Łojasiewicz Inequality of Area-Preserving Curve Shortening Flow

Dyne Kim

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1 Introduction

The **area-preserving curve shortening flow (APCSF)** [2] $X : [0,T) \times \mathbb{S}^1 \to \mathbb{R}^2$ is a geometric flow of plane curves $\gamma_t(-) = X(t,-)$ evolving according to the equation

$$X_t = (k - \bar{k})N \quad , \quad k_t = k^2(k_{\theta\theta} + k - \bar{k}),$$

where k is curvature, θ is tangent angle, N is inward unit normal, and $\bar{k} = 2\pi/L$ is average of curvature. It is one-dimensional version of volume-preserving mean curvature flow (VPMCF) [9]. Its name is due to the fact that the area of the curves $\{\gamma_t\}$ are preserved, but their lengths are decreasing along the flow; this follows from the fact that it is a gradient flow of length functional under additional resetriction taht the enclosed area must be preserved. (See Remark 4.2 for detailed arguments.)

In 1986, Michael Gage [2] introduced APCSF as a variant of curve shortening flow (CSF) [8] and proved that the APCSF starting from a C^2-simple closed convex plane curve exists for all time $t > 0$ (in other words, it has an immortal solution) and converge exponentially smoothly to a circle with the same area. This clearly implies uniform and Hausdorff exponential convergence results.

Gradient inequalities are inequalities which bound the increment of a functional by the increment of its derivative. In other words, functional $\mathscr{E}$ satisfies gradient inequality in V with respect to function ϕ if

$$\phi(\mathscr{E}(v)) \leq \|\mathscr{E}'(v)\| \qquad (\forall v \in V).$$

One of the most famous type of gradient inequalities is the **Łojasiewicz-Simon type (LS-type)**, which has the following form near the critical point φ of the functional:

$$c \cdot |\mathscr{E}(u) - \mathscr{E}(\varphi)|^\theta \leq \|\mathscr{E}'(u)\| \qquad (\forall u \in U, \|u - \varphi\| < \sigma).$$

Here, c, σ, θ are positive constants; constant $\theta \in [1/2, 1)$ is called as **Łojasiewicz exponent**. The term 'Łojasiewicz-Simon type' is named after Polish mathematician Stanisław Łojasiewicz [4], who first proved inequality of this type for analytic functions on finite-dimensional Euclidean spaces, and Leon Simon [5], who first generalized Łojasiewicz's result to functionals on infinite-dimensional function spaces. If the inequality holds for some exponent θ_0, then it holds for any $\theta \in [\theta_0, 1)$ (with possibly different constants c, σ); Therefore, inequality with Łojasiewicz exponent $1/2$ is the strongest result, and therefore called **optimal** or **critical exponent** case.

After Almgren and Allard [12] suggested the possibility to inspect geometric properties of minimal hypersurfaces by inspecting their tangent cones, Leon Simon [5] first introduced Łojasiewicz-Simon inequalities to the field of geometric analysis in order to prove convergence result of some evolution equation which implies that a minimal surface in Euclidean space which has a singularity at 0 and which has a multiplicity 1 tangent cone with an isolated singularity at 0 asymptotically converges to the tangent cone near the singularity.

Simon's frameworks based on Łojasiewicz-Simon inequalities can be applied to several evolution equations including geometric flows, and therefore have wide range of profound consequences in the field of geometric analysis. For example, Felix Schulze [13] proved uniqueness of compact tangent flows in mean curvature flow (MCF); this is closely related to the singularity analysis of MCF, as we can see in Tobias Colding and William Minicozzi's paper [14]. Song Sun and Yuanqi Wang [15] have shown convergence of Kähler-Ricci flow near the Kähler-Einstein metric to a Kähler-Einstein metric. Paul Feehan studied relations between optimal Łojasiewicz-Simon inequality and the Morse-Bott property of a functional [11, 16], and have shown that this results are applicable for some specific kinds of Yang-Mills energy functions [11].

In this paper, we will first characterized convex simple closed plane curves using their radii of curvature functions in $L^2(\mathbb{S}^1)$. Although curvature functions are traditionally used in studying CSF and its variants, the set of curvature functions that define closed curves does not form a linear subspace but rather a Banach submanifold. To avoid these complications, we used radii of curvature functions instead.

Next, We will introduce the $L - \lambda A$ functional on $L^2(\mathbb{S}^1)$, where L is curve length, A is the enclosed area, and λ is a Lagrange multiplier. Intuitively, this functional enables the minimization of length while preserving area. Since APCSF is the gradient flow of length on the abstract space of plane curves constrained by area preservation, the $L - \lambda A$ functional is thus closely related to APCSF.

We then proved the optimal Łojasiewicz-Simon inequality with a critical exponent of $1/2$ for the $L - \lambda A$ functional using the Lyapunov-Schmidt reduction for Morse-Bott functionals on Banach spaces, developed by Paul Feehan [11].

Finally, we will provide alternative proofs of the Hausdorff and uniform exponential convergence results for APCSF using the framework based on Łojasiewicz-Simon inequality. While these results have been well-known since Michael Gage first introduced APCSF [2], this alternative proof offers new significance, as Łojasiewicz-Simon inequalities had not been applied to APCSF previously.

The main theorems of this paper are the following:

Theorem 1.1 (Łojasiewicz-Simon Inequality for $L - \lambda A$ functional). *For $L - \lambda A : L^2(\mathbb{S}^1) \to \mathbb{R}$ and its gradient map $d(L - \lambda A) : L^2(\mathbb{S}^1) \to H^2(\mathbb{S}^1)$, there exists constants $c, \sigma > 0$ such that*

$$c \cdot |(L - \lambda A)(f) - \pi\lambda^{-1}|^{1/2} \leq \|d(L - \lambda A)_f\|_{H^2} \qquad (\forall f \in L^2(\mathbb{S}^1) \text{ s.t. } \|f - \lambda^{-1}\|_{L^2} < \sigma). \tag{1.1}$$

Theorem 1.2 (Łojasiewicz-Simon Inequality of APCSF). *There exists a positive real number $\delta > 0$ such that the APCSF, starting from any C^2-initial curve $f_0 = f(0, -)$ satisfying $\|f_0(-) - \lambda^{-1}\|_{H^1} < \delta$, meets the following conditions:*

(i) *It remains within the L^2-neighborhood $\|f(t, -) - \lambda^{-1}\|_{L^2} < \sigma$ described in Theorem 1.1, where the optimal Łojasiewicz-Simon inequality for the $L - \lambda A$ functional holds.*
(ii) *It has positive uniform upper and lower bounds $0 < m \leq f(t, \theta) \leq M$.*
(iii) *Its length $L(t) = L(f(t, -))$ converges exponentially to $2\pi\lambda^{-1}$.*

Theorem 1.3 (Exponential Convergence of APCSF). *The APCSF $\{\gamma_t\}$, starting from an initial curve satisfying the assumptions of Theorem 1.2, converges exponentially to a circle γ_∞ with radius λ^{-1} in both the uniform distance and the Hausdorff distance.*

2 Radii of Curvature and $L - \lambda A$ Functional

For any plane simple closed curve, its Gauss map is injective, and we may use tangent angle θ (the angle between tangent vector at a point on the curve with x-axis) as a parameter of the curve. Moreover, we may identify curve with its curvature function, up to translation.

Proposition 2.1. *A positive-valued continuous function $k : \mathbb{S}^1 \to \mathbb{R}^+$ is a curvature function of some C^2-curve if and only if*

$$\int_0^{2\pi} \frac{e^{i\theta}}{k(\theta)}\, d\theta = 0. \tag{2.1}$$

In this case, parameter equation of the curve which has k as curvature function is

$$\left(\int_0^{\theta} \frac{\cos\phi}{k(\phi)}\, d\phi, \int_0^{\theta} \frac{\sin\phi}{k(\phi)}\, d\phi \right), \tag{2.2}$$

up to translation.

Proof. If k is a curvature function of some C^2-simple closed plane curve, then

$$0 = \int_0^L T\, ds = \int_0^{2\pi} T\, \frac{d\theta}{k(\theta)} = \int_0^{2\pi} \frac{(\cos\theta, \sin\theta)}{k(\theta)}\, d\theta = \left(\int_0^{2\pi} \frac{\cos\theta}{k(\theta)}\, d\theta, \int_0^{2\pi} \frac{\sin\theta}{k(\theta)}\, d\theta \right).$$

Conversely, if k is any continuous function satisfying (2.1), then parameter equation (2.2) actually defines a C^2-closed curve. In particular, it is a simple closed curve, since its Gauss map is injective. Moreover, direct calculation shows that k is the curvature function of this curve. Since all such curves can be represented as

$$(a,b) + \int_0^s T\, ds = (a,b) + \int_0^{\theta} \frac{(\cos\phi, \sin\phi)}{k(\phi)}\, d\phi,$$

(where (a,b) is the unique point where tangent angle is 0.) such parameter equation is unique up to translation. $\square$

Although curvature functions are traditionally used in studying CSF and its variants, there is a problem with curvature functions: they cannot be directly added, since $k_1 + k_2$ may not satisfy $\int (k_1 + k_2)^{-1} e^{i\theta}\, d\theta = 0$ even if $\int k_1^{-1} e^{i\theta}\, d\theta = \int k_2^{-1} e^{i\theta}\, d\theta = 0$. This means the set of curvature functions that define closed curves does not form a linear subspace but rather a Banach submanifold. In order to avoid this complications, we will use **radius of curvature** $f := k^{-1}$ as a graph which characterizes curve.

