Lawvere Cat. Theory

Section - 9

* $A \trianglelefteq B$ means there is atleast one map from A to B

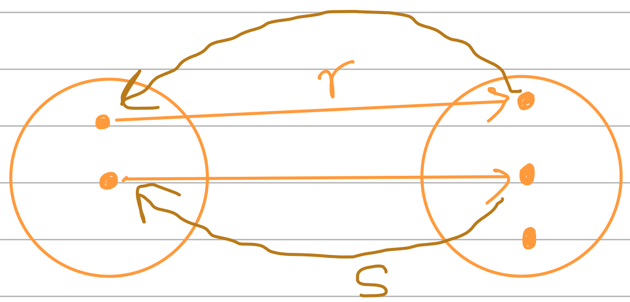
for sets $\Rightarrow$ this also means if A has a point then B has too

Def: A is a **retract** of B means that there are maps

$$A \xrightarrow{r} B \xrightarrow{s} A \quad \text{with} \quad r \circ s = 1_A$$

we can write this as $A \leq_R B$

* We can say that A has at most as many points as B.



* Note that if $n(A) > n(B)$ then $\exists$ at least two arrows meeting at one element in so we cannot send those 2 arrows back to A to make 1_A

$\Rightarrow$ Idempotents as records of retracts

If for a retract $A \xrightarrow{s} B \xrightarrow{r} A$ so $r \circ s = 1_A$

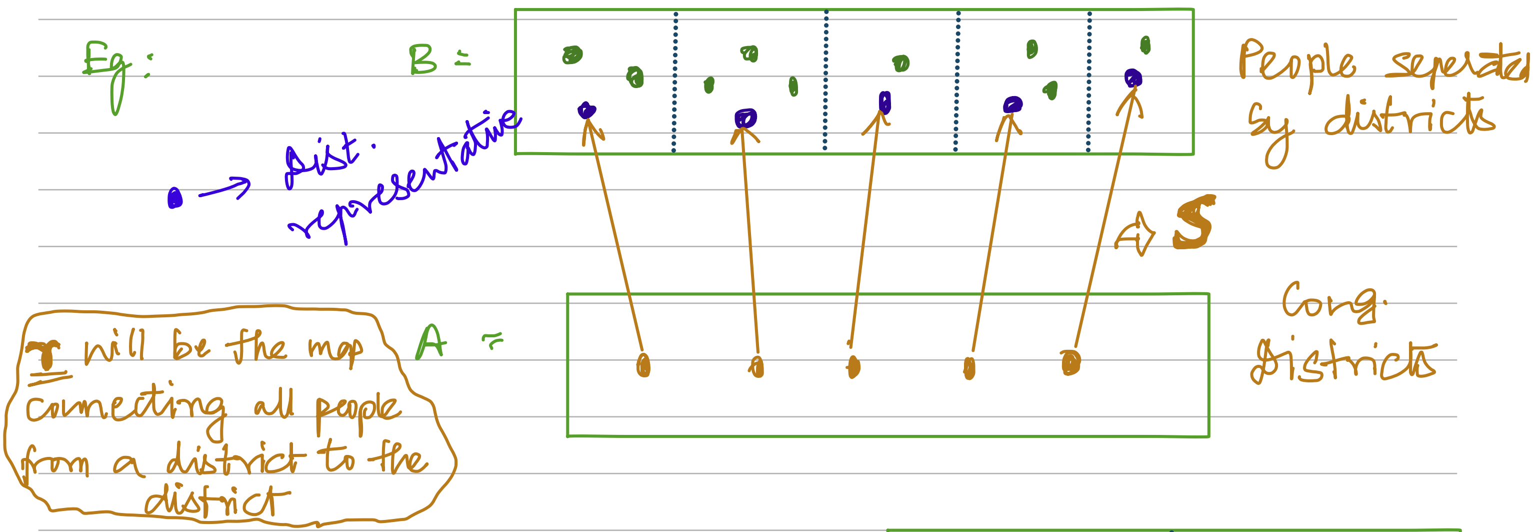
then there is a endomap of B $\Rightarrow$ $s \circ r = e$ which is Idempotent for idempotent,

