Lawvere Notes

Category

Recipe of a Category:

Let C be a category

- Objects are $Ob(C)$
- Morphisms are $Hom(C)$
- Composition exists for morphisms

Need to follow two laws:

- Associativity in composition
- Identity morphisms must exist

Finsets: Category with objects: finite sets, morhpisms: functions

Identity Law:

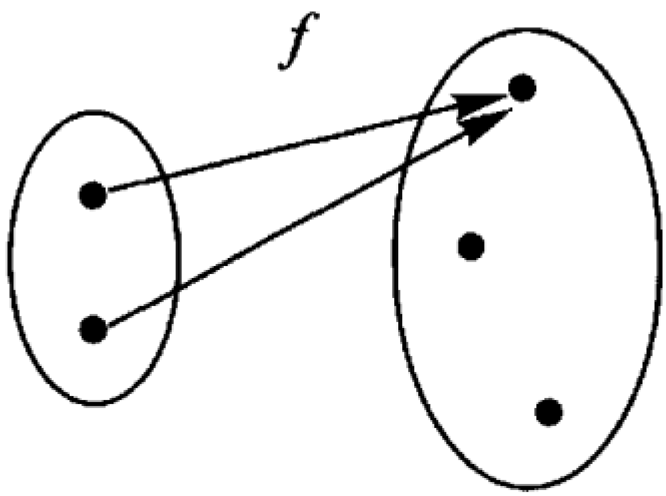
$$f : A \to B \implies f \circ 1_A = f \text{ and } 1_B \circ f = f$$

Associative Law:

$$h \circ (f \circ g) = (h \circ f) \circ g$$

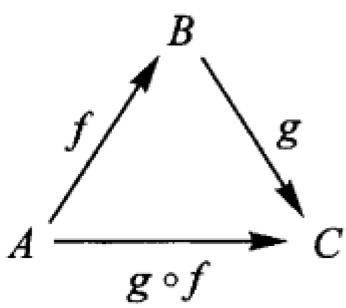
Diagrams

Internal:



You get to see the meat of what the map does to the internal components of the object; the focus is on the specific effect that a map has

External:



You don't need to know what goes on inside; you focus on what the maps do and how they are related

Some terms

1. **Point**: For a set X, a point is the map f where $f : 1 \to X$ (1 is a set having only 1 element)
2. **Isomorphism:** An invertible map (iso = same, morph = form) (has a unique inverse)

   $$f : A \to B \text{ is isomorphic if } \exists g : B \to A \text{ s.t. } f \circ g = 1_B \text{ and } g \circ f = 1_A$$

   Here A and B are isomorphic
   Reflexive: $A \cong A$
   Symmetric: If $A \cong B$, then $B \cong A$
   Transitive: $A \cong B$ and $B \cong C$ then $A \cong C$
3. **Automorphism**: An isomorphism whose domain is the same as the co-domain

4. **Retraction:** For a map $f : A \to B$ if there exists a map $g : B \to A$ such that $g \circ f = 1_A$ then g is a retraction for f

   Such an f is called a ***monomorphism*** (mono = one) (one-one mapping) (injective) which leads to a property in f

   $$f \circ x_1 = f \circ x_2 \implies x_1 = x_2 \forall x_1, x_2 : T \to A$$

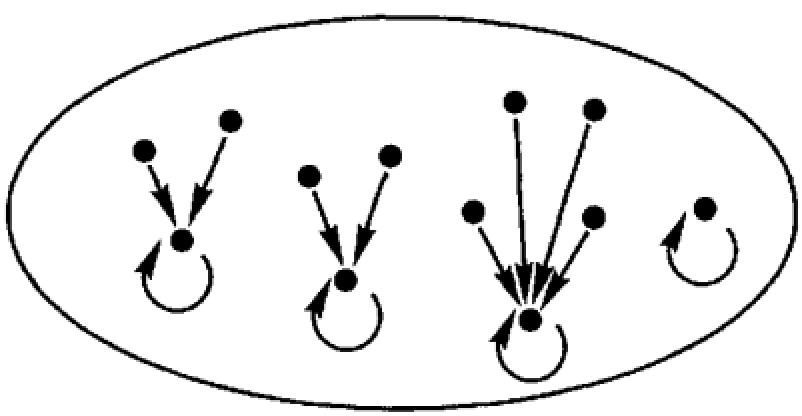
   Pretty intuitive if you ask me, whenever there is a one-one mapping from A to B going back from B to A shouldn't be a problem, just follow the arrows back and for whichever item in B is not mapped to you can arbitrarily chose how to map it back without stopping the retract from existing. If multiple items in A map to the same item in B then you do not need the inputs to be necessarily equal.
5. **Section:** For a map $f : A \to B$ if there exists a map $g : B \to A$ such that $f \circ g = 1_B$ then g is a section for f

