Exercise 1:
Show that if both f as above and also

$$\boxed{Y \circlearrowleft \beta} \xrightarrow{g} \boxed{Z \circlearrowleft \gamma}$$

are maps in $\mathbf{S}^{\circlearrowleft}$, then the composite $g \circ f$ in $\mathbf{S}$ actually defines another map in $\mathbf{S}^{\circlearrowleft}$. Hint: What should the domain and the codomain (in the sense of $\mathbf{S}^{\circlearrowleft}$) of this third map be? Transfer the definition (given for the case f) to the cases g and $g \circ f$; then calculate that the equations satisfied by g and f imply the desired equation for $g \circ f$.

$$X \circlearrowleft^{\alpha} \xrightarrow{f} Y \circlearrowleft^{\beta} \xrightarrow{g} Z \circlearrowleft^{\gamma}$$

Domain: $X \circlearrowleft^{\alpha}$ Codomain: $Z \circlearrowleft^{\gamma}$

For g, $g\beta = \gamma g$ For gf, $gf\alpha = \gamma gf$

Exercise 2:
What can you prove about an idempotent which has a retraction?

$$A \xrightarrow{\alpha} B \xrightarrow{r} A$$

$\alpha\alpha = \alpha$

$r\alpha = 1_A$

$r\alpha\alpha = \alpha$

$r\alpha = \alpha$

$\boxed{1_A = \alpha}$

Exercise 3:
A finite set A has an even number of elements iff (i.e. if and only if) there is an involution on A with *no fixed points*; A has an odd number of elements iff there is an involution on A with just *one* fixed point. Here we rely on known ideas about numbers – but these properties can be used as a *definition* of oddness or evenness that can be verified without counting if the structure of a real situation suggests an involution. The map 'mate of' in a group A of socks is an obvious example.

Sus

Let us exemplify the above types of endomaps on the set

$$\mathbb{Z} = \{\ldots -3, -2, -1, 0, 1, 2, 3, \ldots\}$$

of all (positive and negative) whole numbers, considered as an object of $\mathcal{S}$.

Exercise 4:
If $\alpha(x) = -x$ is considered as an endomap of $\mathbb{Z}$, is α an involution or an idempotent? What are its fixed points?

Exercise 5:
Same questions as above, if instead $\alpha(x) = |x|$, the absolute value.

Exercise 6:
If α is the endomap of $\mathbb{Z}$, defined by the formula $\alpha(x) = x + 3$, is α an automorphism? If so, write the formula for its inverse.

Exercise 7:
Same questions for $\alpha(x) = 5x$.

4) $\alpha\alpha(x) = x \Rightarrow \alpha\alpha = 1_{\mathbb{Z}}$: Involution

$\alpha(x) = x \Rightarrow \boxed{x = 0}$

5) $\alpha\alpha(x) = ||x|| = |x| = \alpha(x)$

$\Rightarrow \alpha\alpha = \alpha$: Idempotent

$\alpha(x) = x \quad \checkmark \ \boxed{x \geq 0}$

6) Yes, $\beta(x) = x - 3$

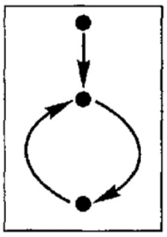
7) $\beta(x) = \frac{x}{5}$, $x^{d} = 0$

Exercise 8:
Show that both $\mathcal{C}^e$, $\mathcal{C}^\theta$ are subcategories of the category above, i.e. that either an idempotent or an involution will satisfy $\alpha^3 = \alpha$.

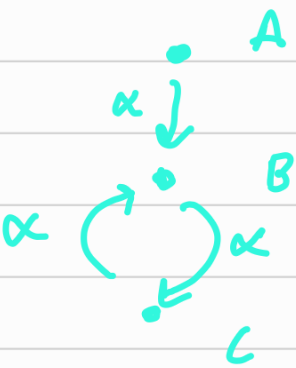
$\mathcal{C}^e$: $\alpha^2 = \alpha$ $\qquad$ $\mathcal{C}^\theta$: $\alpha^2 = 1_A$

$\alpha^3 = \alpha(\alpha\alpha) = \alpha\alpha = \alpha$ $\qquad$ $\alpha^3 = \alpha \cdot \alpha^2 = \alpha$

