Lawvere - Cat. Theory

Examples of category

(1) $S^{\circlearrowleft}$: Category of endomaps of sets

$$\boxed{X^{\circlearrowleft \alpha} \xrightarrow{f} Y^{\circlearrowleft \beta}}$$

B/w two objects of $S^{\circlearrowleft}$ is an S-map and it satisfies,

$$\boxed{f \circ \alpha = \beta \circ f}$$

Two 'subcategories' of $S^{\circlearrowleft}$:

$$S^{\circlearrowleft} \supset S^{e}$$

$$S^{\circlearrowleft} \supset S^{\leftrightarrows}$$

$S^{e} \rightarrow$ category whose obj. are all idempotent endomaps of sets

$S^{\leftrightarrows} \rightarrow$ automorphisms / permutation

CATEGORIES OF ENDOMAP :

We can generalise this to any category

if C is any category :

$$C^{\circlearrowleft} \supset C^{\leftrightarrows} \supset C^{\theta} \supset C$$

$$C^{\circlearrowleft} \supset C^{e} \supset C$$

auto morphisms $\supset$ involutions

$\supset$ $\cup$

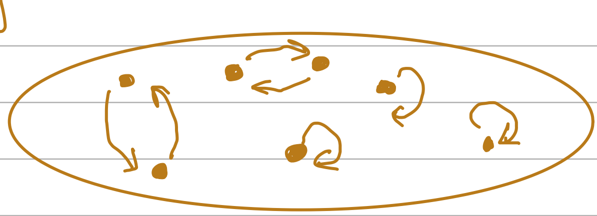
endomaps

$\supset$ idempotents $\supset$ Identities

* An endomap that is both automorphism and idempotents is 'Identity'

$\theta \Rightarrow$ Involutions $\longrightarrow$ $\theta \circ \theta = 1_A$

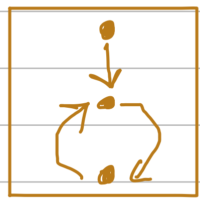
Internal diagram looks like this θ:



A

☆ Based on no. of '2-cycles' & fixed point we can tell about even / odd $n(A)$.

Example of $\alpha \circ \alpha \circ \alpha = \alpha$:



$\Rightarrow$ Irreflexive graphs :

Represented by : $S^{\downdownarrows}$

* S^{2} itself is a subcategory of $S^{\downdownarrows}$

Diagram of maps :

$$\begin{tikzcd} X \arrow[d, shift right, "s"'] \arrow[d, shift left, "t"] \\ P \end{tikzcd} \longrightarrow \text{an object in } S^{\downarrow\downarrow}$$

$X \to$ set of arrows ; $P \to$ set of dots

$s \to$ source of a map 'x' in X

$t \to$ target of a map 'x'

The map looks like

$$\begin{tikzcd} X \arrow[d, shift right, "s"'] \arrow[d, shift left, "t"] & & Y \arrow[d, shift right, "s'"'] \arrow[d, shift left, "t'"] \\ P & & Q \end{tikzcd} \xrightarrow{f}$$

A map in $S^{\downarrow\downarrow}$ is defined to be any 'pair' of S-maps

$$X \xrightarrow{f_A} Y \quad \text{and} \quad P \xrightarrow{f_D} Q$$

for which,

$$f_D \circ s = s' \circ f_A$$

and $f_D \circ t = t' \circ f_A$ holds

$$\boxed{\begin{tikzcd} X \arrow[r, "f_A"] \arrow[d, shift right, "s"'] \arrow[d, shift left, "t"] & Y \arrow[d, shift right, "s'"'] \arrow[d, shift left, "t'"] \\ P \arrow[r, "f_D"'] & Q \end{tikzcd}}$$

* Endomaps as special graphs :

As said $S^{\circlearrowright}$ is a subcategory of $S^{\downdownarrows}$

$$S^{\downdownarrows} \xleftarrow{\ I\ } S^{\circlearrowright}$$

we are doing this arbitrarily

consider,

$$\begin{tikzcd} X \arrow[d, shift right, "1_X"'] \arrow[d, shift left, "\alpha"] \\ X \end{tikzcd}$$

Note : x in first X is arrow and x in below X is representing a dot.

