Brouwer's theorem notes

Aneed

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1 Introduction

1.1 Why Brouwer's theorem?

- Inorder to have a smooth transition from category of sets and proving theorems by geometrical notations to using categorical ways to prove a statement , the author has used Brouwer's fixed point theorems as a tool.
- In this session , we shall learn what is Brouwer's theorem first and then we will extend our idea to objectification of concepts.

1.2 Category used by Brouwer

- Brouwer proved these theorems about maps between familiar objects such as circle,disc etc.., This setting is nothing but category of topological spaces and continuous maps.
- Instead of having a confinement on a category by having a precise description , the author tries to list "facts" where these theorems work and generalise to all categories obeying these "facts".These "facts" are called as **axioms**.
- The axioms are chosen such that it reflects our idea about *cohesive sets* and *continuous maps* which can be roughly said as $f(p)$ won't jump from one position to a far away point when you gradually move point p

2 Brouwer's fixed point theorems

(1) Let I be a line segment (endpoints inclusive) and $f : I \rightarrow I$ is a continuous endomap , then this endomap must have a fixed point i.e $\forall$ f $\exists$ x : f(x)=x

Example :

If a person travels in a road uniformly , then there will be a isomorphic map from I(time) to R(road position).

$$\text{u : I} \rightarrow R$$

$$u^{-1}\text{: I} \rightarrow R$$

Let's assume another person travelling non-uniformly m : I $\rightarrow R$. The endomap $u^{-1} \circ m$ is a endomap of object I(time) which just maps the time at which the first person reached a point to the time stamps at which the second person reached the same exact position.

2) Let D be a closed disk (the plane figure consisting of all the points inside or on a circle), and f a continuous endomap of D. Then f has a fixed point.

3) Similarly for a 3 dimensional case , any continuous endomap of a solid ball has a fixed point.

2.1 Banach fixed point theorem

- While discussing fixed point theorems the author tries to give a special case of Brouwer fixed point theorem with a beautiful example.
- The example is that , If we place a map of a table on the table itself , we can infer that the map itself will have a picture of itself i.e map will show the table with map and if we proceed through this iteration at the end , we will have a fixed point.
- This theorem proves Brouwer fixed point theorem for a special case of **contraction of distances**. This won't be valid if the map is larger than table.

3 Brouwer's retraction theorems

I) Consider the inclusion map* j : E → I of the two-point set as boundary of the interval I. There is no continuous map which is a retraction for j

- This means that there is no **continuous** map r : I → E such that r∘j $= id_E$
- The explanation is that we can't map all points of a line to its end without ripping apart the domain. So there is no continuous retraction possible for an inclusion map from E to I.

II) Consider the inclusion map j : C → D of the circle C as the boundary of the disk D into the disk. There is no continuous map which is a retraction for j

- Again , the reason is that we can't map the points on the circle to its boundary without ripping the circle.

III) Similarly for the three dimension , Consider the inclusion map j : C → D of the circle C as the boundary of the disk D into the disk. There is no continuous map which is a retraction for j

4 Proof of fixed point theorem

- What Brouwer tried to prove was **(1)** is nothing but the equivalent of **(I)** and similarly the equations **(2)** , **(3)** are equivalents of **(II)** and **(III)** respectively.
- So we are going to deduce fixed point theorem from retraction theorem.For the sake of simplicity without the loss of generality we are going to take 2D case to prove it.
- **To prove** :

If there is no continuous retraction of the disk to its boundary then every continuous map from the disk to itself has a fixed point.

- From here , the author tries to play with the statement and prove it contra positively . The contra-positive logic can be explained as

$$A \implies B \sim (not\, B) \implies (not\, A)$$

- So the contra-positive equivalent of our "To Prove" statement will be

Given a continuous endomap of the disk with no fixed points, one can construct a continuous retraction of the disk to its boundary.

4.1 Proof

Assumption : There is a **continuous** endomap with no fixed point i.e $\exists\, f\ :\ \forall\, x\ ,\ f(x) \neq x$

- Let j : C → D be the inclusion map and we have to find a continuous retraction to this map.
- Let the endomap with no fixed point be f : D → D. So we can infer that f(x) and x in D will be distinct i.e they won't coincide as f(x) ≠ x.
- Now , we can draw a arrow starting from f(x) to x and make this arrow fall on C and call it as **r(x)**.
- As the endomap f is continuous the map r created by the arrow from f(x) to x will also be continuous.
- We also infer that this map **r** will map the set C i.e the boundary to itself. So , this map will satisfy the condition $r \circ j\ =\ id_C$.

*Inclusion map is nothing but the map between a subset to superset with each element of the subset mapping to itself

- $\therefore$ r : D $\rightarrow$ C is the continuous retraction map for the inclusion map j.