Exercise 9:
In $\mathcal{S}$, consider the endomap α of a three-element set defined by the internal picture



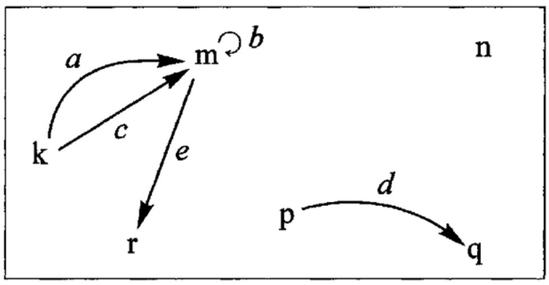
Show that it satisfies $\alpha^3 = \alpha$, but that it is *not* idempotent and that it is *not* an involution.



$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[rrr, bend right, "\alpha"'] & B \arrow[r, "\alpha"] & C \arrow[r, "\alpha"] & B \end{tikzcd}$$

$\underline{\alpha^2 \neq \alpha}$ $(\alpha^2 : A \to C$
$\alpha : A \to B)$

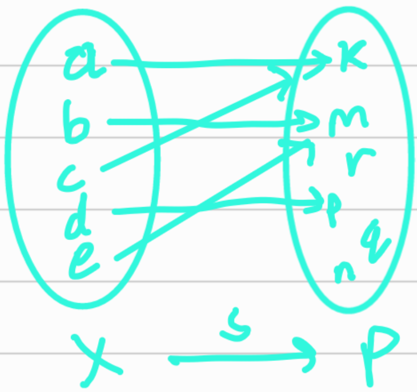
$\alpha^2 \neq 1_A \neq 1_B \neq 1_C$

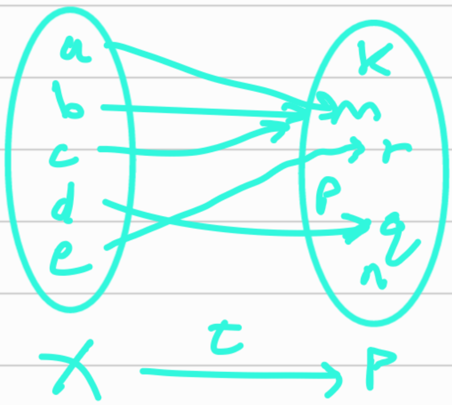


Exercise 10:
Complete the specification of the two maps

$$X \xrightarrow{s} P \quad \text{and} \quad X \xrightarrow{t} P$$

which express the source and target relations of the graph pictured above. Is there any element of X at which s and t take the same value in P? Is there any element to which t assigns the value k?





for b, $s(b) = t(b) = m$

No arrow points to k.

Exercise 11:
If f is as above and if

$$\begin{tikzcd} Y \arrow[d, shift right, "s'"'] \arrow[d, shift left, "t'"] \arrow[r, "g"] & Z \arrow[d, shift right, "s''"'] \arrow[d, shift left, "t''"] \\ Q & R \end{tikzcd}$$

is another map of graphs, show that the pair $g_A \circ f_A$, $g_D \circ f_D$ of $\boldsymbol{S}$-composites is also an $\boldsymbol{S}^{\downdownarrows}$-map.

$$\begin{tikzcd} X \arrow[d, shift right, "s"'] \arrow[d, shift left, "t"] \arrow[r, "f_A"] & Y \arrow[d, shift right, "s'"'] \arrow[d, shift left, "t'"] \arrow[r, "g_A"] & Z \arrow[d, shift right, "s''"'] \arrow[d, shift left, "t''"] \\ P \arrow[r, "f_D"'] & Q \arrow[r, "g_D"'] & R \end{tikzcd} \qquad \xrightarrow{f} \quad \xrightarrow{g}$$

For f, $\quad s' f_A = f_D s$ $\qquad$ For g, $\quad s'' g_A = g_D s'$

$\qquad t' f_A = f_D t$ $\qquad\qquad t'' g_A = g_D t'$

Just replace with $g_A f_A$ and $g_D f_D$?

