Łojasiewicz Inequality of Area-Preserving Curve Shortening Flow

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Abstract

In this paper, we present an alternative proof of the uniform and Hausdorff exponential convergence results for Michael Gage's area-preserving curve shortening flow (APCSF) using Leon Simon's framework based on Łojasiewicz-Simon inequalities. We introduce the $L - \lambda A$ functional on $L^2(\mathbb{S}^1)$ and establish the optimal Łojasiewicz-Simon inequality for it to achieve the desired convergence.

1 Introduction

The **area-preserving curve shortening flow (APCSF)** $X : [0, T) \times \mathbb{S}^1 \to \mathbb{R}^2$ is a geometric flow for plane curves, where the evolving family of simple closed curves $\gamma_t(-) = X(t, -)$ satisfies the equation

$$X_t = (k - \bar{k})N \quad , \quad k_t = k^2(k_{\theta\theta} + k - \bar{k}),$$

where k is the curvature, θ is the turning angle, N is the inward unit normal, and $\bar{k} = 2\pi/L$ represents the average curvature. This flow is the one-dimensional case of the volume-preserving mean curvature flow (VPMCF) as studied in [9]. The term "area-preserving" arises because the enclosed area of the evolving curves $\{\gamma_t\}$ remains constant, even as their lengths decrease. This behavior stems from the fact that APCSF is a gradient flow of the length functional under the constraint of area preservation. (See Subsection 2.2 for detailed arguments.)

The APCSF was first introduced by Michael Gage [2] as a variant of the classical curve shortening flow (CSF) [8]. Gage demonstrated that the APCSF, starting from any convex simple closed C^2-plane curve, exists for all time $t > 0$ and converges exponentially in the smooth topology to a circle enclosing the area as the initial curve. This result also implies exponential convergence in the uniform and Hausdorff metrics.

Let L denote the length of a plane curve and A its enclosed area. In this paper, we first introduce the $L - \lambda A$ functional on $L^2(\mathbb{S}^1)$, where $\lambda > 0$ is a Lagrange multiplier. We then establish the optimal Łojasiewicz-Simon inequality for this functional in a $L^2(\mathbb{S}^1)$-neighborhood of constant function λ^{-1}, which is a critical point of the functional. Next, we show that the APCSF, starting from an initial curve sufficiently close to λ^{-1}, remains within a small $H^1(\mathbb{S}^1)$-neighborhood of λ^{-1}, and we prove the exponential convergence of the curve length. Finally, we provide an alternative proof of the uniform and Hausdorff exponential convergence of the APCSF by interpreting it as the Łojasiewicz convergence theorem derived from the optimal Łojasiewicz-Simon inequality. Notably, our results require only the assumption of the long-term existence of the flow, which was shown by Gage [2].

This paper is organized as follows. In Subsection 2.2, we begin by characterizing convex simple closed plane curves through their reciprocal of curvature (radius of curvaure) functions in $L^2(\mathbb{S}^1)$. While curvature functions are traditionally employed in the study of CSF and its variants, the set of curvature functions defining closed curves forms a Banach submanifold rather than a linear subspace. To address these complexities, we instead utilize reciprocal of curvature functions.

Subsequently, in Section 3, we introduce the $L - \lambda A$ functional on $L^2(\mathbb{S}^1)$, where L denotes the curve length, A denotes the enclosed area, and $\lambda > 0$ serves as a Lagrange multiplier.

Definition 1.1. On $L^2(\mathbb{S}^1)$, we define $Q : L^2(\mathbb{S}^1) \to L^2(\mathbb{S}^1)$ and $L - \lambda A : L^2(\mathbb{S}^1) \to \mathbb{R}$ as

$$Qf(\theta) := f(\theta) - \frac{1}{\pi}\sin\theta \int_0^{2\pi} f(\phi)\sin\phi\, d\phi - \frac{1}{\pi}\cos\theta \int_0^{2\pi} f(\phi)\cos\phi\, d\phi, \tag{1.1}$$

$$(L - \lambda A)(f) := L(Qf) - \lambda A(Qf) = \int_0^{2\pi} Qf\, d\theta - \lambda \int_0^{2\pi} Qf(\theta)\cos\theta \int_0^{\theta} Qf(\phi)\sin\phi\, d\phi\, d\theta. \tag{1.2}$$

Here, we employ the orthogonal projection Q to eliminate the pathological sin and cos components. Intuitively, this functional facilitates the minimization of curve length while preserving the enclosed area. Since APCSF represents the gradient flow of the length functional within the space of plane curves under the constraint of area preservation, the $L - \lambda A$ functional is intrinsically linked to APCSF.

We then establish the optimal Łojasiewicz-Simon inequality with a critical exponent of $1/2$ for the $L-\lambda A$ functional. This is achieved using the Lyapunov-Schmidt reduction method for Morse-Bott functionals on Banach spaces, as developed by Paul Feehan [11].

Theorem 1.2 (Łojasiewicz-Simon inequality for the $L-\lambda A$ functional)**.** *Let $\lambda > 0$ be any fixed constant. For $L-\lambda A : L^2(\mathbb{S}^1) \to \mathbb{R}$ and its gradient map $d(L-\lambda A) : L^2(\mathbb{S}^1) \to H^2(\mathbb{S}^1)$, there exists constants $c, \sigma > 0$ depending on λ such that*

$$c \cdot |(L - \lambda A)(f) - \pi\lambda^{-1}|^{1/2} \leq \|d(L - \lambda A)_f\|_{H^2} \qquad (\forall f \in L^2(\mathbb{S}^1)\ s.t.\ \|f - \lambda^{-1}\|_{L^2} < \sigma). \tag{1.3}$$

Finally, in Sections 4 and 5, we provide alternative proofs of the Hausdorff and uniform exponential convergence results for APCSF within the framework of the Łojasiewicz-Simon inequality. Since the Łojasiewicz-Simon inequality applied here is optimal, with a critical exponent of 1/2, the associated Łojasiewicz convergence theorem guarantees an exponential convergence rate. This is in contrast to non-optimal inequalities, which only yield polynomial convergence rates.

Theorem 1.3 (Łojasiewicz-Simon Inequality of APCSF)**.** *There exists a positive real number $\delta > 0$, depending on constants $\lambda > 0$ and $c, \sigma > 0$ described in Theorem 1.2, such that the APCSF, starting from any C^2-initial curve with enclosed area $A_0 = \pi\lambda^{-2}$ and whose reciprocal of curvature $f_0 = f(0, -)$ satisfies $\|f_0(-) - \lambda^{-1}\|_{H^1} < \delta$, fulfills the following conditions:*

(i) *It remains within the $L^2(\mathbb{S}^1)$-neighborhood $\|f(t, -) - \lambda^{-1}\|_{L^2} < \sigma$ described in Theorem 1.2, where the optimal Łojasiewicz-Simon inequality for the $L - \lambda A$ functional holds.*
(ii) *It has positive uniform upper and lower bounds $0 < m \leq f(t, \theta) \leq M$.*
(iii) *Its length $L(t) = L(f(t, -))$ converges exponentially to $2\pi\lambda^{-1}$.*

Theorem 1.4 (Łojasiewicz Convergence Theorem for APCSF)**.** *The APCSF $\{\gamma_t\}$, starting from an initial curve satisfying the assumptions of Theorem 1.3, converges exponentially to a circle γ_∞ with radius λ^{-1} in both the uniform distance and the Hausdorff distance.*

Although these convergence results are well-known, our alternative proof offers new insights by applying the Łojasiewicz-Simon inequality to APCSF for the first time. This approach suggests the potential to extend Simon's framework to a broader class of geometric flows.

