

Revision of Concepts

Terminal object: An object in \mathcal{C} to which we will have a unique morphism from all objects in \mathcal{C}

Eg: Any one element ~~in~~ set in a category of sets is a terminal object.

* We don't care about number of terminal objects as they are all isomorphic

Eg of a category without terminal object:



(No unique morphism)

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Initial object: Dual of terminal object. an object

I s.t. $\forall X \in \mathcal{C} \exists$ morphism $I \rightarrow X$

* An initial object in \mathcal{C} is a terminal object \mathcal{C}^{op}

Empty set: set \emptyset is initial object

→ The set with the base point ~~is~~ has both initial and terminal object to be the same.

+ Initial objects are also unique.

Products

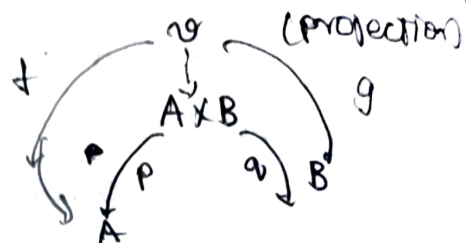
Given objects A and B in \mathcal{C} is, a product of them is given by



Note: Universal properties have a unique morphism

Cartesian product of sets:

$$A \times B = \{(a,b) \mid a \in A, b \in B\}$$



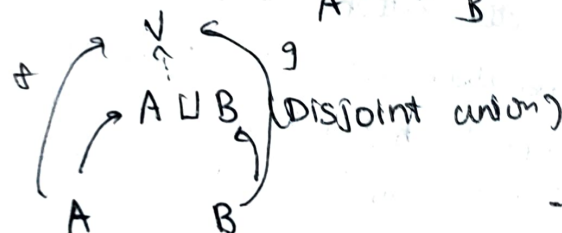
* There can be an object isomorphic to the previous product. ($A \times B \cong B \times A$)

We don't actually care about the object but the whole commutation diagram

Co-products:



(Projection/Injection maps)

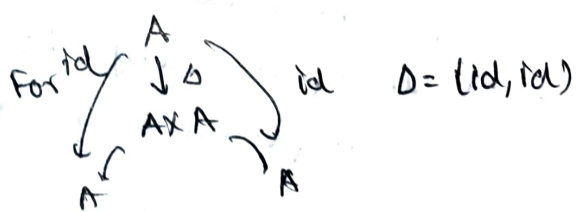
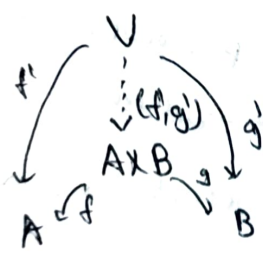


(Disjoint union)

A gets mapped to its copy and so does B

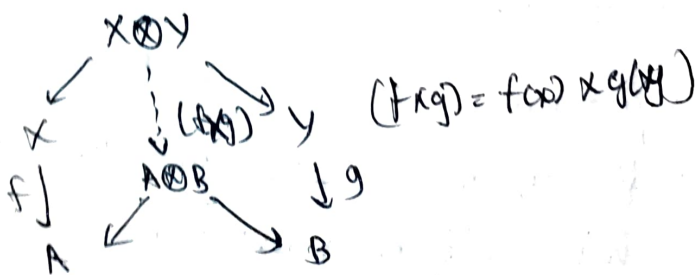
→ This coproduct if exist will have "unique" morphism

Facts



$$D = (id, id)$$

|||



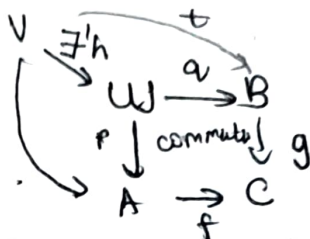
$$(f \otimes g) = f \otimes g$$

* Products get interesting for monoids i.e. terminal object. $(A, 1)$



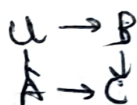
The $U \rightarrow A$ is f and it satisfies perfectly

Pull-back: Universal diagram



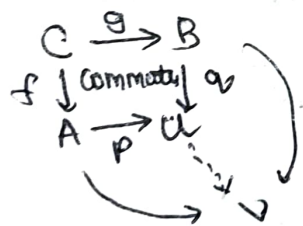
Pullback of g along f , f along g $A \times_C B$ is given by the commuting square

Set $\&$ commuting $V \rightarrow B$
 \downarrow $A \rightarrow C$ $\exists! h$



Eg in set: It is same as "product" but with the additional condition of $f(a) = g(b)$

Pushouts: (Dual of pull-backs)



Eg in sets: U is disjoint union but with the condition $p(f(c)) = q(g(c)) \ \& \ c \in \emptyset$

Limit: A limit for a diagram in \mathcal{C} is a universal cone over it.

What is a cone? cone over the diag

\rightarrow An object u , $\& \ x \in \mathcal{C} \exists$ an morphism

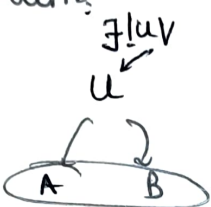
$u \rightarrow x$ s.t. $u \rightarrow x$ commutes $\&$ morphisms $x \rightarrow y$

Universal cone? means given any cone V
 $\exists! V \xrightarrow{h} u$ (smiley face) everything gets factorised

i.e. $V \xrightarrow{h} u$ commutes $\forall x \in \mathcal{C}$
 $\searrow \quad \swarrow$
 x

Let's discuss about the properties:

* Terminal objects: If u is terminal, limit is an empty diagram.

* Product: $\exists! u$

 (Two object limit)

* Pullback: $\exists! u$

 (Three object limit)

* Equaliser:

→ A diagram in \mathcal{C} of shape I is a functor $D: I \rightarrow \mathcal{C}$,
 I is a category.

$$\bullet \rightarrow \bullet \equiv A \rightarrow B$$

→ Define a functor $\Delta u: I \rightarrow \mathcal{C} \quad \forall i \in I$
 $i \mapsto u$
 $F \mapsto f_u$

A cone over D with vertex u is precisely a
 natural transformation $\Delta u \rightarrow D$