Week Nine

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Applications of Category Theory

Applied Category Theory

- **Join of Systems**: The join of two systems, A and B (denoted as $A \vee B$), is performed by combining their connections. Specifically, $A \vee B$ has a connection between two points x and y if there exists a sequence of points $z_1, \ldots, z_n$ such that:
  - x is connected to z_1,
  - z_i is connected to z_{i+1} for all i,
  - z_n is connected to y,

  and each of these conditions holds in at least one of the systems, A or B.

  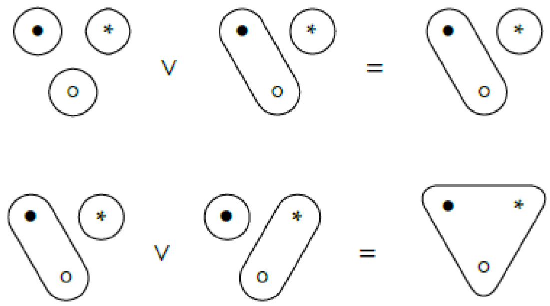

  Figure 1: Join of two objects

  Alternatively, this can be understood as taking the *transitive closure* of the union of the connections in A and B.

- **Generative Effects**: When extracting information from a system, one must often simplify or abstract away details. For example, storing a real number typically involves rounding to a certain precision. However, when details are relevant to the system's operations, such simplifications can lead to unexpected results. This phenomenon, where surprises arise from simplifications or abstractions, is known as *generative effects*.

  For instance, consider a function ϕ that returns `True` if there is a connection between a black dot (representing an infected person) and a star (representing a susceptible person), and `False` otherwise. If we apply this function to two separate systems, we might observe that the black dot is not connected to the star in either system. However, after joining the two systems, ϕ might return `True`, revealing an indirect connection between the black dot and the star that wasn't apparent in the individual systems. This demonstrates that the join operation is not preserved by our function, as the function's behavior changes based on whether we observe the individual systems or the joined system.

- **Ordering Systems**: We define an order on systems, denoted $A \leqslant B$, such that if x is connected to y in A, then x is also connected to y in B.

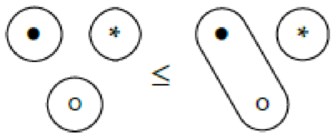

Figure 2: B can be anything as A has no connections

Using this notion of order, we can represent relationships between systems using arrow diagrams, known as *Hasse diagrams* as seen below:

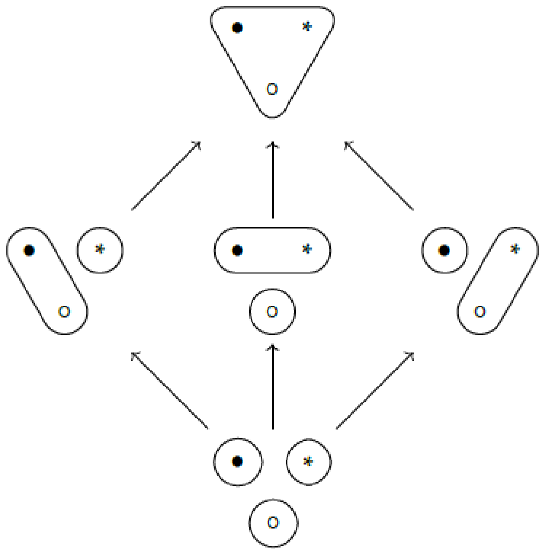

Figure 3: Hasse Diagram for the set of points $\{\bullet, \circ, \star\}$

Here, each arrow from system A to system B represent $A \leqslant B$. In particular, the joined system $A \vee B$ is the smallest system that is greater than both A and B, meaning $A \leqslant A \vee B$ and $B \leqslant A \vee B$.

As a simple example, consider the set $\{\texttt{True}, \texttt{False}\}$ with the order $\texttt{False} < \texttt{True}$.

```python
print(False<=False)
print(False<=True)
print(True<=False)
print(True<=True)
```

Figure 4: Python code

```
True
True
False
True
```

Figure 5: Corresponding Output

- **Disjoint Union**: For sets X and Y, the disjoint union $X \sqcup Y$ consists of pairs of the form (x, a) or (y, b), where $x \in X$ and $y \in Y$. This construction ensures that elements from X and Y are kept distinct in the union.