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2 Preliminaries

2.1 Łojasiewicz-Simon Inequalities

Gradient inequalities are inequalities which estimate the increment of a functional by the increment of its gradient map. Their rigorous definitions are as follows:

Definition 2.1 (Gradient Maps; Huang [1], Definition 2.1.1)**.** Let $\mathscr{X}, \mathscr{Y}$ be Banach spaces with continuous embedding $\mathscr{Y} \subset \mathscr{X}^*$. Let $\mathscr{U} \subset \mathscr{X}$ be an open subset. A continuous map $\mathscr{M} : \mathscr{U} \to \mathscr{Y}$ is called a **gradient map** if there exists a C^1 functional $\mathscr{E} : \mathscr{U} \to \mathbb{R}$ such that $\mathscr{M}(u) = \mathscr{E}'(u)$ for all $u \in \mathscr{U}$:

$$\mathscr{E}'(u)h = \langle \mathscr{M}(u), h \rangle \qquad (\forall u \in \mathscr{U}, h \in \mathscr{X}), \tag{2.1}$$

where $\langle -, - \rangle$ is the canonical bilinear form on $\mathscr{X}^* \times \mathscr{X}$. Correspondingly, $\mathscr{E}$ is called a **potential** for the map $\mathscr{M}$.

Definition 2.2 (Gradient Inequalities; Huang [1], Definition 2.2.2)**.** A functional $\mathscr{E}$ satisfies **gradient inequality** in $\mathscr{V}$ with respect to function ϕ if

$$\phi(\mathscr{E}(v)) \leq \|\mathscr{E}'(v)\| \qquad (\forall u \in \mathscr{V}). \tag{2.2}$$

A particularly significant class of gradient inequalities is the **Łojasiewicz-Simon (LS) type**, which takes the form:

$$c \cdot |\mathscr{E}(u) - \mathscr{E}(\varphi)|^\theta \leq \|\mathscr{E}'(u)\| \qquad (\forall u \in U, \|u - \varphi\| < \sigma), \tag{2.3}$$

where $c > 0, \sigma > 0$, and $\theta \in [1/2, 1)$ are positive constants, and θ is referred to as the **Łojasiewicz exponent**.

Łojasiewicz-Simon inequalities are named after Polish mathematician Stanisław Łojasiewicz [4], who first established such inequalities for analytic functions on finite-dimensional Euclidean spaces, and Leon Simon [5], who first generalized Łojasiewicz's result to functionals on infinite-dimensional function spaces. These inequalities have become a cornerstone of modern geometric analysis due to their wide applicability in studying the behavior of gradient flows.

If the Łojasiewicz-Simon inequality holds for some exponent θ_0, then it also holds for any $\theta \in [\theta_0, 1)$ (possibly with different constants c and σ). Therefore, the inequality with Łojasiewicz exponent $\theta = 1/2$ is the strongest result and is referred to as **optimal** or **critical exponent** case.

The Łojasiewicz inequality for a functional is a powerful tool for proving the convergence of gradient flows associated with the functional. To illustrate its application, consider a real-analytic function $F : \mathbb{R}^n \to \mathbb{R}$ defined on a finite-dimensional space. For any critical point x_∞ of F, Łojasiewicz's original work [4] established the following inequality:

$$c \cdot |F(x) - F(x_\infty)|^\theta \leq |\nabla F(x)|. \tag{2.4}$$

The (negative) gradient flow of F is given by

$$x'(t) = -\nabla F(x(t)). \tag{2.5}$$

The **Łojasiewicz convergence theorem** for F states that this gradient flow converges to a critical point, and also provides the convergence rate.

Theorem 2.3 (Finite-Dimensional Łojasiewicz Convergence Theorem [4, 17])**.** *Let $F : \mathbb{R}^n \to \mathbb{R}$ be a real-analytic function, and let $x = x(t) : [1, \infty) \to \mathbb{R}^n$ be an integral curve satisfying $x'(t) = -\nabla F$, where $x(t)$ has a limit point x_∞. Then the total length of the integral curve is finite, and the curve converges to the limit point x_∞ as $x \to \infty$:*

$$\lim_{t \to \infty} x(t) = x_\infty. \tag{2.6}$$

Moreover, the convergence rate depends on the Łojasiewicz exponent θ: it is polynomial in the non-optimal cases when $\theta > 1/2$, and exponential in the optimal case $\theta = 1/2$.

Proof. Since x_∞ is a limit point of $x(t)$ and f is non-increasing along the curve, x_∞ must be a critical point for F. Therefore, the inequality (2.4) holds in a neighborhood of x_∞. Assume that $x(t)$ remains within a sufficiently small neighborhood of x_∞ where the Łojasiewicz inequality is valid; using the fact that x_∞ is a limit point, we can easily extend this situation to the general case.

Differentiating $F(x(t))$, we find

$$-\frac{d}{dt} F(x(t)) = -\nabla F(x(t)) \cdot x'(t) = |\nabla F(x(t))|^2 \geq c^2 \cdot |F(x(t)) - F(x_\infty)|^{2\theta}. \tag{2.7}$$

Letting $\tilde{F}(t) = F(x(t)) - F(x_\infty) \geq 0$, the above inequality becomes

$$-\frac{d\tilde{F}}{dt} \geq c^2 \tilde{F}(t)^{2\theta}. \tag{2.8}$$

Integrating this differential over $[t_0, \infty)$, we obtain

$$\tilde{F}(t) \leq \begin{cases} Ct^{-1/(2\theta-1)} & (\theta > 1/2) \\ C\exp(-c^2 t) & (\theta = 1/2) \end{cases}. \tag{2.9}$$

Next, consider the total length of the curve $x(t)$:

$$\begin{aligned} L(x(-)) &= \int_1^\infty |x'(t)|\, dt = \int_1^\infty |\nabla F(x(t))|\, dt = \int_1^\infty (-\tilde{F}'(t))^{1/2}\, dt \\ &\leq \left(-\int_1^\infty \tilde{F}'(t) \cdot t^{1+\varepsilon}\, dt\right)^{1/2} \left(\int_1^\infty t^{-1-\varepsilon}\, dt\right)^{1/2} \end{aligned} \tag{2.10}$$

Here, $\varepsilon > 0$ is a constant to be determined below. The second integral of the last line is clearly finite. To ensure the first integral is finite and to analyze the convergence rate, we estimate

$$\begin{aligned} I_1 = -\int_{t_1}^{t_2} \tilde{F}'(t) \cdot t^{1+\varepsilon}\, dt &= -\tilde{F}(t) \cdot t^{1+\varepsilon}\Big|_{t_1}^{t_2} + (1+\varepsilon) \int_{t_1}^{t_2} \tilde{F}(t) \cdot t^{\varepsilon}\, dt \\ &\leq \tilde{F}(t_1) t_1^{1+\varepsilon} + (1+\varepsilon) \int_{t_1}^\infty \tilde{F}(t) \cdot t^{\varepsilon}\, dt. \end{aligned} \tag{2.11}$$

Case I. Non-optimal ($\theta < 1/2$). Let $2\varepsilon = 1/(2\theta - 1) - 1$. Then

$$I_1 \leq C t_1^{-\varepsilon} + (1+\varepsilon) \int_{t_1}^\infty C t^{-1-\varepsilon}\, dt = \frac{1+2\varepsilon}{\varepsilon} C t_1^{-\varepsilon}.$$

This demonstrates the t_1-uniform boundedness of the length and polynomial convergence of the integral curve.
Case II. Optimal ($\theta = 1/2$). There exist constants $\varepsilon > 0, C' > 0$ such that

$$C\exp(-c^2 t) \cdot t^{1+\varepsilon} \leq C\exp(-c^2 t) \cdot t^{\varepsilon} \leq C' \exp(-\tfrac{1}{2} c^2 t). \qquad (\forall t \in [1, \infty)) \tag{2.12}$$

Then,

$$I_1 \leq C' \exp(-\tfrac{1}{2} c^2 t_1) + (1+\varepsilon) \int_{t_1}^\infty C' \exp(-\tfrac{1}{2} c^2 t)\, dt = \left(1 + \frac{2(1+\varepsilon)}{c^2}\right) C' \exp(-\tfrac{1}{2} c^2 t_1). \tag{2.13}$$

This demonstrates t_1-uniform boundedness of the length and exponential convergence of the integral curve.