Exercise 12:
If we denote the result of the above process by $I(f)$, then $I(g \circ f) = I(g) \circ I(f)$ so that our insertion I preserves the fundamental operation of categories.

$$\begin{tikzcd} X \arrow[d, shift right, "1_X"'] \arrow[d, shift left, "\alpha"] \arrow[r, "f"] & Y \arrow[d, shift right, "1_Y"'] \arrow[d, shift left, "\beta"] \arrow[r, "g"] & Z \arrow[d, shift right, "1_Z"'] \arrow[d, shift left, "\gamma"] \\ X & Y & Z \end{tikzcd}$$

Exercise 13:
(Fullness) Show that if we are given any $\mathcal{S}^{\downdownarrows}$-morphism

$$\begin{tikzcd} X \arrow[r, "f_A"] \arrow[d, shift right, "1_X"'] \arrow[d, shift left, "\alpha"] & Y \arrow[d, shift right, "1_Y"'] \arrow[d, shift left, "\beta"] \\ X \arrow[r, "f_D"'] & Y \end{tikzcd}$$

between the special graphs that come via I from endomaps of sets, then it follows that $f_A = f_D$, so that the map itself comes via I from a map in $\mathcal{S}^{\circlearrowleft}$.

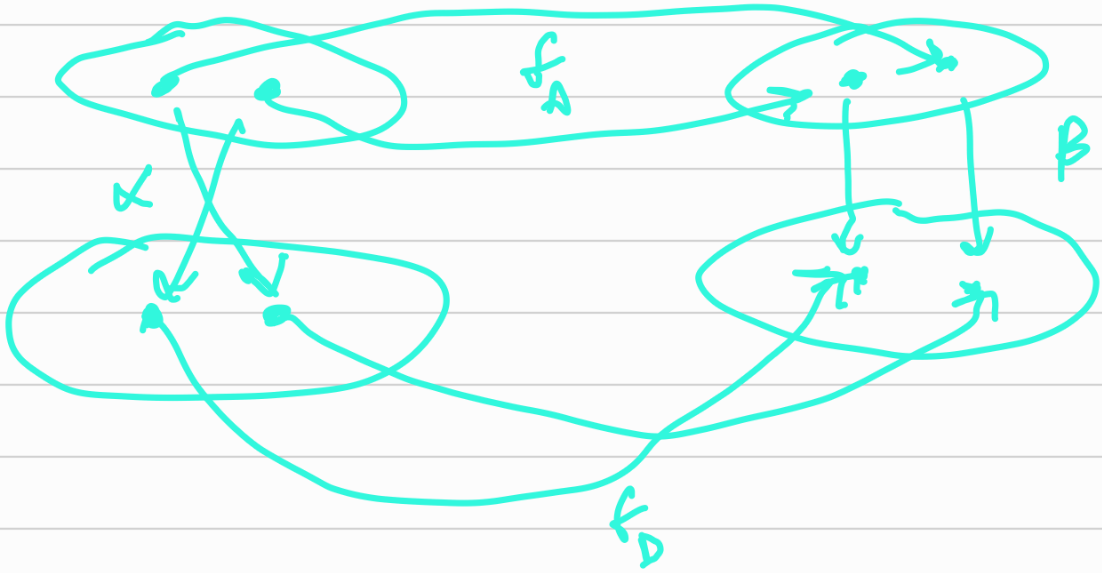
Fullness: morphisms are equal

$$1_Y f_A = f_D 1_X \Rightarrow f_A = f_D$$

Exercise 14:
Give an example of $\mathcal{S}$ of two endomaps and two maps as in

$$\begin{tikzcd} X \arrow[r, "f_A"] \arrow[d, "\alpha"'] & Y \arrow[d, "\beta"] \\ X \arrow[r, "f_D"'] & Y \end{tikzcd}$$

which satisfy the equation $f_D \circ \alpha = \beta \circ f_A$, but for which $f_A \neq f_D$.



Exercise 15:
In a reflexive graph, the two endomaps $e_1 = is$, $e_0 = it$ of the set of arrows are not only idempotent, but even satisfy *four* equations:

$$e_k e_j = e_j \quad \text{for} \quad k, j = 0, 1$$

$$\begin{tikzcd} X \arrow[d, bend right, "s"'] \arrow[d, bend left, "t"] \\ P \arrow[u, "i"] \end{tikzcd}$$

$si = 1_P$
$ti = 1_P$

$e_1 e_1 = isis = is = e_1$
$e_0 e_0 = itit = it = e_0$

To prove: $e_0 e_0 = e_0$ ✓ $\quad e_0 e_1 = e_1$ ✓ (III)
$e_1 e_0 = e_0$ (II) $\quad e_1 e_1 = e_1$

