Categories of Structured Sets

The Category of Endomaps of Sets

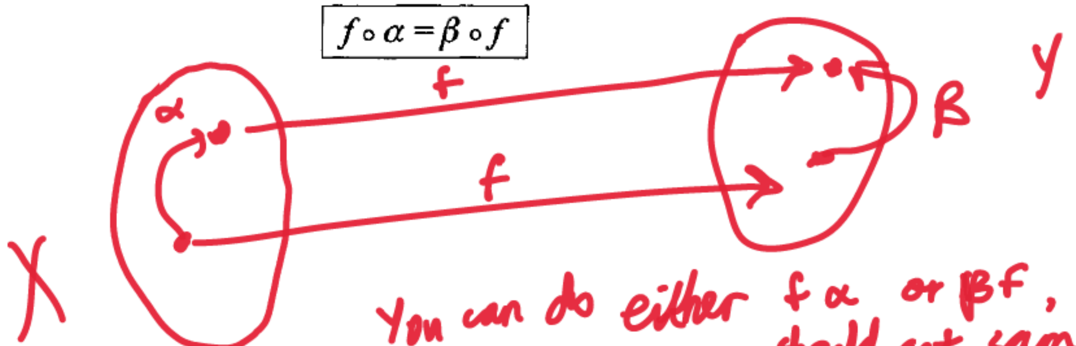
One thing we must notice till now is that each point in a set is identical, we really don't bother distinguishing one from another. All that changes when we talk about the endomaps of sets-interesting structure often arises in this context. This category of endomaps, call it S', what are its objects? And how are maps defined? Let's answer all this below.

For any objects X and Y, α: $X \to X$ and β: $Y \to Y$ we define a specific map f as follows:

$$\boxed{X \circlearrowleft \alpha} \xrightarrow{\quad f \quad} \boxed{Y \circlearrowleft \beta}$$

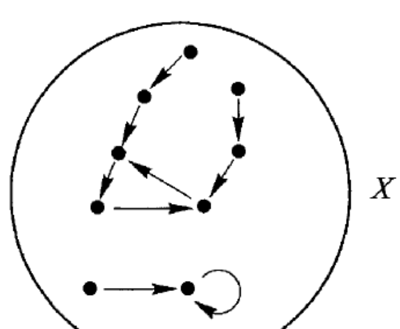
Note that this map is not arbitrary. Since morphisms must "preserve structure", such an *equivariant map* f must satisfy the following:

$$\boxed{f \circ \alpha = \beta \circ f}$$



You can do either fα or βf, you should get same

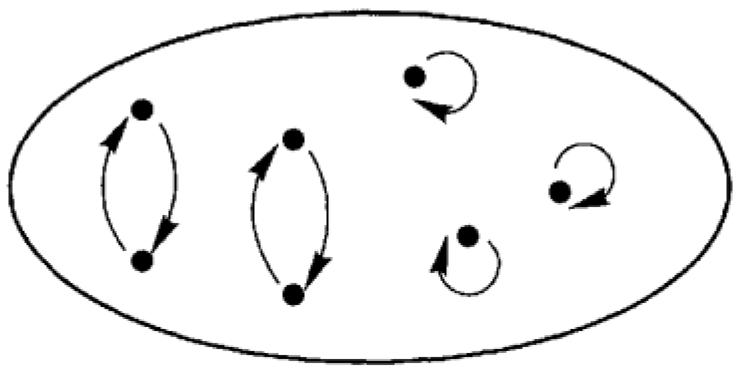
The internal diagram of an object may look like



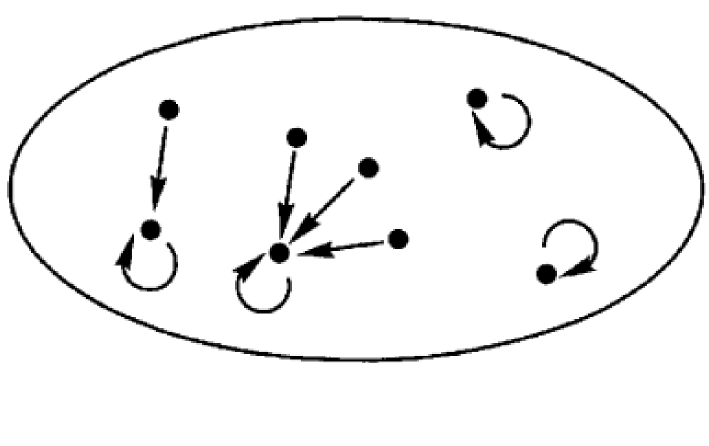
In the above diagram, the idea is that when we apply α to any point in the set, the "arrow" will point towards another point (as the mapping is an endomap so the source and target elements belong to the same set). So we may observe chains of arrows, cycles, fixed points, etc. It should be noted that if two objects in this category are isomorphic, then the structure is mirrored (same number of same length cycles, same chains, etc.).

