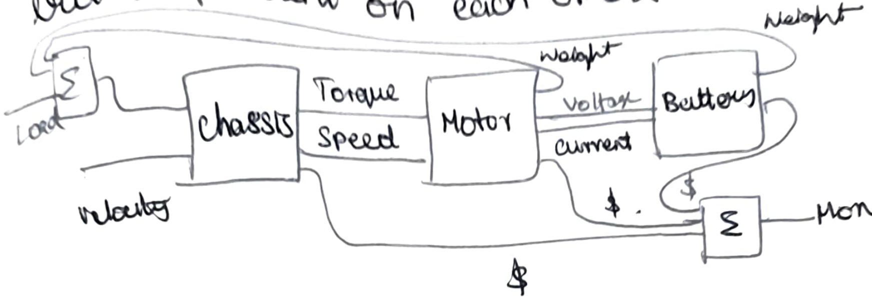
Collaborative design: Profunctors, categorification & monoidal categories.

- We will have a hierarchical order in designing a system. ostensibly independent but dependent on each other



- This is called as co-design diagrams. Each wires represents a preorder of resources

→ Each of the boxes in a co-design diagram corresponds to what we call a feasibility relation. $(p, r) \in P \times R$, P = preorder of resources to be produ-
R = Preorder of to be required

Box returns True/False.

- It defines a fn $\phi: P \times R \to \text{Bool}$.

$\phi: P \times R \to \text{Bool}$.

(i) $\phi(p, r) = \text{true}$ and $p' \leq p$ $\Rightarrow \phi(p', r) = \text{true}$.

(ii) $\phi(p, r) = T$ and $r \leq r'$ $\Rightarrow \phi(p, r') = T$.

- A co-design problem is represented by a co-design diagram. We will understand this using Bool-profunctor (bridge connecting preorders).

Enriched pro-functors:

1) Feasibility relationships as Bool-profunctors.

$x \le y$ represents availability of x given y.

If $5W \le 10W$ (If we are given with 10W, 5W is available)

ஞாபகம் உண்டா: Any preorder can be conceived as a Bool category.

$\mathbb{B}$ cat → Preorder $\mathbb{B}$ functors → Monotone map

$\mathbb{B}$ profunctors → feasibility relation.

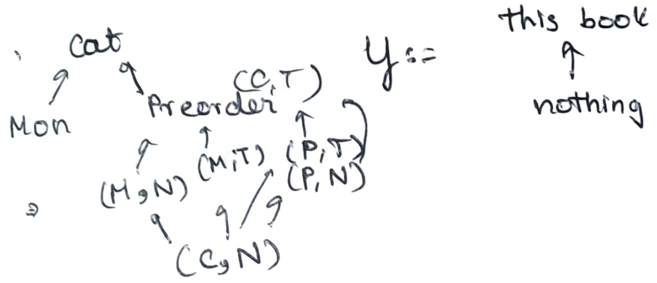
Defn: Let $\mathcal{X} = (X, \le_X)$ and $\mathcal{Y} = (Y, \le_Y)$ be preorders. A feasibility relation, $\phi : \mathcal{X}^{op} \times \mathcal{Y} \to \text{Bool}$ denoted by $\phi : \mathcal{X} \nrightarrow \mathcal{Y}$

Given $x \in X, y \in Y$ if $\phi(x, y) = T$ then x can be obtained given y

If $x' \le_X x$ and $y \le_Y y'$ $\phi(x', y') \le_{Bool} \phi(x', y')$

44) $\mathcal{X} :=$ $\mathcal{Y} :=$

1. $\mathcal{X}^{op} \times \mathcal{Y}$:



2) $\wedge : \mathcal{X} \nrightarrow \mathcal{Y}$ $\wedge(x, y) = T$ means my aunt can explain x given y. $(x, y) \in \{(P, T), (C, T)\}$

4.7) $\Rightarrow$ satisfies $b \wedge c \le d$ iff $b \le (c \Rightarrow d)$

b can be anything if $c \Rightarrow d$ is True

but b must be F if $c = T$ $d = F$

$F \wedge T \le F$ ★

Bool is a quantale (has all joins $\vee$) and a closure operation, $\Rightarrow : \mathbb{B} \times \mathbb{B} \to \mathbb{B}$

