not retract of D.