Monoidal pre-orders and Enrichment

Goal: methods of transforming one set of resources into another. (deriving new from old)

$\mathcal{V}$-category is a set of objects, which one may think of as points on map and $\mathcal{V}$ "structures the question" of getting from a to b.

Eg: Bool: $a \to b$ has a true / false answer

Cost - $a \to b$ has an answer $d \in [0, \infty)$

Defn: (Need: Combining preorder with combining elements)

- A symmetric monoidal ~~order~~ structure on a preorder $(X, \leq)$ consists of

(i) an element $I \in X$ called monoidal unit.

(ii) a fn $\otimes : X \times X \to X$ called monoidal product.

satisfying: $(\otimes(x_1, x_2) = x_1 \otimes x_2)$

Monotonicity a) $\forall\, x_1, x_2, y_1, y_2 \in X$, if $x_1 \leq y_1$ and $x_2 \leq y_2$

$$x_1 \otimes x_2 \leq y_1 \otimes y_2$$

unitality b) $\forall\, x \in X$, $I \otimes x = x$ and $x \otimes I = x$

associativity c) $\forall\, x, y, z \in X$ $(x \otimes y) \otimes z = x \otimes (y \otimes z)$

symmetry d) $\forall\, x, y \in X$ $x \otimes y = y \otimes x$

Eg: $(\mathbb{R}, \leq, +, 0)$ is a symmetric monoidal preorder

Eg: $\frac{1}{2} \leq 2$ Monotonicity fails

2.6) A monoid consists of set M, a fn $*: M\times M \to M$ called the monoid multiplication, and an element $e \in M$ called the monoid unit, $*(m,n) = m*n$ s.t

$m*e = m \quad e*m = m \quad (m*n)*p = m*(n*p)$

and if $(m*n = n*m) \Rightarrow$ commutative monoid

Every set S determines a discrete preorder. $Disc_S$

(Note this is not commutative)

Ex: If $(M, *, e)$ is commutative monoid. $(Disc_M, =, *, e)$ is symmetric. (H.P)

2.9) Poker game.

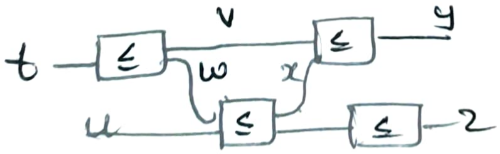
Introducing wiring diagram.

→ Building new relationships from old.

→ In a preorder without monoidal structure, only relationship is $\leq$.

We denote $x \leq y$ by 

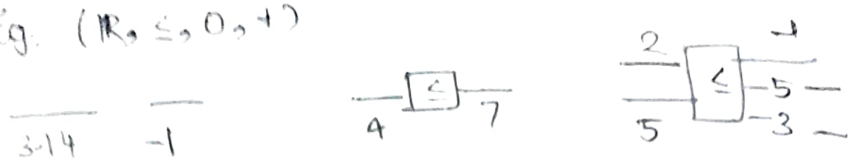
Symmetric monoidal structure allows us to combine parallely -



We call wires as elements and boxes represent relationship

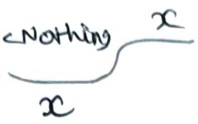
boxes + wires ⇒ icons

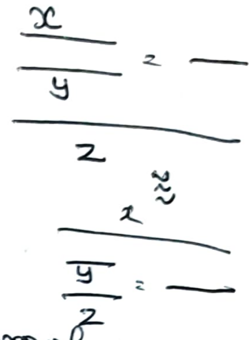
eg. $(\mathbb{R}, \leq, 0, +)$



Transitivity 

Monoid: 

Identity. 

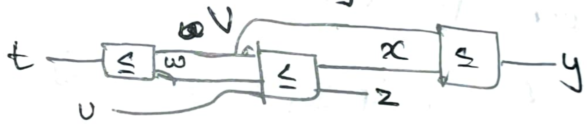
Associativity: 

Commutative: 

→ This can be used for graphical proof

$t \leq v+w \qquad w+u \leq x+z \qquad v+x \leq y$

⇓



2.20) Applied examples:

Resource theory: How resources are exchanged in a given arena.

Static notion: decides what buys/gives what once and for all.

$2H_2O + 2Na \to 2NaOH + H_2(g)$

We consider it as single symmetric monoidal preorder $\boxed{(Mat, \to, 0, +)}$

Mat: set of all collections of atoms and molc.