   Such an f is called an ***epimorhpism*** (epi = onto, after) (the map's coming after all of the items in B leaving no one alone I suppose) (surjective) which leads to a property in f

   $$x_1 \circ f = x_2 \circ f \implies x_1 = x_2 \forall x_1, x_2 : B \to T$$

   Pretty intuitive if you ask me, whenever there is a onto mapping from A to B then post-compositions are equal iff the maps composing are equal themselves because every element from B would be mapped further. Composition doesn't care about the intermediate stops you make in the flow of mapping.
6. **Idempotent Map:** A map $e : X \to X$ such that $e \circ e = e$

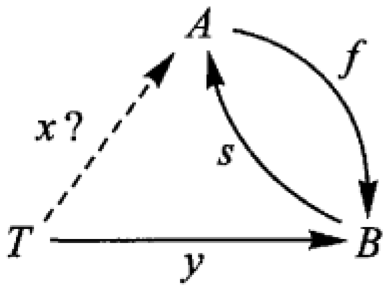
   Often an idempotent is of the form $r \circ s$ where $s : X \to A, r : A \to X$ s.t. $s \circ r = 1_A$

   An idempotent works because of the existence of fixed points; and whenever split and idempotent, it's as good as setting A to be the object containing the fixed points.

   The only idempotent with an inverse is the identity map
7. Involution: A map $f : A \to A$ s.t. $f \circ f = 1_A$

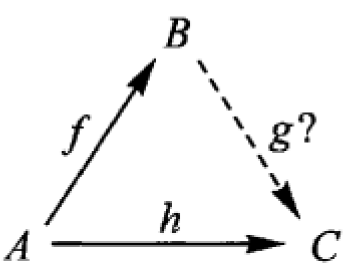
Some results

1. If a map $f : A \to B$ has a section, then $\exists s \text{ s.t. } f \circ s = 1_B$. Now pre-compose with y to get $f \circ s \circ y = y \implies f \circ x = y$
   There exists such an $x : T \to A$ for any map $y : T \to B$
   

2. If a map $f : A \to B$ has a retraction, then $\exists r \text{ s.t. } r \circ f = 1_A$. Now post-compose with y to get $y \circ r \circ f = y \implies x \circ f = y$
   There exists such an $x : B \to T$ for any map $y : A \to T$
3. "The order of going back is always in reverse" - words to live by?
   Sections and retracts show transitivity whenever the objects involved possess sections and retracts.

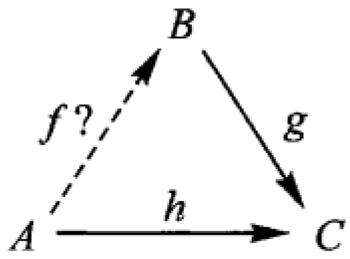
Determination and Choice Problems

Determination (Falling)



- g is uniquely determined by f where $g \circ f = h$
- The retraction problem is a determination problem in g as the retraction where $h = 1_A$ and $C = A$

Choice (Lifting)



- f is a choice for which $g \circ f = h$
- The retraction problem is a determination problem in g as the retraction where $h = 1_A$ and $C = A$

Misc Fun Stuff

Pick's formula: Area = #(Interior Points) + #(Boundary Points)/2 - 1

Sorting, Stacking, Combining - What maps are often used for if you think about it and determining / choosing boils down to going back

"the main use of 'describing the smaller in terms of the larger' occurs in other categories. It often happens that even though B is bigger, it is 'structurally simpler' than A" (splitting of idempotents, Boolean algebra 2^n states (B) and an n-length binary string (A))