In both cases, t_1-uniform boundedness implies that the total length of the curve is finite, leading to the convergence of $x(t)$ to x_∞. This completes the proof. □

This framework extends naturally to infinite-dimensional settings. The infinite-dimensional versions of Łojasiewicz convergence theorem, derived from Łojasiewicz-Simon inequalities for functionals on infinite-dimensional spaces, provide a robust method for proving the convergence of gradient flows, and flows closely related to the gradients of such functionals.

In 1981, Allard and Almgren [12] made a significant contribution to the study of minimal surfaces by introducing a method to analyze their geometric properties through tangent cones. They showed how minimal surfaces asymptotically approach their tangent cones as the radius shrinks to zero, providing profound insights into the structure of minimal surfaces near singularities. Their concepts of tangent cones and their uniqueness have also been widely applied in the singularity theory of various geometric flows, such as mean curvature flow and Ricci flow.

Building on this, Leon Simon [5] introduced the Łojasiewicz-Simon inequalities to geometric analysis in 1983, combining them with Allard and Almgren's methods to study the behavior of minimal surfaces near singularities. Simon employed a technique called Lyapunov-Schmidt reduction, which simplifies the proof of gradient inequalities for functionals defined on infinite-dimensional spaces by reducing them to proofs for related functionals on finite-dimensional spaces. This approach allowed Simon to derive the infinite-dimensional Łojasiewicz-Simon inequality from Łojasiewicz's original finite-dimensional inequality [4]. Using this framework, Simon proved an infinite-dimensional version of the Łojasiewicz convergence theorem and applied it to analyze evolution equations. He demonstrated that a minimal surface in Euclidean space with a singularity at 0 and a multiplicity-1 tangent cone with an isolated singularity at 0 asymptotically converges to the tangent cone near the singularity.

Simon's framework has far-reaching implications in geometric analysis and has been successfully applied to various evolution equations, including geometric flows.

Felix Schulze [13] used it to establish the uniqueness of compact tangent flows in mean curvature flow (MCF), a result closely tied to singularity analysis, as explored by Tobias Colding and William Minicozzi [14]. Jonathan Zhu and Ao Sun [20, 21, 22] applied it to study various types of self-shrinkers in MCF. Jingyi Chen and John Ma [24] demonstrated that the entropy of Lagrangian self-shrinking tori can only take finitely many values, and used this result to construct a piecewise Lagrangian MCF.

Song Sun and Yuanqi Wang [15] proved the convergence of the Kähler-Ricci flow near a Kähler-Einstein metric. Building on this, Xiuxiong Chen, Song Sun, and Bing Wang [18] established the existence of Kähler-Einstein metrics on K-stable Fano manifolds. Robert Haslhofer [25] proved the Łojasiewicz-Simon inequality for Perelman's λ-functional and used it to demonstrate the convergence of Ricci flows constructed near a Ricci-flat metric, thereby proving the stability of Ricci-flat metrics.

Alessandro Carlotto, Otis Chodosh, and Yanir Rubinstein [23] applied Łojasiewicz-Simon techniques to control the convergence rate of the Yamabe flow. Maria Colombo, Luca Spolaor, and Bozhidar Velichkov [19] introduced a modified inequality, referred to as the constrained Łojasiewicz inequality, which they used to analyze the asymptotic behavior of solutions to parabolic variational inequalities.

Finally, Paul Feehan investigated the relationship between optimal Łojasiewicz-Simon inequalities and the Morse-Bott property of functionals [11, 16], demonstrating their applicability to specific types of Yang-Mills energy functionals [11].

The references presented here highlight just a selection of the many works that build on the Łojasiewicz-Simon inequality.

In Section 5, we apply a version of the optimal Łojasiewicz convergence theorem for the $L - \lambda A$ functional to establish the exponential convergence of APCSF.

2.2 Area-Preserving Curve Shortening Flow

The area-preserving curve shortening flow (APCSF), the central focus of this paper, is the one-dimensional version of the volume-preserving mean curvature flow (VPMCF), itself a modification of the mean curvature flow (MCF) designed to preserve enclosed volume. Below, we introduce the definitions of these flows for clarity and context.

Definition 2.4 (The Mean Curvature Flow [7])**.**

(a) The **mean curvature flow (MCF)** is a geometric flow of hypersurfaces $\{M_t\}_t = \{X(t,-)\}_t$ in $\mathbb{R}^{n+1}$ governed by

$$X_t = HN. \tag{2.14}$$

Here, H is mean curvature scalar and N is inward unit normal of the hypersurface $M_t = X(t,-)$.

(b) The **volume-preserving mean curvature flow (VPMCF)** [9] is geometric flow of hypersurfaces in $\mathbb{R}^{n+1}$ governed by

$$X_t = (H - \bar{H})N, \tag{2.15}$$

where $\bar{H} = \fint_{M_t} H \, d\mathcal{H}^n$ is the average mean curvature.

(c) The **curve shortening flow (CSF)** [8] is mean curvature flow of curves $\{\gamma_t\}_t = \{X(t,-)\}_t$ in $\mathbb{R}^2$. Here, the mean curvature H simplifies to the curvature k, resulting in the evolution equation

$$X_t = kN. \tag{2.16}$$

(d) The **area-preserving curve shortening flow (APCSF)** [2] is a VPMCF of curves in $\mathbb{R}^2$.

$$X_t = kN - \frac{2\pi}{L}N = (k - \bar{k})N, \tag{2.17}$$

where $\bar{k} = 2\pi/L = \fint_{\gamma_t} k\, ds$ is the average curvature.

Since the mean curvature vector corresponds to the Laplacian of a hypersurface, MCF can be viewed as a heat equation for hypersurfaces. This parabolic behavior ensures the short-term existence of both MCF and VPMCF [8, 9]. The stationary points of MCF are minimal surfaces, characterized by zero mean curvature. Therefore, MCF naturally serves as a powerful tool for studying minimal surfaces.

For instance, Brian White employed MCF to investigate the formation of minimal surfaces [26] and established a connection between the total curvature of minimal surface boundaries and the singularities observed in MCF [27]. Kyeongsu Choi and Christos Mantoulidis [28] explored closed ancient solutions to gradient flows of elliptic functionals, including MCF, classifying ancient solutions originating from the unstable manifold of a minimal submanifold. Alexander Mramor and Alec Payne [41] constructed embedded ancient solutions to MCF that are mean convex but not strictly convex, revealing a strong relationship between such ancient solutions and certain classes of unstable minimal hypersurfaces.

The VPMCF has constant mean curvature (CMC) surfaces as their stationary points. These surfaces are critical points of the area functional constrained by volume preservation, making them a natural generalization of minimal surfaces. Consequently, methods used to study minimal surfaces via MCF can often be extended to study CMC surfaces using VPMCF. For example, Jacopo Tenan [34] demonstrated that spacetime VPMCF, a variant of VPMCF adapted to General Relativity, converges to a constant spacetime-curvature limit, which he applied to define the center of mass of isolated systems.