→ (order relation)

2.21) a) $Na \to NaOH$ $\quad x \to y \quad z \to w$

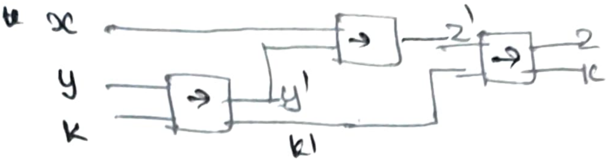
$x + z \to y + w$

b) Unitality: 0 (No material) c) Associativity

d) Symmetry ☺

Including catalysis:

$y + k \to y' + k' \quad x + y' \to z' \quad z' + k' \to z + k$



"You can trash anything you want, and it disappears from the view"

Adding this will yield $x \leq I \ \forall x \in X$ (Discard axiom)

$H_2O + NaCl \to H_2O$

Informatics: (Information can be copied)

$-2 \leq 0$
$-2 \leq \cancel{-2} \leq 2$
$-2 \leq$

In wiring diagram,

copying is given by 

Copy axiom (f) $x \leq x + x$

Abstract examples:

Booleans: $\mathbb{B} = \{T, F\}$ with false $\leq$ true

Two symmetric monoidal structures $(\wedge, *)$ operations.

$\wedge$	F	T
F	F	F
T	F	T

$\cong$

$*$	0	1
0	0	0
1	0	1

Bool := $(\underset{\text{set}}{\mathbb{B}}, \underset{\text{preorder}}{\leq}, \underset{e}{\text{true}}, \wedge)$...

→ It will be highly useful in enrichment.

Enriching in monoidal preorder means, "letting V structure the qn of getting a to b.

Eg. Look at $Bool = (\mathbb{B}, \leq, T, \wedge)$. The set $\mathbb{B}$ will translate into saying that "getting from a to b is a T/F qn"

Another monoidal preorder is $(\overset{\vee}{\mathbb{B}}) := (\mathbb{B}, \leq, F, \vee)$

$\vee$	F	T
F	F	T
T	T	T

Ex: Monoidal structures on preorder $(\mathbb{N}, \leq)$

① $(\mathbb{N}, \leq, 0, +)$ $(\mathbb{N}, \leq, 1, *)$

divisibility order on $\mathbb{N}$ $m|n \Rightarrow$ m divides n

$1|m \; \forall m.$ $(1 \leq 2 \leq 4 \leq 16 \ldots)$

and let the product is $*$

$\therefore$ (i) $x_1|y_1, x_2|y_2 \Rightarrow (x_1 * x_2)|(y_1 * y_2)$ ✓

(ii) $1|y$.

2.35) $P(S)$ is the power set. $A \leq B$ if $A \subseteq B$

And monoidal product is $\wedge$.

(i) $A \leq B$ $C \subseteq D$ $A \cap C \subseteq B \cap D$ ✓

(ii) $A \cap \overset{S}{\phi} = A$ $S \cap A = A$ (iii)

2.36) $Prop^{\mathbb{N}}$ set of all mathematical stat.

"n is prime" $\in Prop^{\mathbb{N}}$ $P \leq Q$ if $\forall n \in \mathbb{N}$ $P \Rightarrow Q$

Monoidal unit $= T$

	F	T
F	T	T
T	F	T

Monoidal preorder cost:

Eg: Lawvere's monoidal preorder, cost.

→ Consider the preorder $([0,\infty), \geq)$ m.U: 0   m.p: +

$$Cost ::= ([0,\infty), \geq, 0, +)$$

opposite of monoidal preorder: $(X, \leq)^{op} ::= (X, \geq)$

Prop: If $(X, \leq)$ is symmetric monoidal preorder then so is its opp.

2.39) $e \otimes x = x$ - ..

2.40)

Monoidal monotone maps:

Let $\mathcal{P} = (P, \leq_P, I_P, \otimes_P)$ and $\mathcal{Q} = (Q, \leq_Q, I_Q, \otimes_Q)$ be monoidal preorders. A monoidal monotone $f: (P, \leq_P) \to (Q, \leq_Q)$ satisfies

(a) $I_Q \leq_Q f(I_P)$

(b) $f(P_1) \otimes_Q f(P_2) \leq_Q f(P_1 \otimes_P P_2)$

$\forall P_1, P_2 \in P$

It is strengthened by

(a') $I_Q \cong f(I_P)$

(b') $f(P_1) \otimes_Q f(P_2) \cong f(P_1 \otimes_P P_2)$

→ strong monoidal monotone

2.42) $i : (\mathbb{N}, \leq, 0, +) \to (\mathbb{R}, \leq, 0, +)$ where $i(n) = n$ is a monoidal monotone.

$i(1+1) \quad i(3+2) \leq i(5)$

(i) Monotonic ✓ (ii) Even strict monoidal

However $f : (\mathbb{R}, \leq, 0, +) \to (\mathbb{N}, \leq, 0, +)$ is floor fn.

It is just monoidal monotone.

→ Between Bool and Cost, ∃ a monoidal monotone.

$g : \text{Bool} \to \text{Cost} \quad g(F) := \infty \quad g(T) := 0$

$\geq$ is the operator.