Consider any positive-valued continuous function $f : \mathbb{S}^1 \to \mathbb{R}$ which satisfies

$$\int_0^{2\pi} f e^{i\theta}\, d\theta = 0.$$

Such functions correspods to following curve up to translation:

$$X(\theta) = \left(\int_0^{\theta} f \cos\phi\, d\phi, \int_0^{\theta} f \sin\phi\, d\phi \right).$$

Note that arc length derivative becomes

$$ds = k^{-1}\, d\theta = f\, d\theta \quad , \quad \frac{d}{ds} = k\frac{d}{d\theta} = f^{-1}\frac{d}{d\theta}.$$

Proposition 2.2. *Length and area of the curve corresponding to given radius of curvature f is*

$$L = \int_0^{2\pi} f\, d\theta \quad , \quad A = \int_0^{2\pi} f(\theta)\cos\theta \int_0^{\theta} f(\phi)\sin\phi\, d\phi\, d\theta.$$

Proof. Length formula directly follows from the fact that $ds = f\, d\theta$. Area is given by

$$A = \oint y\, dx = \int_0^{2\pi} y(\theta)\frac{dx}{d\theta}\, d\theta = \int_0^{2\pi} f(\theta)\cos\theta \int_0^{\theta} f(\phi)\sin\phi\, d\phi\, d\theta. \qquad \square$$

Along APCSF, area is constant but length is strictly decreasing; its asymptotic limit (circle with prescribed area) has the least length L among curves with same area A. In order to minimize length among curves with prescribed area, we may consider Lagrange multiplier method. Therefore, we will consider functional $L - \lambda A$ with Lagrange multiplier $\lambda \in \mathbb{R}^+$.

Definition 2.3. On $L^2(\mathbb{S}^1)$, define

$$Qf = f - \frac{1}{\pi}\sin\theta \int_0^{2\pi} f(\phi)\sin\phi\, d\phi - \frac{1}{\pi}\cos\theta \int_0^{2\pi} f(\phi)\cos\phi\, d\phi.$$

Namely, Q is the orthogonal projection whose kernel is $\mathrm{span}\{\sin, \cos\}$.

We will use this projection Q in order to avoid pathology originating from L^2-functions which do not correspond to closed curves.

Definition 2.4. We define the functional $L - \lambda A : L^2(\mathbb{S}^1) \to \mathbb{R}$ as

$$(L - \lambda A)(f) = L(Qf) - \lambda A(Qf) = \int_0^{2\pi} Qf\, d\theta - \lambda \int_0^{2\pi} Qf(\theta)\cos\theta \int_0^{\theta} Qf(\phi)\sin\phi\, d\phi\, d\theta.$$

In particular, when f corresponds to a curve, then it coincides with the $L - \lambda A$ value of the curve corresponding to f; This is the reason why we abuse the notation '$L - \lambda A$'.

Theorem 2.5. *Under the isomorphism $(L^2(\mathbb{S}^1))^* \cong L^2(\mathbb{S}^1)$, the first variation of the $L - \lambda A$ functional corresponds to the following function:*

$$d(L - \lambda A)_f = \lambda\, \mathrm{Im}\left[e^{i\theta} \int_0^{\theta} (Qf(\phi) - \lambda^{-1})e^{-i\phi}\, d\phi\right].$$

It is zero exactly when $Qf = \lambda^{-1}$ almost everywhere; therefore, a function f is a critical point iff $f - \lambda^{-1} \in \mathrm{span}\{\sin, \cos\}$. Range of the map $f \mapsto d(L - \lambda A)_f$ is codimension 2 closed subspace of $H^2(\mathbb{S}^1)$. Moreover, we can estimate its norm by

$$\|d(L - \lambda A)\|_{H^2}^2 \le 25\lambda^2 \|Qf - \lambda^{-1}\|_{L^2}^2.$$

The second variation of the $L - \lambda A$ functional corresponds to the following map:

$$g \mapsto d^2(L - \lambda A)_f(g, -) = \lambda\, \mathrm{Im}\left[e^{i\theta} \int_0^{\theta} Qg(\phi)e^{-i\phi}\, d\phi\right].$$

Its kernel is $\mathrm{span}\{\sin, \cos\}$.

Proof. First, we may use definition of derivative to calculate first variation as an element of dual space:

$$\begin{aligned}
d(L - \lambda A)_f(g) &= \lim_{t\to 0} t^{-1}[(L - \lambda A)(f + tg) - (L - \lambda A)(f)] \\
&= \lim_{t\to 0} \frac{1}{t}\left[\int_0^{2\pi} (Q(f + tg) - Qf)\, d\theta - \lambda \int_0^{2\pi} Q(f + tg)(\theta)\cos\theta \int_0^{\theta} Q(f + tg)(\phi)\sin\phi\, d\phi\, d\theta\right. \\
&\quad \left. + \lambda \int_0^{2\pi} Qf(\theta)\cos\theta \int_0^{\theta} Qf(\phi)\sin\phi\, d\phi\, d\theta\right] \\
&= \int_0^{2\pi} Qg\, d\theta - \lambda \int_0^{2\pi} Qg(\theta)\cos\theta \int_0^{\theta} Qf(\phi)\sin\phi\, d\phi + Qf(\theta)\cos\theta \int_0^{\theta} Qg(\phi)\sin\phi\, d\phi\, d\theta \\
&= \int_0^{2\pi} Qg\, d\theta - \lambda \int_0^{2\pi} Qg(\theta)\cos\theta \int_0^{\theta} Qf(\phi)\sin\phi\, d\phi - Qg(\theta)\sin\theta \int_0^{\theta} Qf(\phi)\cos\phi\, d\phi\, d\theta \\
&= \lambda \int_0^{2\pi} Qg(\theta)\, \mathrm{Im}\left[e^{i\theta} \int_0^{\theta} (Qf(\phi) - \lambda^{-1})e^{-i\phi}\, d\phi\right] d\theta
\end{aligned}$$

Since Q is self-adjoint and is identity map on orthogonal complement of span$\{\sin, \cos\}$, this linear functional corresponds to the following L^2-function:

$$\begin{aligned}
d(L-\lambda A)_f &= Q\left[1-\lambda\cos\theta\int_0^\theta Qf(\phi)\sin\phi\, d\phi+\lambda\sin\theta\int_0^\theta Qf(\phi)\cos\phi\, d\phi\right]\\
&= \lambda Q\,\mathrm{Im}\left[e^{i\theta}\int_0^\theta (Qf(\phi)-\lambda^{-1})e^{-i\phi}\, d\phi\right]\\
&= \lambda\,\mathrm{Im}\left[e^{i\theta}\int_0^\theta (Qf(\phi)-\lambda^{-1})e^{-i\phi}\, d\phi\right]
\end{aligned}$$

(Here, we used that $(Qf-\lambda^{-1})(\phi)$ has no $e^{\pm i\phi}$ component, and therefore the function in the square bracket does not have $e^{\pm i\phi}$ component.)

If we differentiate above formula once more, we obtain second variation formula:

$$\begin{aligned}
d^2(L-\lambda A)_f(g,-) &= \lim_{t\to 0} t^{-1}[d(L-\lambda A)_{f+tg}-d(L-\lambda A)_f]\\
&= \lim_{t\to 0}\frac{1}{t}\lambda\,\mathrm{Im}\left[e^{i\theta}\int_0^\theta (Q(f+tg)(\phi)-\lambda^{-1})e^{-i\phi}\, d\phi-e^{i\theta}\int_0^\theta (Qf(\phi)-\lambda^{-1})e^{-i\phi}\, d\phi\right]\\
&= \lambda\,\mathrm{Im}\left[e^{i\theta}\int_0^\theta Qg(\phi)e^{-i\phi}\, d\phi\right].
\end{aligned}$$

In order to locate critical points, we will identify an L^2-function with its Fourier series:

$$S_{Qf-\lambda^{-1}}(\theta)=\sum_{-\infty}^{\infty} a_n e^{in\theta} \quad , \quad a_n=\frac{1}{2\pi}\int_0^{2\pi}(Qf(\theta)-\lambda^{-1})e^{-in\theta}\, d\theta$$