(II): $e_1 e_0 = isit$
$= it = e_0$

(III): $e_0 e_1 = itis$
$= is = e_1$

Exercise 16:
Show that if f_A, f_D in $\boldsymbol{\mathcal{S}}$ constitute a map of reflexive graphs, then f_D is determined by f_A and the internal structure of the two graphs.

$$\begin{tikzcd} X \arrow[r, "f_A"] \arrow[d, bend right, "s"'] \arrow[d, bend left, "t"] & Y \arrow[d, bend right, "s'"'] \arrow[d, bend left, "t'"] \\ P \arrow[u, "i"] \arrow[r, "f_D"'] & Q \arrow[u, "i'"] \end{tikzcd}$$

$si = 1_P \quad s'i' = 1_Q$
$ti = 1_P \quad t'i' = 1_Q$

$\boxed{f_A i = i' f_D}$ (?)

Exercise 17:
Consider a structure involving two sets and four maps as in

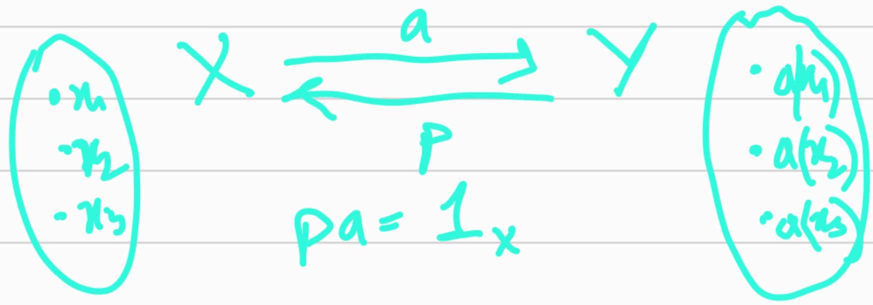
$$\begin{tikzcd} M \arrow[loop right, "\varphi"] \arrow[d, shift right, "\mu'"'] \\ F \arrow[u, shift right, "\varphi'"'] \arrow[loop right, "\mu"] \end{tikzcd}$$

(no equations required)

(for example $M = males$, $F = females$, φ and φ' are *father*, and μ and μ' are *mother*). Devise a rational definition of *map* between such structures in order to make them into a category.

$$\begin{tikzcd} M \arrow[loop above, "\varphi"] \arrow[d, bend right, "\mu'"'] \\ F \arrow[u, bend right, "\varphi'"'] \arrow[loop below, "\mu"] \end{tikzcd}$$

Exercise 18:
If a has a retraction, then a is injective. (Assume $pa = 1_X$ and $ax_1 = ax_2$; then try to show by calculation that $x_1 = x_2$.)

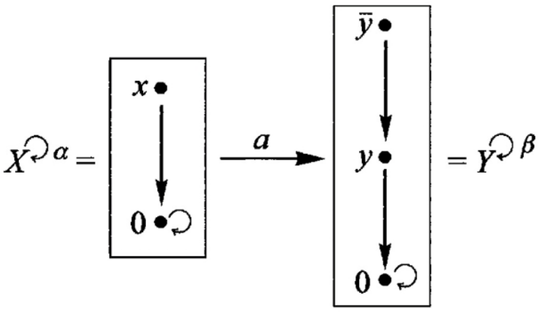


$pa = 1_X$

$a(x_1) = a(x_2)$

$p(a(x_1)) = p(a(x_2))$

$\Rightarrow x_1 = x_2$



Let $ax = y$ and $a0 = 0$, with X, Y, α, and β as pictured above.

Exercise 19:
Show that a is a map $\boxed{X^{\circlearrowleft \alpha}} \xrightarrow{a} \boxed{Y^{\circlearrowleft \beta}}$ in $\mathbf{S}^{\circlearrowleft}$.

Exercise 20:
Show that a is *injective*.

Exercise 21:
Show that, as a map $X \xrightarrow{a} Y$ in $\mathbf{S}$, a has exactly two retractions p.

Exercise 22:
Show that neither of the maps p found in the preceding exercise is *a map* $\boxed{Y^{\circlearrowleft \beta}} \longrightarrow \boxed{X^{\circlearrowleft \alpha}}$ in $\mathbf{S}^{\circlearrowleft}$. Hence a has no retractions in $\mathbf{S}^{\circlearrowleft}$.

