Visiting isomorphism again: A map f is called isomorphism if there exists another map f^{-1} which is both a retraction and a section for f.

* There exists a unique inverse (f_1, f_2 are both inverse of f)

$f_1 \circ f = id_A \quad f \circ f_2 = id_B \quad f_1 \circ id_B = f_1$

$f_1 \circ (f \circ f_2) = f_1 \Rightarrow (f_1 \circ f) \circ f_2 = f_1 \Rightarrow id_A \circ f_2 = f_1 \Rightarrow \boxed{f_2 = f_1}$

- Having an isomorphism gives us the advantage to define a characteristic between two objects.

Eg: In set, isomorphic $\Rightarrow$ same number without counting

- Number of isomorphisms from A to B = Number of automorphisms of A

Let us define a relation sort of thing to go from a Automorphism to an isomorphism (F)

$Aut(A) \xrightarrow{F} Isom(A,B)$

$A \underset{f^{-1}}{\overset{f}{\rightleftarrows}} B \underset{g^{-1}}{\overset{g}{\rightleftarrows}} C \qquad (g \circ f)^{-1} = f^{-1} \circ g^{-1}$

$$\begin{tikzcd} & A \arrow[dr, "f"] & \\ A \arrow[ur, "\alpha"] \arrow[rr, "f \circ \alpha"'] & & B \end{tikzcd}$$

because

$f \circ \alpha \rightarrow$ This is indeed an isomorphism

Proof: Define G s.t. $Isom(A,B) \xrightarrow{G} Aut(A)$

$$\begin{tikzcd} & B \arrow[dr, "f^{-1}"] & \\ A \arrow[ur, "g"] \arrow[rr, "f^{-1} \circ g"'] & & A \end{tikzcd}$$

f^{-1} and g are not related

So now we will prove $F(\alpha)$, $G(g)$ has some relation.

$$(F\circ S)(g) = F(S(g)) = F(f^{-1}\circ g) = f\circ(f^{-1}\circ g)$$
$$= (f\circ f^{-1})\circ g = id_B \circ g = g$$

Which means the function $F(A\to B)$ and $S(A\to A)$ on composition gives you a isomorphism given as an input.

$$(S\circ F)(\alpha) = S(F(\alpha)) = S(f\circ\alpha) = f^{-1}\circ(f\circ\alpha) = \alpha$$

What we have shown here is a function/map between a isomorphism and automorphism has an inverse.

* Automorphism is just shifting the elements around in a specified way.

→ We can use it to describe our second example of category ⇒ category of permutations.

→ An object of this category is a set A with a given automorphism α.

$$\begin{tikzcd} A \arrow[loop above, "\alpha"] \arrow[r, "f"] & B \arrow[loop above, "\beta"] \end{tikzcd}$$

Here a map preserves the given automorphisms α, β.

$$\boxed{f\circ\alpha = \beta\circ f}$$ ★★

$$\begin{tikzcd} & B \arrow[loop above, "\beta"] \arrow[dr, "g"] & \\ A \arrow[loop left, "\alpha"] \arrow[ur, "f"] \arrow[rr, "?"'] & & C \arrow[loop right, "\gamma"] \end{tikzcd}$$

Assuming $f\circ\alpha = \beta\circ f$, $g\circ\beta = \gamma\circ g$

To prove: $? \circ\alpha = \gamma\circ ?$ ⇒ $(g\circ f)\circ\alpha = \gamma\circ(g\circ f)$

$$(g\circ f)\circ\alpha = g\circ(f\circ\alpha) = (g\circ\beta)\circ f = \gamma\circ(g\circ f) \quad \text{H.P}$$

Multiplication : composition :: Division : ?

→ In normal division, most of the time, we will get a unique soln and with '0', we get no, more than one soln. But in maps, thats the typical soln.

