4. Co-Design

⇒ Feasibity relation:

For every pair $(p, r) \in P \times R$ where,

P - Preorder of resources to be produced

R - " " required

We define a function $\phi : P \times R \rightarrow \text{Bool}$ s.t.

a) $\phi(p, r) = \text{true}$ & $p' \leq p \Rightarrow \phi(p', r) = \text{true}$.

b) $\phi(p, r) = \text{true}$ & $r \leq r' \Rightarrow \phi(p, r') = \text{true}$

THINK (a), (b) using meanings of P, R.

The problems involving composite of some feasibility reactions is called a co-design problem

⇒ Enriched profunctors:

$x \leq y$ means availability of x given y

Bool profunctors as feasibility relations

Definition 4.2. Let $\mathcal{X} = (X, \leq_X)$ and $\mathcal{Y} = (Y, \leq_Y)$ be preorders. A *feasibility relation* for $\mathcal{X}$ given $\mathcal{Y}$ is a monotone map

$$\Phi : \mathcal{X}^{\text{op}} \times \mathcal{Y} \rightarrow \textbf{Bool}. \tag{4.3}$$

We denote this by $\Phi : \mathcal{X} \nrightarrow \mathcal{Y}$.

Given $x \in X$ and $y \in Y$, if $\Phi(x, y) = \texttt{true}$ we say *x can be obtained given y*.

Eg: refer Ex 4.4.

Since, Bool is a quantale ⇒ it has all joins & a closure operation say here '⇒'

$\Rightarrow : \mathbb{B} \times \mathbb{B} \rightarrow \mathbb{B}$

$b \wedge c \leq d$ iff $b \leq (c \Rightarrow d)$

'$\Rightarrow$' is like implies here

Gen: **Definition 4.8.** Let $\mathcal{V} = (V, \leq, I, \otimes)$ be a (unital commutative) quantale,[1] and let $\mathcal{X}$ and $\mathcal{Y}$ be $\mathcal{V}$-categories. A $\mathcal{V}$-*profunctor* from $\mathcal{X}$ to $\mathcal{Y}$, denoted $\Phi\colon \mathcal{X} \nrightarrow \mathcal{Y}$, is a $\mathcal{V}$-functor

$$\Phi\colon \mathcal{X}^{\mathrm{op}} \times \mathcal{Y} \to \mathcal{V}.$$

It is same as $\phi : Ob(\mathcal{X}) \times Ob(\mathcal{Y}) \to V$ s.t.

$$\mathcal{X}(x', x) \otimes \phi(x, y) \otimes \mathcal{Y}(y, y') \leq \phi(x', y')$$

proof: $\mathcal{V}$-functors from $\mathcal{X}^{op} \times \mathcal{Y}$ to $\mathcal{V}$ follows,

$$(\mathcal{X}^{op} \times \mathcal{Y})((x, y), (x', y')) \leq \mathcal{V}(\phi(x, y), \phi(x', y'))$$

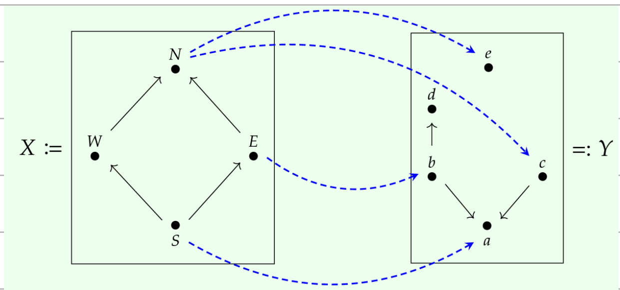
$$\underbrace{\mathcal{X}^{op}(x, x') \otimes \mathcal{Y}(y, y')}_{\text{product } \mathcal{V}\text{-cat}} \leq \underbrace{\phi(x, y) \multimap \phi(x', y')}_{\mathcal{V} \text{ is enriched in itself}}$$

$(\mathcal{X}(x, y) = \mathcal{X}^{op}(y, x))$

$$\mathcal{X}(x', x) \otimes \mathcal{Y}(y, y') \leq \phi(x, y) \multimap \phi(x', y')$$

(hom-element & $\otimes$ symm.) $\mathcal{X}(x', x) \otimes \phi(x, y) \otimes \mathcal{Y}(y, y') \leq \phi(x', y')$

Eg: Bool profunctor as bridges b/w cities



if there is a way to go from x in $\mathcal{X}$ to y in $\mathcal{Y}$ then

$\phi(x, y) =$ true else false

for that construct $X \nrightarrow Y$

⇒ Composing profunctors :