Here, $\bar{a}_n=a_{-n}$ since f is real function; $a_{\pm 1}=0$ since we applied projection Q. This yields

$$e^{i\theta}\int_0^\theta (Qf(\phi)-\lambda^{-1})e^{-i\phi}\, d\phi=e^{i\theta}\sum\int_0^\theta a_n e^{i(n-1)\phi}\, d\phi=\sum_{n\neq\pm 1}\frac{a_n}{i(n-1)}(e^{in\theta}-e^{i\theta}).$$

$d(L-\lambda A)_f$ is constant multiple of imaginary part of this function.

$$\begin{aligned}
S_{d(L-\lambda A)_f} &= \frac{\lambda}{2i}\sum_{n\neq\pm 1}\frac{a_n}{i(n-1)}(e^{in\theta}-e^{i\theta})-\frac{\lambda}{2i}\sum_{n\neq\pm 1}-\frac{\bar{a}_n}{i(n-1)}(e^{-in\theta}-e^{-i\theta})\\
&= -\frac{\lambda}{2}\sum_{n\neq\pm 1}\left[\frac{a_n}{n-1}+\frac{\bar{a}_{-n}}{-n-1}\right]e^{in\theta}+\frac{\lambda}{2}\sum_{n\neq\pm 1}\frac{a_n e^{i\theta}+\bar{a}_n e^{-i\theta}}{n-1}\\
&= -\lambda\sum_{|n|\geq 2}\frac{a_n e^{in\theta}}{n^2-1}+\frac{\lambda}{2}\sum_{n\neq\pm 1}\frac{a_n e^{i\theta}}{n-1}-\frac{\lambda}{2}\sum_{n\neq\pm 1}\frac{a_n e^{-i\theta}}{n+1}
\end{aligned}$$

So, its Fourier coefficients $\{b_n\}$ are

$$b_0=0 \quad , \quad b_1=\frac{\lambda}{2}\sum_{n\neq 1}\frac{a_n}{n-1} \quad , \quad b_{-1}=-\frac{\lambda}{2}\sum_{n\neq\pm 1}\frac{a_n}{n+1} \quad , \quad b_n=-\lambda\frac{a_n}{n^2-1} \quad (|n|\geq 2)$$

It shows that $d(L-\lambda A)_f=0$ iff $a_n=0$ for $|n|\neq 1$; this corresponds to the functions $f-\lambda^{-1}\in$ span$\{\sin,\cos\}$. In other words, the critical set is $\mathrm{Crit}(L-\lambda A)=\lambda^{-1}+\mathrm{span}\{\sin,\cos\}$. (This also shows that the kernel of $d^2(L-\lambda A)_f$ is span$\{\sin,\cos\}$, since $d^2(L-\lambda A)_f$ has the same form except that $f-\lambda^{-1}$ replaced to g.)

Moreover, the sequence $\{-n^2 b_n\}$ is also square-summable; namely, $d(L-\lambda A)_f$ is in $H^2(\mathbb{S}^1)$. Conversely, if any sequence $\{b_n\}$ corresponding to a H^2-function is given, then it is $d(L-\lambda A)_f$ for some f if and only if

$$b_0=0 \quad , \quad b_1+b_{-1}=-\sum_{|n|\geq 2} b_n.$$

In this case, Fourier coefficients $\{a_n\}$ of f are

$$a_n = -\lambda^{-1}(n^2-1)b_n \quad (|n| \geq 2) \quad , \quad a_{\pm 1} \text{ arbitrary} \quad , \quad a_0 = -\lambda^{-1}\sum nb_n.$$

Therefore, range of the map $f \mapsto d(L-\lambda A)_f$ is codimension 2 closed subspace of $H^2(\mathbb{S}^1)$.

In addition, we can estimate $\|d(L-\lambda A)\|_{H^2}^2$:

$$\begin{aligned}
\|d(L-\lambda A)\|_{H^2}^2 &= \pi \sum_{n\neq 0}(n^4+n^2+1)|b_n|^2 \\
&= \pi\lambda^2 \sum_{|n|\geq 2} \frac{n^4+n^2+1}{(n^2-1)^2}|a_n|^2 + 3\pi \cdot \frac{\lambda^2}{4}\left|\sum_{n\neq\pm 1}\frac{a_n}{n-1}\right|^2 + 3\pi \cdot \frac{\lambda^2}{4}\left|\sum_{n\neq\pm 1}\frac{a_n}{n+1}\right|^2 \\
&\leq 3\pi\lambda^2 \sum_{|n|\geq 2}|a_n|^2 + \frac{3\pi\lambda^2}{4}\sum_{n\neq\pm 1}|a_n|^2 \left[\sum_{n\neq 1}\frac{1}{(n-1)^2} + \sum_{n\neq -1}\frac{1}{(n+1)^2}\right] \\
&= 3\pi\lambda^2 \sum_{|n|\geq 2}|a_n|^2 + \frac{3\pi\lambda^2}{4}\cdot\frac{4\pi^2}{6}\sum_{n\neq\pm 1}|a_n|^2 \\
&\leq (3\pi + \tfrac{1}{2}\pi^3)\lambda^2 \sum_{n\neq\pm 1}|a_n|^2 \\
&\leq 25\lambda^2\|Qf-\lambda^{-1}\|_{L^2}^2.
\end{aligned}$$

□

3 Lyapunov-Schmidt Reduction

Lyapunov-Schmidt reduction is a process which reduces a functional defined on infinite-dimensional space into reduced functional defined on some finite-dimensional space; then gradient inquality on reduced functional implies gradient inequality on original functional. In particular, this can be used to prove infinite-dimensional Łojasiewicz-Simon inequality from finite-dimensional Łojasiewicz inequality [5]. We will only give main statements here; for detailed proof, see [11].

Definition 3.1 (Gradient Maps [1])**.** Let $\mathscr{X}, \mathscr{Y}$ be Banach spaces with continuous embedding $\mathscr{Y} \subset \mathscr{X}^*$. Let $\mathscr{U} \subset \mathscr{X}$ be an open subset. A continuous map $\mathscr{M} : \mathscr{U} \to \mathscr{Y}$ is called a **gradient map** if there exists a C^1 functional $\mathscr{E} : \mathscr{U} \to \mathbb{R}$ such that $\mathscr{M}(u) = \mathscr{E}'(u)$ for all $u \in \mathscr{U}$:

$$\mathscr{E}'(u)h = \langle \mathscr{M}(u), h\rangle \qquad (\forall u \in \mathscr{U}, h \in \mathscr{X}), \tag{3.1}$$

where $\langle -, -\rangle$ is the canonical bilinear form on $\mathscr{X}^* \times \mathscr{X}$. Correspondingly, $\mathscr{E}$ is called a **potential** for the map $\mathscr{M}$.

Gradient inequalities are inequalities which bound the increment of a functional by the increment of its derivative.

Definition 3.2 (Gradient Inequalities [1])**.** A functional $\mathscr{E}$ satisfies **gradient inequality** in $\mathscr{V}$ with respect to function ϕ if

$$\phi(\mathscr{E}(v)) \leq \|\mathscr{E}'(v)\| \qquad (\forall u \in \mathscr{V}). \tag{3.2}$$

It satisfies **Łojasiewicz-Simon type (LS-type)** gradient inequality if there exist some constants $c, \sigma > 0$ and $\theta \in [1/2, 1)$ (called the **Łojasiewicz exponent**) such that for all $u \in \mathscr{U}$ satisfying $\|u-\varphi\| < \sigma$,

$$c \cdot |\mathscr{E}(u) - \mathscr{E}(\varphi)|^\theta \leq \|\mathscr{E}'(u)\|. \tag{3.3}$$

In this case, the function ϕ has the form

$$\phi(x) = c \cdot |x-a|^\theta \quad , \quad a = \mathscr{E}(\varphi).$$

If the inequality holds for some exponent θ_0, then it holds for any $\theta \in [\theta_0, 1)$ (with possibly different constants c, σ); Therefore, inequality with Łojasiewicz exponent $1/2$ is the strongest result, and therefore called **optimal** or **critical exponent** case.

Definition 3.3 (Morse-Bott Functional on Banach Spaces [10]). Let $\mathscr{B}$ be a smooth Banach manifold, $\mathscr{E}:\mathscr{B}\to\mathbb{R}$ be a C^2-function, and $\operatorname{Crit}\mathscr{E}:=\{x\in\mathscr{B}:\mathscr{E}'(x)=0\}$ be the critical set. A smooth submanifold $\mathscr{C}\hookrightarrow\mathscr{B}$ is called a **nondegenerate critical submanifold of** $\mathscr{E}$ if $\mathscr{C}\subset\operatorname{Crit}\mathscr{E}$ and

$$(T\mathscr{C})_x=\operatorname{Ker}\mathscr{E}''(x)\qquad(\forall x\in\mathscr{C}),\tag{3.4}$$

where $\mathscr{E}''(x):(T\mathscr{B})_x\to(T\mathscr{B})_x^*$ is the Hessian of $\mathscr{E}$ at the point $x\in\mathscr{C}$. Then,

(a) $\mathscr{E}$ is a **Morse-Bott function** if $\operatorname{Crit}\mathscr{E}$ consists of nondegenerate critical submanifolds.
(b) $\mathscr{E}$ is **Morse-Bott at a point** $x_0\in\mathscr{B}$ if there is an open neighborhood $\mathscr{U}\subset\mathscr{B}$ of x_0 such that $\mathscr{U}\cap\operatorname{Crit}\mathscr{E}$ is a relatively open smooth submanifold of $\mathscr{B}$ and (3.4) holds at x_0.

