Week Twelve

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25^{th} December

- **Co-design diagrams** are similar to a *UWD*, each *boxes* represent **feasibility relations** (*design constraints* in the below figure), each *wire* represents a **preorder of resources** ($x \leqslant y$ represents *availability of x given y*): the wire on the left represent a **team's output** (which should be greater than or equal to the usage, hence, represented by '$\leqslant$'), the wire on the right represents the **team's input** requirements to generate output.

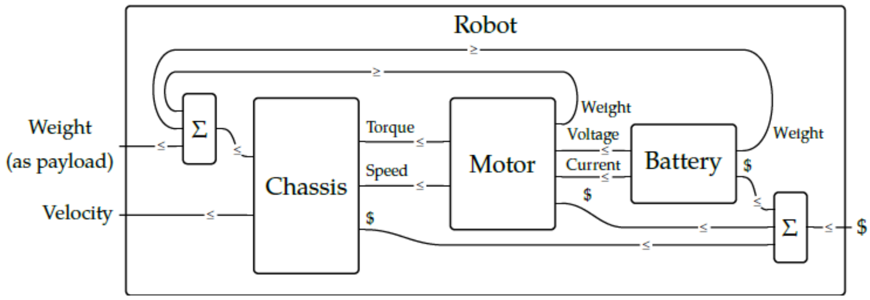

Figure 1: Example for a co-design diagram (Eq 4.1)

A *feasibility relation* matches resource production with requirements. $\forall (p, r) \in P \times R$, where P and R are the preorders of resources to be **produced** and **required** respectively, the box says **true** or **false** for that pair.

Hence, feasibility relations define a function $\Phi : P \times R \longrightarrow \textbf{Bool}$ as:

(a) $(\Phi(p, r) \;\;\&\;\; p' \leqslant p) \implies \Phi(p', r)$, ie, if p amount of produce can be made given r, you can also produce less $p' \leqslant p$ with the same resources r.

(b) $(\Phi(p, r) \;\;\&\;\; r \leqslant r') \implies \Phi(p', r)$, ie, if p amount of produce can be made given r, with $r' \geqslant r$ resources, you can produce p.

- Let $\mathcal{X} = (X, \leqslant_X)$ and $\mathcal{Y} = (Y, \leqslant_Y)$ be preoders. A **feasibility relation** for $\mathcal{X}$ given $\mathcal{Y}$ is a monotone map:

$$\Phi : \mathcal{X}^{op} \times \mathcal{Y} \longrightarrow \textbf{Bool}$$

We denote this by $\Phi : \mathcal{X} \nrightarrow \mathcal{Y}$. Given $x \in X$ and $y \in Y$, if $\Phi(x, y)$, we say *x can be obtained given y*.

This map is said to be monotone because by definition:

$$x' \leqslant_X x \;\;\&\;\; y \leqslant_Y y' \implies \Phi(x, y) \leqslant_{\textbf{Bool}} \Phi(x', y').$$

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- **$\mathcal{V}$-profunctor**: Let $\mathcal{V} = (V, \leqslant, I, \otimes)$ be a **quantale** (a closed symmetric monoid with all joins existing), and let $\mathcal{X}$ and $\mathcal{Y}$ be $\mathcal{V}$-categories. A **$\mathcal{V}$-profunctor** $\Phi : \mathcal{X} \nrightarrow \mathcal{Y}$ is a $\mathcal{V}$-functor:

$$\Phi : \mathcal{X}^{op} \times \mathcal{Y} \longrightarrow \mathcal{V}$$

- **Bool**-profunctors and **Cost**-profunctors can be interpreted as bridges. See ex 4.11, 4.13. Also see **feasibility matrix** (ex 4.12).

  Profunctor can be optained via **matrix multiplication**. (See remark 4.16)

- The category **Feas** has objects as *preorders* and morphisms as *feasibility relations* (**Bool**-profunctor) and their composition is given by using $\wedge$ in place of $\otimes$ in the composite equation given in the below point.