Hence we proved that ,

Given a continuous endomap with no fixed point , there exists a continuous retraction map for the inclusion map from boundary to it's domain.

Key note : Absence of a retraction map for a map has a huge significance in proving theorems.

5 Excercise

5.1 Generalisation of fixed point theorem

Question :

Let j : C $\rightarrow$ D be, as before, the inclusion of the circle into the disk. Suppose that we have two continuous maps $D \underset{g}{\overset{f}{\rightrightarrows}} D$ and that g satisfies g $\circ$ j = j. Use the retraction theorem to show that there must be a point x in the disk at which f(x) = g(x).

Solution :

- The solution is similar to the geometrical proof of fixed point theorem from retraction theorem.
- Similarly let us take the contra-positive statement of our "To prove" statement

  For a given continuous map g and f with f(x) $\neq$ g(x) $\forall$ x in D , there is a continuous retraction for the inclusion map j : C $\rightarrow$ D.

- **Assumption : There is a map g with f(x) $\neq$ g(x) $\forall$ x in D**
- Let us draw the domain with a point , f(x) and g(x) . Let us draw an arrow starting from f(x) to g(x) and make this arrow fall on the boundary of the circle.
- The map formed by these points are from co-domain of g(x) which is D $\rightarrow$ C.Let us denote this map as some r(g(x))
- We can infer that if g(x) is on the boundary then r(g(x)) map to the boundary itself as g(j(x)) is j(x) by the condition given and also j(x) is an inclusion map.
- From the above statement , we can infer that r(g(x)) is nothing but the retraction of the inclusion map j.
- As it violates the retraction theorem , we conclude that our assumption is wrong. $\forall$ continuous endomaps f and g in D $\exists$ x such that f(x) = g(x)

5.2 Deducing retraction theorem from fixed point theorem

Question 1) :

Suppose that A is a 'retract' of X, i.e. there are maps $A \underset{r}{\overset{s}{\rightleftarrows}} X$ with r$\circ$s=id_A . Suppose also that X has the fixed point property for maps from T, i.e. for every endomap $X \xrightarrow{f} X$, there is a map $T \xrightarrow{x} X$ for which f$\circ$x = x. Show that A also has the fixed point property for maps from T. (Hint: The proof should work in any category, so it should only use the algebra of composition of maps.)

Solution :

- This is a part of our motto to deduce retraction theorem from fixed point theorem.
- **To Prove :** $\forall$ endomap f' of A $\exists$ a map x' such that f'$\circ$x'=x'
- A key thing to know about having a retract of an object is that if there is an endomap for one object we can create an another endomap for the other object.

- Example of the above statement is

  Let f' be an endomap of A i.e $A - f' \to A$. As A is a retract of X , there exist a map from A to X i.e $A - s \to X$ and the retraction map of s from X to A i.e $X - r \to A$
  Therefore an endomap can be created for the object X with the maps f' , s and r which will be **s** ∘ **f'** ∘ **r** i.e (s(f'(r(p)) p ∈ X)

- So we know that for every endomao f' in A , there exist a equivalent endomap in X which is s ∘ f' ∘ r . Also it is given in question that for every endomap f in X , there exist a map x : T → X such that f ∘ x = x.
- From our reasoning f is nothing but r∘f'∘s . So there exist a map x such that $\boxed{\text{s} \circ \text{f'} \circ \text{r} \circ \text{x} = \text{x}}$.
- On taking composition with r on left side in both the equations , we get r ∘ s ∘ f' ∘ r ∘ x = r∘x . On grouping we get as (r ∘ s) ∘ f' ∘ (r ∘ x) = (r ∘ x).
- We know that r∘s = id_A. So finally we get as $\boxed{\text{f'} \circ (\text{r}\circ\text{x}) = (\text{r}\circ\text{x})}$.
- On taking (r∘x) as x' , we get for every endomap f' of A , there exist x' = (r∘x) , represented $T - x' \to A$ such that f'∘x'=x'. Hence we proved the given question's statement.

Question 2)

Use the result of the preceding exercise, and the fact that the antipodal map* has no fixed point, to deduce each retraction theorem from the corresponding fixed point theorem.

Solution :

- Let us take X to be the domain and A to be the boundary and T to be a point object. From the preceding question we proved that if A(C) is a retract of X(D) and there is a fixed point in any endomap in X(D) then there must be a fixed point in any endomap of A(C).
- But antipodal map is a clear example of an endomap of A(C) which does not have a fixed point whereas any endomap in X(D) has a endomap as per fixed point theorem.
- Hence we conclude that A(C) is **not** the retract of X(D) which is nothing but the equivalent form of retraction theorem.