Brouwer's Theorems

Fixed point theorem

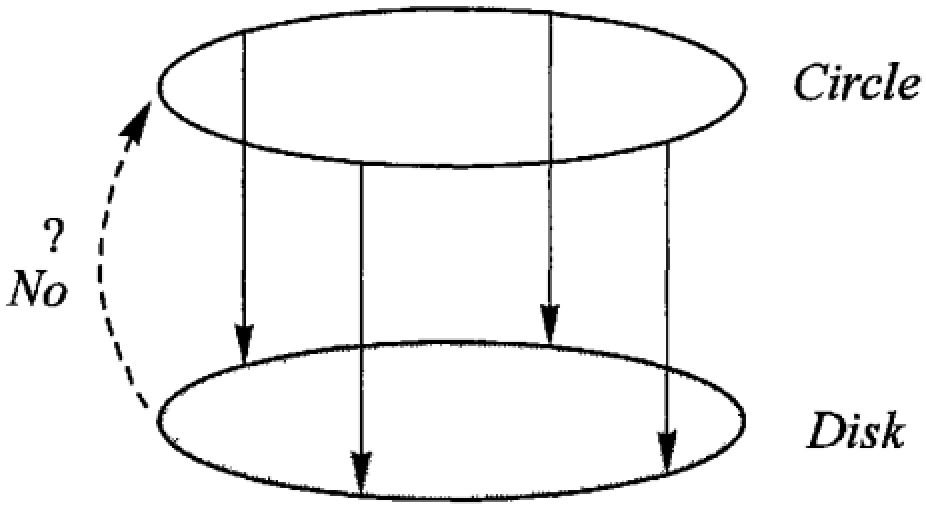
For a continuous endomap $f : D^n \to D^n$ (where D^n is the closed n-dimensional disc, closed i.e. boundary is also included), **there always exists at least one fixed point**

1. 1D Fixed point theorem: Line segment, cars moving
2. 2D Fixed point theorem: A rotating disc with centre fixed point, throw crumbled circle map
3. and so on ...

Retraction theorem

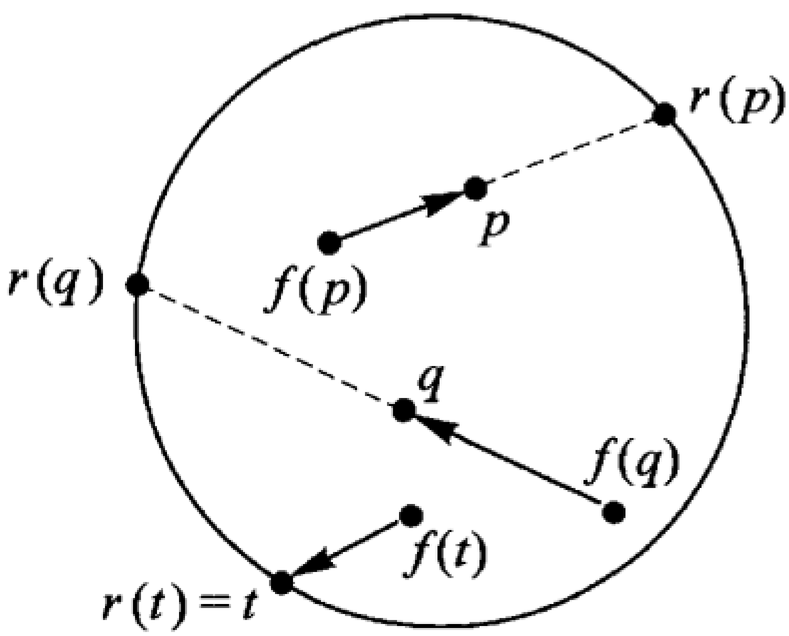
For an inclusion map $i : C^n \to D^n$ (where C^n is the n-dimensional circumference of D^n, the closed n-dimensional disc) (inclusion map places the boundary C on the boundary of D itself)

there exists NO continuous retraction



1. 1D Retraction theorem: Ends of lineseg
2. 2D Retraction theorem: Circle to disc
3. and so on...

Theorem: If there is no continuous retraction of the disk to its boundary then every continuous map from the disk to itself has a fixed point and vice versa.



Proof by contraspositive *(if A then B = if not B then not A)*: assume no fixed-points, apply projection procedure and show the existence of a continuous retract (projecting is continuous since no disruption for nearby points)

Thinking Categorically

We need: Arrows A, a Volume B, its Boundary S and continuous maps across them
Let these things constitute a category $\mathscr{C}$, and we add necessary axioms as properties to the category.
Extract only the useful structure that you want and leave out everything else.
We'll have j be the inclusion map, h be the map from the arrow to where the arrow points (head), p be the map projecting of the arrow to the boundary (projection)

How do we prove it?

