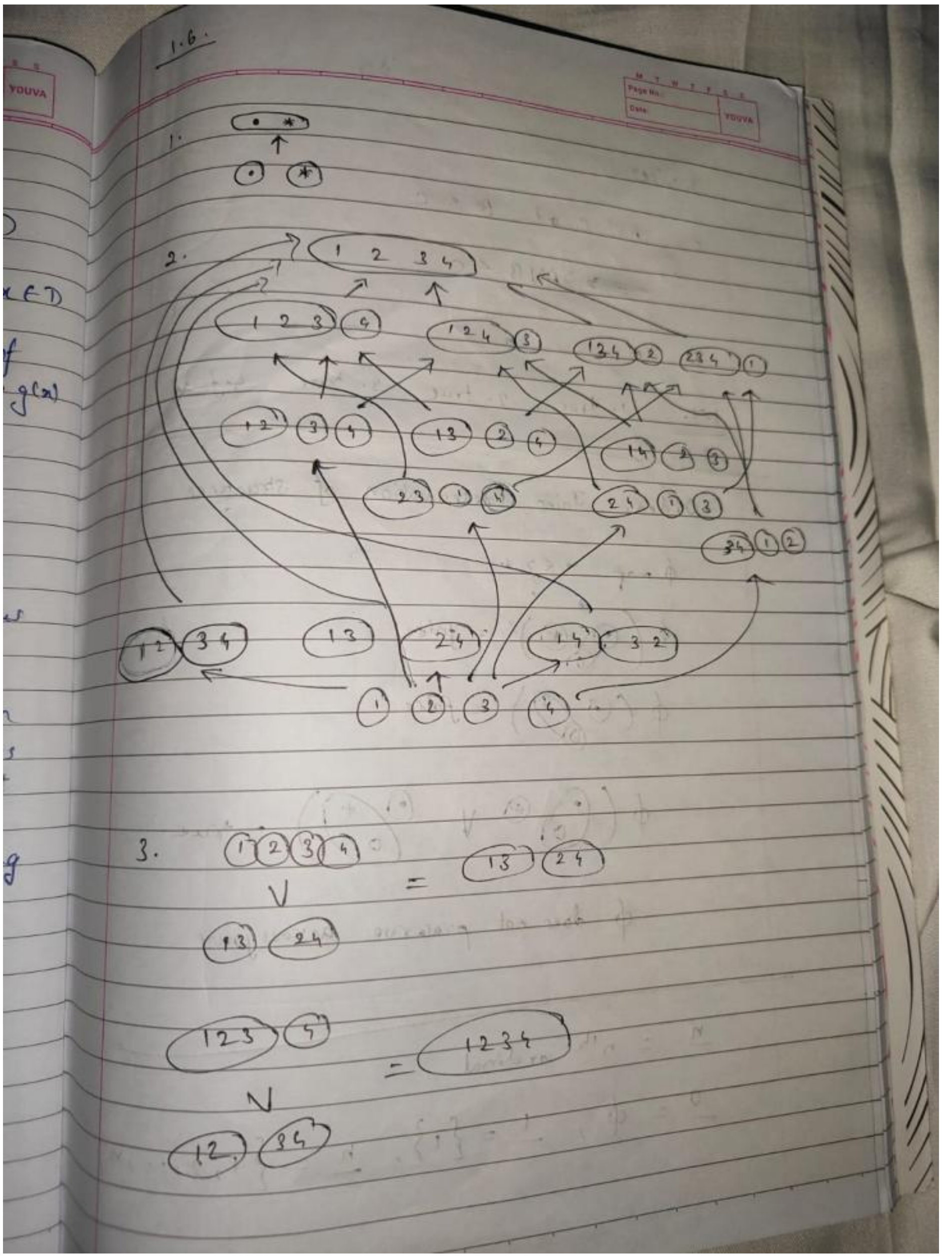
Exercises

08 December 2024 21:00



4. Yes

5. $A \leq C$ and $B \leq C$

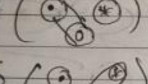
$\Rightarrow A \vee B \leq C$

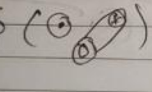
6. Yes

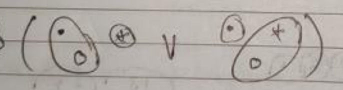
<u>1.7.</u> 1. true 2. true 3. true 4. false

Order, Joins, Preservation of structures

ϕ map $\bullet \overset{?}{\leftrightarrow} *$

ϕ  = false

ϕ  = false

ϕ  = true

ϕ does not preserve across joins

$\underline{n}$ = n^{th} ordinal

$\underline{0} = \phi$, $\underline{1} = \{1\}$, $\underline{n} = \{1, 2, \dots, n\}$