Theorem 3.4 (Optimal Łojasiewicz-Simon Inequality for C^2-Morse-Bott Functions [11]). *Let $\mathscr{X},\mathscr{Y},\mathscr{G},\mathscr{H}$ be Banach spaces with continuous embeddings $\mathscr{X}\subset\mathscr{G}$ and $\mathscr{Y}\subset\mathscr{H}\subset\mathscr{G}^*\subset\mathscr{X}^*$. Let $\mathscr{U}\subset\mathscr{X}$ be an open subset, $\mathscr{E}:\mathscr{U}\to\mathbb{R}$ be a C^2-function, and $x_\infty\in\mathscr{U}$ be a critical point of $\mathscr{E}$, so $\mathscr{E}'(x_\infty)=0$. Let $\mathscr{M}:\mathscr{U}\to\mathscr{Y}$ be a C^1 gradient map for $\mathscr{E}$, and require that $\mathscr{E}$ be Morse-Bott at x_∞, so $\mathscr{U}\cap\operatorname{Crit}\mathscr{E}$ is relatively open smooth submanifold of $\mathscr{X}$ and $K:=\operatorname{Ker}\mathscr{E}''(x_\infty)=T_{x_\infty}\operatorname{Crit}\mathscr{E}$. Suppose that, for each $x\in\mathscr{U}$, the bounded linear operator $\mathscr{M}'(x):\mathscr{X}\to\mathscr{Y}$ has an extension $\mathscr{M}_1(x):\mathscr{G}\to\mathscr{H}$ such that the following map is continuous:*

$$\mathscr{U}\ni x\mapsto\mathscr{M}_1(x)\in\mathscr{L}(\mathscr{G},\mathscr{H}).$$

Assume that $K\subset\mathscr{X}$ has a closed complement $\mathscr{X}_0\subset\mathscr{X}$, that $\mathscr{K}:=\operatorname{Ker}\mathscr{M}_1(x_\infty)\subset\mathscr{G}$ has a closed complement $\mathscr{G}_0\subset\mathscr{G}$ and that $\operatorname{Ran}\mathscr{M}_1(x_\infty)\subset\mathscr{H}$ is a closed subspace. Then there exists constants $c,\sigma>0$ such that for every $x\in\mathscr{U}$ which satisfies $\|x-x_\infty\|_{\mathscr{X}}<\sigma$,

$$c\cdot|\mathscr{E}(x)-\mathscr{E}(x_\infty)|^{1/2}\le\|\mathscr{M}(x)\|_{\mathscr{H}}.\tag{3.5}$$

Proof. See [11], Theorem 2 in Section 3. □

Corollary 3.5 (Łojasiewicz-Simon Inequality for $L-\lambda A$ functional). *For $\mathscr{E}=L-\lambda A:L^2(\mathbb{S}^1)\to\mathbb{R}$ and its gradient map $\mathscr{M}=d(L-\lambda A):L^2(\mathbb{S}^1)\to H^2(\mathbb{S}^1)$, there exists constants $c,\sigma>0$ such that*

$$c\cdot|(L-\lambda A)(f)-\pi\lambda^{-1}|^{1/2}\le\|d(L-\lambda A)_f\|_{H^2}\qquad(\forall f\in L^2(\mathbb{S}^1)\ s.t.\ \|f-\lambda^{-1}\|_{L^2}<\sigma).\tag{3.6}$$

Proof. Let $\mathscr{X}=\mathscr{G}=L^2(\mathbb{S}^1)$ and $\mathscr{Y}=\mathscr{H}=H^2(\mathbb{S}^1)$; then there is a canonical embedding $H^2(\mathbb{S}^1)\subset(L^2(\mathbb{S}^1))^*\cong L^2(\mathbb{S}^1)$. $\mathscr{E}=(L-\lambda A):L^2(\mathbb{S}^1)\to\mathbb{R}$ is C^2-functional defined on $\mathscr{U}=\mathscr{X}=L^2(\mathbb{S}^1)$. Then $\operatorname{Crit}\mathscr{E}=\lambda^{-1}+\operatorname{span}\{\sin,\cos\}$ is translation of linear subspace. Take x_∞ as constant function λ^{-1}. Since $K=\operatorname{Ker}\mathscr{E}''(x_\infty)=\operatorname{span}\{\sin,\cos\}$, it is trivial that $K=T_{x_\infty}\operatorname{Crit}\mathscr{E}=\operatorname{span}\{\sin,\cos\}$. The gradient map $\mathscr{G}$ is $d(L-\lambda A)$ considered as a map $L^2(\mathbb{S}^1)\to H^2(\mathbb{S}^1)\subset(L^2(\mathbb{S}^1))^*$. Moreover, since $\mathscr{X}=\mathscr{G}$ and $\mathscr{Y}=\mathscr{H}$, $\mathscr{M}'(x):\mathscr{X}\to\mathscr{Y}$ has trivial continuous extension $\mathscr{M}_1(x)=\mathscr{M}'(x)$. Since any finite-dimensional subspace of a given Banach space has closed complement in the given Banach space, complement $\mathscr{X}_0=\mathscr{G}_0=L^2(\mathbb{S}^1)/\operatorname{span}\{\sin,\cos\}$ is closed in $L^2(\mathbb{S}^1)$. We have explicitely shown that $\operatorname{Ran}\mathscr{M}_1(x_\infty)$ is closed subspace of codimension 2 in $H^2(\mathbb{S}^1)$. Theorem 3.4 implies the result. □

4 The Area-Preserving Curve Shortening Flow

Definition 4.1.
(a) The **mean curvature flow (MCF)** [7] is a geometric flow of hypersurfaces in $\mathbb{R}^{n+1}$ evolving according to

$$X_t=HN.$$

Here, H is mean curvature scalar and N is inward unit normal of the hypersurface.
(b) The **volume-preserving mean curvature flow (VPMCF)** [9] is geometric flow of hypersurfaces in $\mathbb{R}^{n+1}$ evolving according to

$$X_t=(H-\bar{H})N.$$

Here, $\bar{H}=\fint H\,d\mathcal{H}^n$ is average of mean curvature.

(c) The **curve shortening flow (CSF)** [8] is mean curvature flow of curves in $\mathbb{R}^2$; in this case, mean curvature is simply $H = k$, and the equation becomes

$$X_t = kN.$$

(d) The **area-preserving curve shortening flow (APCSF)** [2] is a VPMCF of curves in $\mathbb{R}^2$:

$$X_t = kN - \frac{2\pi}{L}N = (k - \bar{k})N.$$

Here, $\bar{k} = 2\pi/L = ⨍ k\, ds$ is average of curvature.

Remark 4.2. Consider the space of hypersurfaces in $\mathbb{R}^{n+1}$. The first variation formula shows that gradient of n-volume $\mathcal{V}$ is [7]

$$\nabla\mathcal{V} = -HN.$$

Therefore, mean curvature flow is gradient flow of the n-volume functional $\mathcal{V}$. Hence, along the MCF $\{M_t\}$, n-volume evolves by

$$\frac{d}{dt}\mathcal{V}(M_t) = \frac{d}{dt}\int_{M_t} d\mathcal{H}^n = -\langle\nabla\mathcal{V}, \nabla\mathcal{V}\rangle = -\int_{M_t} H^2\, d\mathcal{H}^n.$$

In particular, its plane curve version - curve shortening flow - is gradient flow of length functional.

Consider an additional restriction that the enclosed $(n+1)$-volume must be preserved. Since $(n+1)$-volume evolves by

$$\frac{d}{dt}\operatorname{Vol}(M_t) = -\int_{M_t} \langle X_t, N\rangle\; d\mathcal{H}^n,$$

the evolution $X_t = (H - \bar{H})N$ can be considered as projection of evolution $X_t = HN$ to the subspace of evolutions which preserve enclosed $(n+1)$-volume. Therefore, VPMCF is a gradient flow of n-volume under the additional restriction that the enclosed $(n+1)$-volume must be preserved. In particular, its plane curve version - APCSF - is a gradient flow of length functional under additional restriction that the enclosed area must be preserved.

It is well known that APCSF starting from C^2-convex simple closed curve exists for infinite time and asymptotically smoothly exponentially converges to a circle. [2] We will use Łojasiewicz inequality to prove Hausdorff and uniform exponential convergence to circle. As a preliminary, we will assume that we know the long-term existence of flow.