Axiom 1: If T is an object in $\mathscr{C}$, and $ha = js$ then $pa = s$

$$\begin{tikzcd} T \arrow[r, "a"] \arrow[d, "s"'] & A \arrow[d, "h"] \arrow[dl, "p"'] \\ S \arrow[r, "j"'] & B \end{tikzcd}$$

(Think of T as being a parameterisation for the arrows)
This axiom brings the property that if an arrow has its head on the boundary then its head is the projection itself.

Axiom 2: If T is any object in $\mathscr{C}$ and $f, g : T \to B$ are any maps, then either there is a point $t : 1 \to T$ with $ft = gt$ or there is a map $a : T \to A$ with $ha = g$

$$\begin{tikzcd} T \arrow[r, "a"] \arrow[d, "s"'] & A \arrow[d, "h"] \arrow[dl, "p"'] \\ S \arrow[r, "j"'] & B \end{tikzcd}$$

When two points in B are "different" there exists an arrow between the two points. This property is being captured by axiom 2. Here a is the arrow from ft to gt

Theorem 1: If $\alpha : B \to A$ satisfies $h\alpha j = j$ then $p\alpha$ is a retraction for j (just put $s = 1_S$ and $T = S$ and belt axiom 1)

Theorem 2: If we have maps $f, g : B \to B$ and $gj = j$ then there is a point $b : 1 \to B$ such that $fb = gb$, or there is a retraction for $j : S \to B$ (belt axiom 2 with $T = B$ and here $ha = g \implies haj = gj = j, so\$p\alpha$ is the retraction)

Setting $g = 1_B$, we get: If $f : B \to B$ then either there is a fixed point or there is a retraction

Tada! We have a proof to Brouwer's theorem

Examples of Categories

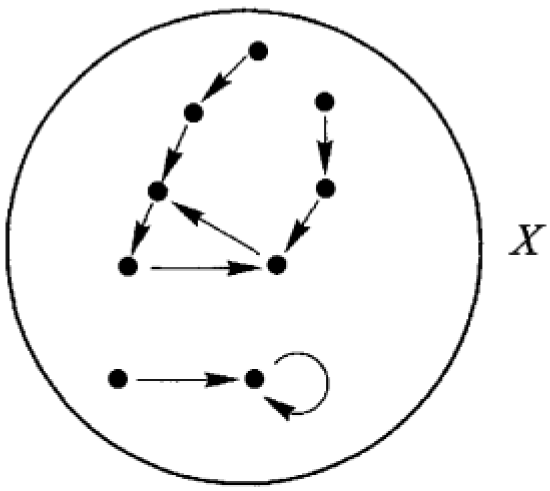
Categories of Endomaps of Sets ($\mathcal{S}^{\circlearrowleft}$)

X is a set equipped with an endomap α
Here $X^{\circlearrowleft\alpha}$, $Y^{\circlearrowleft\beta}$ are two objects of this category and f is a map between objects in this category

$$X^{\circlearrowleft\alpha} \to f \to Y^{\circlearrowleft\beta}$$

We make it so that f satisfies:

1. $X \xrightarrow{f} Y$
2. $f \circ \alpha = \beta \circ f$
   which preserves certain structures such as the ones given below:

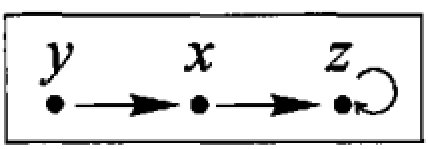


Isomorphisms in $\mathcal{S}^{\circlearrowleft}$ are powerful: imply equal number of points in the two linked sets, equal number of fixed points in the two linked endomaps, equal number and nature of cycles, etc

Naturally, finds applications in dynamical systems and state machines (automata)

Accessibility

Can $\alpha(x) = x'$ for some states x, x'?

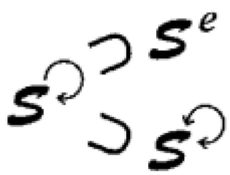


Here except y all are accessible

Convergence to Equilibrium

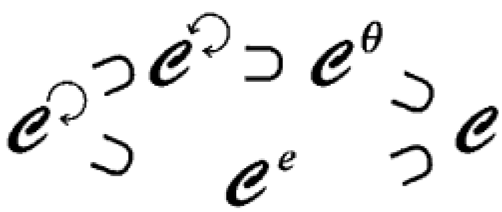
Can $\alpha^{n+1}(x) = \alpha^n(x)$?