- **Composition of $\mathcal{V}$-profunctors**: Let $\mathcal{V}$ be a quantale and $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ be $\mathcal{V}$-categories, and let $\Phi : \mathcal{X} \nrightarrow \mathcal{Y}$ and $\Psi : \mathcal{Y} \nrightarrow \mathcal{Z}$ be $\mathcal{V}$-profunctors. Their **composite** $\Psi \circ \Phi : \mathcal{X} \nrightarrow \mathcal{Z}$ is given by:

$$(\Psi \circ \Phi)(p, r) = \bigvee_{q \in Q} (\Phi(p, q) \otimes \Psi(q, r))$$

  Composition of profunctors is associative. (Page 129)

- For any **skeletal quantale** $\mathcal{V}$, the category $\mathbf{Prof}_{\mathcal{V}}$ has objects as $\mathcal{V}$-categories $\mathcal{X}$, whose morphisms are $\mathcal{V}$-profunctors $\mathcal{X} \nrightarrow \mathcal{Y}$, and with composite defined in the above point.

  Hence, **Feas**:= $\mathbf{Prof}_{\mathbf{Bool}}$.

  The identity morphism is given by the *unit-profunctor* $U_{\mathcal{X}} : \mathcal{X} \nrightarrow \mathcal{X}$,

$$U_{\mathcal{X}}(x, y) := \mathcal{X}(x, y)$$

$$\forall \Phi : \mathcal{P} \nrightarrow \mathcal{Q} \qquad \Phi \circ U_{\mathcal{P}} = \Phi = U_{\mathcal{Q}} \circ \Phi$$

  Proof for the above identity is in page 128.

- A monoidal category is a *categorified* monoidal preorder.

- Let $F : \mathcal{P} \longrightarrow \mathcal{Q}$ be a $\mathcal{V}$-functor. The **companion** of F ($\widehat{F} : \mathcal{P} \nrightarrow \mathcal{Q}$) and the **conjoint** of F ($\check{F} : \mathcal{Q} \nrightarrow \mathcal{P}$) are defined as:

$$\widehat{F}(p, q) := Q(F(p), q) \ \ \& \ \ \check{F}(q, p) := Q(q, F(p))$$

  The **companion** profunctor represents a bridge from $\mathcal{P}$ to $\mathcal{Q}$. Reversing the arrows result in the **conjoint** profunctor representing bridge from $\mathcal{Q}$ to $\mathcal{P}$.

- **$\mathcal{V}$-enriched adjunction** is a pair of $\mathcal{V}$-functors $F : \mathcal{P} \to \mathcal{Q}$ and $G : \mathcal{Q} \to \mathcal{P}$ such that:

$$\mathcal{P}(p, G(q)) \cong \mathcal{Q}(F(p), q)$$

  In this figure, $\forall p \in \mathcal{P} \ \ \& \ \ q \in \mathcal{Q}$, the above condition holds true except for the pair $(1, c)$, hence F and G **do not** form an *enriched* adjunction pair.

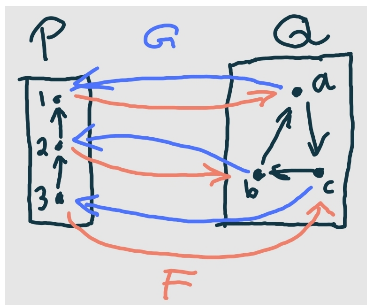

Figure 2: Example

- If $\mathcal{P}$ and $\mathcal{Q}$ are enriched in skeletal quantale $\mathcal{V}$ The companion of the adjoint F is equal to the conjoint of the adjoint G. (see ex 4.41)

  This can be used to prove that: $\widehat{\mathrm{id}} = \check{\mathrm{id}}$.

- A $\mathcal{V}$-profunctor $\Phi : \mathcal{X} \nrightarrow \mathcal{Y}$ can be thought of as a $\mathcal{V}$-category with $\mathcal{X}$ on the left and $\mathcal{Y}$ on the right. This construction is called **Collage of the Profunctor**. (denoted as $\mathbf{Col}(\Phi)$, see definition in page 131)