1.11 $A := \{h, 1\}$, $B := \{1, 2, 3\}$

1. $\{\}$, $\{1\}, \{2\}, \{3\}$
   $\{1,2\}, \{2,3\}, \{1,3\}, \{1,2,3\}$
2. $\{1\} \cup \{1,3\} = \{1,3\}$
3. $(h,1)$, $(h,2)$, $(h,3)$
   $(1,1)$, $(1,2)$, $(1,3)$
4. $(h,1)$, $(1,1)$, $(1,2)$, $(2,2)$, $(3,2)$
5. $h, 1, 2, 3$

Infix notation: Rel. $R \subseteq X \times Y$

For a binary relation $R \subseteq X \times X$
if $a * b \Rightarrow (a,b) \in R$

$5 \leq 6 \iff$ "There is a binary relation R on $\mathbb{R}$ with infix notation $\leq$" where $(5,6) \in R$

$5 \mid 10 \iff$ 5 divides 10

Counting partitions $\approx$ Surjective functions

Partitions of a set $\approx$ Equivalence relations of the set

A preorder relation on a set X is a binary relation on X denoted by $\leq$ s.t.

a) $x \leq x$

b) $x \leq y$, $y \leq z \Rightarrow x \leq z$

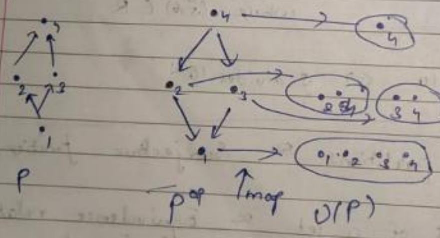
if $x \leq y$ and $y \leq x$ then $x \cong y$

$(X, \leq)$ is a preorder

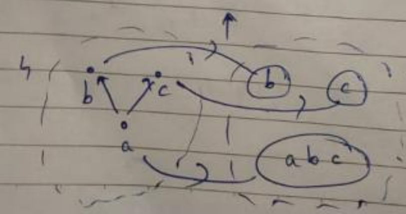
preorders are relations that are reflexive and transmittive

$(X,=)$ discrete preorder $\longrightarrow$ $x \leq x$ ~~if $x \neq y$~~ always and $x \leq y$ is effectively $x \neq y$

$(X, \leq)$ codiscrete preorder $\longrightarrow$ the total binary relation



1.66



Yoneda lemma

upto equivalence knowing x is as good as knowing $\uparrow(x)$

Notes

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Basic Stuff

Example 1.9. Here are some important sets from mathematics—and the notation we will use—that will appear again in this book.

- $\varnothing$ denotes the empty set; it has no elements.
- $\{1\}$ denotes a set with one element; it has one element, 1.
- $\mathbb{B}$ denotes the set of *booleans*; it has two elements, `true` and `false`.
- $\mathbb{N}$ denotes the set of *natural numbers*; it has elements $0, 1, 2, 3, \ldots, 90^{717}, \ldots$.
- $\underline{n}$, for any $n \in \mathbb{N}$, denotes the n^{th} *ordinal*; it has n elements $1, 2, \ldots, n$. For example, $\underline{0} = \varnothing$, $\underline{1} = \{1\}$, and $\underline{5} = \{1, 2, 3, 4, 5\}$.
- $\mathbb{Z}$, the set of *integers*; it has elements $\ldots, -2, -1, 0, 1, 2, \ldots, 90^{717}, \ldots$.
- $\mathbb{R}$, the set of *real numbers*; it has elements like $\pi, 3.14, 5 * \sqrt{2}, e, e^2, -1457, 90^{717}$, etc.

Definition 1.12. Let X and Y be sets. A *relation between X and Y* is a subset $R \subseteq X \times Y$. A *binary relation on X* is a relation between X and X, i.e. a subset $R \subseteq X \times X$.

Definition 1.18. Let A be a set. An *equivalence relation* on A is a binary relation, let's give it infix notation $\sim$, satisfying the following three properties:

(a) $a \sim a$, for all $a \in A$,

(b) $a \sim b$ iff[a] $b \sim a$, for all $a, b \in A$, and

(c) if $a \sim b$ and $b \sim c$ then $a \sim c$, for all $a, b, c \in A$.

Definition 1.21. Given a set A and an equivalence relation $\sim$ on A, we say that the *quotient* $A/\sim$ of A under $\sim$ is the set of parts of the corresponding partition.

Preorders

Definition 1.30. A *preorder relation* on a set X is a binary relation on X, here denoted with infix notation $\leq$, such that

(a) $x \leq x$; and

(b) if $x \leq y$ and $y \leq z$, then $x \leq z$.