Ex: (i) Monotonic ✓ $g(F) \geq g(T)$

(ii) $0 \geq 0$ (iii) $(F,F) : \infty \geq \infty$

$(T,F) : \infty \geq \infty$

$(F,T) : \infty \geq \infty$

$(T,T) : 0 \geq 0$

2.44) $d, U : [0, \infty] \to \mathbb{B}$

$d(x) := \begin{cases} F & \text{if } x > 0 \\ T & \text{if } x = 0 \end{cases} \qquad U(x) = \begin{cases} F & \text{if } x = \infty \\ T & \text{if } x < \infty \end{cases}$

$0 \leq \infty \quad d(0) \leq d(\infty)$

1) Monotonic : $d(0) \geq d(\infty)$ Strict monotonic.

$U(0) \geq U(\infty)$ Strict monotonic

(2)

(a) $d(x_1) \wedge d(x_2) \leq d(x_1 + x_2)$

2.45)

1) $(\mathbb{N}, \leq, 1, *)$ Yes monoidal preorder

2) $(\mathbb{N}, \leq, 0, +) \to (\mathbb{N}, \leq, 1, *)$ $f(n) = 1$

fn $f(m) * f(n) \leq f(m * n)$

$1 \leq 1$ (strict)

3) $(\mathbb{Z}, \leq, *, 1)$ not a preorder

1 $-1 \leq 1$ $2 \nleq 1$

$-2 \leq 1$

(3) Enrichment:

Introduce $\mathcal{V}$ categories: Symmetric monoidal preorder.

$\mathcal{V}$-categories:

$\mathcal{V} = (V, \leq, I, \otimes)$ be a sym. monoidal preorder. A $\mathcal{V}$-category $\mathcal{X}$ will have 2 constituents.

(i) $Ob(\mathcal{X}) \rightarrow$ objects.

(ii) $\forall$ 2 objects x, y $\mathcal{X}(x,y) \in V$ called the hom-object.

Two props: a) $\forall x \in Ob(\mathcal{X})$, $I \leq \mathcal{X}(x,x)$

b) $\forall x, y, z \in Ob(\mathcal{X})$, $\mathcal{X}(x,y) \otimes \mathcal{X}(y,z) \leq \mathcal{X}(x,z)$

$\mathcal{V}$ the base of enrichment for $\mathcal{X}$ or $\mathcal{X}$ is enriched in $\mathcal{V}$.

* Every pre-order can be constructed into a Bool category.

Eg:

$$\begin{tikzcd} & t & \\ & s \arrow[u] & \\ q \arrow[ur] & & r \arrow[ul] \\ & p \arrow[ul] \arrow[ur] & \end{tikzcd}$$

$Ob(\mathcal{X}) = \{p, q, r, s, t\}$

say $\mathcal{X}(x,y) = T$ iff $x \leq y$

	p	q	r	s	t
p	T	T	T	T	T
q	F	T	F	T	T
r	F	F	T	T	T
s	F	F	F	T	T
t	F	F	F	F	T

$\mathcal{X}(x,x) = T$ as in Bool $(\mathbb{B}, \leq, T, \wedge)$

$T \leq \underbrace{\mathcal{X}(x,x)}_{T}$

Preorders as bool categories:

* There is a one-to-one correspondence between preorders and Bool-categories.

Proof: $\mathbb{B} = \{F, T\}$ A set $Ob(\mathcal{X})$ and $\forall x, y \in Ob(\mathcal{X})$ an element $\mathcal{X}(x,y) \in \mathbb{B}$ (T, F)

Preorder: $(X, \leq)$ Let $X := Ob(\mathcal{X})$ and $x \leq y$ if $\mathcal{X}(x,y) = T$

- Cdt of preorder: Reflexivity, transitivity.

By defn of bool category: $I \leq \mathcal{X}(x,x)$

and $x, y, z \in X$ $\mathcal{X}(x,y) \otimes \mathcal{X}(y,z) \leq \mathcal{X}(x,z)$

$\therefore T \leq \mathcal{X}(x,x) \Rightarrow \mathcal{X}(x,x) = T \quad \therefore x \leq x$ ✓

$\mathcal{X}(x,y) \wedge \mathcal{X}(y,z) \leq \mathcal{X}(x,z)$

If $x \leq y$ and $y \leq z \Rightarrow T \wedge T \leq \mathcal{X}(x,z) \Rightarrow \mathcal{X}(x,z) = T$

$\Rightarrow x \leq z$ H.P. ☺

2.50)

$(P, \leq)$ is a preorder. Let $Ob(\mathcal{X}) = P$

and $\mathcal{X}(x,y) = T$ if $x \leq y$

To prove: $I \leq \mathcal{X}(x,x)$ As $\mathcal{X}(x,x) = T$

$\Rightarrow T \leq T$

$\mathcal{X}(x,y) \wedge \mathcal{X}(y,z) \leq \mathcal{X}(x,z)$

$x \leq y \Rightarrow T$ $\Rightarrow T \leq \mathcal{X}(x,z)$ As $x \leq z$ (By Associ-)

$y \leq z \Rightarrow T$ $\mathcal{X}(x,z) = T \geq T$

else $x \nleq y \Rightarrow F \quad \therefore F \leq \mathcal{X}(x,z)$ (Always true)

Lawvere metric spaces:

* Metric spaces offer a precise way to describe spaces of pts, as seperated by some distance.