Exercise 23:
How many of the eight $\mathbf{S}$-maps $Y \longrightarrow X$ (if any) are actually $\mathbf{S}^{\circlearrowleft}$-maps?

$$\boxed{Y^{\circlearrowleft \beta}} \longrightarrow \boxed{X^{\circlearrowleft \alpha}}$$

Exercise 24:
Show that our map a does not have any retractions, even when considered (via the insertion J in Section 7 of this article) as being a map in the 'looser' category $\mathbf{S}^{!}$.

19) To prove: $a\alpha = \beta a$

$a\alpha(x) = a(0) = 0$

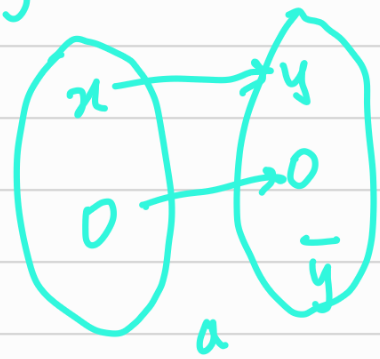
$\beta a(x) = \beta(y) = 0$

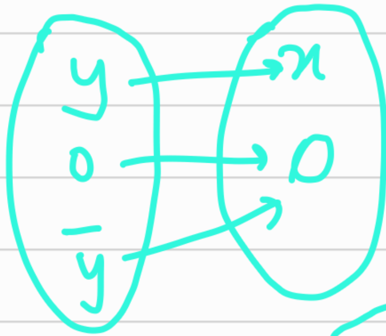
$a\alpha(0) = a(0) = 0$

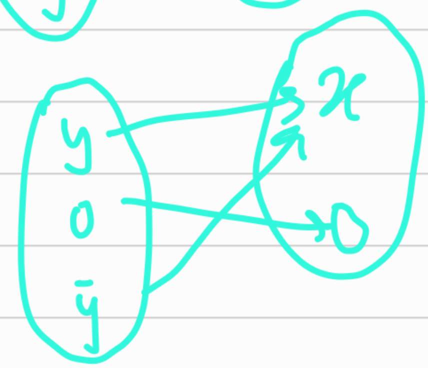
$\beta a(0) = \beta(0) = 0$

QED.

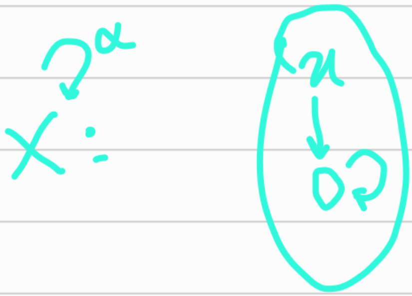
20) Clearly, a is one-one (if $ax_1 = ax_2 \Rightarrow x_1 = x_2$)

21) 

P_1: 

P_2: 

22) To check if $p_k \beta = \alpha p_k$ $\quad k = 1, 2.$

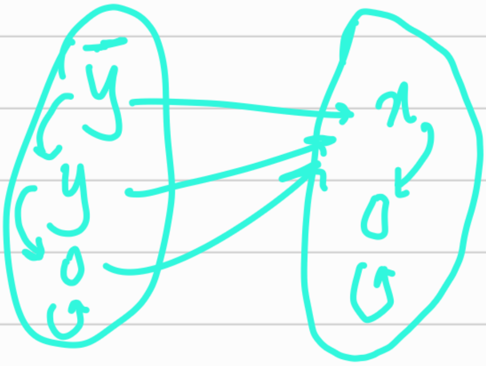


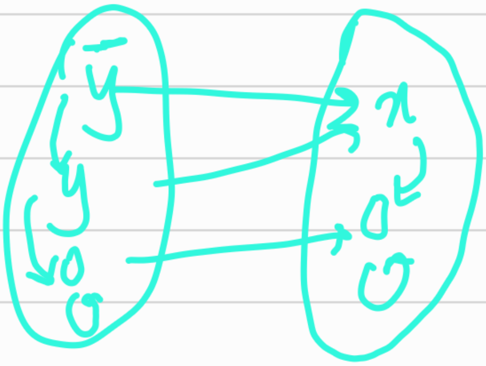
For $k = 1$, $\quad p_1 \beta(\bar{y}) = p_1(y) = x$