$\mathcal{V}$-profunctors:

4.8 : Let $\mathcal{V} = (V, \leq, I, \otimes)$ be a (unital commutative quantale) and let $\mathcal{X}$ and $\mathcal{Y}$ be $\mathcal{V}$-categories. A $\mathcal{V}$-profunctor from $\mathcal{X}$ to $\mathcal{Y}$ denoted $\phi : \mathcal{X} \nrightarrow \mathcal{Y}$ is a $\mathcal{V}$-functor

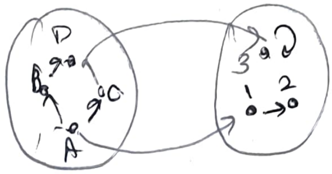
$$\phi : \mathcal{X}^{op} \times \mathcal{Y} \to \mathcal{V}$$

4.9) $\mathcal{V}$-profunctor is same as $\phi : Ob(\mathcal{X}) \times Ob(\mathcal{Y}) \to V$

If $x' \leq x$ or $y \leq y'$ $\phi(x, y) \leq \phi(x', y')$

- **Bool functors** as **bridges**. $\phi : X \nrightarrow Y$

Ex

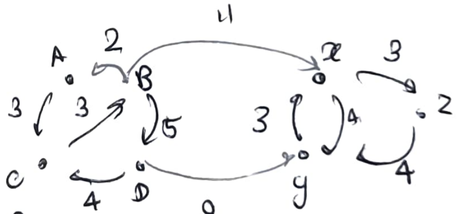


$\phi(A, 2) = \text{True}$

Feasibility matrix

	1	2	3
A			
B			
C			
D			

Cost profunctor :



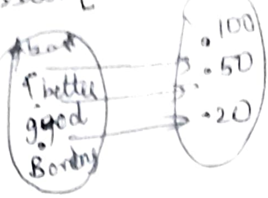
$\phi(B, x) = 11$ $\phi(D, y) = 9$

$\phi(B, y) =$

We can perform matrix multiplication as done in chap-2

Back to co-design diagrams.

chassis: (load × vel.) ↛ (Torque × speed × \$)

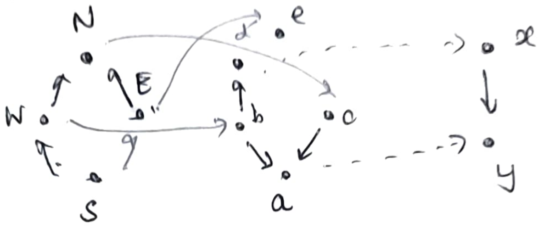


(To get to source from target)

"Feas" preorders, feasibility relation.

Categories of profunctors.

(i) Composing



Φ	a	b	c	d	e
N	T	F	T	F	F
E	T	F	T	F	T
W	T	T	T	T	F
S	T	T	T	T	T

Ψ	x	y
a	F	T
b	T	T
c	F	T
d	T	T
e	F	F

$(\Phi;\Psi)(p,r) := \bigvee_{q \in Q} \Phi(p,q) \wedge \Psi(q,r)$

$\Phi;\Psi$	x	y
N	F	T
E	F	T
W	T	T
S	T	T

Defn: $\Phi: \mathcal{X} \nrightarrow \mathcal{Y}$ $\quad \Psi: \mathcal{Y} \nrightarrow \mathcal{Z}$

$(\Phi;\Psi)(p,r) = \bigvee_{q \in Q} (\Phi(p,q) \otimes \Psi(q,r))$

4.22)

22	24	20	21
16	18	14	15
19	21	17	18
11	13	9	10

Categories $\mathcal{V}$-Prof and Feas:

- Bool and Cost are skeletal quantales

$x \leq y$ and $y \leq x \Rightarrow x = y$.

For any skeletal quantale $\mathcal{V}$, category $\mathbf{Prof}_{\mathcal{V}}$ is defined with objects as $\mathcal{V}$-categories and morphisms as $\mathcal{V}$-profunctors.