Involutions, Identities and Automorphisms

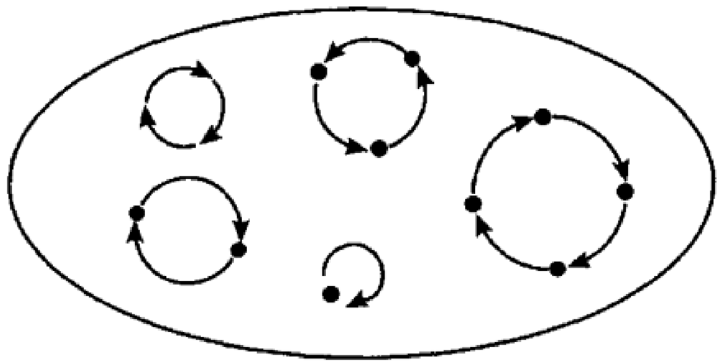
Similar to how we defined these terms for matrices, an involution is a morphism $\theta\colon X \to X$ such that $\theta \circ \theta = 1_X$. It's internal diagram looks like



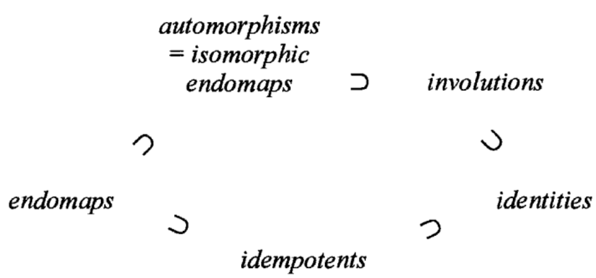
An idempotent is a morphism $e\colon X \to X$ such that $e \circ e = e$. It's internal diagram looks like



An automorphism is an endomap which is invertible, and its internal diagram possesses cycles and fixed points only.



All the above discussed kinds of morphisms may be related as follows:

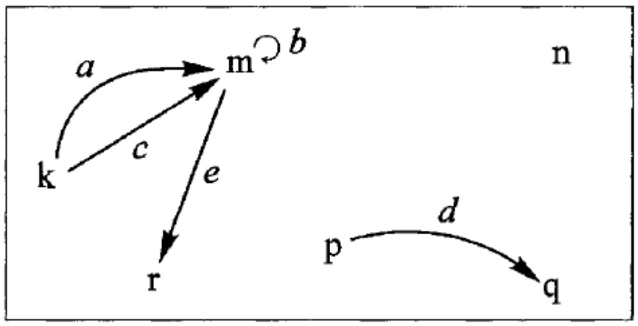


The Category of Irreflexive Graphs

These are *directed* graphs, where a pair of maps point from one set to another. Specifically, we define maps s and t from X, the set of 'arrows' to P, the set of 'dots'.

$$\begin{tikzcd} X \arrow[d, shift right, "s"'] \arrow[d, shift left, "t"] \\ P \end{tikzcd}$$

Hence if x is any arrow in X, then $s(x)$ and $t(x)$ represent the *source* and the *target* of the arrow. One thing should be noticed: each arrow must be assigned a pair of dots, but each dot may not be attached to an arrow (like n in the example below).



Any map f between two such graphs is actually a pair of maps f_A and f_D that respects source and target points, thus making the following diagram commute:

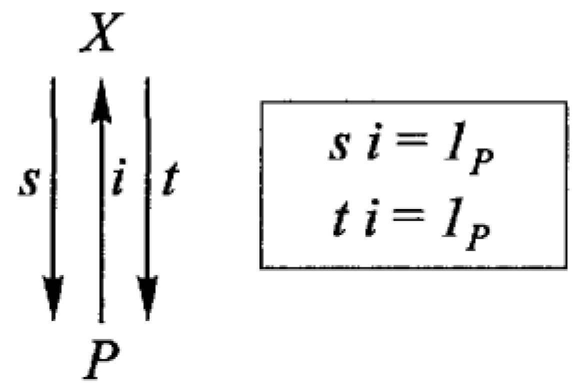
$$\boxed{\begin{array}{c} f_D\, s = s' f_A \\ f_D\, t = t' f_A \end{array}}$$

$$\begin{tikzcd} X \arrow[r, "f_A"] \arrow[d, shift right, "s"'] \arrow[d, shift left, "t"] & Y \arrow[d, shift right, "s'"'] \arrow[d, shift left, "t'"] \\ P \arrow[r, "f_D"'] & Q \end{tikzcd}$$

[Doubt: Did not understand the part about how endomaps are special cases of these irreflexive graphs-how can x be a dot and an arrow at the same time? Once understood, fullness must be discussed.]