$\alpha p_1(\bar{y}) = \alpha(0) = 0 \neq x$

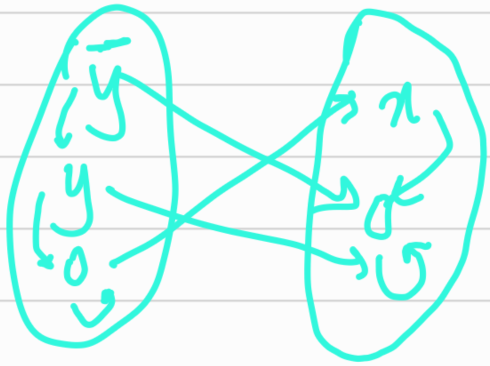
Similarly for $k = 2$.

$\therefore$ a has no retractions in $S^{\circlearrowright}$

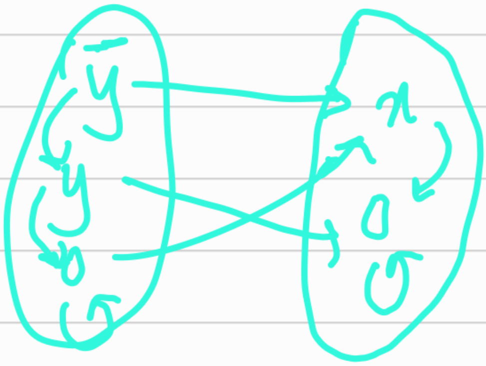
23) 



✗ P_2



✗



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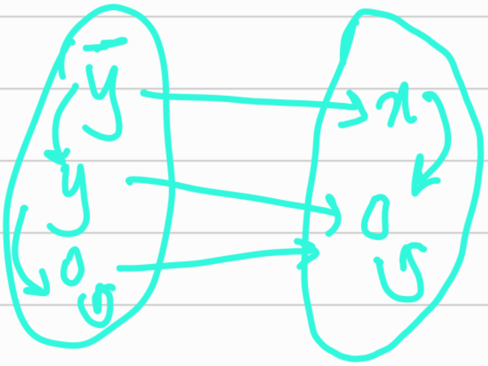
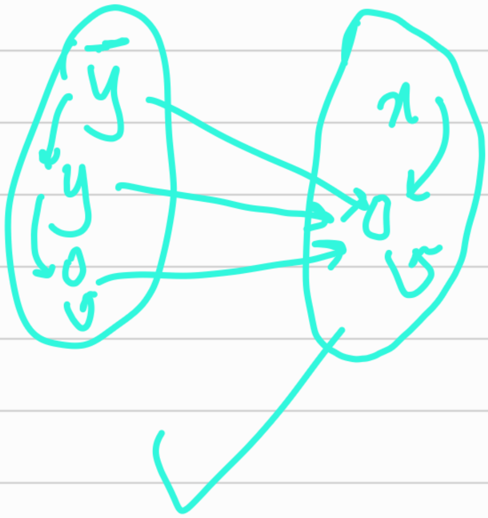
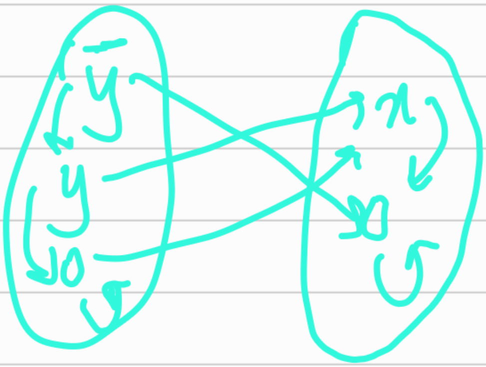


✓





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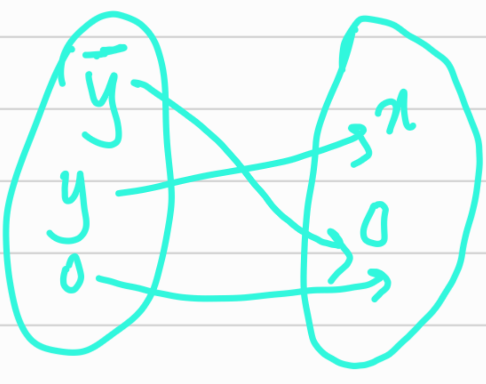