Along the flow, call the curve at time t as $\gamma_t := X(t, -)$; note that subscript t do not mean time-derivative here. Parametrize γ_t by tangent angle. Call its curvature [resp. radius of curvature] at tangent angle $\theta \in \mathbb{S}^1$ as $k(t,\theta)$ [resp. $f(t,\theta)$].

Theorem 4.3. *If we identify curves with its curvature [resp. radius of curvature] function, then APCSF equation translates into*

$$k_t = k^2(k_{\theta\theta} + k - \bar{k}) = k^2k_{\theta\theta} + k^3 - \frac{2\pi k^2}{L}, \tag{4.1}$$

$$f_t = -\frac{k_t}{k^2} = -(f^{-1})_{\theta\theta} - f^{-1} + \frac{2\pi}{L} = \frac{ff_{\theta\theta} - 2f_\theta^2 - f^2}{f^3} + \frac{2\pi}{L}. \tag{4.2}$$

Moreover, this time-evolution satisfies

$$\int_0^{2\pi} f_t e^{i\theta}\, d\theta = 0.$$

Therefore, f still defines closed curve after time-evolution.

Proof. Curvature evolution equation is due to Gage [2]. Call the original parameters as u, t, and let $v = \partial s/\partial u$. Reparametrization of the family of curves amounts to a tangential term in the evolution equation:

$$X_t = \left(k - \frac{2\pi}{L}\right)N + \alpha(u,t)T \tag{4.3}$$

If we differentiate (4.3), change the order of differentiation, use the fact that $X_u = vT$, and compare normal and tangential components, we obtain two component equations:

$$T_t = \left(\frac{1}{v}k_u + \alpha k\right) N \tag{4.4}$$

$$v_t = -vk^2 + v\frac{2\pi k}{L}. \tag{4.5}$$

In order to determine the exact function α which gives reparametrization from u to tangent angle θ, we should require the condition that T is independent on t. (Since θ is defined as an angle between T and x-axis, this means that t and θ are independent parameters.) This requirement implies

$$\alpha = -\frac{1}{vk}k_u. \tag{4.6}$$

Differentiating (4.4) with respect to u and changing the order of differentiation gives

$$0 = \frac{\partial^2 T}{\partial u \partial t} = \frac{\partial^2 T}{\partial t \partial u} = \frac{\partial}{\partial t}(vkN) \tag{4.7}$$

This shows $(vk)_t = 0$. Using (4.5) and (4.6), we obtain

$$k_t = -\frac{1}{v}kv_t = k^3 - \frac{2\pi k^2}{L} + \frac{k}{v}\left(\frac{1}{vk}k_u\right)_u \tag{4.8}$$

If we use $d\theta = kv\, du, \partial/\partial\theta = (vk)^{-1}\partial/\partial u$, this becomes

$$k_t = k^2 k_{\theta\theta} + k^3 - \frac{2\pi k^2}{L}$$

We can directly translate this equation to evolution equation of f using following derivatives:

$$f_t = -k^{-2}k_t \quad , \quad k_\theta = -f^{-2}f_\theta \quad , \quad k_{\theta\theta} = f^{-3}(2f_\theta^2 - ff_{\theta\theta})$$

Therefore, we obtain desired evolution equation of f.

$$f_t = -\frac{k_t}{k^2} = -(f^{-1})_{\theta\theta} - f^{-1} + \frac{2\pi}{L} = \frac{ff_{\theta\theta} - 2f_\theta^2 - f^2}{f^3} + \frac{2\pi}{L}.$$

The last statement follows from

$$\int_0^{2\pi} f_t e^{i\theta}\, d\theta = \int_0^{2\pi} -(f^{-1})_{\theta\theta}e^{i\theta} - f^{-1}e^{i\theta} + \frac{2\pi}{L}e^{i\theta}\, d\theta = \int_0^{2\pi}(f^{-1} - f^{-1})e^{i\theta}\, d\theta + \frac{2\pi}{L}\int_0^{2\pi} e^{i\theta}\, d\theta = 0. \qquad \square$$

Proposition 4.4 (Gage [2]). *Convex curve evolving according to equation* (4.1) *remains convex.*

Proof. Let $W(\theta, t) = k(\theta, t)e^{\mu t}$ for some constant μ. It satisfies

$$W_t = k^2 W_{\theta\theta} + \left(k^2 - \frac{2\pi}{L}k + \mu\right) W \tag{4.9}$$

The coefficient of W is a quadratic polynomial in k whose discriminant is $(4\pi^2/L^2) - 4\mu$. The isoperimetric inequality provides a lower bound for L in terms of area, hence for sufficiently large μ the discriminant is negative and the coefficient of W is positive. Choose μ so that this coefficient is always positive.

Now let $W_{\min}(t) = \inf\{W(\theta, t) : 0 \le \theta \le 2\pi\}$ and suppose that at some time $t > 0$, $W_{\min}(t) = \beta$, where $0 < \beta < W_{\min}(0)$. Let $t_0 = \inf\{t : W_{\min}(t) = \beta\}$. Then β is achieved for the first time at $W(\theta_0, t_0)$ and at this point

$$W_t \le 0, \quad W_{\theta\theta} \ge 0, \quad W = \beta > 0.$$

This contradicts the fact that W satisfies (4.9) and proves that $W_{\min}(t)$ is nondecreasing. Hence

$$k_{\min}(t) = W_{\min}(t) \cdot e^{-\mu t} \ge W_{\min}(0) \cdot e^{-\mu t} > 0. \qquad \square$$

This proposition implies that radius of curvature $1/f$ is well-defined as a finite value whenever the solution k exists. We will use the long-term existence result for evolution equation (4.1) as preliminary:

Proposition 4.5 (Gage [2])**.** *For any positive-valued function $k_0 \in C^2_\theta(\mathbb{S}^1)$, there exists a positive-valued solution $k \in C^1_t C^2_\theta([0,\infty) \times \mathbb{S}^1)$ of the equation (4.1) which has $k_0(-) = k(0,-)$ as initial datum.*

If we combine above two propositions, we obtain long-term existence of $1/f$. We will abuse notation and write $L(t) = L(f(t,-)), L_0 = L(0) = L(f(0,-))$ along the flow.

Proposition 4.6. *Along APCSF, length and area time-evolve by*

$$-L_t = \int (k - \bar{k})\, d\theta = \int (k - \bar{k})^2\, ds \geq 0 \quad , \quad A_t = 0$$

In particular, length is nonincreasing function, and it is decreasing if the curve is not circle. Moreover, area is preserved, and always equals to the initial area A_0.

Proof. These results easily follow from formulae in Remark 4.2; However, we will prove them here using evolution (4.2). Length evolves by

$$\begin{aligned} -L_t &= -\int f_t\, d\theta = \int (f^{-1})_{\theta\theta} + f^{-1} - \frac{2\pi}{L}\, d\theta = \int \frac{1}{f}\, d\theta - \frac{4\pi^2}{L} \\ &= \int (k - \bar{k})\, d\theta = \int k(k - \bar{k})\, ds = \int (k - \bar{k})^2\, ds \geq 0. \end{aligned}$$

This is always nonnegative and is strictly positive when $k \not\equiv \bar{k}$, meaning that k does not correspond to a circle.

Area evolves by

$$\begin{aligned} A_t &= \int_0^{2\pi} f_t \left[\cos\theta \int_0^\theta f(\phi)\sin\phi\, d\phi - \sin\theta \int_0^\theta f(\phi)\cos\phi\, d\phi\right] d\theta \\ &= \int_0^{2\pi} (-(f^{-1})_{\theta\theta} - f^{-1} + 2\pi L^{-1}) \left[\cos\theta \int_0^\theta f(\phi)\sin\phi\, d\phi - \sin\theta \int_0^\theta f(\phi)\cos\phi\, d\phi\right] d\theta \\ &= \int_0^{2\pi} (-f^{-1} + 2\pi L^{-1}) \left[\cos\theta \int_0^\theta f(\phi)\sin\phi\, d\phi - \sin\theta \int_0^\theta f(\phi)\cos\phi\, d\phi\right] \\ &\quad + (f^{-1} - 2\pi L^{-1})_\theta \left[-\sin\theta \int_0^\theta f(\phi)\sin\phi\, d\phi - \cos\theta \int_0^\theta f(\phi)\cos\phi\, d\phi\right] d\theta \\ &= \int_0^{2\pi} (-f^{-1} + 2\pi L^{-1}) \left[\cos\theta \int_0^\theta f(\phi)\sin\phi\, d\phi - \sin\theta \int_0^\theta f(\phi)\cos\phi\, d\phi\right] \\ &\quad - (f^{-1} - 2\pi L^{-1}) \left[-\cos\theta \int_0^\theta f(\phi)\sin\phi\, d\phi + \sin\theta \int_0^\theta f(\phi)\cos\phi\, d\phi - f\right] d\theta \\ &= \int_0^{2\pi} (1 - 2\pi f L^{-1})\, d\theta = 2\pi - 2\pi\frac{L}{L} = 0. \end{aligned}$$

□

Let $\lambda := \sqrt{\pi/A_0}$. Then circle of radius λ^{-1} has area A_0. Since $L(t) \geq 2\pi\lambda^{-1}$ by isoperimetric inequality, $L(t) - 2\pi\lambda^{-1}$ is decreasing nonnegative function along the flow.