Subcategories of $\mathcal{S}^{\circlearrowleft}$



e for idempotents -> subcategory of idempotent endoaps
doublearrow for isomorphims -> subcategory of idempotent isomorphisms

Categories of Endomaps ($\mathscr{C}^{\circlearrowleft}$)

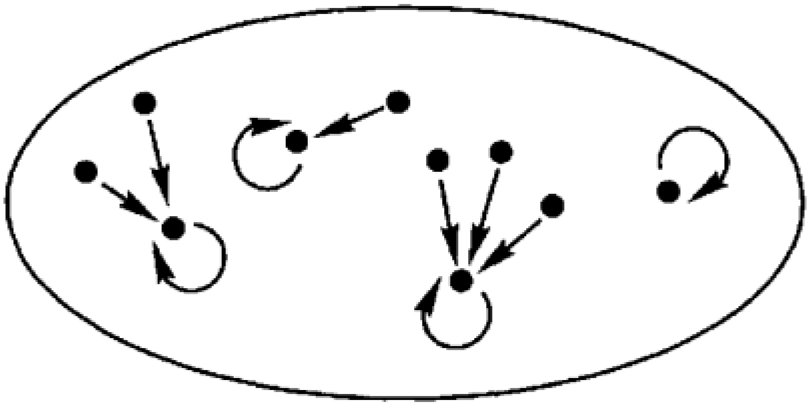


endomaps is the super of:

- automorphisms > involutions > identities
- idempotents > identities

Idempotents

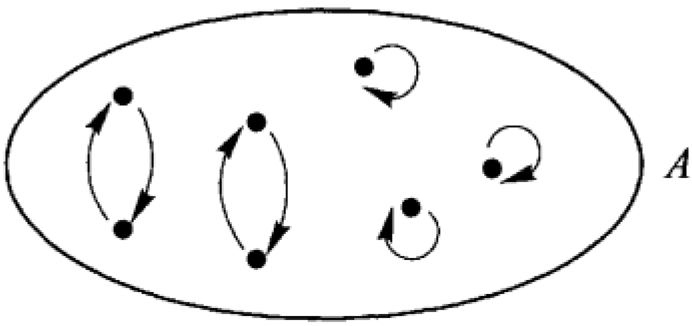
- An idempotent with a retract can only be the identity map
- The only idempotent automorphism is the identity map



-
- Every point is a fixed point or reaches a fixed point in one step

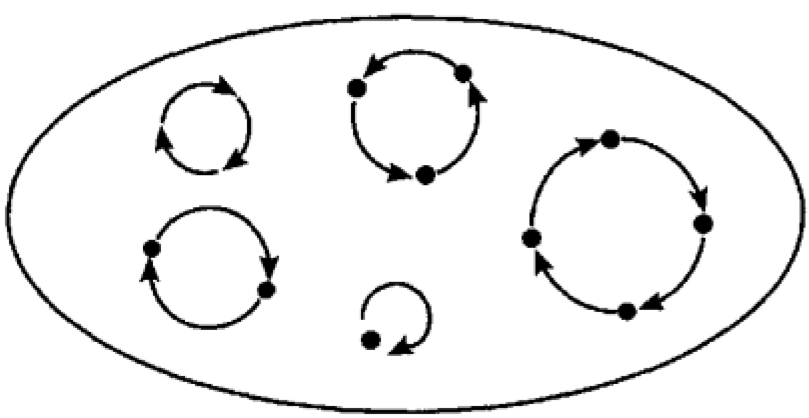
Involutions

When $\mathscr{C} = \mathcal{S}$, involutions are maps with 2-cycles and fixed points only



The existence of involutions helps in formalizing parity in counting, even size if no fixed points, odd size if only one fixed point (slight error made by lawvere is stating iff)