VPMCF also provides alternative approaches to studying MCF. In [33], Ben Lambert and Elena Mäder-Baumdicker analyzed VPMCF singularities to investigate MCF. They demonstrated that type I and type II blowups of VPMCF are related to ancient homothetically shrinking solutions and eternal solutions of MCF, respectively.

In applied mathematics, CSF has been widely used in phase separation, image processing, and computer vision due to its parabolic behavior, which regularizes curves by smoothing irregularities [31, 32]. However, a limitation of CSF is that it always causes curves to shrink. The area-preserving curve shortening flow (APCSF) overcomes this issue, as shown in the work of Italo Capuzzo Dolcetta and Stefano Finzi Vita [30]. In higher dimensions, both MCF and VPMCF have been employed for similar smoothing procedures, particularly in surface denoising and geometric modeling [35, 36, 37].

APCSF can also be regarded as the simplest case ($m = 0$) of **m-th area preserving curvature flows (m-APCF)** [38]:

$$X_t = (-1)^m \partial_s^{2m}(k - \bar{k})N. \tag{2.18}$$

This family of flows includes the well-known **surface diffusion equation** ($m = 1$), which originated in material science [39]:

$$X_t = -k_{ss}N. \tag{2.19}$$

Higher-order cases ($m \geq 2$) have been studied by Scott Parkins and Glen Wheeler [40]. These references represent just a fraction of the extensive research on MCF and its variants.

Since APCSF is the simplest case of both VPMCF and m-APCF, its study can inspire generalizations to these more complex geometric flows. We believe this research indicates that Simon's framework could potentially be applied to the convergence analysis of higher-dimensional VPMCF in various Riemannian manifolds, a problem that is generally more challenging.

In this research, we analyze APCSF using the $L - \lambda A$ functional. This approach leverages the fact that APCSF is the gradient flow of the length functional L under the constraint of area preservation, as detailed in the following remark.

Remark 2.5. Consider the space of compact closed hypersurfaces in $\mathbb{R}^{n+1}$. The first variation formula ([7], Section 1) indicates that gradient of the n-volume $\mathcal{V}$ is given by

$$\nabla \mathcal{V} = -HN. \tag{2.20}$$

Hence, the MCF is the gradient flow of the n-volume functional $\mathcal{V}$. Along the MCF $\{M_t\}$, the n-volume evolves according to

$$\frac{d}{dt}\mathcal{V}(M_t) = \frac{d}{dt}\int_{M_t} d\mathcal{H}^n = -\int_{M_t} H^2\, d\mathcal{H}^n. \tag{2.21}$$

Specifically, in the case of plane curves, the curve shortening flow (CSF) is the gradient flow of the length functional. Now, consider an additional constraint: the enclosed $(n+1)$-volume must remain constant. The evolution of the enclosed $(n+1)$-volume is given by

$$\frac{d}{dt}\operatorname{Vol}(M_t) = -\int_{M_t} \langle X_t, N\rangle \; d\mathcal{H}^n. \tag{2.22}$$

Consequently, the evolution equation $X_t = (H - \bar{H})N$ can be interpreted as the projection of $X_t = HN$ onto the subspace of motions that preserve the enclosed $(n+1)$-volume. This implies that the volume-preserving mean curvature flow (VPMCF) is the gradient flow of the n-volume functional with the additional constraint that the enclosed $(n+1)$-volume remains constant. Similarly, the plane curve version, APCSF, is the gradient flow of the length functional with the added restriction of area preservation.

Generally, the VPMCF asymptotically converges to an n-sphere, which is the solution to the n-dimensional isoperimetric problem. This convergence has been proven for smooth initial surfaces in [9] and for more general star-shaped surfaces in [29]. Notably, this asymptotic convergence is closely related to the gradient flow structure of VPMCF.

In particular, it is well-known that APCSF initiated from a convex simple closed C^2-curve exists for all positive time and asymptotically converges smoothly exponentially to a circle [2]. This clearly implies the Hausdorff and uniform exponential convergence results.

In this study, we leverage Łojasiewicz inequality to provide alternative proof for Hausdorff and uniform exponential convergence of APCSF. As a preliminary, we only assume the long-term existence of the flow.

Let the curve at time t be demonstrated as $\gamma_t := X(t, -)$. Note that the subscript t does not represent a time derivative here. These curves are initially parameterized by $u \in \mathbb{S}^1$.

For any convex simple closed C^2-plane curve, its Gauss map is injective, allowing the turning angle θ (the angle between the tangent vector at a point on the curve and the x-axis) to serve as a natural parameter for the curve. Consequently, we can reparametrize $\{\gamma_t\}$ by their turning angles if convexity is maintained along the flow.

The curvature and reciprocal of curvature of γ_t are denoted as $k(t,\theta)$ and $f(t,\theta)$, respectively. Since convex curves are uniquely determined by their curvature functions up to a translation, as demonstrated in the following proposition, we use curvature (or reciprocal of curvature) functions to characterize these curves:

Proposition 2.6. *A positive-valued continuous function $k : \mathbb{S}^1 \to \mathbb{R}^+$ is the curvature function of a convex simple closed C^2-plane curve if and only if*

$$\int_0^{2\pi} \frac{e^{i\theta}}{k(\theta)}\, d\theta = 0. \tag{2.23}$$

In this case, the parametric equation of the curve with k as its curvature function is given by

$$\left(\int_0^{\theta} \frac{\cos\phi}{k(\phi)}\, d\phi, \int_0^{\theta} \frac{\sin\phi}{k(\phi)}\, d\phi\right), \tag{2.24}$$

up to a translation.

Proof. If k is the curvature function of a convex simple closed C^2-plane curve, then the total tangent vector displacement over one full traversal of the curve must be zero. Denoting the tangent vector by $T = (\cos\theta, \sin\theta)$, we have:

$$0 = \int_0^L T\, ds = \int_0^{2\pi} T\,\frac{d\theta}{k(\theta)} = \int_0^{2\pi} \frac{(\cos\theta, \sin\theta)}{k(\theta)}\, d\theta = \left(\int_0^{2\pi} \frac{\cos\theta}{k(\theta)}\, d\theta, \int_0^{2\pi} \frac{\sin\theta}{k(\theta)}\, d\theta\right).$$

This confirms that the integral condition in equation (2.23) must hold.

Conversely, suppose k is any continuous function satisfying equation (2.23). Then the parametric equation given by (2.24) defines a convex closed C^2-curve. Since the Gauss map of this curve is injective, the curve is simple.

A direct calculation verifies that k is indeed the curvature function of this curve. Specifically, the curve can be represented in terms of arc length s as:

$$(a,b) + \int_0^s T\, ds = (a,b) + \int_0^\theta \frac{(\cos\phi, \sin\phi)}{k(\phi)}\, d\phi,$$

where (a,b) is the unique point on the curve corresponding to the turning angle $\theta = 0$. This representation demonstrates that the parametric equation is unique up to a translation by (a,b). $\square$

Although curvature functions are traditionally used in the study of CSF and its variants, they present a significant limitation: the set of curvature functions that define closed curves is not closed under linear combinations. Specifically, if k_1 and k_2 are curvature functions satisfying $\int k_1^{-1} e^{i\theta}\, d\theta = \int k_2^{-1} e^{i\theta}\, d\theta = 0$, their linear combination $a_1 k_1 + a_2 k_2$ $(a_1, a_2 \in \mathbb{R})$ may fail to satisfy $\int (a_1 k_1 + a_2 k_2)^{-1} e^{i\theta}\, d\theta = 0$. Consequently, the set of curvature functions satisfying (2.23) forms a Banach submanifold rather than a linear subspace.