Defn: (X, d) consists of

(i) A set X, elements called pts.

(ii) A fn $d: (X \times X) \to \mathbb{R}_{\geq 0}$ where $d(x,y)$ is called distance.

Props: (i) $\forall x \in X$ $d(x,x) = 0$

(ii) $\forall x, y \in X$ $d(x,y) = 0 \Rightarrow x = y$

(iii) $\forall x, y \in X$ $d(x,y) = d(y,x)$

(iv) $\forall x, y, z \in X$ $d(x,y) + d(y,z) \geq d(x,z)$

* Lawvere metric space is a Cost category. $\in$ ($Ob(\mathcal{X})$ and $x, y \in Ob(\mathcal{X})$
$\mathcal{X}(x,y) \in [0, \infty)$

Define $X := Ob(\mathcal{X})$ and $\forall$ $\mathcal{X}(x,y) = d(x,y)$

- $0 \geq d(x,x)$ $\forall x \in X$ 😊
- $d(x,y) + d(y,z) \geq d(x,z)$

2.54) $d(x,y) := |y - x|$

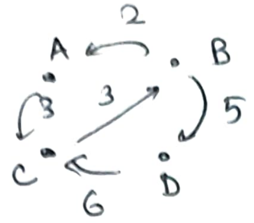
2.55) $(\mathbb{R}_{\geq 0}, \geq, 0, +)$ sym. monoidal preorder ($\approx$ Cost but $[\infty$)

Finite distance

Metric spaces with weighted graphs:

- Weighted graph - edges are labeled with numbers $(w \geq 0)$

Graphs named with $e \in [0, \infty] \rightarrow$ Cost weighted graph.

eg $X :=$ 

$d(?)$	A	B	C	D
A	0	6	3	11
B	2	0	5	5
C	5	3	0	8
D	11	9	6	0

OR

$d(?)$	A	B	C	D
A	0	∞	3	∞
B	2	0	∞	5
C	∞	3	0	∞
D	∞	∞	6	0

V-variations on pre-orders & metric spaces:

2.61) $NMY := \{P, \leq, yes, min\}$ (No, maybe, Yes)

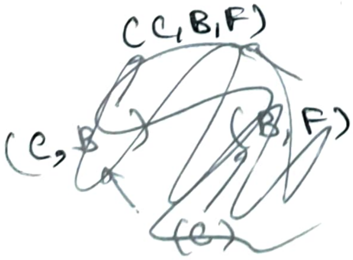
$P := Ob(\mathcal{X})$ $\mathcal{X}(x,y) = \cancel{max(x,y)}$ possibility of getting from x, to y

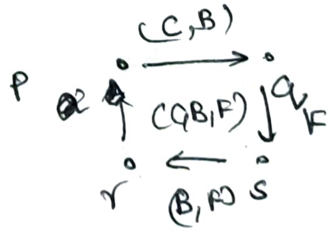
$\mathcal{X}(x,x) \geq yes \Rightarrow \mathcal{X}(x,x) = yes$

$min(\mathcal{X}(x,y), \mathcal{X}(y,z)) \leq \mathcal{X}(x,z)$

2.62) $M := (P(M), \subseteq, M, \cap)$

M category $\mathcal{X}$ tells you all modes that will get you from a all way to b for $a, b \in Ob(\mathcal{X})$





$Ob(\mathcal{X}) := \{p, q, r, s\}$ $\mathcal{X}(x,y) = ?$

$M \subseteq \mathcal{X}(x,x)$ and $\mathcal{X}(x,y) \cap \mathcal{X}(y,z) \subseteq \mathcal{X}(x,z)$

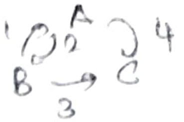
$\mathcal{X}(x,x) = M$

So take all edges

Makes sense

2.63) $\mathcal{W} := (\mathbb{N} \cup \{\infty\}, \leq, \infty, \min)$

1) $1 \to 2 \to 3 \to 4$ - - -



2)

	A	B	C
A	∞	2	4
B	1	∞	3
C	0	0	∞

(Soln wrong?)

Construction of $\mathcal{V}$-categories:

. Any monoidal monotone $\mathcal{V} \to \mathcal{W}$ btw symm. monoidal preorder lets us convert $\mathcal{V}$-categories into $\mathcal{W}$-categories.