6 Objectification of concepts

- Throughout the talk , we haven't figured out what exactly do we mean by continuous maps.
- So what we are going to do next is we will extract all the properties which are needed for our reasoning and we will use that in any category where these properties hold true.

6.1 Category defined

- For the sake of simplicity and without the loss of generality , let us take the case of 3D i.e an inclusion map from S to B.
- Brouwer besides having these objects introduces further more objects into our desired category $\mathscr{C}$
  - An object **A** (Elements of these object are arrows in B i.e endomaps in B)
  - A map $A \xrightarrow{h} B$ (This map assigns each arrow to its head in B)
  - A map $A \xrightarrow{h} S$ (This map assigns each arrow to the point in S that it points to)
- A key thing to note is that our desired category $\mathscr{C}$ allows only continuous maps so that any composition of the maps in $\mathscr{C}$ will also be a continuous map.

*antipodal map maps the boundary points to its exact diametric opposite point

6.2 Axiom 1

Observation :

- If an arrow has its head on the boundary , then its head is the place where it points.
- Mathematically , for any a in **A** if h(a) $\in$ S , then $\boxed{\text{h(a)=p(a)}}$

Axiom :

If **T** is any object in category $\mathscr{C}$, and $T \xrightarrow{a} A$ and $T \xrightarrow{s} S$ are the maps satisfying **h**$\circ$**a**=**j**$\circ$**s** , then **p**$\circ$**a** = **s**.

- The above axiom is nothing but the extension of our observation.
- Instead of just saying head of an arrow which is in object **B** is same as where the arrow is pointing to which is in object **S** , we are saying it as ,
  - If the head of an arrow is in "boundary" : which in turn is defined by if it is in the codomain of our inclusion map j,
  - Then the boundary we choose from the external object **T** is nothing but the composition of choosing arrow and the map p.

Theorem :

If $B \xrightarrow{\alpha} A$ satisfies h$\circ\alpha\circ$j = j, then p$\circ\alpha$ is a retraction for j.

Proof :

- Let us take the object **T** to be **S** and the map s to be id_S
- From the axiom-1 , **h**$\circ$**a**= **j**$\circ id_S$ = **j**. On taking a to be composition of α and j we get h$\circ\alpha\circ$j=j.
- From the result of axiom-1 , (p$\circ\alpha$)$\circ$j=id_S. So the retraction of j is (p$\circ\alpha$).

Corollary :

- It is just a special case of our theorem , the equation h$\circ\alpha\circ$j = j will be valid if h$\circ\alpha$ is id_B.
- So the corollary says if h$\circ\alpha$=id_B , then p$\circ\alpha$ is the retraction of j.

6.3 Axiom 2

Observation :

- If we take any two points in B , there are two possibilities , either both are distinct or both coincide.

Axiom :

If **T** is any object in $\mathscr{C}$, and $T \underset{g}{\overset{f}{\rightrightarrows}} B$ are any maps, then either there is a point $\mathbf{1} \xrightarrow{t} T$ with f$\circ$t = g$\circ$t, or there is a map $T \xrightarrow{\alpha} A$ with h$\circ\alpha$ = g

- The above axiom is just the mathematical way of telling our observation.
- TO denote two points in **B** , we have two maps f and g from the object **T** in the category $\mathscr{C}$.
- So for every point in **T** they either coincide i.e f$\circ$t = g$\circ$t or they should be distinct i.e there should be an arrow between them.
- If there is an arrow between them , then there should be a map from **T** to **A** (whose elements are arrows) and our map 'g' can be represented as a head of the arrows which are mapped from **T**.

Theorem :

Suppose we have maps $B \overset{f}{\underset{g}{\rightrightarrows}} B$ and g∘j = j, then either there is a point $\mathbf{1} \xrightarrow{b} T$ with f∘b = g∘b, or there is a retraction for $S \xrightarrow{j} B$

Proof :

- If we take **T=B** in axiom-2 , then map t becomes b and the results will be either f∘b = g∘b or h∘α = g.
- On taking composition of j on right side , we get h∘α∘j = g∘j.
- We know that g∘j = j so our equation becomes h∘α∘j = j.
- From our theorem-1 , if this equation holds true then the retraction of j is p∘α.

Corollary :

- On taking g to be id_B , we get the theorem to be ,

  If $B \xrightarrow{f} B$, then either there is a fixed point for for there is a retraction for $S \xrightarrow{j} B$

- This is nothing but our **Brouwer fixed point theorem**.

6.4 Advantage of mapification of concepts :

- The way in which by objectifying certain concepts as maps in a category, the combining of concepts becomes composition of maps!
- Then we can condense a complicated argument into simple calculations using the associative law
- Such a study will be helpful because this example is a model for the method of 'thinking categorically.'