The Category of Reflexive Graphs

These are the same as the above directed dual-arrow graphs except that now the object possesses an extra map i going in the reverse direction (hence the name reflexive).



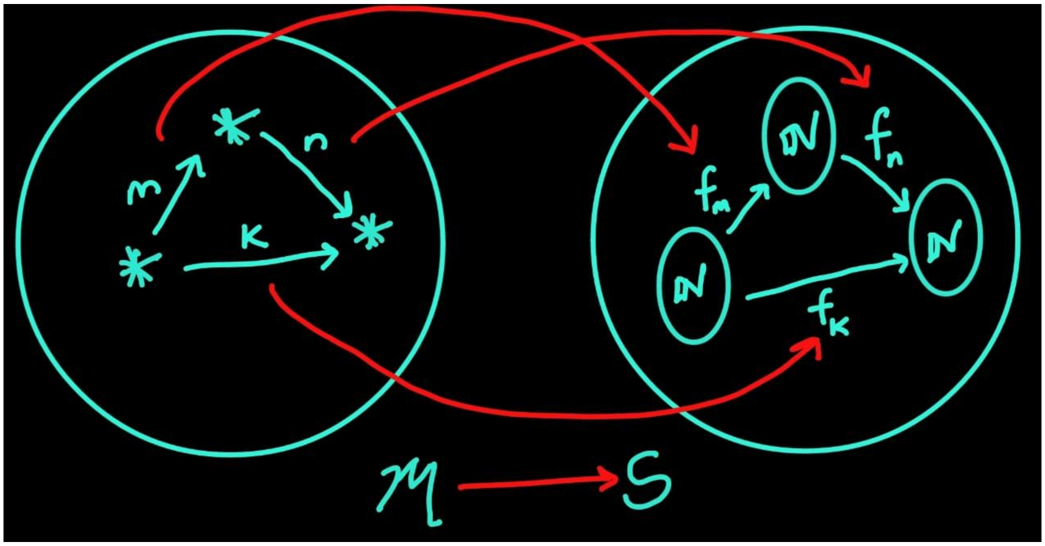
Monoids

Monoids are categories with exactly one object. Let's call this object "$\ast$". Can we really expect to do anything interesting with monoids? Turns out, yes we can. Remember that although we may only have one object, we can create endomaps and expect to see some structure appear.

Say each map $\ast \to \ast$ is identified by some natural number n and a composition of maps may be taken as multiplication. This just means that $n \circ m = nm$ where m is another map. Here the identity map $1_\ast = 1$ because $n \circ 1_\ast = 1_\ast \circ n = n$.

Now if we define a map from this monoid M to the category of finite sets and maps S (such a map between two categories is called a *functor*-more on this later) such that each map n in M corresponds to the map f_n in S and $f_n\colon \mathbb{N} \to \mathbb{N}$, where $\mathbb{N}$ is the set of natural numbers. Furthermore,

$$f_n(x) = nx$$



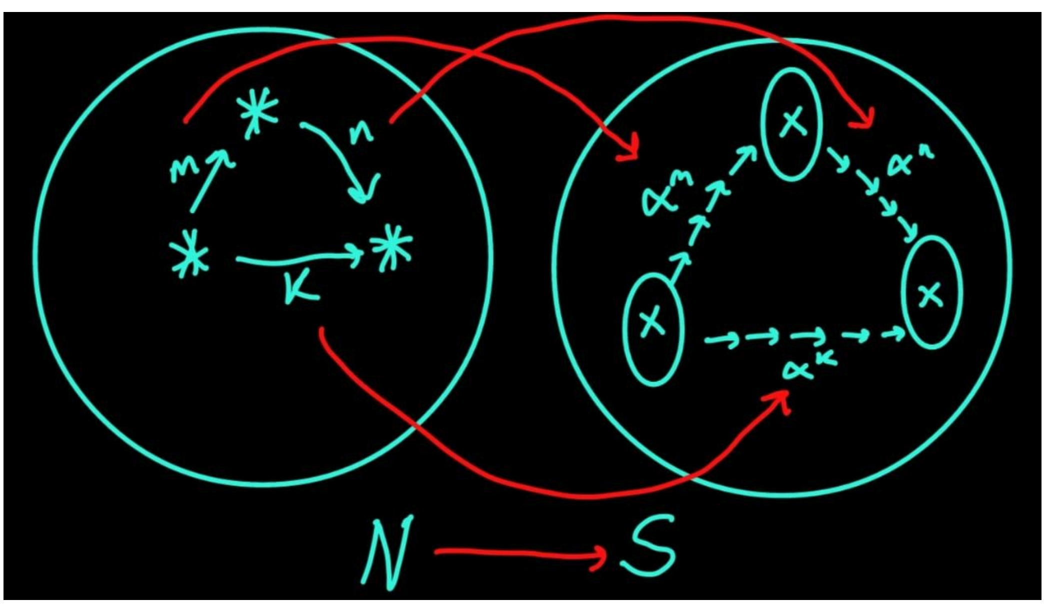
Let's see another example, one where composition of maps in the monoid is addition instead of multiplication. In this monoid N, $1_{\ast} = 0$. Define a functor $F\colon N \to S$,