Construction: $\mathcal{V}$-category: $\mathcal{C}$ and $\mathcal{W}$-category: $\mathcal{C}_f$ defn of $\mathcal{C}_f$ is

- $Ob(\mathcal{C}_f) = Ob(\mathcal{C})$. $\forall c, d \in Ob(\mathcal{C})$

$$\mathcal{C}_f(c, d) := f(\mathcal{C}(c, d))$$

Proof:

Need: $I_\mathcal{W} \leq \mathcal{C}_f(c, c)$; $\mathcal{C}_f(c, c) = f(\mathcal{C}(c, c))$

From monoidal m. cdt, $I_\mathcal{W} \leq f(I_\mathcal{V})$

Need: $\mathcal{C}_f(c, d) \otimes_\mathcal{W} \mathcal{C}_f(d, e) \leq \mathcal{C}_f(c, e)$

$$f(p) \otimes_\mathcal{W} f(q) \;\cancel{\leq f(\mathcal{C}(c, e))}$$

$\mathcal{C}(c,d) \otimes \mathcal{C}(d,e) \leq \mathcal{C}(c,e)$

$$\begin{array}{c} \rotatebox{90}{$\leq$} \\ f(p \otimes_\mathcal{V} q) \\ \rotatebox{90}{$\leq$} \\ f(\mathcal{C}(c, e)) \\ \mathcal{C}_f(c, e) \end{array}$$

2.65) $f(x) := \begin{cases} T & x = 0 \\ F & x > 0 \end{cases}$ $\quad f: [0, \infty] \to \{T, F\}$

f is monotonic preserves monoidal product, con...

2.67) Cost category: $Ob(\mathcal{X}) = \cancel{\text{Ob}}\ \{a, b, c, d\}$

$\mathcal{X}(a,b) = d(a,b)$

From this, $Ob(\mathcal{X}_c) = \{a, b, c, d\}$: $\mathcal{X}_c(a,b) = f(d(a,b))$

$\therefore$ $\mathcal{X}_c(a,a) = T$ $\quad$ $\mathcal{X}_c(a,b) = F$ if $a \neq b$

Eg: {US, Spain, Boston} $\quad$ $d(U,B) \neq 0$
$d(B,U) = 0$

$$\begin{matrix} U & \\ \uparrow & S \\ B & \end{matrix}$$

2.68): g: Cost $\to$ Bool $\quad$ $g(x) = \begin{cases} T & \text{if } x < \infty \\ F & \text{if } x = \infty \end{cases}$

2) $X_g(a,b) = T$ (mostly) and F if there is no path. ~~Also~~

$f \to$ presence $\quad$ $g \to$ connectedness

Enriched functors

Let $\mathcal{X}$ and $\mathcal{Y}$ be $\mathcal{V}$-categories. A $\mathcal{V}$ functor from $\mathcal{X}$ to $\mathcal{Y}$, $F: \mathcal{X} \to \mathcal{Y}$

(i) a fn $F: Ob(\mathcal{X}) \to Ob(\mathcal{Y})$

(ii) $\forall x_1, x_2 \in Ob(\mathcal{X})$, $\mathcal{X}(x_1, x_2) \leq \mathcal{Y}[F(x_1), F(x_2)]$

Eg: (Remember Preorder $\Leftrightarrow$ Bool category)
$(\mathcal{X}(x_1, x_2) = T$ iff $x_1 \leq x_2)$

$\therefore$ Monotone maps $\Leftrightarrow$ Bool-functors

$\downarrow$ $\quad$ $\downarrow$

$x_1 \leq x_2 \Rightarrow f(x_1) \leq f(x_2)$

$(X, \leq_X) \to (Y, \leq_Y)$ $\quad$ For $x_1, x_2 \in X$
$x_1 \leq x_2$
$F(x_1) \leq_Y F(x_2)$

2.72) Cost-categories. Lipschitz fn. (X, d_X) and (Y, d_Y) $F: X \to Y$ such that $\forall x_1, x_2 \in X$ $d_X(x_1, x_2) \geq d_Y(F(x_1), F(x_2))$

2.73) Opposite of a $\mathcal{V}$-cat is $\mathcal{X}^{op}$

(i) $Ob(\mathcal{X}^{op}) := Ob(\mathcal{X})$

(ii) $\forall x, y \in \mathcal{X}$, $\mathcal{X}^{op}(x, y) := \mathcal{X}(y, x)$

$\to d(x,y) \leq d'(y,x)$

Dagger: If $\exists$ identity fn : $\mathcal{X} \to \mathcal{X}^{op}$

Skeletal: If $I \leq \mathcal{X}(x,y)$ and $I \leq \mathcal{X}(y,x)$ then $x = y$