Automorphisms

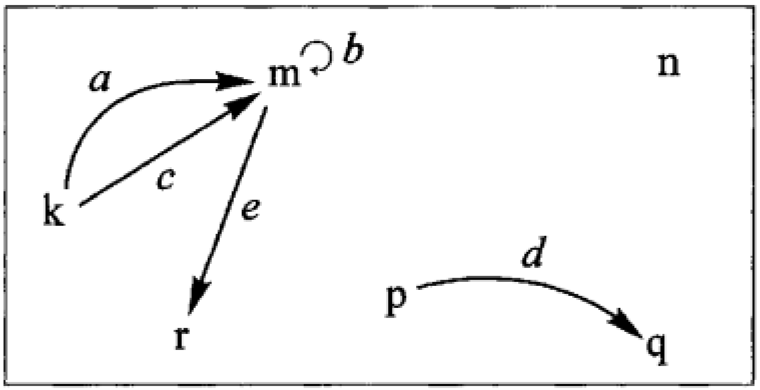


-
    - Cycles of any length, but no branches

Category of Irreflexive Directed Multi Graphs ($\mathcal{S}^{\downarrow\downarrow}$)

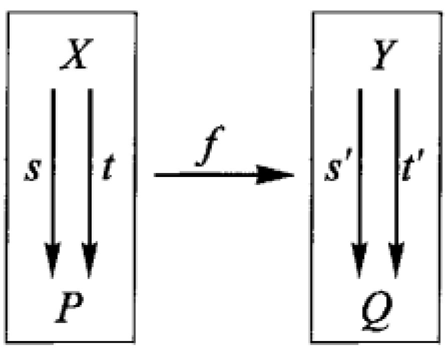
An object of this category is any pair of sets with a parallel pair of maps s, t
$X \xrightarrow{s} P$, $X \xrightarrow{t} P$
Here, X is the set of arrows, P is the set of dots of the graph and s is the source map, t is the target map

$$X = \{a, b, c, d\}, P = \{k, m, n, p, q, r\}$$



Here $s(c) = k, t(c) = m$ and so on
Note that every arrow needs to be mapped to a dot, but every arrow does not need to be mapped to an arrow; Domain, Co-domain of s, t

Maps in this category preserve structure:



f which are maps in the category $\mathcal{S}^{\downarrow\downarrow}$ consist of a pair of $S-$maps $X \xrightarrow{f_A} Y$, $P \xrightarrow{f_D} Q$ which respect $f_D t = t' f_A$, $f_D s = s' f_A$

"f preserves the source and target relations"

Isomorphisms preserve total number of arrows and dots, number of loops and the number of components (a connected subgraph)

$\mathcal{S}^{\circlearrowright}$ is a subcategory of $\mathcal{S}^{\downarrow\downarrow}$

$\mathcal{S}^{\circlearrowright} \xrightarrow{I} \mathcal{S}^{\downarrow\downarrow}$, I is the inclusion map across the categories

Here the source is x and the target is $\alpha(x)$ with $X = P$. It is a special graph whose number of arrows is the same as the number of dots.

"The internal picture of an endomap is a special case of the internal picture of a graph"

$$\begin{tikzcd} X \arrow[d, shift right, "1_X"'] \arrow[d, shift left, "\alpha"] \\ X \end{tikzcd}$$

Hence the conditions for an object being a part of the category of graphs is satisfied (source-target existence)

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, shift right, "1_X"'] \arrow[d, shift left, "\alpha"] & Y \arrow[d, shift right, "1_Y"'] \arrow[d, shift left, "\beta"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

Hence the condition for a map being a part of the category of graphs is satisfied (structure preserving maps)

$$I(g \circ f) = I(g) \circ I(f)$$

can be shown by adding one more map to the above diagram

Fullness

A $\mathcal{S}^{\downarrow\downarrow}-$map itself is an inclusion of a $\mathcal{S}^{\circlearrowright}-$map

Category of Maps of Sets ($\mathcal{S}^{\downarrow}$)

It is a subcategory of $\mathcal{S}^{\downarrow\downarrow}$ whose:

- object is 'a single map between two sets'

- map is a 'commutative square of maps'
  This does not satisfy fullness w.r.t endomaps

