Pullbacks (Fibred Product)

In the category Set a ‘pullback’ is a subset of the cartesian product of two sets.
Say X and Y are your two sets and then P is a subset of (X $\times$ Y) such that the square commutes.

$$\begin{tikzcd}
P \arrow[r, "p_2"] \arrow[d, "p_1"'] & Y \arrow[d, "g"] \\
X \arrow[r, "f"'] & Z
\end{tikzcd}$$

Its akin to finding a "common part" between the two sets X and Y which are related by some Z (Subset of the cartesian product)

Suppose:

- Z={1, 2, 3}
- X={a, b} with f(a) = 1, f(b) = 2
- Y={x, y} with g(x) = 1, g(y) = 1

Then:

- P = {(a, x), (a, y)}

Given f, g the pullback is P with p1, p2
To be universal, P must be equal upto a morphism to every Q

$$
\begin{tikzcd}
Q \arrow[dr, dashed, "u"] \arrow[drr, bend left, "q_2"] \arrow[ddr, bend right, "q_1"'] & & \\
& P \arrow[r, "p_2"] \arrow[d, "p_1"'] & Y \arrow[d, "g"] \\
& X \arrow[r, "f"'] & Z
\end{tikzcd}
$$

Pushouts (Fibred Sum)

In the category Set a ‘pushout’ is a quotient of the disjoint unionof two sets.
Say X and Y are your two sets and then P is a subset of (X $\times$ Y) such that the square commutes.

Suppose:

- Z={1, 2}
- X={a, b, c} with f(1) = a, f(2) = b
- Y={x, y, z} with f(1) = x, f(2) = y

Then:

- P = { {(a, 0), (x, 1)}, {(b, 0), (y, 1)}, {(c, 0)}, {(z, 1)} }

Its akin to "gluing together" two sets X and Y which are related by some Z
(Take disjoint union such that elements related by Z are identified together in a subset)

$$
\begin{tikzcd}
P & Y \arrow[l, "i_2"'] \\
X \arrow[u, "i_1"] & Z \arrow[l, "f"] \arrow[u, "g"']
\end{tikzcd}
$$

Given f, g the pushout is P with p1, p2

To be universal, P must be equal upto a morphism to every Q

$$
\begin{tikzcd}
Q & & \\
& P \arrow[ul, dashed, "u"] & Y \arrow[l, "i_2"'] \arrow[ull, bend right, "j_2"'] \\
& X \arrow[u, "i_1"] \arrow[uul, bend left, "j_1"] & Z \arrow[l, "f"] \arrow[u, "g"']
\end{tikzcd}
$$

Spans

Any diagram of this form is called a span: $Y \leftarrow X \rightarrow Z$

A span is nothing but a diagram relating two objects in a category via some other object

The colimit of a span is a pushout

Cospans

A cospan is a span in the opposite category of whatever category the span was defined in.

Opposite category is denoted by C^{op} which means all morphisms have their dom, codom reversed.

The limit of a cospan is a pullback

I hate the co reversal in this mess!

Limits and Colimits

Generalizations of these to universal properties.

Limits restrict unique morphisms in such cones

$$
\begin{tikzcd}
 & N \arrow[d, dashed, "u"] \arrow[ddl, bend right, "\psi_X"'] \arrow[ddr, bend left, "\psi_Y"] & \\
 & L \arrow[dl, "\phi_X"'] \arrow[dr, "\phi_Y"] & \\
F(X) \arrow[rr, "F(f)"'] & & F(Y)
\end{tikzcd}
$$

Co-limits restrict unique morphisms in unpside-down cones

$$
\begin{tikzcd}
F(X) \arrow[rr, "F(f)"] \arrow[dr, "\phi_X"] \arrow[ddr, bend right, "\psi_X"'] & & F(Y) \arrow[dl, "\phi_Y"'] \arrow[ddl, bend left, "\psi_Y"] \\
 & L \arrow[d, dashed, "u"] & \\
 & N &
\end{tikzcd}
$$

Think of limits as the terminal objects in the category of cones and co-limits as the initial objects in that category.