* The no. of arrows = no. of dots and
* if arrow named 'x', the dot too.
* but the (target arrow named x) is the dot named $\alpha(x)$

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, shift right, "1_X"'] \arrow[d, shift left, "\alpha"] & Y \arrow[d, shift right, "1_Y"'] \arrow[d, shift left, "\beta"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

It satisfies

* $f \circ 1_X = 1_Y \circ f \Rightarrow f = f$

also, [* $f \circ \alpha = \beta \circ f$]

so, $S^{\downdownarrows} \supset S^{\circlearrowright} \supset S^{e}$

$\supset S^{\leftrightarrows}$

⇒ The simpler category $S^{\downarrow}$:

* object is single map b/w two sets.

"commutative square of maps" in S

$$\begin{tikzcd} X \arrow[r, "f_A"] \arrow[d, "\alpha"'] & Y \arrow[d, "\beta"] \\ X \arrow[r, "f_B"'] & Y \end{tikzcd}$$

satisfies : $f_B \circ \alpha = \beta \circ f_A$ and $f_A \neq f_B$.

⇒ Reflexive graphs :

* graphs with 3^{rd} structural map i

obj :

$$\begin{tikzcd} X \arrow[d, shift right=3, "s"'] \arrow[d, shift left=3, "t"] \\ P \arrow[u, "i" description] \end{tikzcd}$$

and,

$$s \circ i = 1_P$$

$$t \circ i = 1_P$$

* it satisfies $e_k e_j = e_j$ for $k, j \in [0,1]$

and $e_1 = is$ and $e_0 = it$

Summary :

$$\begin{tikzcd} & & & S^{e} \arrow[d, phantom, "\subset"] & \\ \text{Reflexive graphs} \arrow[r, "U"] & S^{\downarrow i \downarrow} & \supset & S^{\circlearrowleft} \arrow[ll] \arrow[dll] \arrow[d, "V"] & \supset \; S^{\circlearrowleft\circlearrowleft} \\ & S^{\downarrow i} \arrow[u, phantom, "\cup"] & & S & \end{tikzcd}$$

$V, U \to$ forgetful functors

Ex 17

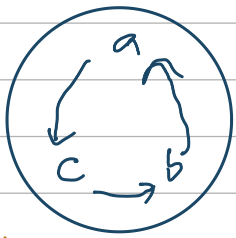
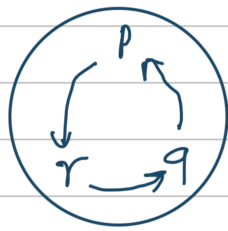
$$\begin{tikzcd} M \arrow[loop above, "\phi_1"] \arrow[r, "f_A"] \arrow[d, shift right, "\mu_1'"'] & M \arrow[loop above, "\phi_2"] \arrow[d, shift right, "\mu_2'"'] \\ F \arrow[loop below, "\mu_1"] \arrow[r, "f_{B}"] \arrow[u, shift right, "\phi_1'"'] & F \arrow[loop below, "\mu_2"] \arrow[u, shift right, "\phi_2'"'] \end{tikzcd}$$

Map should satisfy :

① $f_{B}\, \mu_1 \mu_1' \phi_1 = \mu_2 \mu_2' \phi_2 f_A$

② $f_A \phi_1 \phi_1' \mu_1 = \phi_2 \phi_2' \mu_2 f_{B}$

Pg 158

Ex 9 : $X^{\circlearrowleft \alpha} =$  $Y^{\circlearrowleft \beta} =$ 

$X^{\circlearrowleft \alpha} \xrightarrow{f} Y^{\circlearrowleft \beta}$ $\quad f \circ \alpha = \beta \circ f$

	$f:$ $a \to p$	$c \to r$	$b \to q$
$f \circ \alpha$	r	q	p
$\beta \circ f$	r	q	p

total isomorphisms : 6

$ff^{-1} = 1_y \qquad f^{-1}f = 1_x \qquad X \circlearrowleft^{\alpha} \xrightarrow{f} Y \circlearrowleft^{\beta}$

$Y \circlearrowleft^{\beta} \xrightarrow{f^{-1}} X \circlearrowleft^{\alpha}$

$f \circ \alpha = \beta \circ f$

to prove: $f^{-1} \circ \beta = \alpha \circ f^{-1}$

$f \circ \alpha \circ 1_x = 1_y \circ \beta \circ f$

$f \circ (\alpha \circ f^{-1} \circ f) = f \circ (f^{-1} \circ \beta \circ f)$

$\Rightarrow \quad (\alpha \circ f^{-1}) \circ f = (f^{-1} \circ \beta) \circ f$

$\Rightarrow \quad \boxed{\alpha \circ f^{-1} = f^{-1} \circ \beta}$

$\Rightarrow$ Categories of Diagrams:

In view of dynamical systems, $\quad X \circlearrowleft^{\alpha} \xrightarrow{f} Y \circlearrowleft^{\beta}$

* X, Y are two states
* α, β (the endomap) changes them.