To circumvent these complications, we use the reciprocal of curvature (radius of curvature) $f = k^{-1}$, as a more convenient representation that characterizes the curve.

Consider any positive-valued continuous function $f : \mathbb{S}^1 \to \mathbb{R}$ satisfying

$$\int_0^{2\pi} f e^{i\theta}\, d\theta = 0. \tag{2.25}$$

Such functions correspond to the following curve, defined up to a translation:

$$X(\theta) = \left(\int_0^\theta f \cos\phi\, d\phi, \int_0^\theta f \sin\phi\, d\phi \right). \tag{2.26}$$

Note that the arc length differential is given by

$$ds = k^{-1}\, d\theta = f\, d\theta \quad , \quad \frac{d}{ds} = k \frac{d}{d\theta} = f^{-1} \frac{d}{d\theta}. \tag{2.27}$$

Proposition 2.7. *The length $L(f)$ and enclosed area $A(f)$ of the curve corresponding to a given reciprocal of curvature f are given by*

$$L(f) = \int_0^{2\pi} f\, d\theta \quad , \quad A(f) = \int_0^{2\pi} f(\theta) \cos\theta \int_0^\theta f(\phi) \sin\phi\, d\phi\, d\theta. \tag{2.28}$$

Proof. The formula for the length follows directly from the relationship $ds = f\, d\theta$. The formula for the area is obtained by

$$A(f) = \oint y\, dx = \int_0^{2\pi} y(\theta) \frac{dx}{d\theta}\, d\theta = \int_0^{2\pi} f(\theta) \cos\theta \int_0^\theta f(\phi) \sin\phi\, d\phi\, d\theta. \qquad \square$$

Now, APCSF can also be described entirely in terms of curvature or reciprocal of curvature functions.

Theorem 2.8. *If we identify convex curves with their curvature k (or, equivalently, reciprocal of curvature f) functions, k and f evolve under APCSF as*

$$k_t = k^2 (k_{\theta\theta} + k - \bar{k}) = k^2 k_{\theta\theta} + k^3 - \frac{2\pi k^2}{L}, \tag{2.29}$$

$$f_t = -\frac{k_t}{k^2} = -(f^{-1})_{\theta\theta} - f^{-1} + \frac{2\pi}{L} = \frac{f f_{\theta\theta} - 2 f_\theta^2 - f^2}{f^3} + \frac{2\pi}{L}. \tag{2.30}$$

Moreover, this time-evolution satisfies the constraint

$$\int_0^{2\pi} f_t e^{i\theta}\, d\theta = 0.$$

This ensures that $f(t, \theta)$ continues to define a closed curve throughout the evolution.

Proof. The curvature evolution equation, originally derived by Gage ([2], Section 3), is as follows. Let the original parameters be t and u, and denote $v = \partial s/\partial u$. Reparametrizing the family of curves introduces a tangential term in the evolution equation:

$$X_t = \left(k - \frac{2\pi}{L}\right) N + \alpha(t,u)T. \tag{2.31}$$

Differentiating this equation, rearranging the order of differentiation, and using the fact that $X_u = vT$, we obtain two component equations by separating the normal and tangential terms:

$$T_t = \left(\frac{1}{v}k_u + \alpha k\right) N \tag{2.32}$$

$$v_t = -vk^2 + v\frac{2\pi k}{L}. \tag{2.33}$$

To determine the specific form of α that reparametrizes the curve from u to the turning angle θ, we impose the condition that T is independent of t. Since θ is defined as the angle between T and the x-axis, this ensures that t and θ are independent parameters. This requirement implies

$$\alpha = -\frac{1}{vk}k_u. \tag{2.34}$$

Differentiating (2.32) with respect to u, and rearranging the order of differentiation, gives

$$0 = \frac{\partial^2 T}{\partial u \partial t} = \frac{\partial^2 T}{\partial t \partial u} = \frac{\partial}{\partial t}(vkN). \tag{2.35}$$

This shows that $(vk)_t = 0$. Using (2.33) and (2.34), we derive

$$k_t = -\frac{1}{v}kv_t = k^3 - \frac{2\pi k^2}{L} + \frac{k}{v}\left(\frac{1}{vk}k_u\right)_u. \tag{2.36}$$

Rewriting this in terms of θ by using the relationships $d\theta = kv\ du$ and $\partial/\partial\theta = (vk)^{-1}\partial/\partial u$, the equation simplifies to

$$k_t = k^2 k_{\theta\theta} + k^3 - \frac{2\pi k^2}{L}.$$

This equation can be translated into the evolution equation for f using the following derivatives:

$$f_t = -k^{-2}k_t \quad , \quad k_\theta = -f^{-2}f_\theta \quad , \quad k_{\theta\theta} = f^{-3}(2f_\theta^2 - ff_{\theta\theta})$$

Substituting these relations, we obtain the desired evolution equation for f:

$$f_t = -\frac{k_t}{k^2} = -(f^{-1})_{\theta\theta} - f^{-1} + \frac{2\pi}{L} = \frac{ff_{\theta\theta} - 2f_\theta^2 - f^2}{f^3} + \frac{2\pi}{L}.$$

Finally, the last statement follows from

$$\int_0^{2\pi} f_t e^{i\theta}\, d\theta = \int_0^{2\pi} -(f^{-1})_{\theta\theta}e^{i\theta} - f^{-1}e^{i\theta} + \frac{2\pi}{L}e^{i\theta}\, d\theta = \int_0^{2\pi} (f^{-1} - f^{-1})e^{i\theta}\, d\theta + \frac{2\pi}{L}\int_0^{2\pi} e^{i\theta}\, d\theta = 0. \qquad \square$$

Proposition 2.9 (Gage [2], Lemma 3.1)**.** *Convex curve evolving according to equation* (2.29)*, which is reformulation of* (2.17)*, remains convex.*

Proof. Let $W(\theta,t) = k(\theta,t)e^{\mu t}$, where μ is a constant to be determined below. It satisfies

$$W_t = k^2 W_{\theta\theta} + \left(k^2 - \frac{2\pi}{L}k + \mu\right) W \tag{2.37}$$

The coefficient of W is a quadratic polynomial in k, whose discriminant is $(4\pi^2/L^2) - 4\mu$. By the isoperimetric inequality, there is a lower bound for L in terms of area, which ensures that for sufficiently large

μ, the discriminant becomes negative. This implies that the coefficient of W is positive. We now choose μ such that this positivity is guaranteed for all times.

Next, define $W_{\min}(t) = \inf\{W(\theta,t) : 0 \leq \theta \leq 2\pi\}$. Suppose that at some time $t > 0$, $W_{\min}(t) = \beta$, where $0 < \beta < W_{\min}(0)$. Let $t_0 = \inf\{t : W_{\min}(t) = \beta\}$. Then β is achieved for the first time at $W(\theta_0, t_0)$, and at this point,

$$W_t \leq 0, \quad W_{\theta\theta} \geq 0, \quad W = \beta > 0.$$

This setup contradicts the fact that W satisfies (2.37), thereby proving that $W_{\min}(t)$ is nondecreasing. Consequently,

$$k_{\min}(t) = W_{\min}(t) \cdot e^{-\mu t} \geq W_{\min}(0) \cdot e^{-\mu t} > 0. \qquad \square$$

This proposition ensures that the reciprocal of curvature, $f = k^{-1}$, is well-defined and finite whenever the solution k exists. As a preliminary, we will rely on the long-term existence result for the evolution equation (2.29):

Proposition 2.10 (Gage [2], Theorem 4.1)**.** *For any positive-valued function $k_0 \in C^2_\theta(\mathbb{S}^1)$, there exists a positive-valued solution $k \in C^1_t C^2_\theta([0,\infty) \times \mathbb{S}^1)$ of the equation (2.29) which has $k_0(-) = k(0,-)$ as initial datum.*

By combining the above two propositions, we conclude the long-term existence of $f = k^{-1}$.