5 Łojasiewicz Inequality of APCSF

Theorem 5.1 (Łojasiewicz-Simon Inequality of APCSF)**.** *There exists a positive real number $\delta > 0$ such that the APCSF, starting from any C^2-initial curve $f_0 = f(0,-)$ satisfying $\|f_0(-) - \lambda^{-1}\|_{H^1} < \delta$, meets the following conditions:*

(i) *It remains within the L^2-neighborhood $\|f(t,-) - \lambda^{-1}\|_{L^2} < \sigma$ described in Corollary 3.5, where the optimal Łojasiewicz-Simon inequality for the $L - \lambda A$ functional holds.*

(ii) *It has positive uniform upper and lower bounds* $0 < m \le f(t,\theta) \le M$.
(iii) *Its length* $L(t) = L(f(t,-))$ *converges exponentially to* $2\pi\lambda^{-1}$.

Proof. Fix any C^2-initial curve $f(0,-)$ which satisfies $\|f(0,-) - \lambda^{-1}\|_{H^1} < \frac{1}{10}\lambda^{-1}$, and let T be the supremum of real number which satisfies

$$\|f(t,-) - \lambda^{-1}\|_{H^1} \le \frac{1}{8}\lambda^{-1} \qquad (\forall t \in [0,T)).$$

Then, $T > 0$ by time-continuity of solution $f(t,\theta)$ and T depends on $f(0,-)$. Sobolev inequalities imply

$$\|f(t,-) - \lambda^{-1}\|_{L^\infty} \le \left(\sqrt{2\pi} + \frac{1}{\sqrt{2\pi}}\right) \|f(t,-) - \lambda^{-1}\|_{H^1} < \frac{1}{2}\lambda^{-1},$$

and therefore there exist uniform upper and lower bounds for f and its inverse k.

$$\frac{1}{2}\lambda^{-1} \le f(t,\theta) \le \frac{3}{2}\lambda^{-1} \quad , \quad m := \frac{2}{3}\lambda \le k(t,\theta) \le 2\lambda =: M \qquad (\forall t \in [0,T)\ \forall \theta \in \mathbb{S}^1).$$

We will estimate several values and norms at time $t \in [0,T)$ using initial values.
Step I. Estimate of $L(t) - 2\pi\lambda^{-1}$.

$$\begin{aligned}
L(t) - 2\pi\lambda^{-1} \le L_0 - 2\pi\lambda^{-1} &= \int (f(0,\theta) - \lambda^{-1})\, d\theta \\
&\le \sqrt{2\pi}\left[\int (f(0,\theta) - \lambda^{-1})^2\, d\theta\right]^{1/2} \\
&= \sqrt{2\pi}\|f(0,-) - \lambda^{-1}\|_{L^2}
\end{aligned}$$

Moreover, we may also estimate

$$L(0) - L(t) \le L_0 - 2\pi\lambda^{-1} \le \sqrt{2\pi}\|f(0,-) - \lambda^{-1}\|_{L^2}.$$

Step II. Estimate of $-L_t$.

$$\begin{aligned}
-L_t(t) = -\left.\frac{dL}{dt}\right|_t &= \int \frac{1}{f(t,\theta)} - \frac{2\pi}{L(t)}\, d\theta \\
&= -\int \frac{\lambda}{f}(f - \lambda^{-1})\, d\theta + \frac{2\pi}{L\lambda^{-1}}(L - 2\pi\lambda^{-1}) \\
&= \int \frac{\lambda^2}{f}(f - \lambda^{-1})^2 - \lambda^2(f - \lambda^{-1})\, d\theta + \frac{2\pi}{L\lambda^{-1}}(L - 2\pi\lambda^{-1}) \\
&= \int \frac{\lambda^2}{f}(f - \lambda^{-1})^2\, d\theta - 2\pi\frac{\lambda^2}{L}(L - 2\pi\lambda^{-1})^2
\end{aligned}$$

Therefore, we obtain

$$m\lambda^2\|f(t,-) - \lambda^{-1}\|_{L^2}^2 - 2\pi\frac{\lambda^2}{L_0}(L(t) - 2\pi\lambda^{-1})^2 \le -L_t(t) \le M\lambda^2\|f(t,-) - \lambda^{-1}\|_{L^2}^2.$$

Step III. Estimate of $\|f(t,-) - \lambda^{-1}\|_{L^2}^2$.

$$\begin{aligned}
\frac{d}{dt}\|f(t,-) - \lambda^{-1}\|_{L^2}^2 &= 2\int (f - \lambda^{-1}) f_t\, d\theta \\
&= 2\int \frac{f f_{\theta\theta} - 2f_\theta^2}{f^2} + \cancelto{0}{(-1 + 2\pi L^{-1} f)} - \cancelto{0}{\lambda^{-1}(f^{-1})_{\theta\theta}} + \lambda^{-1}\left(f^{-1} - \frac{2\pi}{L}\right) d\theta \\
&= 2\int \cancelto{0}{(f^{-1} f_\theta)_\theta} - \frac{f_\theta^2}{f^2} + \lambda^{-1}\left(f^{-1} - \frac{2\pi}{L}\right) d\theta \\
&= -2\lambda^{-1} L_t - 2\int \frac{f_\theta^2}{f^2}\, d\theta \\
&\le -2\lambda^{-1} L_t.
\end{aligned}$$

Integration yields

$$\begin{aligned}\|f(t,-)-\lambda^{-1}\|_{L^2}^2 &\le 2\lambda^{-1}(L_0-L(t))+\|f(0,-)-\lambda^{-1}\|_{L^2}^2\\ &\le 2\lambda^{-1}(L_0-2\pi\lambda^{-1})+\|f(0,-)-\lambda^{-1}\|_{L^2}^2\\ &\le \|f(0,-)-\lambda^{-1}\|_{L^2}(2\sqrt{2\pi}\lambda^{-1}+\|f(0,-)-\lambda^{-1}\|_{L^2})\\ &\le 6\lambda^{-1}\|f(0,-)-\lambda^{-1}\|_{L^2}.\end{aligned}$$

Step IV. Estimate of $\|k(t,-)-\bar{k}(t)\|_{L^2}^2$.

$$\begin{aligned}\|k(t,-)-\bar{k}(t)\|_{L^2}^2 &= \int\left(\frac{1}{f(t,\theta)}-\frac{2\pi}{L(t)}\right)^2\, d\theta=\int f^{-2}L^{-2}\left(L-2\pi f\right)^2\, d\theta\\ &\le \frac{M^2}{L^2}\int\left[(L-2\pi\lambda^{-1})-2\pi(f-\lambda^{-1})\right]^2\, d\theta\\ &\le 2\frac{M^2}{L^2}\left[2\pi(L-2\pi\lambda^{-1})^2+4\pi^2\int(f-\lambda^{-1})^2\, d\theta\right]\\ &\le 2M^2\lambda^2\left[\frac{(L_0-2\pi\lambda^{-1})^2}{2\pi}+\|f(t,-)-\lambda^{-1}\|_{L^2}^2\right]\\ &\le 2M^2\lambda^2\left[\|f(0,-)-\lambda^{-1}\|_{L^2}^2+6\lambda^{-1}\|f(0,-)-\lambda^{-1}\|_{L^2}\right]\\ &\le 13M^2\lambda\|f(0,-)-\lambda^{-1}\|_{L^2}.\end{aligned}$$

Step V. Estimate of $\|k_\theta(0,-)\|_{L^2}^2$.

$$\|k_\theta(0,-)\|_{L^2}^2=\int[(f^{-1})_\theta]^2\, d\theta=\int\frac{f_\theta^2}{f^4}\, d\theta\le M^4\int f_\theta^2\, d\theta=M^4\|f_\theta(0,-)\|_{L^2}^2$$

Step VI. Estimate of $\|k_\theta(t,-)\|_{L^2}^2$.

$$\begin{aligned}\frac{d}{dt}(\|k(t,-)-\bar{k}(t)\|_{L^2}^2-\|k_\theta(t,-)\|_{L^2}^2) &= 2\int(k-\bar{k})(k_t-\bar{k}_t)\, d\theta-2\int k_\theta k_{\theta t}\, d\theta\\ &= 2\int(k_{\theta\theta}+k-\bar{k})k_t\, d\theta-2\bar{k}_t\int(k-\bar{k})\, d\theta\\ &= 2\int k^2(k_{\theta\theta}+k-\bar{k})^2\, d\theta-4\pi\frac{L_t^2}{L^2}\\ &\ge -4\pi\frac{L_t^2}{L^2}\ge-\pi^{-1}M\lambda^4\|f(t,-)-\lambda^{-1}\|_{L^2}^2\cdot(-L_t)\\ &\ge -2M\lambda^3\|f(0,-)-\lambda^{-1}\|_{L^2}\cdot(-L_t)\end{aligned}$$