→ Inverse : reciprocals:

- If $f: A \xrightarrow{f} B$, an inverse for f is a map $B \xrightarrow{g} A$ satisfying both $g \circ f = id_A$, $f \circ g = id_B$
- Any map f will have atmost one inverse.
- The process of following maps by a particular isomorphism is itself a reversible process.

$$\begin{tikzcd} & B \arrow[dr, "?"] & \\ A \arrow[ur, "f"] \arrow[rr, "h"'] & & C \end{tikzcd} \qquad \begin{tikzcd} & B \arrow[dr, "s"] \arrow[rrd, bend left, "?"] & & \\ A \arrow[ur, "f"] \arrow[rr, "id_A"'] & & A \arrow[r, "h"'] & C \end{tikzcd}$$

$h \circ s = h \circ f^{-1}$

To really appreciate isomorphism, we need to look into other categories (other than set)

- Let us see the category where object is algebraic category is a set ~~and~~ with a combining rule (a operation)

Eg: $(\mathbb{R}, +)$, $(\mathbb{R}^+, \times)$. Map here is from an object $(A, *) \to (A', *')$ which respects combining rule

$f(a * b) = (fa) *' (fb)$

Eg: cubing on $(\mathbb{R}, \times) \xrightarrow{c} (\mathbb{R}, \times)$ $\quad c(a \times b) = c(a) \times c(b)$

exp on $(\mathbb{R}, +) \xrightarrow{exp} (\mathbb{R}^+, \times)$ $\quad exp(a+b) = exp(a) \times exp(b)$

$e^{a+b} = e^a * e^b$

- Here every single map chosen is an isomorphism

$h \circ d = h(d(\mathbb{R}, +)) = h((\mathbb{R}, +)) = (\mathbb{R}, +)$

* In algebraic category, we need not have sets in object ~~eg~~

Eg: ({odd, even}, +) $\xrightarrow{f}$ ({positive, negative}, ×)

Category with geometry: (Euclid's category)

* Object is any polygonal figure drawn in a plane. map from F to F' is a map that preserves distances. Basically you are picking up the rigid material and transform it.

* In topology, rubber sheet geometry maps preserve continuity

Determination problem:

* Scientists investigate using determination of one quantity from the other.

$$\begin{tikzcd} & \text{Temp} \arrow[d, "g=?"] \\ \text{State} \arrow[ur, "f"] \arrow[r, "h"] & \text{Vol} \end{tikzcd}$$

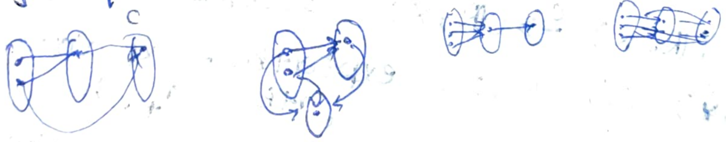
Basically is the map h is determined by f or not

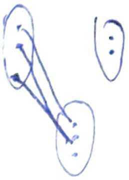
$\boxed{\text{Is } h \text{ determined by } f\ ?}$

As it gives retraction as a solution, we can say that inputs are clearly defined.

For any singleton sets a_1, a_2 of A

$f \circ a_1 = f \circ a_2 \Rightarrow h \circ a_1 = h \circ a_2$





Constant map: Object B is a singleton set

$$\begin{tikzcd} & B \arrow[dr, "g?"] & \\ A \arrow[ur, "f"] \arrow[rr, "h"'] & & C \end{tikzcd}$$

Choice problems:

$$\begin{tikzcd} & B \arrow[dr, "g"] & \\ A \arrow[ur, dashed, "f=?"] \arrow[rr, "h"'] & & C \end{tikzcd}$$

Find f such that $h = g \circ f$

For any a in A, atleast one b exist in B for which $h(a) = g(b)$

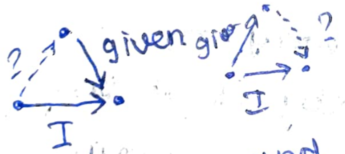
- $A \xrightarrow{f} B$ is a section of $B \xrightarrow{g} A$ if $g \circ f = id_A$

With section. We can have solution to choice problem without any constraint on h

$$\begin{tikzcd} & B \arrow[dr, shift left, "g"] & \\ A \arrow[ur, dashed, "f"] \arrow[rr, "h"'] & & C \arrow[ul, shift left, "s"] \end{tikzcd}$$

$g \circ s = id_C$

Sections alone don't give the solution to choice problem. ~~We~~ There can be other ones also

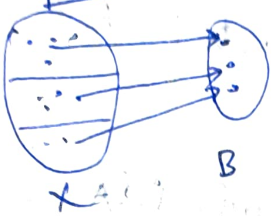


Retraction and section are pairs that are interchangable. If s is a section of f, then f is a retraction of s.