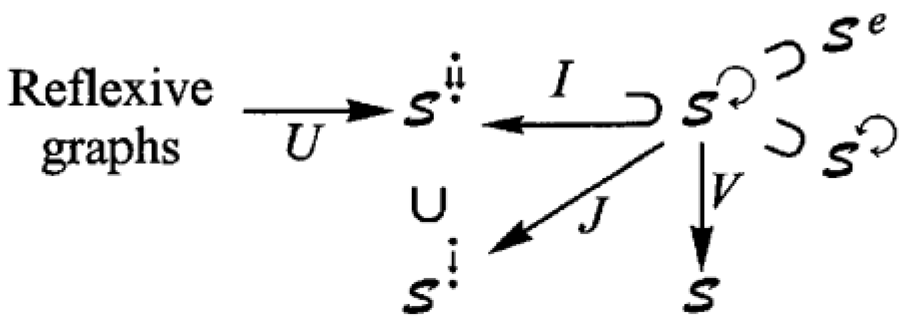
The structure preserved by an $\mathcal{S}^{\downarrow}-$map is looser than of an $\mathcal{S}^{\circlearrowright}-$map

Category of Reflexive Graphs

An object of this category is any pair of sets with a parallel pair of maps s, t
$X \xrightarrow{s} P$, $X \xrightarrow{t} P$
and a map i which acts as a section for both s, t

A map would be the same as those of irreflexive graphs, the only addition being an added map across the sections

Summary of these Examples



Horseshoe = Subcategory
I, J = Insertion
U, V = Forgetful functors

(What's up with Exercise 16, cohesion and tearing of sets)

Retractions and Injectivity

In the category of sets, the existence of a retraction is equivalent to a map being injective

The double implication does not hold in other categories, only the one-way implication that is, existence of retraction implies injectivity hold true, injectivity may not imply existence of a retraction (example Brouwer's retraction theorems)

More on the Category of Endomaps ($\mathcal{S}^{\circlearrowright}$)

Isomorphisms

Useful prop: Say we have $X^{\circlearrowleft\alpha} \xrightarrow{f} Y^{\circlearrowleft\beta}$ in $\mathcal{S}^{\circlearrowright}$, then if f has an inverse in $\mathcal{S}$ then the same inverse is also a valid map and works in $\mathcal{S}^{\circlearrowright}$ as well

New Terms

Monoids

A category with exactly one object (*) is a monoid

Functors

A "structure preserving" map from one category to another

Let $\mathcal{M}$ be a monoid whose object is $\mathbb{N}$ and maps f_n are multiplication by n that is $f_n(x) = nx$
Here $f_m \circ f_n = f_m n$
The identity is f_1

Let $\mathcal{N}$ be a monoid whose whose object is $\mathbb{N}$ and maps f_n are addition by n that is $f_n(x) = n + x$
Here $f_m \circ f_n = f_{m+n}$
The identity is f_0

One can construct a functor from any monoid to sets

A set with an endomap can be constructor from $\mathcal{N}$ using a functor h

$$h(*) = X, h(n) = \alpha^n, h(0) = 1_X$$

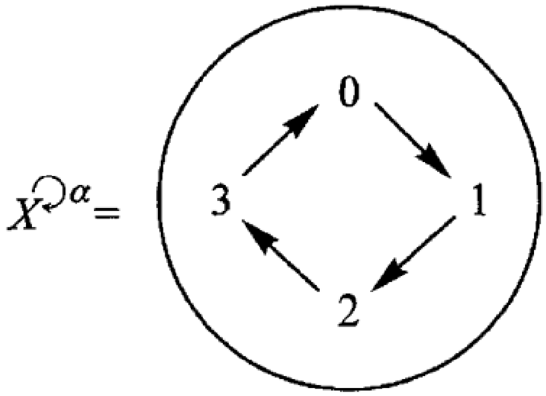
Maps preserve positive Properties

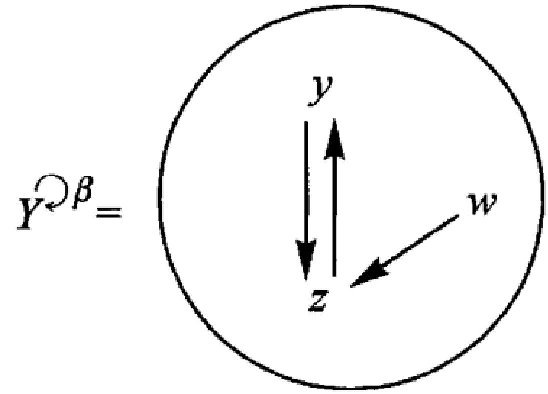
Accessibility is a (positive property?) and is always **preserved** by maps in $\mathcal{S}^{\circlearrowright}$
Non-existence of fixed points is a (negative property?) and can be **reflected** by maps in $\mathcal{S}^{\circlearrowright}$

Positive usually preserved, Negative usually reflected

How are positive, negative properties determined through??