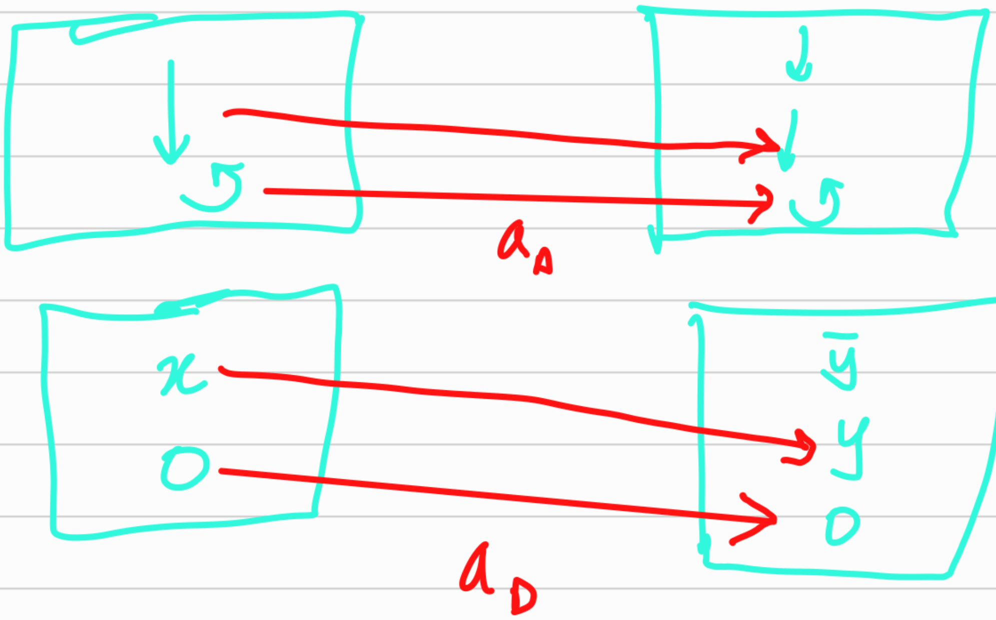
✗ P_1

2h)

$$\begin{tikzcd} X \arrow[d, "\alpha"] & Y \arrow[d, "\beta"] \\ X & Y \end{tikzcd}$$

Sus

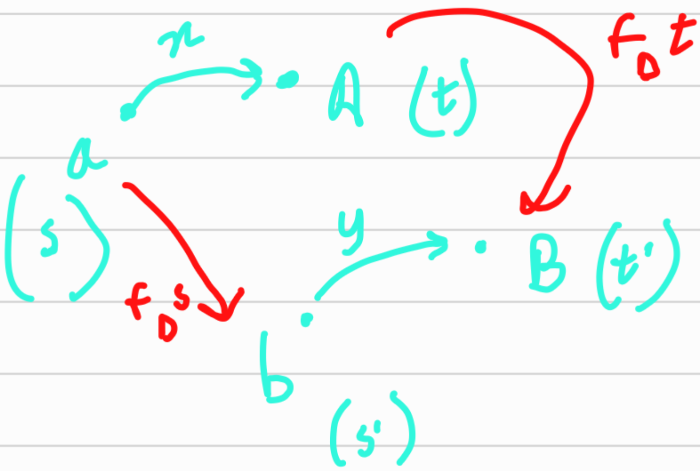
How is a_A and a_D defined?



Exercise 25:
Show that for any two graphs and any $\boldsymbol{S}^{\downarrow\downarrow}$-map between them

$$\begin{tikzcd} X \arrow[r, "f_A"] \arrow[d, shift right, "s"'] \arrow[d, shift left, "t"] & Y \arrow[d, shift right, "s'"'] \arrow[d, shift left, "t'"] \\ P \arrow[r, "f_D"'] & Q \end{tikzcd}$$

the equation $f_D \circ s = f_D \circ t$ can only be true when f_A maps every arrow in X to a *loop* (relative to s', t') in Y.



$B = f_D t$
$b = f_D s$

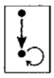
For $f_D t = f_D s$, $b = B$ so y is a loop.