If Φ & Ψ are two profunctors b/w $\mathcal{P}$ to $\mathcal{Q}$ & $\mathcal{Q}$ to $\mathcal{R}$ then,

$$\Phi ; \Psi (p, r) := \bigvee_{q \in Q} \Phi(p, q) \otimes \Psi(q, r)$$

Meaning : (for bool)
we can get from p in $\mathcal{P}$ to r in $\mathcal{R}$ by 'AND' ing the the way from p to q, q to r for all possible ('OR' ways) q's in $\mathcal{Q}$

Theorem 4.23. For any skeletal quantale $\mathcal{V}$, there is a category $\mathbf{Prof}_{\mathcal{V}}$ whose objects are $\mathcal{V}$-categories $\mathcal{X}$, whose morphisms are $\mathcal{V}$-profunctors $\mathcal{X} \nrightarrow \mathcal{Y}$, and with composition defined as in Definition 4.21.

Definition 4.24. We define $\mathbf{Feas} := \mathbf{Prof}_{\mathbf{Bool}}$.

* Unit Profunctor : $U_{\mathcal{X}} : \mathcal{X} \nrightarrow \mathcal{X}$ (for a $\mathcal{V}$ category $\mathcal{X}$)

if $U_{\mathcal{X}}(x, y) = \mathcal{X}(x, y)$

Lemma 4.31. Serial composition of profunctors is associative. That is, given profunctors $\Phi \colon \mathcal{P} \to \mathcal{Q}$, $\Psi \colon \mathcal{Q} \to \mathcal{R}$, and $\Upsilon \colon \mathcal{R} \to \mathcal{S}$, we have

$$(\Phi \mathbin{⨟} \Psi) \mathbin{⨟} \Upsilon = \Phi \mathbin{⨟} (\Psi \mathbin{⨟} \Upsilon).$$

⇒ companion & Conjoint of a $\mathcal{V}$-functor :

Definition 4.34. Let $F \colon \mathcal{P} \to \mathcal{Q}$ be a $\mathcal{V}$-functor. The *companion of F*, denoted $\widehat{F} \colon \mathcal{P} \nrightarrow \mathcal{Q}$ and the *conjoint of F*, denoted $\check{F} \colon \mathcal{Q} \nrightarrow \mathcal{P}$ are defined to be the following $\mathcal{V}$-profunctors:

$$\widehat{F}(p, q) := \mathcal{Q}(F(p), q) \quad \text{and} \quad \check{F}(q, p) := \mathcal{Q}(q, F(p))$$

Eg :

Example 4.37. Consider the function $+ \colon \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, sending a triple (a, b, c) of real numbers to $a + b + c \in \mathbb{R}$. This function is monotonic, because if $(a, b, c) \le (a', b', c')$—i.e. if $a \le a'$ and $b \le b'$, and $c \le c'$—then obviously $a + b + c \le a' + b' + c'$. Thus it has a companion and a conjoint.

Its companion $\widehat{+} \colon (\mathbb{R} \times \mathbb{R} \times \mathbb{R}) \nrightarrow \mathbb{R}$ is the function that sends (a, b, c, d) to `true` if $a + b + c \le d$ and to `false` otherwise.

$\hat{F}((a, b, c), k) = \hat{F}(d, k)$ acc. to def. of F.

* For $F : \mathcal{P} \to \mathcal{Q}$ and $G : \mathcal{Q} \to \mathcal{P}$ be 2 $\mathcal{V}$-functors then if $\hat{F} = \check{G}$ then F, G are $\underbrace{\mathcal{V}\text{-adjoints}}$

if $\mathcal{P}(p, G(q)) \cong \mathcal{Q}(F(p), q)$

$\Rightarrow$ Collage of a profunctor:

Definition 4.42. Let $\mathcal{V}$ be a quantale, let $\mathcal{X}$ and $\mathcal{Y}$ be $\mathcal{V}$-categories, and let $\Phi\colon \mathcal{X} \nrightarrow \mathcal{Y}$ be a $\mathcal{V}$-profunctor. The *collage of* Φ, denoted $\mathbf{Col}(\Phi)$ is the $\mathcal{V}$-category defined as follows:

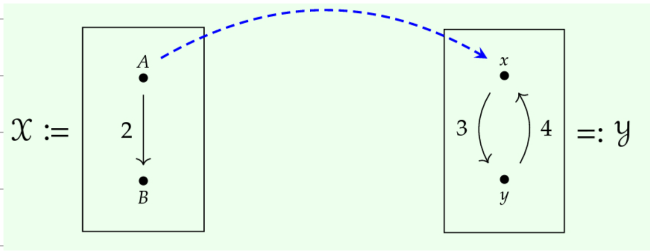
(i) $\mathrm{Ob}(\mathbf{Col}(\Phi)) := \mathrm{Ob}(\mathcal{X}) \sqcup \mathrm{Ob}(\mathcal{Y})$;