$$e \circ e = e$$

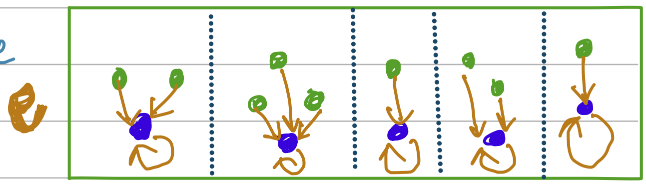
hence, $$s \circ r \circ s \circ r = s \circ (r \circ s) \circ r$$

$$= s \circ 1_A \circ r = s \circ r = e$$

Eg:



The endomap of B will be (which will be idempotent)



Def: In any category if $B \xrightarrow{e} B$ is an idempotent map, a splitting of e consists of two maps

$$A \underset{r}{\overset{s}{\rightleftarrows}} B \quad \text{with} \quad r \circ s = 1_A \quad \text{and} \quad s \circ r = e$$

* We have actually got some A by means of B and it's idempotent endomap, this means 'describing the smaller in terms of the larger'.

* Also we are basically choosing a section by sorting of B into A sorts (refer museum example) pg 103 and a dual to this - bird watcher example.

Note: for $A \xrightarrow{j} X \xrightarrow{p} A$ & $p \circ j = 1_A$

① In finite sets, no. of sections of p = $\prod_{1 \xrightarrow{a} A} \#(p^{-1}a)$ Chad's formula
{# → no. of} (p must be surjective)

② no. of retractions of j = $(\#A)^{(\#X - \#A)}$ Danilo's formula
(j must be injective)

* No. of elements of A = no. of fixed points in the endomap α

See pg 118 for some examples.

Section - 10

* Brouwer's Fixed Points Theorems :

(1) Let I be a line segment, including its endpoints and suppose that

$f: I \to I$ is a continuous endomap

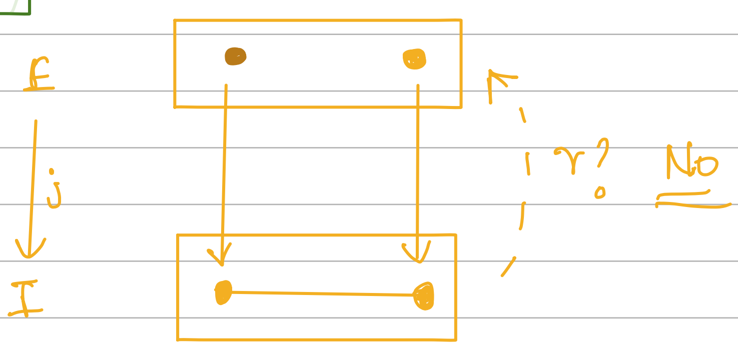
Then this map must have a fixed point : a point x in I for which $f(x) = x$ (pg 120 for eg.)

2) Let D be a closed disk and if f is a conti. endomap of D. Then f has a fixed point ($f(x)=x$)

3) Also, any continuous endomap of a solid ball has a fixed point.

⇒ Retraction Theorems:

1) Consider the map $j: E \xrightarrow{j} I$ of the two-point set E as boundary of the interval I. There is no continuous map which is a retraction of j



☆ Same for a 'disk and a ring' and 'ball and outer sphere'

This is the connection b/w fixed point & retraction theorems

* If there is no continuous retraction of a disk to its boundary, then every continuous map from the disk to itself has a fixed point

↓ contrapositive

* Given a continuous endomap of a disk with no fixed points, one can construct a continuous retraction o the disk to its boundary.

see proof in pg 125

The proof is done through contrapositive

* $A \Rightarrow B$ is true IFF Not $B \Rightarrow$ Not A is true

Law of contraposition