Exercise 26:
There is an 'inclusion' map $\mathbb{Z} \xrightarrow{f} \mathbb{Q}$ in $\mathcal{S}$ for which

1. $\boxed{\mathbb{Z}^{\circlearrowright 5\times()}} \xrightarrow{f} \boxed{\mathbb{Q}^{\circlearrowright 5\times()}}$ is a map in $\mathcal{S}^{\circlearrowright}$, and
2. $\mathbb{Q}^{\circlearrowright 5\times()}$ is an automorphism, and
3. f is *injective*.

Find the f and prove the three statements.

Exercise 27:
Consider our standard idempotent

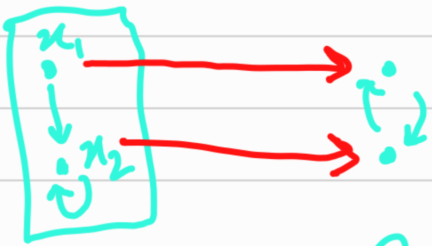
$X^{\circlearrowright\alpha} =$ 

and let $Y^{\circlearrowright\beta}$ be any *automorphism*. Show that any $\mathcal{S}^{\circlearrowright}$-map $X^{\circlearrowright\alpha} \xrightarrow{f} Y^{\circlearrowright\beta}$ must be non-injective, i.e. must map *both* elements of X to the *same* (fixed) point of β in Y.

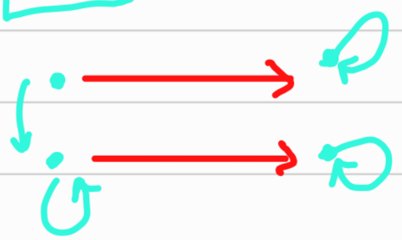
Exercise 28:
If $X^{\circlearrowright\alpha}$ is any object of $\mathcal{S}^{\circlearrowright}$ for which there exists an *injective* $\mathcal{S}^{\circlearrowright}$-map f to some $Y^{\circlearrowright\beta}$ where β is in the subcategory of *automorphisms*, then α itself must be injective.

26) $f = 1_{\mathbb{Z}}$ satisfies $f(5z) = 5f(z)$

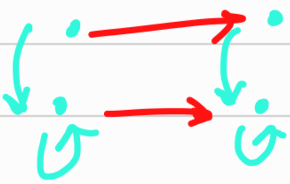
27) α : idempotent, β : automorphism

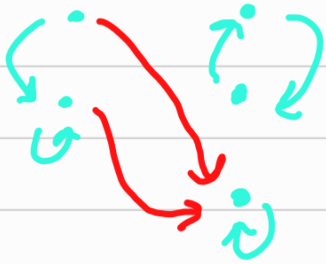


$f\alpha = \beta f$ does not hold

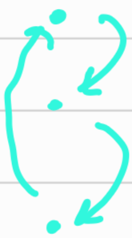


β is not an automorphism

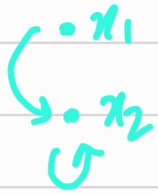


But  or  satisfy all conditions

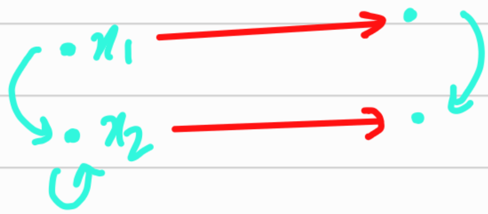
(not rigorous)

28) $Y^{\circlearrowright\beta}$ looks like  or  etc.

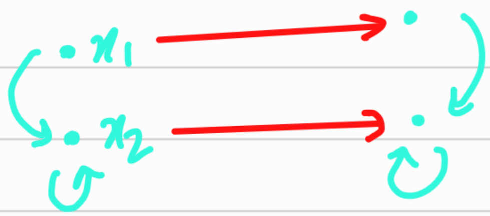
Each element in X^{2^α} is mapped to one element in Y^{2^β} by f.
If there is a fixed point in X^{2^α} as shown below,



then $\beta f(x_1) = f\alpha(x_1)$ will require the following diagram



Now considering $\beta f(x_2) = f\alpha(x_2)$ then,



But that is not possible as β is an automorphism.
The above logic holds for >1 fixed points in X^{2^α} as well.
So X^{2^α} does not have fixed points and it is injective.