Revision of Concepts

Terminal object: An object in $\mathcal{C}$ to which we will have a unique morphism from all objects in $\mathcal{C}$

Eg: Any one element ~~in~~ set in a category of sets is a terminal object.

* We don't care about number of terminal objects as they are all isomorphic

Eg of ~~a~~ category without terminal object:

 (No unique morphism)

Initial object: Dual of terminal object. an object I s.t $\forall X \in \mathcal{C}$ $\exists$ morphism $I \to X$

* An initial object in $\mathcal{C}$ is a terminal object $\mathcal{C}^{op}$

Empty set: set ϕ is initial object

→ The set with the base point ~~is~~ has both initial and terminal object to be the same.

* Initial objects are also unique.

Products

Given objects A and B in $\mathcal{C}$ ~~is~~, a product of them is given by

$$\begin{tikzcd} & U \arrow[dl, "p"'] \arrow[dr] & & V \arrow[ll] \arrow[dl, "\exists ! h"] \arrow[d, "g"] \\ A & & B & \end{tikzcd}$$

unique factorisation "h"

Note: Universal properties have a unique morphism

Cartesian product of sets:

$$A \times B = \{(a,b) \mid a \in A, b \in B$$

(projection)

$$\begin{tikzcd} & V \arrow[d] \arrow[dl, bend right, "f"'] \arrow[dr, bend left, "g"] & \\ A & A \times B \arrow[l, "p"] \arrow[r, "q"'] & B \end{tikzcd}$$

* There can be an object isomorphic to the previous product. $(A \times B \cong B \times A)$

We don't actually care about the object but the whole commutation diagram

Co-products:

$$\begin{tikzcd} & V & \\ & U \arrow[u, dashed] & \\ A \arrow[ur, "p"] \arrow[uur, bend left, "f"] & & B \arrow[ul, "q"'] \arrow[uul, bend right, "g"'] \end{tikzcd}$$

(~~Projection~~ insertion maps)

$$\begin{tikzcd} & V & \\ & A \sqcup B \arrow[u, dashed] & \\ A \arrow[ur] \arrow[uur, bend left, "f"] & & B \arrow[ul] \arrow[uul, bend right, "g"'] \end{tikzcd}$$

(Disjoint union)

A gets mapped to its copy and so does B

→ This coproducts if exist will have "unique" morphism

Facts:

$$\begin{tikzcd} & V \arrow[d, dashed, "(f', g')"] \arrow[dl, bend right, "f'"'] \arrow[dr, bend left, "g'"] & \\ A & A \times B \arrow[l, "f"] \arrow[r, "g"'] & B \end{tikzcd}$$

For

$$\begin{tikzcd} & A \arrow[d, "\Delta"] \arrow[dl, bend right, "id"'] \arrow[dr, bend left, "id"] & \\ A & A \times A \arrow[l] \arrow[r] & A \end{tikzcd}$$

$\Delta = (id, id)$

|||

$$\begin{tikzcd} & X \otimes Y \arrow[dl] \arrow[dr] \arrow[dd, dashed, "(f \times g)"] & \\ X \arrow[dd, "f"'] & & Y \arrow[dd, "g"] \\ & A \otimes B \arrow[dl] \arrow[dr] & \\ A & & B \end{tikzcd}$$

$(f \times g) = f(x) \times g(y)$

* Products get interesting for monoids i.e terminal object. $(A, ①)$

$$\begin{tikzcd}
 & v \arrow[dl, "f"'] \arrow[d] \arrow[dr, "1'"] & \\
A & A \arrow[l, "id"] \arrow[r, "1'"'] & ①
\end{tikzcd}$$

The $v \to A$ is f and it satisfies perfectly

Pull-back: Universal diagram

$$\begin{tikzcd}
v \arrow[dr, "\exists! h"] \arrow[drr, bend left, "t"] \arrow[ddr, bend right] & & \\
 & W \arrow[r, "q"] \arrow[d, "p"'] \arrow[dr, phantom, "\text{commutes}"] & B \arrow[d, "g"] \\
 & A \arrow[r, "f"'] & C
\end{tikzcd}$$

Pullback of g along f, f along g $A \times_C B$ is given by the commuting square

$$\begin{tikzcd}
u \arrow[r] \arrow[d] & B \arrow[d] \\
A \arrow[r] & C
\end{tikzcd}$$

s.t $\forall$ commuting

$$\begin{tikzcd}
v \arrow[r] \arrow[d] & B \arrow[d] \\
A \arrow[r] & C
\end{tikzcd}$$

$\exists! \, h$

Eg in set: It is same as "product" but with the additional condition of $f(a) = g(b)$

Pushouts: (Dual of pull-backs)

$$\begin{tikzcd}
C \arrow[r, "g"] \arrow[d, "f"'] \arrow[dr, phantom, "\text{commutes}"] & B \arrow[d, "q"] \arrow[ddr, bend left] & \\
A \arrow[r, "p"'] \arrow[drr, bend right] & u \arrow[dr, dashed] & \\
 & & v
\end{tikzcd}$$

Eg in sets: u is disjoint union but with the condition $p(f(c)) = q(g(c)) \; \forall c \in C$

Limit: A limit for a diagram in $\mathcal{C}$ is a universal cone over it.

What is a cone? Cone over the diag

→ An object u, $\forall x \in \mathcal{C} \; \exists$ an morphism $u \to x$ s.t

$$\begin{tikzcd}
 & x \arrow[d] \\
U \arrow[ur] \arrow[r] & y
\end{tikzcd}$$

commutes $\forall$ morphisms $x \to y$

Universal cone? means given any cone V $\exists!\ V \xrightarrow{h} U$ ☺ everything gets factorised

i.e
$$\begin{tikzcd} V \arrow[r, "h"] \arrow[dr] & U \arrow[d] \\ & X \end{tikzcd}$$
commutes $\forall\, X \in \mathcal{C}$

Lets discuss about the properties:

* Terminal objects: If U is terminal, limit is an empty diagram.

* Product:
$$\begin{tikzcd} & V \arrow[d, "\exists!"'] & \\ & U \arrow[dl] \arrow[dr] & \\ A & & B \end{tikzcd}$$
(Two object limit)

* Pullback:
$$\begin{tikzcd} V \arrow[dr, "\exists!"] & & \\ & U \arrow[r] \arrow[d] & B \arrow[d] \\ & A \arrow[r] & C \end{tikzcd}$$
(Three object limit)

* Equaliser:

→ A diagram in $\mathcal{C}$ of shape I is a functor $D: I \to \mathcal{C}$, I is a category.

$\bullet \to \bullet \equiv A \to B$

→ Define a functor $\Delta u : I \to \mathcal{C}$ $\forall\, i \in I$

$i \to u$

$F \to 1_u$

A cone over D with vertex u is precisely a natural transformation $\Delta u \to D$