Dynamix

* Learning Category theory.

  * Better methods for understanding and using mathematical concepts.

$$\begin{tikzcd} \boxed{\text{spaa}} \arrow[r, "\text{level}"] \arrow[d, "\text{shadow}"] & \boxed{\text{line}} \\ \boxed{\text{Plane}} & \end{tikzcd}$$

Multiplication "and"

$$\begin{tikzcd} \boxed{\text{Stat}} \arrow[r, "A"] \arrow[d, "B"] & \boxed{1} \\ \boxed{2} & \end{tikzcd}$$

→ Before giving definition for category, let us famialarize ourselves with an example. category of sets.

Category of sets / maps: * Object here is finite set (collection

* Internal diagram:  * A map here has three things: domain, codomain, rule with $a \to b$

* endo map: representing map between same object.

  Eg: Identity map: A  (Id_A)

* Dynamics of category: two maps are combined - Composition

  $A \xrightarrow{g} A \xrightarrow{f} B$: $A \xrightarrow{f \circ g} B$ (Domain of f a same object as co-domain of g)

Data for a category: Objects, maps, identity, composition.

Rules for a category:

* Identity laws: $A \xrightarrow{id_A} A \xrightarrow{g} B = A \xrightarrow{g} B$
* Associative laws: $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D = A \xrightarrow{h \circ (g \circ f) = (h \circ g) \circ f} D$

Useful sort of set: singleton set with 1 element

* A function is defined by what the rule accomplish

• In category map means getting one category from the other one.

Test for equality of points: A point set '1'

* For each pt $1 \xrightarrow{a} A$, $f \circ a = g \circ a \Rightarrow f = g$
  i.e every element in domain has same output in both fns

$\boxed{f \circ g : f \text{ after } g}$

* A single map can arise as a composition of function in several ways:

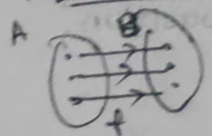
$$\begin{tikzcd} & \text{No} \arrow[dr, "\text{Add } 1"] & \\ \text{No} \arrow[ur, "\text{Add } 1"] \arrow[rr, "\text{Adding } 2"] \arrow[dr, "\text{Add } 3"'] & & \text{No} \\ & \text{No} \arrow[ur, "\text{subtract } 1"'] & \end{tikzcd} \qquad \begin{tikzcd} & B \arrow[dr, "g"] & \\ A \arrow[ur, "f"] \arrow[rr, "g \circ f"'] & & C \end{tikzcd}$$

• External diagrams show more info than any other diagram
• In associative law, we care about putting brakes on the methods. The order is not affected.
* Number of maps from set A to set B $= (|B|)^{(|A|)}$

$A \xrightarrow{f} B : id_B \circ f = f$; $B \xrightarrow{g} C : g \circ id_B = g$

$\underbrace{f}_{\text{Codomain}} \circ \underbrace{g}_{\text{Domain}}$;

Isomorphism: (Before counting, resemblance was important)



(What is special about this map?)

→ Crucial property of f is that there is an inverse map g for the map f, meaning

$g \circ f = id_A \quad f \circ g = id_B$

• A map $A \xrightarrow{f} B$ is called an isomorphism if there is a map $B \xrightarrow{g} A$ for which,
$g \circ f = id_A$, $f \circ g = id_B$. Two objects are isomorphic if there is atleast one isomorphism $A \xrightarrow{f} B$.

- Isomorphic or equinumerous or same-size has three properties:
  - Reflexive: A is isomorphic to A
  - Symmetric: If A is isomorphic to B, then B is isomorphic to C
  - Transitive: $A \to B$, $B \to C$ then $A \to C$ ✓

- Translating real-life curves to algebra and using inverse to apply it back.

- Inverse if exist is unique $B \xrightarrow{g} A$, $B \xrightarrow{k} A$ are both inverses for $A \xrightarrow{f} B \Rightarrow g = k$

$g \circ f = id_A$; $f \circ k = id_B$; $\Rightarrow (g \circ f) \circ k = id_A \circ k$

$\Rightarrow g \circ (f \circ k) = k \Rightarrow g \circ id_B = k \Rightarrow \boxed{g = k}$

- $3 \times x = 4$ is same as $x = 4 \times (3)^{-1}$; (This is a kind of idea for inverse)

- If f has an inverse, $f \circ h = f \circ k \Rightarrow h = k$; $h \circ f = k \circ f \Rightarrow h = k$

Determination problem:

$$\begin{tikzcd} & B \arrow[dr, "g?"] & \\ A \arrow[ur, "f"] \arrow[rr, "h"'] & & C \end{tikzcd}$$

$g?$ st $g \circ f = h$

Choice problem:

$$\begin{tikzcd} & B \arrow[dr, "g"] & \\ A \arrow[ur, "f?"] \arrow[rr, "h"'] & & C \end{tikzcd}$$

$f = ?$ s.t $g \circ f = h$

- Special case of determination prob. is retraction.