For convenience, we will use the notation $L(t) = L(f(t,-)), L_0 = L(0) = L(f(0,-))$ to represent the length of the curve along the flow.

Proposition 2.11. *Along the APCSF, the length and area of the curve evolve as follows:*

$$-L_t = \int (k - \bar{k})\, d\theta = \int (k - \bar{k})^2\, ds \geq 0 \quad , \quad A_t = 0 \tag{2.38}$$

In particular, the length is a nonincreasing function of time and strictly decreases if the curve is not a circle. Moreover, the enclosed area remains preserved throughout the evolution and is always equal to the initial area A_0.

Proof. These results follow directly from the formulae in Remark 2.5. However, we will derive them explicitly here using the evolution equation (2.30). The evolution of the enclosed area is given by

$$\begin{aligned}
A_t &= \int_0^{2\pi} f_t \left[\cos\theta \int_0^\theta f(\phi)\sin\phi\, d\phi - \sin\theta \int_0^\theta f(\phi)\cos\phi\, d\phi\right] d\theta \\
&= \int_0^{2\pi} (-(f^{-1})_{\theta\theta} - f^{-1} + 2\pi L^{-1}) \left[\cos\theta \int_0^\theta f(\phi)\sin\phi\, d\phi - \sin\theta \int_0^\theta f(\phi)\cos\phi\, d\phi\right] d\theta \\
&= \int_0^{2\pi} (-f^{-1} + 2\pi L^{-1}) \left[\cos\theta \int_0^\theta f(\phi)\sin\phi\, d\phi - \sin\theta \int_0^\theta f(\phi)\cos\phi\, d\phi\right] \\
&\quad + (f^{-1} - 2\pi L^{-1})_\theta \left[-\sin\theta \int_0^\theta f(\phi)\sin\phi\, d\phi - \cos\theta \int_0^\theta f(\phi)\cos\phi\, d\phi\right] d\theta \\
&= \int_0^{2\pi} (-f^{-1} + 2\pi L^{-1}) \left[\cos\theta \int_0^\theta f(\phi)\sin\phi\, d\phi - \sin\theta \int_0^\theta f(\phi)\cos\phi\, d\phi\right] \\
&\quad - (f^{-1} - 2\pi L^{-1}) \left[-\cos\theta \int_0^\theta f(\phi)\sin\phi\, d\phi + \sin\theta \int_0^\theta f(\phi)\cos\phi\, d\phi - f\right] d\theta \\
&= \int_0^{2\pi} (1 - 2\pi f L^{-1})\, d\theta = 2\pi - 2\pi \frac{L}{L} = 0.
\end{aligned}$$

The evolution of the length is given by

$$\begin{aligned}
-L_t &= -\int f_t\, d\theta = \int (f^{-1})_{\theta\theta} + f^{-1} - \frac{2\pi}{L}\, d\theta = \int \frac{1}{f}\, d\theta - \frac{4\pi^2}{L} \\
&= \int (k - \bar{k})\, d\theta = \int k(k - \bar{k})\, ds = \int (k - \bar{k})^2\, ds \geq 0.
\end{aligned}$$

The inequality is strict when $k \not\equiv \bar{k}$, indicating that the curve is not a circle. $\square$

Let $\lambda := \sqrt{\pi/A_0}$. A circle with radius λ^{-1} has area A_0. By the isoperimetric inequality, the length $L(t)$ always satisfies $L(t) \geq 2\pi\lambda^{-1}$. Therefore, $L(t) - 2\pi\lambda^{-1}$ is nonnegative function that decreases monotonically along the flow.

3 The $L - \lambda A$ Functional and Its Łojasiewicz-Simon Inequality

3.1 The $L - \lambda A$ Functional

Along the APCSF, the enclosed area remains constant while the curve length decreases strictly. The asymptotic limit of the flow, a circle with the prescribed area, achieves the minimum length L among all curves with the same area A. To minimize the length of curves with a fixed area, we employ an analogous approach to the finite-dimensional Lagrange multiplier method. Consequently, we define the functional $L - \lambda A$, where $\lambda > 0$ is a positive constant to be determined, serving as the Lagrange multiplier.

To consistently define this functional for all functions in $L^2(\mathbb{S}^1)$, we apply an orthogonal projection. This removes the pathological sin and cos components, which are L^2-functions that do not correspond to closed curves.

Definition 3.1. The operator $Q : L^2(\mathbb{S}^1) \to L^2(\mathbb{S}^1)$ is defined as the orthogonal projection onto the orthogonal complement of $\operatorname{span}\{\sin, \cos\}$. Explicitly, Q is given by

$$Qf(\theta) := f(\theta) - \frac{1}{\pi}\sin\theta \int_0^{2\pi} f(\phi)\sin\phi\, d\phi - \frac{1}{\pi}\cos\theta \int_0^{2\pi} f(\phi)\cos\phi\, d\phi. \tag{3.1}$$

Definition 3.2. We define the functional $L - \lambda A : L^2(\mathbb{S}^1) \to \mathbb{R}$ as

$$(L - \lambda A)(f) := L(Qf) - \lambda A(Qf) = \int_0^{2\pi} Qf\, d\theta - \lambda \int_0^{2\pi} Qf(\theta)\cos\theta \int_0^{\theta} Qf(\phi)\sin\phi\, d\phi\, d\theta. \tag{3.2}$$

In particular, when f represents a convex simple closed C^2-curve, the functional $L - \lambda A$ evaluated at f coincides with the $L - \lambda A$ value of the corresponding curve. This justifies our abuse of the notation $L - \lambda A$.

Theorem 3.3 (First and Second Variation Formulae of $L - \lambda A$)**.** *Under the isomorphism $(L^2(\mathbb{S}^1))^* \cong L^2(\mathbb{S}^1)$, the first variation of the $L - \lambda A$ functional corresponds to the following function:*

$$d(L - \lambda A)_f = \lambda \operatorname{Im}\left[e^{i\theta}\int_0^{\theta} (Qf(\phi) - \lambda^{-1})e^{-i\phi}\, d\phi\right]. \tag{3.3}$$

This derivative vanishes if and only if $Qf = \lambda^{-1}$ almost everywhere. Consequently, f is a critical point if and only if $f - \lambda^{-1} \in \operatorname{span}\{\sin, \cos\}$. The range of the map $f \mapsto d(L - \lambda A)_f$ is a codimension-2 closed subspace of $H^2(\mathbb{S}^1)$. Additionally, the norm of $d(L - \lambda A)$ can be estimated as follows:

$$\|d(L - \lambda A)\|_{H^2}^2 \leq 25\lambda^2 \|Qf - \lambda^{-1}\|_{L^2}^2. \tag{3.4}$$

The second variation of the $L - \lambda A$ functional corresponds to the following map:

$$g \mapsto d^2(L - \lambda A)_f(g, -) = \lambda \operatorname{Im}\left[e^{i\theta}\int_0^{\theta} Qg(\phi)e^{-i\phi}\, d\phi\right]. \tag{3.5}$$

Its kernel is $\operatorname{span}\{\sin, \cos\}$.