- Stacking helps us analyze the results in a very effective manner

* In category theory, we are talking about a abstract set which is little more than a number. It allows us to carry rich structures.

<u>Sorting</u>: For a map $X \xrightarrow{g} B$, 'g' gives rise to a sorting of X into B 'sorts' or map g is $\boxed{\text{sorting of } X \text{ by } B}$



B valued property on X
g is a stacking of elements of X into B stacks

* Another word for it is "fibering", X is divided into B fibers. If one fiber is empty, the map won't have any section.

* When no fiber/sort is empty, section exist and we use the word "partitioning"

All these terms emphasize that $X \to B$ produces a structure in domain and map is a B-valued property.

Eg: All creatures $\xrightarrow{S}$ Species $\xrightarrow{g}$ Genera

"We have discussed the one way of looking at maps". The other way is looking at co-domain. Map is a family of domain elements of codomain. Naming/listing a part of a co-domain by codomain

Definition of map

naming, listing, exemplifying, parametrizing of a part of co-domain by domain	sorting, stacking, fibering, partitioning of domain by co-domain (If stack has 0 elements no section)

* We are trying to reflect the reality by thinking. The result of which is knowledge accumulated to scien
* One of particular science is philosophy, reflecting the relationship between thinking and reality.
* Within scientific thinking, relation between objective and subjective.

* In objective. we will have clear an image as possible of realility and move in itself. independent of our particular thoughts.

Isomorphism and Coordinates

* Isomorphism from a known object A to an object X allows us to know X very well.

$A \underset{\text{coordinate}}{\overset{\text{plot}}{\rightleftarrows}} B$ coordinate $\circ$ plot $= Id_A$ plot $\circ$ coordinate $= id_B$

Eg: isomorphism from set R to line L
$R \xrightarrow{\text{plot}} L$ plot(0) = P (P is origin), measuring stick, +ve dirn
* "Naming" of points on line. "coordinate" assigns to each point its numerical name
* coordinate assigns to each pt its numerical name

$IR^2 \underset{\text{coordinate}}{\overset{\text{plot}}{\rightleftarrows}}$ plane ; $IR \underset{\text{seed}}{\overset{\text{rank}}{\rightleftarrows}}$ Player Player assigned a rank is an isomorphism

* Once we fixed an isomorphism $A \xrightarrow{f} X$, it is harmless to treat A and X as same object. We have f and f^{-1} to translate. $X \xrightarrow{f^{-1}} A \xrightarrow{G} Y$. $X \xrightarrow{g} Y$ can be given by
* We use it when one is a "better-known" object than other.
* The category of structured objects requires some respect.

Abuses of isomorphism:

* First abuse is that having an additional structure In A for (A → X) be meaningful in X.

Eg: IR → Point — Addition of two numbers another one but points don't follow.

* Second abuse is involving one familiar o and two objects X and Y coordinatized by

Eg: Adding distance and mass just because they are characterized by numbers.

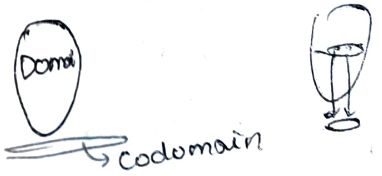
To decide about calculations, we should think in large objective category but if required do it in subjective category and translate the results to objective

Revisiting section:

* Section defines the codomain clearly i.e each fiber has atleast one element.

For a map 'f' find 'g' such that $f \circ g = 1$

* Section is short form of cross-sectio



* A section can also be called a choice of representatives.

Revisiting retraction

* Retraction defines domain clearly i.e each element has a unique parameter

For a map 'f', find 'g' such that $\boxed{g \circ f = id_A}$

→ Again, having a section 's' means that s has a retraction.

If a map has section, codomain is small, if it has retraction, domain is small

• Domain of section is smaller ($f \circ s = id_B$) and codomain of retraction is smaller ($r \circ f = id_A$). So we found a way to indicate "smaller" without actually "counting".

Retracts and comparisons:

* How do we tell A is "atmost as big as" B.

def: $A \rightarrowtail B$ means $\exists$ atleast 1 map from A to B