* A map from $X \circlearrowleft^{\alpha}$ to $Y \circlearrowleft^{\beta}$ sends the state x of the 1st system to a state which transforms under the dynamics β 'in the same way' that x transforms under the dynamics α

Ex 1: $\quad x' = \alpha^3(x) \qquad X \circlearrowleft^{\alpha} \xrightarrow{f} Y \circlearrowleft^{\beta}$ is a map in $S^{\circlearrowleft}$

$y = f(x)$ and $y' = \beta^3(y)$

to prove: $f(x') = y'$

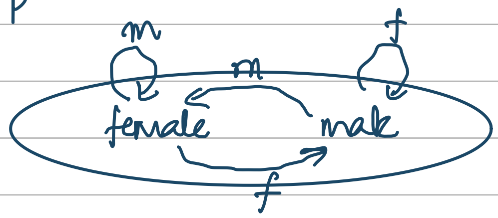
$f(x') = f \circ \alpha \circ \alpha \circ \alpha \circ x \Rightarrow \beta \circ f \circ \alpha \circ \alpha \circ x$

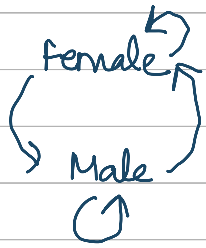
$f \circ \alpha = \beta \circ f \qquad = \beta \circ \beta \circ f \circ \alpha \circ x = \beta \circ \beta \circ \beta \circ f(x)$

$= \beta^3(y) = y'$

⇒ Family Trees : $S^{2.2}$

Ex 2: m G p ⊃f

$G =$ 



- Doubt -

* Like we constructed many categories from S we can generalise to C, we got C^2, $C^{\downarrow\downarrow}$ ....

⇒ Monoids :

* Single obj. category ⇒ all maps are endo

Ex : M is category with obj as '$*$'

maps are $* \xrightarrow{0} *$, $* \xrightarrow{1} *$, $* \xrightarrow{2} *$ . . . Interpreted as a set

$0 \rightarrow$ multiply with 0.

$$\mathbb{N} \xrightarrow{f_n} \mathbb{N} \qquad f_n = \underbrace{n}_{\text{map}} \times \underbrace{x}_{\text{obj in init } \mathbb{N}}$$

for other category with same logic, $n \neq m$ say. $\mathbb{N} \xrightarrow{f_m} \mathbb{N}$

$M \longrightarrow S$

* identity is f_1

* $f_n \circ f_m (x) = f_{nm} (x)$

then

⇒ This is a functor.

En for another category n $\quad * \xrightarrow{n} * \quad S \xrightarrow{g_n} S$

* 'n' is adding n to x

$$1_* = 0 \quad (\text{identity})$$

$$g_0 = 1_S \quad ; \quad g_n \circ g_m = g_{n+m}$$

$$g_n = n + x$$

Session - 14

En 1: $\quad x_1, x_2$ are $\cdot$s in X $\quad y_1 = f(x_1), \; y_2 = f(x_2)$

if $\alpha(x_1) = \alpha(x_2)$

$$f \circ \alpha \circ x_1 = \beta \circ f \circ x_1$$

$$f \circ \alpha \circ x_1 = \beta \circ y_1$$

$$f \circ \alpha \circ x_2 = \beta \circ y_1$$

$$\beta \circ f \circ x_2 = \beta \circ y_1$$

proved $\quad \beta \circ y_2 = \beta \circ y_1$

En : 3

if $\alpha(x) = x$ $\quad$ prove $\quad \beta(y) = y$

$$f \circ x = \beta \circ f \circ x = \beta \circ y$$

$$y = \beta \circ y$$

Section - 15

* All maps in $S^{\circlearrowleft}$ preserve positive properties
* Some maps in $S^{\circlearrowleft}$ don't preserve -ve properties.

Example of a +ve property is,

in $X \overset{\alpha}{\curvearrowleft} \xrightarrow{f} Y \overset{\beta}{\curvearrowleft}$, $\alpha(\bar{x}) = x$ for some $\bar{x}$ is a +ve property,

i.e. it is preserved under f, $\exists$ $\beta(\bar{y}) = f(x)$ for a $\bar{y}$