Universal Properties

Describes the nature of an object in terms of how its related to everything else (the universe)

Things to keep in mind

"Our goal is to understand everything in terms of maps and their composition, so
we should ask ourselves: what special property do singleton sets have? We want the
answer to involve maps."

"to say that Chad is 'universally admired' means:
For every person X in the world, X admires Chad."

Terminal Objects

S is a terminal object of C if for each object X in a category C, there exists only one C-map from X to S

If S_1 and S_2 are terminal objects, then they are isomorphic in C and there exists only one unique isomorphism between them
Since there is only one isomorphism between them, it is better to call the terminal object 1

Category	Terminal object	'Points of X' means...
$\mathcal{C}$	T	map $T \longrightarrow X$
$\mathcal{S}$	$\boxed{\bullet}$	element of X
$\mathcal{S}^{\circlearrowright}$ endomaps of sets	$\boxed{\bullet\circlearrowright}$	fixed point or equilibrium state
$\mathcal{S}^{\downdownarrows}$ irreflexive graphs	 or $\boxed{p \bullet\circlearrowright a}$	?

Points

A map from a terminal object 1 to any other object X

Points can be used to probe things inside an object
Points are often useful in "separating" maps (showing they are distinct)

A "point" in the category of dynamical systems ends up being a "fixed point"

Initial (Coterminal) Objects

S is an initial object of C if for each object X in a category C, there exists only one C-map from S to X

In the category of sets, the null set is the initial object and the map that exists to every other set is the empty function (mapping from nothingness)

If S_1 and S_2 are initial objects, then they are isomorphic in C and there exists only one unique isomorphism between them
Since there is only one isomorphism between them, it is better to call the terminal object 0

Null object

If an object is both initial and terminal, it is called the null object or zero object for that category

Products

P is a product of B_1, B_2 if for every X in the category and a pair of maps f_1, f_2 there exists a unique map $< f_1, f_2 >$ (determined uniquely using f_1, f_2) from X to P such that the below diagram commutes

$$
\begin{tikzcd}
 & & B_1 \\
X \arrow[rru, bend left, "f_1"] \arrow[rrd, bend right, "f_2"'] \arrow[r, "<f_1, f_2>"] & P \arrow[ru, "p_1"'] \arrow[rd, "p_2"] & \\
 & & B_2
\end{tikzcd}
$$

p_1, p_2 are projection maps

Definition: *Suppose that A and B are objects in a category $\mathcal{C}$. A* **product** *of A and B (in $\mathcal{C}$) is*

1. *an object P in $\mathcal{C}$, and*
2. *a pair of maps, $P \xrightarrow{p_1} A$, $P \xrightarrow{p_2} B$, in $\mathcal{C}$ satisfying:*

for every object T and every pair of maps $T \xrightarrow{q_1} A$, $T \xrightarrow{q_2} B$, there is exactly one map $T \xrightarrow{q} P$ for which $q_1 = p_1 \circ q$ and $q_2 = p_2 \circ q$.