(ii) For any $a, b \in \mathrm{Ob}(\mathbf{Col}(\Phi))$, define $\mathbf{Col}(\Phi)(a, b) \in \mathcal{V}$ to be

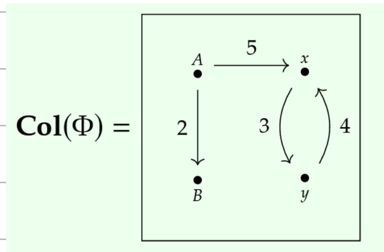
$$\mathbf{Col}(\Phi)(a,b) := \begin{cases} \mathcal{X}(a,b) & \text{if } a, b \in \mathcal{X} \\ \Phi(a,b) & \text{if } a \in \mathcal{X}, b \in \mathcal{Y} \\ \varnothing & \text{if } a \in \mathcal{Y}, b \in \mathcal{X} \\ \mathcal{Y}(a,b) & \text{if } a, b \in \mathcal{Y} \end{cases}$$

There are obvious functors $i_{\mathcal{X}}\colon \mathcal{X} \to \mathbf{Col}(\Phi)$ and $i_{\mathcal{Y}}\colon \mathcal{Y} \to \mathbf{Col}(\Phi)$, sending each object

Eg:



then,



$\Rightarrow$ Categorification:

$\hookrightarrow$ Basically adding structures

$\mathbb{N}$: $5 + 3 = 8$

Categorified $\mathbb{N}$: $\bar{5} \sqcup \bar{3} \cong \bar{8}$

(using FinSet)

* CHANGES TO WIRING DIAGRAM : replaced $\leq$ in preorder with morphisms remaining all main ideas are same.

$\Rightarrow$ Monoidal Categories:

S.M.C :

Rough Definition 4.45. Let $\mathcal{C}$ be a category. A *symmetric monoidal structure* on $\mathcal{C}$ consists of the following constituents:

(i) an object $I \in \mathrm{Ob}(\mathcal{C})$ called the *monoidal unit*, and

(ii) a functor $\otimes: \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, called the *monoidal product*

subject to well-behaved, natural isomorphisms

(a) $\lambda_c: I \otimes c \cong c$ for every $c \in \mathrm{Ob}(\mathcal{C})$,

(b) $\rho_c: c \otimes I \cong c$ for every $c \in \mathrm{Ob}(\mathcal{C})$,

(c) $\alpha_{c,d,e}: (c \otimes d) \otimes e \cong c \otimes (d \otimes e)$ for every $c, d, e \in \mathrm{Ob}(\mathcal{C})$, and

(d) $\sigma_{c,d}: c \otimes d \cong d \otimes c$ for every $c, d \in \mathrm{Ob}(\mathcal{C})$, called the *swap map*, such that $\sigma \circ \sigma = \mathrm{id}$.

A category equipped with a symmetric monoidal structure is called a *symmetric monoidal category*.

↗ strict if = instead of $\cong$

Note:

(Set, {1}, ×) is a monoidal cat. for S, T in Set

$$S \times T = \{(s,t) \mid s \in S, t \in T\}$$

lly

① $S \times (T \times U) = \{(s,(t,u)) \mid$ " $u \in U\}$

② $(S \times T) \times U = \{((s,t),u) \mid$ " $\}$

but for monoidal preorder:

$$(p \otimes q) \otimes r = p \otimes (q \otimes r)$$

& here we say,

$$① \cong ②$$

⇒ categories enriched in a SMC:

We have already read about V-cat. but before V was req. to be a preorder & now it is categorised.

Rough Definition 4.51. Let $\mathcal{V}$ be a symmetric monoidal category, as in Definition 4.45. To specify a *category enriched in* $\mathcal{V}$, or a $\mathcal{V}$*-category*, denoted $\mathcal{X}$,

(i) one specifies a collection $\mathrm{Ob}(\mathcal{X})$, elements of which are called *objects*;

(ii) for every pair $x, y \in \mathrm{Ob}(\mathcal{X})$, one specifies an object $\mathcal{X}(x,y) \in \mathcal{V}$, called the *hom-object* for x, y;