The first condition is called *reflexivity* and the second is called *transitivity*. If $x \leq y$ and $y \leq x$, we write $x \cong y$ and say x and y are *equivalent*. We call a pair $(X, \leq)$ consisting of a set equipped with a preorder relation a *preorder*.

Example 1.52 (Partitions). We talked about getting a partition from a preorder; now let's think about how we might order the set $\mathsf{Prt}(A)$ of *all partitions* of A, for some set A. In fact, we have done this before in Eq. (1.5). Namely, we order on partitions by fineness: a partition P is *finer* than a partition Q if, for every part $p \in P$ there is a part $q \in Q$ such that $A_p \subseteq A_q$. We could also say that Q is *coarser* than P.

Recall from Example 1.26 that partitions on A can be thought of as surjective functions out of A. Then $f : A \twoheadrightarrow P$ is finer than $g : A \twoheadrightarrow Q$ if there is a function $h : P \to Q$ such that $f \mathbin{⨾} h = g$.

Example 1.56 (Product preorder). Given preorders $(P, \leq)$ and $(Q, \leq)$, we may define a preorder structure on the product set $P \times Q$ by setting $(p, q) \leq (p', q')$ if and only if $p \leq p'$ and $q \leq q'$. We call this the *product preorder*. This is a basic example of a more general construction known as the product of categories.

Example 1.58 (Opposite preorder). Given a preorder $(P, \leq)$, we may define the opposite preorder $(P, \leq^{\text{op}})$ to have the same set of elements, but with $p \leq^{\text{op}} q$ if and only if $q \leq p$.

Monotone Maps

Definition 1.59. A *monotone map* between preorders $(A, \leq_A)$ and $(B, \leq_B)$ is a function $f\colon A \to B$ such that, for all elements $x, y \in A$, if $x \leq_A y$ then $f(x) \leq_B f(y)$.

Example 1.54 (Upper sets). Given a preorder $(P, \leq)$, an *upper set* in P is a subset U of P satisfying the condition that if $p \in U$ and $p \leq q$, then $q \in U$. "If p is an element then so is anything bigger." Write $\mathsf{U}(P)$ for the set of upper sets in P. We can give the set U an order by letting $U \leq V$ if U is contained in V.

For example, if $(\mathbb{B}, \leq)$ is the booleans (Example 1.34), then its preorder of uppersets $\mathsf{U}(\mathbb{B})$ is

$$\begin{tikzcd} \{\texttt{true}, \texttt{false}\} \\ \{\texttt{true}\} \arrow[u] \\ \varnothing \arrow[u] \end{tikzcd}$$

The subset $\{\texttt{false}\} \subseteq \mathbb{B}$ is not an upper set, because $\texttt{false} \leq \texttt{true}$ and $\texttt{true} \notin \{\texttt{false}\}$.

Example 1.68. Recall from Example 1.52 that given a set X we define $\mathsf{Prt}(X)$ to be the set of partitions on X, and that a partition may be defined using a surjective function $s\colon X \twoheadrightarrow P$ for some set P.

Any surjective function $f\colon X \twoheadrightarrow Y$ induces a monotone map $f^*\colon \mathsf{Prt}(Y) \to \mathsf{Prt}(X)$, going "backwards." It is defined by sending a partition $s\colon Y \twoheadrightarrow P$ to the composite $f \mathbin{⨟} s\colon X \twoheadrightarrow P$.[7]

Meet and Join

Definition 1.81. Let $(P, \leq)$ be a preorder, and let $A \subseteq P$ be a subset. We say that an element $p \in P$ is a *meet* of A if

(a) for all $a \in A$, we have $p \leq a$, and

(b) for all q such that $q \leq a$ for all $a \in A$, we have that $q \leq p$.

We write $p = \bigwedge A$, $p = \bigwedge_{a \in A} a$, or, if the dummy variable a is clear from context, just $p = \bigwedge_A a$. If A just consists of two elements, say $A = \{a, b\}$, we can denote $\bigwedge A$ simply by $a \wedge b$.

Similarly, we say that p is a *join* of A if

(a) for all $a \in A$ we have $a \leq p$, and

(b) for all q such that $a \leq q$ for all $a \in A$, we have that $p \leq q$.

We write $p = \bigvee A$ or $p = \bigvee_{a \in A} a$, or when $A = \{a, b\}$ we may simply write $p = a \vee b$.

Definition 1.92. We say that a monotone map $f \colon P \to Q$ *preserves meets* if $f(a \wedge b) \cong f(a) \wedge f(b)$ for all $a, b \in P$. We similarly say f *preserves joins* if $f(a \vee b) \cong f(a) \vee f(b)$ for all $a, b \in P$.