Integration yields following inequality:

$$\begin{aligned}\|k(t,-)-\bar{k}(t)\|_{L^2}^2-\|k_\theta(t,-)\|_{L^2}^2\ge{}&\|k(0,-)-\bar{k}(0)\|_{L^2}^2-\|k_\theta(0,-)\|_{L^2}^2\\ &-2M\lambda^3\|f(0,-)-\lambda^{-1}\|_{L^2}\cdot(L_0-L(t))\end{aligned}$$

In particular, this implies

$$\begin{aligned}\|k_\theta(t,-)\|_{L^2}^2 &\le \|k(t,-)-\bar{k}(t)\|_{L^2}^2+\|k_\theta(0,-)\|_{L^2}^2+2M\lambda^3\|f(0,-)-\lambda^{-1}\|_{L^2}\cdot(L_0-L(t))\\ &\le 13M^2\lambda\|f(0,-)-\lambda^{-1}\|_{L^2}+M^4\|f_\theta(0,-)\|_{L^2}^2+6M\lambda^3\|f(0,-)-\lambda^{-1}\|_{L^2}^2\\ &\le 52\lambda^3\|f(0,-)-\lambda^{-1}\|_{L^2}+16\lambda^4\|f(0,-)-\lambda^{-1}\|_{H^1}^2\\ &\le 54\lambda^3\|f(0,-)-\lambda^{-1}\|_{H^1}\end{aligned}$$

Step VII. Estimate of $\|f_\theta(t,-)\|_{L^2}^2$.

$$\|f_\theta(t,-)\|_{L^2}^2=\int f^4(f^{-1})_\theta^2\, d\theta\le m^{-4}\int k_\theta^2\, d\theta=m^{-4}\|k_\theta(t,-)\|_{L^2}^2\le 274\lambda^{-1}\|f(0,-)-\lambda^{-1}\|_{H^1}$$

This proves that f stays in the given H^1-neighborhood (on which Łojasiewicz inequality holds) along the flow if f was initially contained in an appropriate (possibly smaller) H^1-neighborhood of λ^{-1}.

Step VIII. Estimate of $\|f(t,-)-\lambda^{-1}\|_{H^1}^2$.

$$\begin{aligned}\|f(t,-)-\lambda^{-1}\|_{H^1}^2 &= (\|f(t,-)-\lambda^{-1}\|_{L^2}+\|f_\theta(t,-)\|_{L^2})^2\\ &\le 2(\|f(t,-)-\lambda^{-1}\|_{L^2}^2+\|f_\theta(t,-)\|_{L^2}^2)\\ &\le 2(6\lambda^{-1}\|f(0,-)-\lambda^{-1}\|_{L^2}+274\lambda^{-1}\|f(0,-)-\lambda^{-1}\|_{H^1})\\ &\le 560\lambda^{-1}\|f(0,-)-\lambda^{-1}\|_{H^1}.\end{aligned}$$

Step IX. Let

$$\sigma' := \min\left\{\sigma, \frac{c}{225}, \frac{\lambda^{-1}}{10}\right\},$$

where c,σ are as in the Łojasiewicz inequality. Let

$$\delta := \frac{1}{560}\lambda(\sigma')^2.$$

Therefore, we may conclude that for any C^2-initial curve $f(0,-)$ which satisfies $\|f(0,-)-\lambda^{-1}\|_{H^1}<\delta$, then $\|f(t,-)-\lambda^{-1}\|_H^1<\sigma'$ for any $t\in[0,T)$. Here, $T>0$ depends on $f(0,-)$, and it is the supremum of real numbers which satisfy

$$\|f(t,-)-\lambda^{-1}\|_{H^1}\le\lambda^{-1}/8 \qquad (\forall t\in[0,T)).$$

Assume that $T<\infty$. Then there exists some $t'>T$ arbitrarily close to T which satisfies

$$\|f(t',-)-\lambda^{-1}\|_{H^1}>\lambda^{-1}/8.$$

However, it is contradiction to the continuity of $f(t,-)$ since $\|f(t,-)-\lambda^{-1}\|_H^1<\sigma'\le\lambda^{-1}/10$ holds for any $t<T$. This implies $T=\infty$ for any C^2-initial curve $f(0,-)$ with $\|f(0,-)-\lambda^{-1}\|_{H^1}<\delta$. Therefore, for any C^2-initial curve $f(0,-)$ with $\|f(0,-)-\lambda^{-1}\|_{H^1}<\delta$, $\|f(t,-)-\lambda^{-1}\|_H^1<\sigma'$ for any $t\in[0,\infty)$.

Step X. For such solution $f(t,\theta)$, we may apply Łojasiewicz inequality:

$$\begin{aligned}c(L(t)-2\pi\lambda^{-1}) &\le \|d(L-\lambda A)_{f(t,-)}\|_{H^2}^2\le 25\lambda^2\|f(t,-)-\lambda^{-1}\|_{L^2}^2\\ &\le 25m^{-1}(-L_t+2\pi\lambda^2L_0^{-1}(L(t)-2\pi\lambda^{-1})^2).\end{aligned}$$

Since

$$L(t)-2\pi\lambda^{-1}\le\sqrt{2\pi}\|f(0,-)-\lambda^{-1}\|_{L^2}\le\sqrt{2\pi}\sigma'\le\frac{\sqrt{2\pi}c}{225}\le\frac{c}{75}=\frac{cm(2\pi\lambda^{-1})}{100\pi}\le\frac{cmL_0}{100\pi},$$

we obtain

$$\tfrac{1}{75}\lambda c(L(t)-2\pi\lambda^{-1})=\tfrac{1}{50}mc(L(t)-2\pi\lambda^{-1})\le\tfrac{1}{25}mc(L(t)-2\pi\lambda^{-1})-2\pi\lambda^2L_0^{-1}(L(t)-2\pi\lambda^{-1})^2\le -L_t.$$

Integration yields

$$L(t)-2\pi\lambda^{-1}\le(L_0-2\pi\lambda^{-1})\exp(-\tfrac{1}{75}\lambda ct).$$

□

6 Convergence of APCSF

Recall that for closed plane curves γ_1,γ_2, **Hausdorff distance** between them is defined as

$$\mathrm{dist}(\gamma_1,\gamma_2)=\max\left(\sup_{x\in\gamma_1}\inf_{y\in\gamma_2}|x-y|+\sup_{x\in\gamma_2}\inf_{y\in\gamma_1}|x-y|\right)$$

(Here, we slightly abuse notation by letting γ_1 denote $\mathrm{Im}(\gamma_1)$.)

Theorem 6.1 (Exponential Convergence of APCSF). *The APCSF $\{\gamma_t\}$, starting from an initial curve satisfying the assumptions of Theorem 1.2, converges exponentially to a circle γ_∞ with radius λ^{-1} in both the uniform distance and the Hausdorff distance.*

Proof. Let u be the original parametrization of the flow $X(t,u)$. Then $\gamma_t(u) = X(t,u)$ and

$$X_t = (k - \bar{k})N.$$

Therefore, trajectory $x(-) = X(-,u)$ starting at $x_1 = X(t_1,u) = \gamma_{t_1}(u)$ satisfies

$$x(t) = x_1 + \int_{t_1}^{t} X_t(t', x(t'))\, dt'$$

$$|\gamma_{t_2}(u) - \gamma_{t_1}(u)| = |x(t_2) - x(t_1)| \leq \int_{t_1}^{t_2} |X_t|(t', x(t'))\, dt' \leq \int_{t_1}^{t_2} \|X_t(t', -)\|_{L^\infty}\, dt'$$

Therefore,

$$\|\gamma_{t_2} - \gamma_{t_1}\|_{L^\infty} \leq \int_{t_1}^{t_2} \|X_t(t', -)\|_{L^\infty}\, dt'$$

Assume that it is possible to bbound $\int_{t_1}^{t_2} \|X_t(t', -)\|_{L^\infty}\, dt'$ by some exponential function $C_1 \exp(-C_2 t_1)$. Then $\{\gamma_t\}$ is uniformly Cauchy, so it converges exponentially uniformly to a continuous curve γ_∞:

$$\|\gamma_t - \gamma_\infty\|_{L^\infty} \leq C_1 \exp(-C_2 t).$$

Such uniform limit γ_∞ must be a circle of radius λ^{-1} by equality condition of the isoperimetric inequality, since its length is

$$L(\gamma_\infty) = \lim_{t\to\infty} L(t) = 2\pi\lambda^{-1}.$$

Moreover, any continuous curves γ_1, γ_2 defined on parameter domain $u \in \mathbb{S}^1$ satisfy

$$\sup_{x\in\gamma_1} \inf_{y\in\gamma_2} |x - y| \leq \|\gamma_1 - \gamma_2\|_{L^\infty},$$

since for any $x = \gamma_1(u) \in \gamma_1$ there exists $y := \gamma_2(u)$ which satisfies $|x-y| = |\gamma_1(u) - \gamma_2(u)| \leq \|\gamma_1 - \gamma_2\|_{L^\infty}$. If we interchange the roles of γ_1, γ_2 and then take maximum, we obtain

$$\mathrm{dist}(\gamma_1, \gamma_2) \leq \|\gamma_1 - \gamma_2\|_{L^\infty}.$$

Therefore, uniform exponential convergence implies Hausdorff exponential convergence.

$$\mathrm{dist}(\gamma_t, \gamma_\infty) \leq \|\gamma_t - \gamma_\infty\|_{L^\infty} \leq C_1 \exp(-C_2 t).$$

Hence, it suffices to show

$$\int_{t_1}^{t_2} \|X_t(t', -)\|_{L^\infty}\, dt' \leq C_1 \exp(-C_2 t_1)$$

in order to prove both Hausdorff and uniform exponential convergence.