Proof. To compute the first variation, we use the definition of the Gateaux derivative. For any $g \in L^2(\mathbb{S}^1)$,

$$
\begin{aligned}
d(L-\lambda A)_f(g) &= \lim_{t\to 0} t^{-1}[(L-\lambda A)(f+tg)-(L-\lambda A)(f)] \\
&= \lim_{t\to 0}\frac{1}{t}\left[\int_0^{2\pi}(Q(f+tg)-Qf)\,d\theta - \lambda\int_0^{2\pi}Q(f+tg)(\theta)\cos\theta\int_0^{\theta}Q(f+tg)(\phi)\sin\phi\,d\phi\,d\theta\right. \\
&\quad \left. +\lambda\int_0^{2\pi}Qf(\theta)\cos\theta\int_0^{\theta}Qf(\phi)\sin\phi\,d\phi\,d\theta\right] \\
&= \int_0^{2\pi}Qg\,d\theta-\lambda\int_0^{2\pi}Qg(\theta)\cos\theta\int_0^{\theta}Qf(\phi)\sin\phi\,d\phi\,d\theta+\int_0^{2\pi}Qf(\theta)\cos\theta\int_0^{\theta}Qg(\phi)\sin\phi\,d\phi\,d\theta \\
&= \int_0^{2\pi}Qg\,d\theta-\lambda\int_0^{2\pi}Qg(\theta)\cos\theta\int_0^{\theta}Qf(\phi)\sin\phi\,d\phi\,d\theta-\int_0^{2\pi}Qg(\theta)\sin\theta\int_0^{\theta}Qf(\phi)\cos\phi\,d\phi\,d\theta \\
&= \lambda\int_0^{2\pi}Qg(\theta)\,\mathrm{Im}\left[e^{i\theta}\int_0^{\theta}(Qf(\phi)-\lambda^{-1})e^{-i\phi}\,d\phi\right]d\theta.
\end{aligned}
$$

The map $d(L-\lambda A)_f(-) : g \mapsto d(L-\lambda A)_f(g)$ can be regarded as a linear functional $L^2(\mathbb{S}^1)\to\mathbb{R}$, which belongs to the dual space $(L^2(\mathbb{S}^1))^*$. Using the canonical Hilbert space isomorphism $(L^2(\mathbb{S}^1))^* \cong L^2(\mathbb{S}^1)$, this linear functional corresponds to the following L^2-function:

$$
\begin{aligned}
(d(L-\lambda A)_f)(\theta) &= Q\left[1-\lambda\cos\theta\int_0^{\theta}Qf(\phi)\sin\phi\,d\phi+\lambda\sin\theta\int_0^{\theta}Qf(\phi)\cos\phi\,d\phi\right] \\
&= \lambda Q\,\mathrm{Im}\left[e^{i\theta}\int_0^{\theta}(Qf(\phi)-\lambda^{-1})e^{-i\phi}\,d\phi\right] \\
&= \lambda\,\mathrm{Im}\left[e^{i\theta}\int_0^{\theta}(Qf(\phi)-\lambda^{-1})e^{-i\phi}\,d\phi\right]
\end{aligned}
$$

Here, we used the fact that $(Qf-\lambda^{-1})(\phi)$ does not have $e^{\pm i\phi}$ components, which ensures that the expression inside the square bracket also does not have $e^{\pm i\phi}$ components.

Differentiating the above formula once more, we derive the second variation formula:

$$
\begin{aligned}
(d^2(L-\lambda A)_f(g,-))(\theta) &= \lim_{t\to 0}t^{-1}[(d(L-\lambda A)_{f+tg})(\theta)-(d(L-\lambda A)_f)(\theta)] \\
&= \lim_{t\to 0}\frac{1}{t}\lambda\,\mathrm{Im}\left[e^{i\theta}\int_0^{\theta}(Q(f+tg)(\phi)-\lambda^{-1})e^{-i\phi}\,d\phi-e^{i\theta}\int_0^{\theta}(Qf(\phi)-\lambda^{-1})e^{-i\phi}\,d\phi\right] \\
&= \lambda\,\mathrm{Im}\left[e^{i\theta}\int_0^{\theta}Qg(\phi)e^{-i\phi}\,d\phi\right].
\end{aligned}
$$

In order to locate critical points, we identify an L^2-function with its Fourier series:

$$
S_{Qf-\lambda^{-1}}(\theta)=\sum_{n=-\infty}^{\infty}a_ne^{in\theta} \quad , \quad a_n=\frac{1}{2\pi}\int_0^{2\pi}(Qf(\theta)-\lambda^{-1})e^{-in\theta}\,d\theta
$$

Here, $\bar{a}_n = a_{-n}$ because f is a real-valued function, and $a_{\pm 1}=0$ due to the application of the projection Q. This leads to the following formula:

$$
e^{i\theta}\int_0^{\theta}(Qf(\phi)-\lambda^{-1})e^{-i\phi}\,d\phi = e^{i\theta}\sum_{n=-\infty}^{\infty}\int_0^{\theta}a_ne^{i(n-1)\phi}\,d\phi=\sum_{n\neq\pm1}\frac{a_n}{i(n-1)}(e^{in\theta}-e^{i\theta}).
$$

$d(L-\lambda A)_f$ is constant multiple of imaginary part of this function.

$$
\begin{aligned}
S_{d(L-\lambda A)_f} &= \frac{\lambda}{2i}\sum_{n\neq\pm1}\frac{a_n}{i(n-1)}(e^{in\theta}-e^{i\theta})-\frac{\lambda}{2i}\sum_{n\neq\pm1}-\frac{\bar{a}_n}{i(n-1)}(e^{-in\theta}-e^{-i\theta}) \\
&= -\frac{\lambda}{2}\sum_{n\neq\pm1}\left[\frac{a_n}{n-1}+\frac{\bar{a}_{-n}}{-n-1}\right]e^{in\theta}+\frac{\lambda}{2}\sum_{n\neq\pm1}\frac{a_ne^{i\theta}+\bar{a}_ne^{-i\theta}}{n-1} \\
&= -\lambda\sum_{|n|\geq 2}\frac{a_ne^{in\theta}}{n^2-1}+\frac{\lambda}{2}\sum_{n\neq\pm1}\frac{a_ne^{i\theta}}{n-1}-\frac{\lambda}{2}\sum_{n\neq\pm1}\frac{a_ne^{-i\theta}}{n+1}
\end{aligned}
$$

Thus, its Fourier coefficients $\{b_n\}$ are given by

$$b_0 = 0 \quad , \quad b_{\pm 1} = \pm\frac{\lambda}{2}\sum_{n\neq\pm 1}\frac{a_n}{n\mp 1} \quad , \quad b_n = -\lambda\frac{a_n}{n^2-1} \quad (|n|\geq 2).$$

It shows that $d(L-\lambda A)_f = 0$ iff $a_n = 0$ for $|n| \neq 1$; this corresponds to the functions $f - \lambda^{-1} \in \operatorname{span}\{\sin,\cos\}$. In other words, the critical set is $\operatorname{Crit}(L-\lambda A) = \lambda^{-1} + \operatorname{span}\{\sin,\cos\}$. (This also shows that the kernel of $d^2(L-\lambda A)_f$ is $\operatorname{span}\{\sin,\cos\}$, since $d^2(L-\lambda A)_f$ has the same form except that $f-\lambda^{-1}$ replaced to g.)

Moreover, the sequence $\{-n^2 b_n\}$ is square-summable, which implies that $d(L-\lambda A)_f$ belongs to $H^2(\mathbb{S}^1)$.