1] Skeletal dagger cost-cat is extended metric space

If $0 \geq d(x,y)$ and $0 \geq d(y,x)$ then it $x = y$ (MUST)

dagger: $d(x,y) \leq d(y,x)$ and $d(y,x) \leq d(x,y)$

2] Hence proved

Product V-categories:

$\mathcal{V} = (V, \leq, I, \otimes)$ sym. monoidal preorder

X, Y are V-cats. We can define a new cat.

defn: $\mathcal{V}$-category $\mathcal{X} \times \mathcal{Y}$

(i) $Ob(\mathcal{X} \times \mathcal{Y}) = Ob(\mathcal{X}) \times Ob(\mathcal{Y})$

(ii) $(\mathcal{X} \times \mathcal{Y})((x, y), (x', y')) := \mathcal{X}(x, x') \otimes \mathcal{Y}(y, y')$

2.75)

(i) $\forall (x,y) \in Ob(\mathcal{X},\mathcal{Y})$ we have $I \leq (\mathcal{X}\times\mathcal{Y})((x,y),(x,y))$

$\mathcal{X}((x,x)) \otimes \mathcal{Y}((y,y)) = I \otimes I = I$

(ii) $\mathcal{X}(x_1,x_2) \otimes \mathcal{Y}(y_1,y_2) \otimes \mathcal{X}(x_2,x_3) \otimes \mathcal{Y}(y_2,y_3)$

Symmetry * (Used here)

2.76)

$\mathcal{X} := \overset{A}{\bullet} \xrightarrow{2} \overset{B}{\bullet} \xrightarrow{3} \overset{C}{\bullet}$

$\boxed{p \rightleftarrows q} := \mathcal{Y}$ (5, 8)

$Ob(\mathcal{X}\times\mathcal{Y}) = \{(A,p),(A,q),(B,p),(B,q),(C,p),(C,q)\}$

$(\mathcal{X}\times\mathcal{Y})((x_1,y_1),(x_2,y_2)) = d_{\mathcal{X}}(x_1,x_2) + d_{\mathcal{Y}}(y_1,y_2)$

$$\begin{tikzcd}
(A,p) \arrow[r, "2"] \arrow[d, shift left, "5"] & (B,p) \arrow[r, "3"] \arrow[d, shift left, "5"] & (C,p) \arrow[d, shift left, "5"] \\
(A,q) \arrow[u, shift left, "8"] \arrow[r, "2"'] & (B,q) \arrow[u, shift left, "8"] \arrow[r, "3"'] & (C,q) \arrow[u, shift left, "8"]
\end{tikzcd}$$

2.78) Cost-product $\mathbb{R}\times\mathbb{R}$ $(\mathcal{X}\times\mathcal{Y})((5,6),(-1,4))$

$= d(5,-1) + d(6,4) = 6+2 = 8$ 😊

$\mathcal{X}$	A	B	C
A	0	2	5
B	∞	0	3
C	∞	∞	0

$\mathcal{Y}$	p	q
p	0	5
q	8	0

$\mathcal{X}\times\mathcal{Y}$	
(A,p)	
(A,	

Just add.

2.5] Presented $\mathcal{V}$-cats with matmul:

(It works for Bool Also!!

→ The property required for pre-order is unital, commutative quantale.

Monoidal closed preorders:

- Let us enrich the preorders themselves

Defn: A symm. monoidal preorder $\mathcal{V} = (V, \leq, I, \otimes)$ is symm closed if $\forall\, v, w \in V$ $\exists$ $v \multimap w \in V$ ~~st~~ (hom element)

$$(a \otimes v) \leq w \text{ iff } a \leq (v \multimap w) \quad (\dagger)$$

$(\forall\, a, v, w \in V)$

closed? ([illegible]) $p \leq j(p)$ and $j(j(p)) \cong j(p)$

$(\dagger)$ looks like a Galois connection result.

2.82) Prove that a monoidal pre-order is closed iff ~~$\forall$~~ any $v \in V$; $(- \otimes v): V \to V$ given by multiplying with v has a right joint, which is $(v \multimap -): V \to V$

(1) $(- \otimes v): V \to V$ is monotone.

$a_1 \leq a_2 \Rightarrow a_1 \otimes v \leq a_2 \otimes v$ (Property)

$v \leq v$

(2) $\mathcal{V}$ is closed, $\forall\, v, w \in V$ we have

$((v \multimap w) \otimes v) \leq w$

($f(g(q)) \leq q$)

(If $f(p) \leq p$

Let $p = g(q)$

$f(g(q)) \leq q$

As $p \leq g(q)$

$f(p) \leq f(g(q))$

2.83) $\mathbf{Cost} = ([0, \infty], \geq, 0, +)$

$x, y \in [0, \infty]$, $x \multimap y := \max(0, y - x)$

$a + x$ ~~$\otimes$~~ $\geq y$ iff $a \geq y - x$

iff $\max(0, a) \geq \max(0, y - x)$

iff $a \geq x \multimap y$

2.84) Bool $= (\mathbb{B}, \leq, T, \wedge)$ is closed. $w = F$; $a \wedge v = F$