We have $|X_t|(u,t) = |k(\theta(u),t) - \bar{k}(t)|$. Change parametrization from u to θ; this does not affect the L^∞-norm. Sobolev inequalities imply

$$\|k(t,-) - \bar{k}(t)\|_{L^\infty} \leq \left(\sqrt{2\pi} + \frac{1}{\sqrt{2\pi}}\right) \max(\|k(t,-) - \bar{k}(t)\|_{L^2}, \|k_\theta(t,-)\|_{L^2}) \leq 3\|k(t,-) - \bar{k}(t)\|_{H^1}.$$

We know that

$$\begin{aligned}
\|k(t,-) - \bar{k}(t)\|_{L^2}^2 &\leq \pi^{-1} M^2 \lambda^2 (L(t) - 2\pi\lambda^{-1})^2 + 2M^2\lambda^2 \|f - \lambda^{-1}\|_{L^2}^2(t) \\
&\leq (\pi^{-1} M^2 \lambda^2 + 4\pi M^2 m^{-1} \lambda^2 L_0^{-1})(L(t) - 2\pi\lambda^{-1})^2 + 2M^2 m^{-1}(-L_t) \\
&\leq (\pi^{-1} M^2 \lambda^2 + 4\pi M^2 m^{-1} \lambda^2 L_0^{-1})(L_0 - 2\pi\lambda^{-1})^2 \exp(-\tfrac{2}{75}\lambda c t) + 2M^2 m^{-1}(-L_t) \\
&\leq 14\lambda^4 (L_0 - 2\pi\lambda^{-1})^2 \exp(-\tfrac{2}{75}\lambda c t) + 12\lambda(-L_t) \\
&\leq 100\lambda^4 \|f(0,-) - \lambda^{-1}\|_{L^2}^2 \exp(-\tfrac{2}{75}\lambda c t) + 12\lambda(-L_t) \\
&\leq \lambda^2 \exp(-\tfrac{2}{75}\lambda c t) + 12\lambda(-L_t),
\end{aligned}$$

and

$$
\begin{aligned}
\|k_\theta(t,-)\|_{L^2}^2 &= \int k_\theta^2\,d\theta \le m^{-1}\int kk_\theta^2\,d\theta = -\frac{1}{2m}\int k^2 k_{\theta\theta}\,d\theta = -\frac{1}{2m}\int k^2(k^{-2}k_t-(k-\bar{k}))\,d\theta \\
&\le \frac{M^2}{2m}\left|\int (k^{-1})_t\,d\theta\right| + \frac{1}{2m}\int k^3(k-\bar{k})\,ds = \frac{M^2}{2m}\left|\frac{d}{dt}\int \frac{1}{k}\,d\theta\right| + \frac{1}{2m}\int (k^3-\bar{k}^3)(k-\bar{k})\,ds \\
&= -\frac{M^2}{2m}\frac{dL}{dt} + \frac{1}{2m}\int (k^2+k\bar{k}+\bar{k}^2)(k-\bar{k})^2\,ds \le \left(\frac{M^2}{2m}+\frac{3M^2}{2m}\right)(-L_t) \\
&= 2M^2m^{-1}(-L_t) = 12\lambda(-L_t).
\end{aligned}
$$

Therefore, we can conclude that

$$
\begin{aligned}
\|k(t,-)-\bar{k}(t)\|_{H^1}^2 &= (\|k(t,-)-\bar{k}(t)\|_{L^2} + \|k_\theta(t,-)\|_{L^2})^2 \\
&\le 2\|k(t,-)-\bar{k}(t)\|_{L^2}^2 + 2\|k_\theta(t,-)\|_{L^2}^2 \\
&\le 48\lambda(-L_t) + 2\lambda^2\exp(-\tfrac{2}{75}\lambda ct)
\end{aligned}
$$

Integration yields

$$
\begin{aligned}
\int_{t_1}^{t_2} \|k(t,-)-\bar{k}(t)\|_{L^\infty}\,dt &\le 3\int_{t_1}^{t_2} \|k(t,-)-\bar{k}(t)\|_{H^1}\,dt \\
&\le 3\left(\int_{t_1}^{\infty} \|k(t,-)-\bar{k}(t)\|_{H^1}^2 (t+\varepsilon^{-1/\varepsilon})^{1+\varepsilon}\,dt\right)^{1/2}\left(\int_{t_1}^{\infty}(t+\varepsilon^{-1/\varepsilon})^{-1-\varepsilon}\,dt\right)^{1/2} \\
&\le 3\left(\int_{t_1}^{\infty} \|k(t,-)-\bar{k}(t)\|_{H^1}^2 (t+\varepsilon^{-1/\varepsilon})^{1+\varepsilon}\,dt\right)^{1/2}\left(\varepsilon^{-1}(t_1+\varepsilon^{-1/\varepsilon})^{-\varepsilon}\right)^{1/2} \\
&\le 3\left(\int_{t_1}^{\infty} 48\lambda(-L_t)(t+\varepsilon^{-1/\varepsilon})^{1+\varepsilon} + 2\lambda^2(t+\varepsilon^{-1/\varepsilon})^{1+\varepsilon}\exp(-\tfrac{2}{75}\lambda ct)\,dt\right)^{1/2}
\end{aligned}
$$

Here, $\varepsilon > 0$ is some fixed positive number. Integral of the first term can be estimated as follows:

$$
\begin{aligned}
\int_{t_1}^{\infty} (-L_t)(t+\varepsilon^{-1/\varepsilon})^{1+\varepsilon}\,dt &= -\left[(L(t)-2\pi\lambda^{-1})(t+\varepsilon^{-1/\varepsilon})^{1+\varepsilon}\right]_{t_1}^{\infty} \\
&\quad + (1+\varepsilon)\int_{t_1}^{\infty}(L(t)-2\pi\lambda^{-1})(t+\varepsilon^{-1/\varepsilon})^{\varepsilon}\,dt \\
&\le (L_0-2\pi\lambda^{-1})(t_1+\varepsilon^{-1/\varepsilon})^{1+\varepsilon}\exp(-\tfrac{1}{75}\lambda ct_1) \\
&\quad + (1+\varepsilon)(L_0-2\pi\lambda^{-1})\int_{t_1}^{\infty}(t+\varepsilon^{-1/\varepsilon})^{\varepsilon}\exp(-\tfrac{1}{75}\lambda ct)\,dt
\end{aligned}
$$

There are some constants $C_0, C_2 > 0$ which satisfies

$$
\begin{aligned}
2\lambda^2(t+\varepsilon^{-1/\varepsilon})^{1+\varepsilon}\exp(-\tfrac{2}{75}\lambda ct) &\le C_0e^{-2C_2t} \\
48\lambda(1+\varepsilon)(L_0-2\pi\lambda^{-1})(t+\varepsilon^{-1/\varepsilon})^{\varepsilon}\exp(-\tfrac{1}{75}\lambda ct) &\le C_0e^{-2C_2t} \\
48\lambda(L_0-2\pi\lambda^{-1})(t+\varepsilon^{-1/\varepsilon})^{1+\varepsilon}\exp(-\tfrac{1}{75}\lambda ct) &\le C_0e^{-2C_2t}
\end{aligned}
$$

Therefore, there exists some $C_1 > 0$ depends on C_0, C_2, λ such that

$$
3\left(\int_{t_1}^{\infty} 2M^2m^{-1}(-L_t)(t+\varepsilon^{-1/\varepsilon})^{1+\varepsilon} + C(t+\varepsilon^{-1/\varepsilon})^{1+\varepsilon}\exp(-\tfrac{1}{3}m\lambda^2ct)\,dt\right)^{1/2} \le C_1e^{-C_2t}.
$$

Thus, for any $0 \le t_1 \le t_2$,

$$
\int_{t_1}^{t_2} \|k-\bar{k}\|_{L^\infty}\,dt \le C_1\exp(-C_2t_1).
$$

This completes the proof. $\square$

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