Definition 1.93. We say that a monotone map $f \colon P \to Q$ *has a generative effect* if there exist elements $a, b \in P$ such that

$$f(a) \vee f(b) \not\cong f(a \vee b).$$

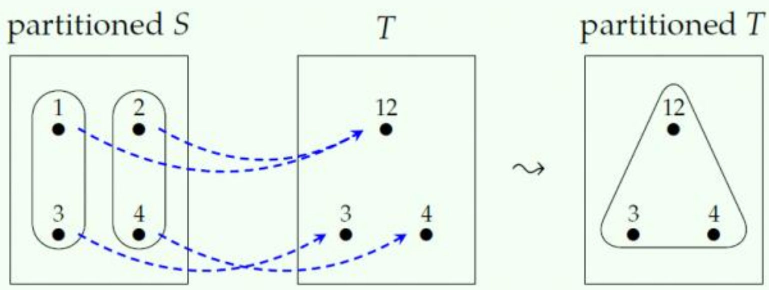
Galois Connections

Definition 1.95. A *Galois connection* between preorders P and Q is a pair of monotone maps $f \colon P \to Q$ and $g \colon Q \to P$ such that

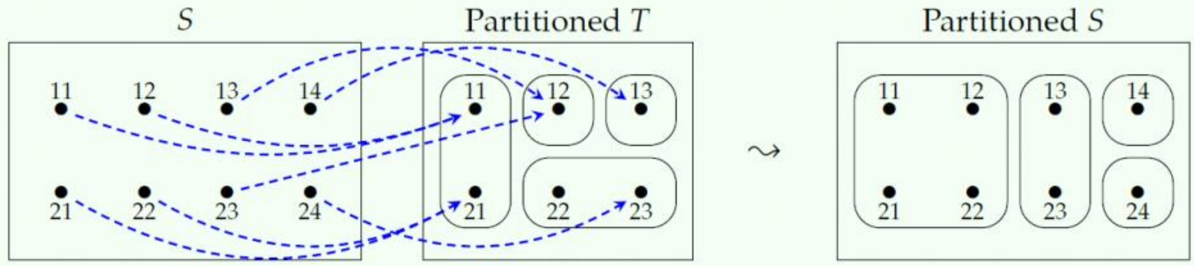
$$f(p) \leq q \quad \text{if and only if} \quad p \leq g(q). \tag{1.96}$$

We say that f is the *left adjoint* and g is the *right adjoint* of the Galois connection.

Example 1.102. Let $S = \{1, 2, 3, 4\}$, $T = \{12, 3, 4\}$, and $g \colon S \to T$ by $g(1) := g(2) := 12$, $g(3) := 3$, and $g(4) := 4$. The partition shown left below is translated by $g_!$ to the partition shown on the right.



Example 1.104. Let S, T be as below, and let $g: S \to T$ be the function shown in blue. Here is a picture of how g^* takes a partition on T and "pulls it back" to a partition on S:



Proposition 1.107. Suppose that $f: P \to Q$ and $g: Q \to P$ are monotone maps. The following are equivalent

(a) f and g form a Galois connection where f is left adjoint to g,

(b) for every $p \in P$ and $q \in Q$ we have

$$p \leq g(f(p)) \qquad \text{and} \qquad f(g(q)) \leq q. \tag{1.108}$$

Go left and then right or right and then left, no matter what you preserve x-> adjunctions
Here x is monotonicity

Preventing Generative Effects

Proposition 1.111 (Right adjoints preserve meets)**.** Let $f: P \to Q$ be left adjoint to $g: Q \to P$. Suppose $A \subseteq Q$ any subset, and let $g(A) := \{g(a) \mid a \in A\}$ be its image. Then if A has a meet $\bigwedge A \in Q$ then $g(A)$ has a meet $\bigwedge g(A)$ in P, and we have

$$g\left(\bigwedge A\right) \cong \bigwedge g(A).$$

That is, right adjoints preserve meets. Similarly, left adjoints preserve joins: if $A \subseteq P$ is any subset that has a join $\bigvee A \in P$, then $f(A)$ has a join $\bigvee f(A)$ in Q, and we have

$$f\left(\bigvee A\right) \cong \bigvee f(A).$$

Theorem 1.115 (Adjoint functor theorem for preorders)**.** Suppose Q is a preorder that has all meets and let P be any preorder. A monotone map $g: Q \to P$ preserves meets if and only if it is a right adjoint.

Similarly, if P has all joins and Q is any preorder, a monotone map $f: P \to Q$ preserves joins if and only if it is a left adjoint.