$a \wedge v \leq w$ iff $\underbrace{v \multimap w}_{\text{what}} \geq a$

F T → F T $v = F$ F T → F T $v = T$

$v \multimap w = F$

$T \multimap F : F$

$T \multimap T : T$

$(\Rightarrow)$ ✓

2.85) $(\mathbb{B}, \leq, F, \vee)$ is not closed ☺

$v = T$ F T → F T

$a \leq p \multimap q$
$\Updownarrow$
$a \vee p \leq q$

$a = F$; p, q be anything

$a = F$ $p = F$, $q = F$ fails

2.86) $H_2O + Na \to NaOH + H_2$ if $H_2O \Rightarrow (Na \multimap (NaOH + H_2))$

2.87) $\mathcal{V} = (V, \leq, I, \otimes, \multimap)$ is a symmetric monoidal preorder that is closed.

$(- \otimes v) : V \to V$ is L.A to $v \multimap - : (V, \leq) \to (V, \leq)$

$(a \otimes v \leq w \Rightarrow a \leq v \multimap w)$

(Prese⊗rves join)

∴ b) If $A \subseteq V$ and $\bigvee_{a \in A} a$ exists then

$$(v \otimes \bigvee_{a \in A} a) \cong \bigvee_{a \in A} (v \otimes a)$$

$f : (- \otimes v)$
$g : (v \multimap -)$

c) $f(g(q)) \leq q$ $(v \multimap w) \otimes v \leq w$

d) $v \cong I \multimap v$ $(g(f(p)) \geq p)$

$v \otimes I \leq v \Rightarrow v \leq I \multimap v$

$I \otimes (I \multimap v) \leq v$ ∴ $\boxed{v \cong I \multimap v}$

e) Proposition.

Quantales:

Defn: A unital commutative quantale is a symm. monoidal closed preorder $\mathcal{V} = (V, \leq, I, \otimes, \multimap)$ that has all joins.
i.e $\bigvee A$ exists $\forall A \subseteq V$.

2.91) Is cost a unital comm. quantale?

$A \subseteq [0,\infty]$ join $j \Rightarrow \bigvee A \geq a$ and if $b \in [0,\infty]$ s.t $b \geq a \; \forall a \in A$ then $b \geq \bigvee A$ but here preordering is $\geq$ ★★★

- So join will be L.U.B.

$\bigvee \phi = \infty$

2.92) $\bigvee \phi = ?$ a) $\mathcal{V} = \text{Bool} = (\mathbb{B}, \leq, T, \wedge)$

$\boxed{\bigvee \phi = T}$ b) $\bigvee \phi = \infty$

2.93) $\bigvee \phi = T$ $\quad \bigvee \{F\} = F$ $\quad \bigvee \{T\} = T$ ...

Quantale ✓

2.94) S be a set and the preorder $(P(S), \subseteq, S, \cap)$ is it quantale

$A \in \{\{1\}, \{1,2\}\}$ $\quad \bigvee \phi = \phi$

$\bigvee a \Rightarrow$ union of all elements

First closed? $\quad a \cap v \leq w \Leftrightarrow a \leq v \multimap w$

$A \cap B \subseteq C \Leftrightarrow A \subseteq B \multimap C$

$\bar{B} \cup C$ ★★

for any B

$B = A$

$\bar{B} \in U \setminus \{B\} \cup$

$\bar{B} \cap C$

2.96) $\mathcal{P} = (P, \le)$ be a preorder. It has all joins iff it has all meets.

Proof: Suppose $\mathcal{P}$ has all joins. $\Rightarrow$ $A \subseteq P$

$M_A := \{p \in P \mid p \le a\ \forall a \in A\}$ $m_A := \bigvee_{p \in M_A} p$ be their join. m_A is a meet for A.

① $m_A \le a\ \forall a \in A$ If $p \le a$ then $m_A \ge p$ (join)

∴ A Quantale has all meets and joins.

2.97) If $\mathcal{X}$ is a $\mathcal{V}$-category with objects X and $U, V \subseteq X$, we define Hausdorff dist. as $d(U,V) := \sup_{u \in U} \inf_{v \in V} d(u,v)$

② op

$$\mathcal{X}(U,V) = \bigwedge_{u \in U} \bigvee_{v \in V} \mathcal{X}(u,v)$$

If $\mathcal{V}$ = Bool, It answers can I get into V from every U.

If $\mathcal{V} = P(M)$, It tells us modes of transportatio[n] into V from every pt that will get us ~~from~~ into V from every pt in U.