For $f: A \xrightarrow{f} B$, retraction (r) for f is a map $B \xrightarrow{r} A$ such that $r \circ f = id_A$

$$\begin{tikzcd} & B \arrow[dr, "r"] & \\ A \arrow[ur, "f"] \arrow[rr, "id_A"'] & & A \end{tikzcd}$$

For $f: A \xrightarrow{f} B$, section (s) for f is a map $B \xrightarrow{s} A$ such that $f \circ s = id_B$

$$\begin{tikzcd} & A \arrow[dr, "f"] & \\ B \arrow[ur, "s"] \arrow[rr, "id_B"'] & & B \end{tikzcd}$$

- If a map $A \xrightarrow{f} B$ has a section, then for any T and any map $T \xrightarrow{y} B$, there exist a map $T \xrightarrow{x} A$ for which $f \circ x = y$

$$\begin{tikzcd} & B \arrow[dr, "s"] & \\ A \arrow[ur, "f"] \arrow[rr, "id_A"'] & & A \end{tikzcd}$$

$$\begin{tikzcd} & & A \arrow[dr, "f"] & \\ T \arrow[urr, dashed, "x=?"] \arrow[r, "y"'] & B \arrow[ur, "s"] \arrow[rr, "id_B"'] & & B \end{tikzcd}$$

$x = s \circ y \Rightarrow f \circ x = \underbrace{(f \circ s)}_{id_B} \circ y$

• Having a section corresponds to a surjective fn as $T \xrightarrow{y} B$ & $T \xrightarrow{x} A$ st $\boxed{y = f \circ x}$. If it is not surjective the map $f: A \xrightarrow{f} B$ can miss out some elements from B.

→ Section of f is referred to as a choice of representatives as codomain elements can be expressed to represent a output

• A object that could be represented into B can also be transformed into A. $\boxed{f \circ x = y}$

Similarly for retraction, if you can transform into from A to a object, you could do that from B also

$$\begin{tikzcd} & B \arrow[dr, "r"] \arrow[rr, dashed, "y"] & & T \\ A \arrow[ur, "f"] \arrow[rr, "id_A"'] & & A \arrow[ur, "x"'] & \end{tikzcd}$$

$y = x \circ r \Rightarrow y \circ f = x \circ \underbrace{(r \circ f)}_{id_A} = x$

from outputs

In section, outputs are clearly known, and in retraction inputs are clearly known from inputs

• If $A \xrightarrow{f} B$ has a retraction, $f \circ g_1 = f \circ g_2 \Rightarrow \boxed{g_1 = g_2}$ — it shares domain so, inputs are known.

f satisfying this condition is called as injective or is a monomorphism.

• If $A \xrightarrow{f} B$, has a section, $g_1 \circ f = g_2 \circ f \Rightarrow \boxed{g_1 = g_2}$

f satisfying the condition is called as surjective or epimorphism.

$$B \xrightarrow{s} A \xrightarrow{f} B \xrightarrow{r} A$$

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[rr, bend left, "g \circ f"] & B \arrow[l, shift left, "r_1"] \arrow[r, shift left, "g"] & C \arrow[l, shift left, "r_2"] \end{tikzcd}$$

retraction for $g \circ f$ is $r_2 \circ r$

- A endomap (A to A) is called idempotent if $e \circ e = e$
- If r is a retraction of f, $\boxed{e = f \circ r}$ $((f \circ r) \circ (f \circ r)) = f \circ (r \circ f) \circ r = f \circ r$
  $f \circ r$ is an idempotent.
- If s is a section of f, $\boxed{e = s \circ f}$ (an's approach)

→ If f has both retraction and section, then $r = s$

$A \xrightarrow{f} B$; $r \circ f = id_A$ $f \circ s = id_B$ $r \circ f \circ s = r \Rightarrow \boxed{s = r}$

Visiting isomorphism again: A map f is called isomorphism if there exists another map f^{-1} which is both a retraction and a section for f.