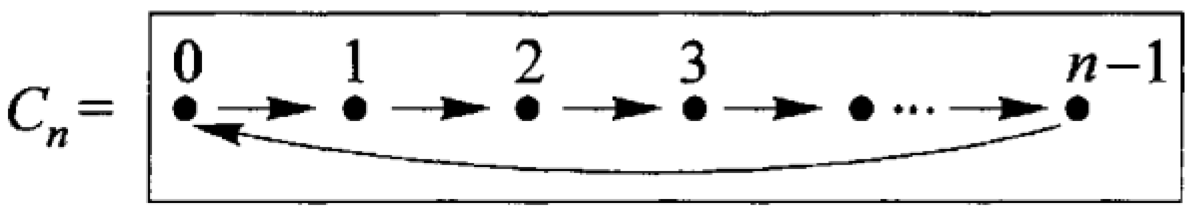
Objectification of Properties





f maps from one endomap to another

Naming elements with a given period



If an element has period n then it also has a period of kn where k is an integer
Each map $C_n \to Y$ names the element $f(0)$ in Y and labels all elements with period n in Y
C_1 corresponds to fixed points

Naming arbitrary elements

Successor map: $\sigma : \mathbb{N} \to \mathbb{N}, \sigma(n) = n + 1$. The element 0 in $\mathbb{N}$ has no positive property, it contains an "arbitrary" element 0

For every element $y \in Y$, there exists a map $\mathbb{N}^{\circlearrowright\sigma} \xrightarrow{f} Y^{\circlearrowright\beta}$, such that $f(0) = y$ defined by $f(1) = f(\sigma(0)) = \beta(y)$ and in general $f(n) = \beta^n(y)$

"Properties about dynamical systems can be probed using simple objects like C_4 and $\mathbb{N}^{\circlearrowright\sigma}$"

If $f : \mathbb{N}^{\circlearrowright\sigma} \to Y^{\circlearrowright\beta}$ corresponds to y,then $f \circ \sigma : \mathbb{N}^{\circlearrowright\sigma} \to Y^{\circlearrowright\beta}$ corresponds to βy

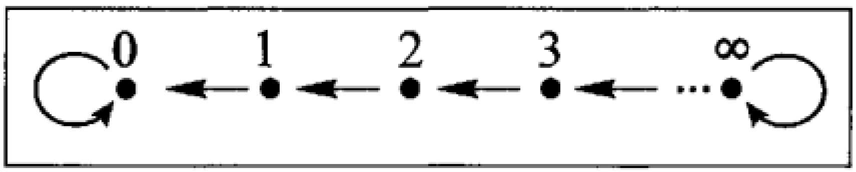
$$\frac{\textit{states of } S \ (= \textit{elements } y_0 \textit{ of } Y)}{\textit{maps of dynamical systems } N \xrightarrow{y} S}$$

An entire dynamical system S can be represented using $N \xrightarrow{\sigma} N \xrightarrow{y} S$ where $N = \mathbb{N}^{\circlearrowright\sigma}$ and $S = Y^{\circlearrowright\beta}$ and $y(n) = \beta^n(y_0)$
A more shorthand notation is $\sigma_S(y) = y \circ \sigma$

"Probing an object using more simple standard objects"

Ω is a useful object to test stability

$\Omega =$



Presentation of Dynamical Systems

1. Label some start points - **generators** - hairs of cycles if they exist, otherwise one point on the cycle
2. Pick a generator and go to the next state until you find some relation like - $\alpha^5(x) = \alpha^3(x)$ and label the newly crossed states as αx, $\alpha^2 x$, $\alpha^3 x$, $\alpha^4 x$
3. Repeat this for all generators
4. You will obtain: a set of labels for each element (**L**), a set of equations identifying α (**R**)(like $\alpha^5(x) = \alpha^3(x)$)
5. In finding a map from one system to another, the set of equations of the endomap should be preserved (for example $f(x) = x'$ and $\alpha^2(x) = x$ then $\alpha^2(x') = x'$ is must)

A family of labels for the elements of the set X (states) and a family of equations which are satisfied by the endomap α (operation) of the dynamical system - **Presentation** of $X^{\circlearrowright\alpha}$