Terminal Objects

→ let C be a category, then a terminal object $T \in C$ is s.t.
$\forall\, X \in C \ \ \exists !$ morphism $f : X \to T$.

→ e.g: For $C = \mathbf{Set}$, T is any one element set.
$C = \mathbf{Top}$, T is a one-point space.
$C = \mathbf{Poset}$, T is the maximal element in the partially ordered set.

→ "Uniqueness" ⟹ unique upto an isomorphism, not always equal elements (as is the case here)

→ Proof of uniqueness : Let T and T' be terminal objects.
⇒ $f : T \to T'$ is a unique morphism
and $g : T' \to T$ is a unique morphism

$$T \xrightarrow{f} T' \xrightarrow{g} T \qquad\qquad T' \xrightarrow{g} T \xrightarrow{f} T'$$

⇒ $gf = 1_T$, $fg = 1_{T'}$
⇒ T and T' are isomorphic.

→ Non-examples : ① $\bullet \rightrightarrows \bullet$ (not unique)
② $\bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \cdots$

Initial Objects (dual to terminal)

→ Let C be a category, then an initial object $I \in C$ is s.t.
$\forall\, X \in C$, $\exists !$ morphism $f : I \to X$

→ An initial object in C is a terminal object in C^{op}.

→ e.g: For $C = \mathbf{Set}$, Φ is initial.
The empty category is initial. (how is empty category valid?)
The category with one object, one morphism is terminal.

Products

→ Given objects A and B in $\mathcal{C}$, a product $A \times B$ is given by

$$\begin{tikzcd} & A \times B \arrow[dl] \arrow[dr] & \\ A & & B \end{tikzcd}$$

s.t. $\forall$ $\begin{tikzcd} & V \arrow[dl, "f"'] \arrow[dr, "g"] & \\ A & & B \end{tikzcd}$ $\exists!$ a factorisation h such that the diagram below commutes.

$$\begin{tikzcd} & V \arrow[d, "h"] \arrow[ddl, bend right, "f"'] \arrow[ddr, bend left, "g"] & \\ & u \arrow[dl, "p"] \arrow[dr, "q"'] & \\ A & & B \end{tikzcd}$$

$u = A \times B$

p, q : projection maps

This diagram commutes. ←

→ e.g: If A and B are sets then u is the Cartesian product

$$\begin{tikzcd} & \underset{(a,b)}{A \times B} \arrow[dl] \arrow[dr] & \\ A_{\ a} & & B_{\ b} \end{tikzcd}$$

$$\begin{tikzcd} & V_{\ v} \arrow[dl, bend right, "f"'] \arrow[dr, bend left, "g"] & \\ A \ \ f(v) & & B \ \ g(v) \end{tikzcd}$$

So,

$$\begin{tikzcd} & V_{\ v} \arrow[d, "h"] \arrow[ddl, bend right, "f"'] \arrow[ddr, bend left, "g"] & \\ & \underset{(f(v),g(v))}{A \times B} \arrow[dl, "p"] \arrow[dr, "q"'] & \\ \underset{f(v)}{A} & & \underset{g(v)}{B} \end{tikzcd}$$

→ In $\mathcal{C} = \text{Vect}$, the product is the direct sum.

→ non e.g: tensor products are not categorical products

Coproducts

→ Given objects A and B, the coproduct u is given by

$$\begin{tikzcd} & u & \\ A \arrow[ur, "p"] & & B \arrow[ul, "q"'] \end{tikzcd}$$

p, q : injection maps

s.t. $\forall$ $\begin{tikzcd} & V & \\ A \arrow[ur, bend left, "f"] & & B \arrow[ul, bend right, "g"'] \end{tikzcd}$, $\exists!\, h$ such that the diagram below commutes.

$$\begin{tikzcd} & V & \\ & u \arrow[u, "h"] & \\ A \arrow[ur, "p"] \arrow[uur, bend left, "f"] & & B \arrow[ul, "q"'] \arrow[uul, bend right, "g"'] \end{tikzcd}$$

→ In $C =$ Set, u is the disjoint union of A and B.