$\mathbf{Feas} := \mathbf{Prof}_{\mathrm{Bool}}$ $\quad U_{\mathcal{X}}: \mathcal{X} \nrightarrow \mathcal{X}$ on a $\mathcal{V}$-category is given by $U_{\mathcal{X}}(x,y) := \mathcal{X}(x,y)$

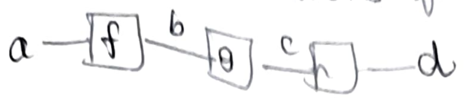
$U_P;\Phi = \Phi = \Phi;U_Q$

$\phi(p,q) = I \otimes \phi(p,q) \leq \mathcal{P}(p,p) \otimes \phi(p,q) \leq \bigvee_{p_1 \in P} \mathcal{P}(p, p_1) \otimes \phi(p_1, q)$

$f(r) \otimes f(w)$

$\leq (U_{\mathcal{P}} ; \phi)(p,q)$

$\therefore$ Feasibility relations form a category.



Monoidal category.

Companions & conjoints: Every $\mathcal{V}$-functor gives rise to two $\mathcal{V}$-profunctors → companion & conjoint

$F: P \to Q$ be functor

Companion: $\hat{F}: \mathcal{P} \nrightarrow \mathcal{Q}$

$\hat{F}(p,q) = \mathcal{Q}(F(p), q)$

Conjoint $\check{F}: \mathcal{Q} \nrightarrow P$

$\check{F}(q,p) := \mathcal{Q}(q, F(p))$

4.38) $t: \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ It will send ~~to~~ (d, a, b, c) to true if $d \leq a+b+c$

$d \leq a+b+c$

$\mathcal{V}$-Adjoints: $\mathcal{V}$-functors $F: P \to \mathcal{Q}$ $G: \mathcal{Q} \to P$

$\mathcal{P}(p, G(q)) \cong \mathcal{Q}(F(p), q)$

$F: P \to Q$ comp is $\hat{F}(p,q) = \mathcal{Q}(F(p), q)$

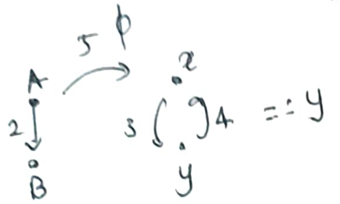
$???$

$G: Q \to P$ conj is $\check{G}(p,q) = P(p, G(q))$

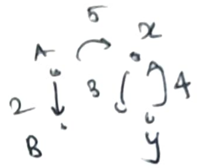
Collage of profunctor: Let $\mathcal{V}$ be a quantale, $\mathcal{X}$ and $\mathcal{Y}$ be $\mathcal{V}$-cats. $\phi: \mathcal{X} \nrightarrow \mathcal{Y}$ be a $\mathcal{V}$-profunctor. Collage of ϕ $\mathrm{Col}(\phi)$ is the $\mathcal{V}$-category

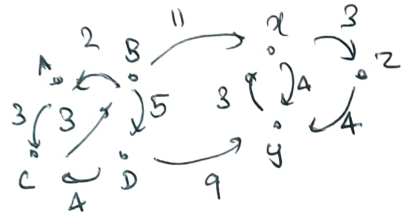
(1) $\mathrm{Ob}(\mathrm{Col}(\phi)) := \mathrm{Ob}(\mathcal{X}) \sqcup \mathrm{Ob}(\mathcal{Y})$

$$\mathrm{Col}(\phi)(a,b) \in \mathcal{V} := \begin{cases} \mathcal{X}(a,b) & \text{if } a,b \in \mathcal{X} \\ \phi(a,b) & \text{if } a \in \mathcal{X}\ b \in \mathcal{Y} \\ \phi & \text{if } a \in \mathcal{Y}\ b \in \mathcal{X} \\ \mathcal{Y}(a,b) & \text{if } a,b \in \mathcal{Y} \end{cases}$$

$\mathcal{X} :=$ 

$=: \mathcal{Y}$

$\mathrm{Col}(\phi)$: 

4.4) 

Categorification: Idea is to add structure (prope

Instead of saying $a \le b$ ($\exists$ a morphism from $a \to b$) in bool we can say the set of morphisms btw a & $b \Rightarrow$ in hom_{set}

Wiring diagrams: 

Monoidal categories: Monoidal preorders $\approx$ Monoidal categories

We will consider $\mathcal{V}$-categories while defining ~~mono~~ $\mathcal{V}$ is a monoidal category.