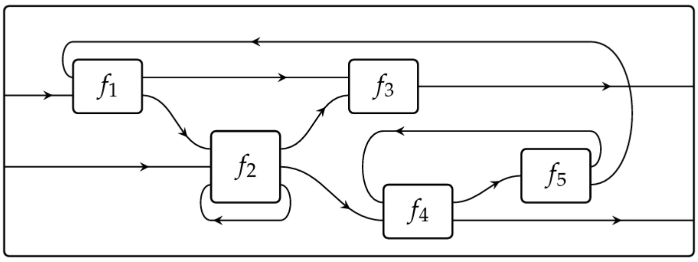
(iii) for every $x \in \mathrm{Ob}(\mathcal{X})$, one specifies a morphism $\mathrm{id}_x: I \to \mathcal{X}(x,x)$ in $\mathcal{V}$, called the *identity element*;

(iv) for each $x, y, z \in \mathrm{Ob}(\mathcal{X})$, one specifies a morphism $\fatsemi: \mathcal{X}(x,y) \otimes \mathcal{X}(y,z) \to \mathcal{X}(x,z)$, called the *composition morphism*.

These constituents are required to satisfy the usual associative and unital laws.

* $I \leq \mathcal{X}(x,x)$ in the previous def. here became $I \to \mathcal{X}(x,x)$ to exist here, same for composition.

⇒ Compact closed categories:
→ monoidal cat. whose wiring diagram allows feedback. Like:



* They have one additional str. than a SMC.

Definition 4.58. Let $(\mathcal{C}, I, \otimes)$ be a symmetric monoidal category, and $c \in \mathrm{Ob}(\mathcal{C})$ an object. A *dual for c* consists of three constituents

(i) an object $c^* \in \mathrm{Ob}(\mathcal{C})$, called the *dual of c*,

(ii) a morphism $\eta_c : I \to c^* \otimes c$, called the *unit for c*,

(iii) a morphism $\epsilon_c : c \otimes c^* \to I$, called the *counit for c*.

These are required to satisfy two equations for every $c \in \mathrm{Ob}(\mathcal{C})$, which we draw as commutative diagrams:

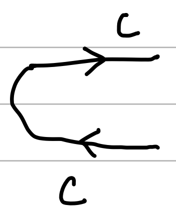
$$\begin{tikzcd} c \arrow[rr, equal] \arrow[d, "\cong"'] & & c \\ c \otimes I \arrow[d, "c \otimes \eta_c"'] & & I \otimes c \arrow[u, "\cong"'] \\ c \otimes (c^* \otimes c) \arrow[rr, "\cong"'] & & (c \otimes c^*) \otimes c \arrow[u, "\epsilon_c \otimes c"'] \end{tikzcd} \qquad \begin{tikzcd} c^* \arrow[rr, equal] \arrow[d, "\cong"'] & & c^* \\ I \otimes c^* \arrow[d, "\eta_c \otimes c^*"'] & & c^* \otimes I \arrow[u, "\cong"'] \\ (c^* \otimes c) \otimes c^* \arrow[rr, "\cong"'] & & c^* \otimes (c \otimes c^*) \arrow[u, "c^* \otimes \epsilon_c"'] \end{tikzcd} \tag{4.59}$$

These equations are sometimes called the *snake equations*.

If for every object $c \in \mathrm{Ob}(\mathcal{C})$ there exists a dual c^* for c, then we say that $(\mathcal{C}, I, \otimes)$ is *compact closed*.

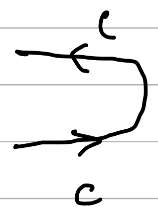
$\xrightarrow{c}$ is same as $\xleftarrow{c^*}$ so,

η_c :



(unit)

ϵ_c :



(counit)

Proposition 4.60. If $\mathcal{C}$ is a compact closed category, then

1. $\mathcal{C}$ is monoidal closed;

and for any object $c \in \mathrm{Ob}(\mathcal{C})$,

2. if c^* and c' are both duals to c then there is an isomorphism $c^* \cong c'$; and
3. there is an isomorphism between c and its double-dual, $c \cong c^{**}$.

for point 1., closed 'cause $c \multimap d$ is given by $c^* \otimes d$

& we can obtain this relation: $\mathcal{C}(b \otimes c, d) \cong \mathcal{C}(b, c \multimap d)$
(in def of closed monoidal preorder)

REFER EXAMPLE 4.61 & 4.62 exercise

* Feas is a compact closed category. → $\mathbf{Prof}_{\mathbf{Bool}}$

Theorem 4.63. Let $\mathcal{V}$ be a skeletal quantale. The category $\mathbf{Prof}_{\mathcal{V}}$ can be given the structure of a compact closed category, with monoidal product given by the product of $\mathcal{V}$-categories.

→ Monoidal products in $\mathrm{Prof}_{\mathcal{V}}$: product categories
→ Monoidal unit : 1 (one object)
→ c^* or duals in $\mathrm{Prof}_{\mathcal{V}}$ are opposite category $\mathcal{X}, \mathcal{X}^{op}$