- **Preorder Relation**: A *preorder* is a binary relation $(X, <)$ on a set X that satisfies the following properties:

  - Reflexivity: $x < x$ for all $x \in X$,
  - Transitivity: If $x < y$ and $y < z$, then $x < z$.

  If $x \leqslant y$ and $y \leqslant x$, we say that x and y are *equivalent* (denoted $x \sim y$).

- **Partial Order**: A *partial order* is a preorder with an additional property called *skeletality*, which states that if $x \cong y$ (i.e., $x \leqslant y$ and $y \leqslant x$), then $x = y$. This condition ensures that there are no distinct, equivalent elements in the ordering, making the order *skeletal*.

Category theory in Circuits

- **Circuits as Labeled Graphs:**

  A circuit of linear resistors can be represented as a graph where the edges are labeled by positive real numbers, called resistances, and the set of nodes contains a subset of terminals.

  Closed circuits cannot be connected to other circuits since they lack input terminals.

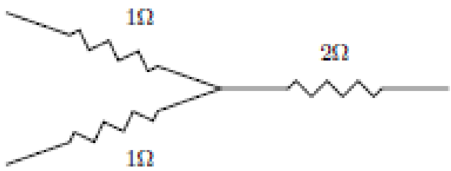

Figure 6: Closed circuit

  On the other hand, open circuits have input terminals, and through these terminals, other open circuits can be connected.

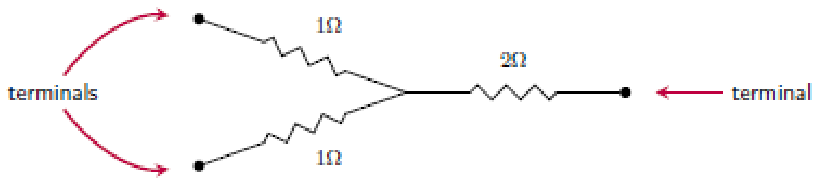

Figure 7: Open circuit graph

  We define this graph as follows: Given a set L of labels, an L-graph is a graph with a function:

$$L \xleftarrow{r} E \underset{t}{\overset{s}{\rightrightarrows}} N$$

  where E is the set of edges, and each edge $e \in E$ has a label $r(e)$ representing its resistance. Let N be the set of nodes, and $s, t : E \to N$ be the source and target maps from E to N, respectively. For resistors, the resistance values lie in $(0, \infty)$, so $L = \mathbb{R}^+$.

  A circuit with boundary over L is the graph defined above, with the additional condition that $\partial N \subseteq N$. We call ∂N the boundary of the circuit, and the elements of ∂N are the terminals.

  A subset $S \subseteq N$ of the nodes of a graph is connected if for every pair of nodes in S, there exists a path between them.

- **Ohm's Law and Kirchhoff's Laws:**

  Let V and I be mappings from an edge $e \in E$ to its corresponding voltage drop $V(e)$ and the intensity of current flow $I(e)$, respectively.

  Ohm's law states that:
  $$V(e) = r(e)I(e)$$

  Kirchhoff's Voltage Law (KVL) states that there exists a potential function $\phi : N \to \mathbb{R}$ (i.e., $\phi \in \mathbb{R}^N$) such that:
  $$V(e) = \phi(t(e)) - \phi(s(e))$$

  A boundary potential is a function $\phi \in \mathbb{R}^{\partial N}$. Since the circuit is open, the boundary potentials are variables that are free for us to choose.

  Kirchhoff's Current Law (KCL) states that:

  - For non-terminal nodes $n \in N \backslash \partial N$:
    $$\sum_{t(e)=n} I(e) = \sum_{s(e)=n} I(e)$$
  - For terminal nodes $n \in \partial N$, we define a function $i \in \mathbb{R}^{\partial N}$ such that:
    $$i(n) = \sum_{t(e)=n} I(e) - \sum_{s(e)=n} I(e)$$

References

[1] A Compositional Framework for Passive Linear Networks

[2] Onine MIT course on Applied Category Theory