where

$F(\ast) = X,\ F(n) = \alpha^{n},\ F(0) = 1_X$ hold. Here $\alpha\colon X \to X$ is an endomap. Note that

$$F(n + m) = F(n) \circ F(m)$$

is obeyed in this particular example.



How do monoids fit into our understanding of dynamical systems? We shall answer this question below soon.

Positive and Negative Properties

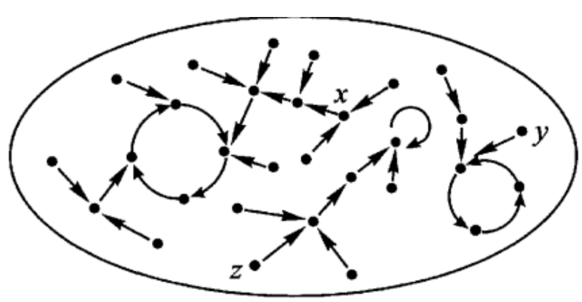
All maps in the category of endomaps preserve positive properties. Some of these maps do not preserve negative properties.

In other words, consider a map $f: X \rightarrow Y$, where X and Y are sets equipped with endomaps. Say X has a fixed point, then it can be shown that Y must also have a fixed point irrespective of the choice of f and according to the above statement having a **fixed point is a positive property.** Another example of a positive property is **accessibility:** a point x is said to be *accessible* if $x = \alpha(x')$ for some x' in X.

Not being a fixed point qualifies as a negative property because if x is not a fixed point then $f(x)$ may or may not be a fixed point.

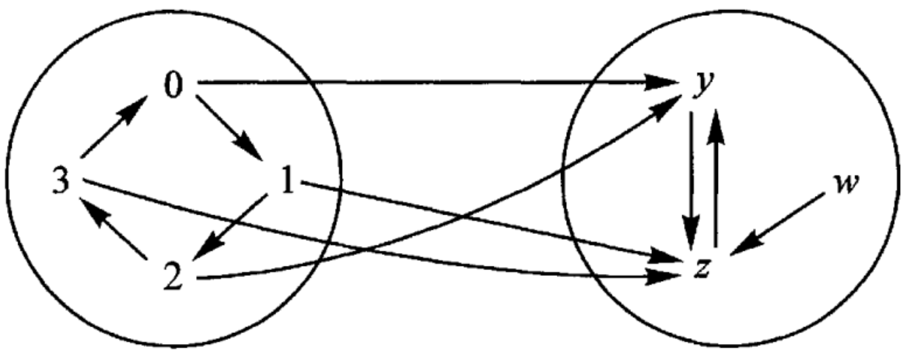
Dynamical Systems

We have indirectly already seen a dynamical system of sorts when we studied endomaps in the Category of Endomaps of Sets. Recall that an internal diagram looked like this:



So if you started at any point (say, x) and kept applying α, you'd be moving from one point to another, thus bringing in the notion of "motion" and "dynamics". You could even end up in cycles or fixed points, just as in a dynamical system. So α becomes like this "button" to press that increments your state by a unit step.

Say you wanted to probe the structure of a dynamical system X (with endomap α). Since cycles turn out to be a positive property, maps from an n-member cycle of elements C_n to X end up pointing only to those elements in X which possess *period*=n ($\alpha^n(x) = x$). But this is obviously a very special case, and we want to probe all of X, not just the elements which possess periodicity. In general, we cannot use a "probing set" that possesses positive properties because we will end up probing only the structure in X that is similar to the positive properties. An example is as below:



Here you can't make a map where an element of C_4 points to w.

A natural example of a set with no positive properties, and one that we might use for probing is

$$N = \mathbb{N}^{\circlearrowleft\sigma} = \boxed{0 \to 1 \to 2 \to 3 \to \dots}$$

Any map from N to X, gives us all the structure in X. Two maps that agree for at least one input-output pair must agree for all input-output pairs. If you want to check for an n-member cycle in X, use C_n instead of N and see whether you can form at least one map.

Now from the last example of a monoid, we can interpret an endomap n in N to be equivalent to the "button" α in X pushed n times. So this map in a monoid n is a measure of time in a discrete dynamical system. Using real numbers ℝ instead of natural numbers, you can describe a continuous dynamical system.