2.98] Suppose $\mathcal{V} = (V, \le, I, \otimes)$ is any symm. monoidal preorder that has all joins (so does meets) then $\mathcal{V}$ is closed $\Rightarrow$ quantale ☺

Proof: $v \multimap w := \bigvee_{\{a \in V \mid a \otimes v \le w\}} a$

ஞாபகம் உண்டா? ~~If $a \in V$~~

If preserving ~~meet~~ join $\Rightarrow$ $g := \{$

$g(q) := \{\bigvee p \mid f(p) \le \ \forall\ p \in P\}$

Matrix multiplication:

* A quantale $\mathcal{V} = (V, \leq, \otimes, I, \multimap)$ is necessary to perform matrix multiplication.

$$(M * N)(i,k) = \sum_j M(i,j) * N(j,k)$$

$$(M * N)(x,z) = \bigvee_{y \in V} M(x,y) \otimes N(y,z)$$

$M : X \times Y \to V \qquad N : Y \times Z \to V$

eg: $V = \text{Bool}$ $\qquad X = \{1,2,3\}$ $\quad Y = \{1,2\}$ $\quad Z = \{1,2,3\}$

$M : X \times Y \to B \quad \begin{pmatrix} F & F \\ F & T \\ T & T \end{pmatrix} \qquad N : Y \times Z \to B = \begin{pmatrix} T & T & F \\ T & F & T \end{pmatrix}$

$$\begin{pmatrix} F & F & F \\ T & F & T \\ T & T & T \end{pmatrix}$$

Identity V-matrix will be

$$I_X(x,y) = \begin{cases} I & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases}$$

103) $(\mathbb{N}, \leq, 1, *)$ $\qquad (B, \leq, T, \wedge)$

$I_{\mathbb{N}} := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad I_B := \begin{pmatrix} T & F \\ F & T \end{pmatrix}$

$\text{Cost} = ([0,\infty], \geq, 0, +) \qquad I_C := \begin{pmatrix} 0 & \infty \\ \infty & 0 \end{pmatrix}$

104) $\mathcal{V} = (V, \leq, I, \otimes, \multimap)$ be a quantale

Prove:

(i) Identity law $I_X * M = M$

$$(I_X * M)_{(x,z)} = \bigvee_y I_X(x,y) \otimes M(y,z)$$

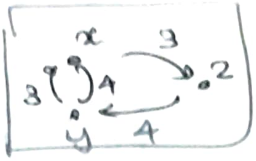
when $y = x$

$$= M(x,z)$$

Associative law:

$(M * N) * P = M * (N * P)$

→ Associativity of $\otimes$ operator.

$Y :=$ 

M_Y	x	y	z
x	0	4	3
y	3	0	∞
z	∞	4	0

M_Y^2	x	y	z
x	0	4	3
y	3	0	∞
z	∞	4	0

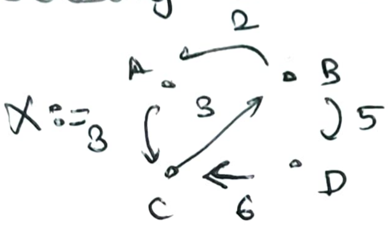
V((0+0), (3+3), (∞,∞))

V (4, 4, 7)   V(3, ∞, 4)

V (

• M_X^N tells us about the shortest path traversing n edges or fewer.

2·1(0 5)

$X :=$ 

(0, ∞, 3, ∞)

M_X	A	B	C	D
A	0	∞	3	∞
B	2	0	∞	5
C	∞	3	0	∞
D	∞	∞	6	∞

(∞, ∞, 6, ∞) C

(∞, 3, 0, ∞) D

(2, 0, ∞, 5)

(∞, 0, ∞

5, ∞, ∞, 11

∞, 5,

∞, 3, 3, ∞

∞, ∞, 0, ∞

∞, 8, ∞, ∞

$\begin{pmatrix}0\\2\\\infty\\\infty\end{pmatrix}\begin{pmatrix}\infty\\0\\3\\\infty\end{pmatrix}\begin{pmatrix}3\\\infty\\0\\6\end{pmatrix}\begin{pmatrix}\infty\\5\\\infty\\\infty\end{pmatrix}$

M_X^2	A	B	C	D
A	0	6	3	∞
B	2	0	5	5
C	5	3	0	8
D	∞	9	6	∞

(5 3 0 8)

(0 ∞ 3 ∞)   (2 0 5 5)   (∞, 9, 6, ∞)

$\begin{pmatrix}0\\2\\5\\\infty\end{pmatrix}\begin{pmatrix}\infty\\0\\3\\9\end{pmatrix}\begin{pmatrix}3\\5\\0\\6\end{pmatrix}\begin{pmatrix}\infty\\5\\8\\\infty\end{pmatrix}$

M_X^3	A	B	C	D
A	0	6	3	11
B	2	0	5	10
C	5	3	0	8
D	11	9	6	14

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