$(\mathrm{Set}, \{1\}, \times)$ $\quad S \times T = \{(s,t) \mid s \in S, t \in T\}$

Associativity must hold true:

$\therefore\ \alpha_{S,T,U}: \{(s,(t,u)) \mid s \in S, t \in T, u \in U\} \xrightarrow{\cong} \{((s,t),u) \mid s \in S, t \in T, u \in U\}$

Let $\mathcal{C}$ be a category. A symmetric monoidal structure on $\mathcal{C}$ will have

(i) $I \in Ob(\mathcal{C})$ (ii) a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$

(a) $\lambda_c : I \otimes c \cong c$ (b) $\rho_c : c \otimes I \cong c$

(c) $\alpha_{c,d,e} : (c \otimes d) \otimes e \cong c \otimes (d \otimes e)$

(d) $\sigma_{c,d} = (c \otimes d) \cong d \otimes e$ (swap map) ($\sigma \circ \sigma = id$) only equal to

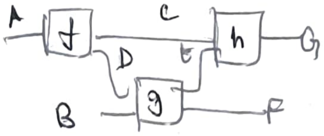
• If $\cong$ replaced by $=$ (strict)

• For Set, $I = \{1\}$ $\otimes := \times$ (product)

$f : (S \to S')$ and $g : (T \to T')$ $(f \times g) : (S \times T) \to (S' \times T')$

$(f \times g)(s, t) = (f(s), g(t))$

4.50)



$A = B = C = D = F = G = \mathbb{Z}$

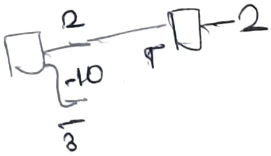
$E = \mathbb{B} = \{T, F\}$

$f_C(a) = |a|$ $f_D(a) = a * 5$ $g_E(d, b) = \text{"} d \leq b \text{"}$

$g_F(d, b) = d - b$ $h(c, e) =$ if (e) c else $1 - e$

(1) $g_E(5, 3)$ (F) $g_F(5, 3) = 2$ (2) $T, -2$

(3) 5 (4) -5 (5) 6



$q : A \times B \to G \times F$ $q_G(-2, 3) = 2$

$q_F = -13$

$q_G(2, 3) =$

Categories enriched in symm. monoidal category:

- $\mathcal{V}$ be a symm. monoidal category. $\mathcal{V}$-category $\mathcal{X}$ has: $Ob(\mathcal{X})$, $\mathcal{X}(x,y) \in \mathcal{V}$ and $id_x : I \to \mathcal{X}(x,x)$

$x,y,z \in Ob(\mathcal{X})$ $\quad \mathcal{X}(x,y) \otimes \mathcal{X}(y,z) \to \mathcal{X}(x,z)$

Let $Obj(\mathcal{X}) = \{a,b,c \ldots\}$

4.52) $\mathcal{V} = (Set, \{1\}, \times)$ $\quad$ $x,y \in Obj(\mathcal{X})$ $\quad \mathcal{X}(x,y) \in$ ~~Singleton set~~ a set

$id_x : \{1\} \to$ That element

Profunctors form a compact closed category

Defn: Let $(C, I, \otimes)$ be symm. monoidal category.

A dual for c has 3 things.

(i) $c^* \in Obj(C)$ (ii) $\eta_c : I \to c^* \otimes c$ (unit for c)

(iii) $\varepsilon_c : c \otimes c^* \to I$ (counit for c)

$$
\begin{tikzcd}
c \arrow[d] \arrow[r, equal] & c \\
c \otimes I \arrow[d, "c \otimes \eta_c"'] & I \otimes c \arrow[u] \\
c \otimes (c^* \otimes c) \arrow[r] & (c \otimes c^*) \otimes c \arrow[u, "\varepsilon_c \otimes c"']
\end{tikzcd}
$$