Conversely, if a sequence $\{b_n\}$ corresponds to an H^2-function, then it represents $d(L-\lambda A)_f$ for some f if and only if the following condition holds:

$$b_0 = 0 \quad , \quad b_1 + b_{-1} = -\sum_{|n|\geq 2} b_n.$$

In this case, Fourier coefficients $\{a_n\}$ of f are given by

$$a_n = -\lambda^{-1}(n^2-1)b_n \quad (|n|\geq 2) \quad , \quad a_{\pm 1} \text{ arbitrary} \quad , \quad a_0 = -\lambda^{-1}\sum nb_n.$$

Thus, the range of the map $f \mapsto d(L-\lambda A)_f$ is codimension-2 closed subspace of $H^2(\mathbb{S}^1)$.

Furthermore, we can estimate $\|d(L-\lambda A)\|_{H^2}^2$:

$$\begin{aligned}
\|d(L-\lambda A)\|_{H^2}^2 &= \pi\sum_{n\neq 0}(n^4+n^2+1)|b_n|^2 \\
&= \pi\lambda^2\sum_{|n|\geq 2}\frac{n^4+n^2+1}{(n^2-1)^2}|a_n|^2 + 3\pi\cdot\frac{\lambda^2}{4}\left|\sum_{n\neq\pm 1}\frac{a_n}{n-1}\right|^2 + 3\pi\cdot\frac{\lambda^2}{4}\left|\sum_{n\neq\pm 1}\frac{a_n}{n+1}\right|^2 \\
&\leq 3\pi\lambda^2\sum_{|n|\geq 2}|a_n|^2 + \frac{3\pi\lambda^2}{4}\sum_{n\neq\pm 1}|a_n|^2\left[\sum_{n\neq 1}\frac{1}{(n-1)^2} + \sum_{n\neq -1}\frac{1}{(n+1)^2}\right] \\
&= 3\pi\lambda^2\sum_{|n|\geq 2}|a_n|^2 + \frac{3\pi\lambda^2}{4}\cdot\frac{4\pi^2}{6}\sum_{n\neq\pm 1}|a_n|^2 \\
&\leq (3\pi + \tfrac{1}{2}\pi^3)\lambda^2\sum_{n\neq\pm 1}|a_n|^2 \\
&\leq 25\lambda^2\|Qf-\lambda^{-1}\|_{L^2}^2.
\end{aligned}$$

□

Remark 3.4. In the above formula, the first variation of the length is given by

$$dL_f(g) = \int_0^{2\pi} Qg(\theta)\, d\theta = \int_0^{2\pi} g\, d\theta,$$

which corresponds to the constant function 1. Notably, this formula exhibits no explicit dependence on f, which seems at odds with the general first variation formula ([42], Theorem 6.3):

$$dL_\gamma(\delta) = -\int_\gamma \langle \delta, D_s\gamma'\rangle\ ds,$$

where γ is the curve, δ is the variation of γ, and s is the arc-length parameter. According to this general formula, the first variation of the length clearly depends on the curve γ. However, this discrepancy is not a contradiction.

Let $f \in L^2(\mathbb{S}^1)$, and let g be its variation. If both f and g are smooth, and do not have sin and cos components, then there exists a curve γ and a corresponding variation δ (up to translation) that relate to f and g as follows:

$$\begin{aligned}
\gamma(\theta) &= \left(\int_0^\theta f(\phi)\cos(\phi)\, d\phi, \int_0^\theta f(\phi)\sin(\phi)\, d\phi\right) + (c_1, c_2), \\
\delta(\theta) &= \left(\int_0^\theta g(\phi)\cos(\phi)\, d\phi, \int_0^\theta g(\phi)\sin(\phi)\, d\phi\right) + (d_1, d_2),
\end{aligned}$$

where c_i, d_i are constants. In this context, f represents the reciprocal of curvature of γ.

Using the relation $ds = k^{-1}\, d\theta = f\, d\theta$, we can reparameterize the formula in terms of θ:

$$dL_\gamma(\delta) = -\int_0^{2\pi} \langle \delta, D_s\gamma' \rangle\, f\, d\theta$$

Furthermore, we compute $\gamma' = T$ and $D_s\gamma' = kN = f^{-1}N$, where $N = (-\sin\theta, \cos\theta)$ is the inward normal vector. Applying integration by parts, we obtain:

$$\begin{aligned} dL_\gamma(\delta) &= -\int_0^{2\pi} \langle \delta, N \rangle\ d\theta \\ &= \int_0^{2\pi} \left[\sin(\theta)\int_0^\theta g(\phi)\cos(\phi)\, d\phi - \cos(\theta)\int_0^\theta g(\phi)\sin(\phi)\, d\phi\right]\ d\theta \\ &= \int_0^{2\pi} -\left[-\cos(\theta)g(\theta)\cos(\theta)\, d\phi - \sin(\theta)g(\theta)\sin(\theta)\right]\ d\theta \\ &= \int_0^{2\pi} g(\theta)\, d\theta, \end{aligned}$$

which coincides with the result derived earlier. (Here, we neglected the integral constant terms d_i which do not affect the result.)

In our case, reciprocal of curvature f and its variation g are not the direct coordinates of the curve. Therefore, the same variation g in the reciprocal of curvatures can represent different actual variations of curves. The dependence on the curve arises not from the formula for $dL_f(g)$ itself, but rather from the process of interpreting δ in terms of g. This interpretation involves integration with respect to θ, a parameter that strongly depends on the shape of the curve γ.

3.2 Łojasiewicz-Simon Inequality for the $L - \lambda A$ Functional

Since the gradient map $d(L-\lambda A)_f$ of $L - \lambda A$ functional gains regularity by mapping $L^2(\mathbb{S}^1)$ to $H^2(\mathbb{S}^1)$, the typical Lyapunov-Schmidt reduction methods—which require embedding into Hilbert spaces and/or rely on the Fredholm property—cannot be directly applied. Instead, we employ the Lyapunov-Schmidt reduction framework for Morse-Bott functionals on Banach spaces, as developed by Paul Feehan [11]. Here, we present only the main results; for detailed proofs, we refer the reader to Feehan's original work.

We begin by presenting the definition of Morse-Bott functionals on Banach spaces.

Definition 3.5 (Morse-Bott Functional on Banach Spaces; Austin-Braam [10], Section 3.1)**.** Let $\mathscr{B}$ be a smooth Banach manifold, $\mathscr{E} : \mathscr{B} \to \mathbb{R}$ be a C^2-function, and $\operatorname{Crit}\mathscr{E} := \{x \in \mathscr{B} : \mathscr{E}'(x) = 0\}$ be the critical set. A smooth submanifold $\mathscr{C} \hookrightarrow \mathscr{B}$ is called a **nondegenerate critical submanifold of** $\mathscr{E}$ if $\mathscr{C} \subset \operatorname{Crit}\mathscr{E}$ and

$$(T\mathscr{C})_x = \operatorname{Ker}\mathscr{E}''(x) \qquad (\forall x \in \mathscr{C}), \tag{3.6}$$

where $\mathscr{E}''(x) : (T\mathscr{B})_x \to (T\mathscr{B})_x^*$ is the Hessian of $\mathscr{E}$ at the point $x \in \mathscr{C}$. Then,

(a) $\mathscr{E}$ is a **Morse-Bott functional** if $\operatorname{Crit}\mathscr{E}$ consists of nondegenerate critical submanifolds.

(b) $\mathscr{E}$ is **Morse-Bott at a point** $x_0 \in \mathscr{B}$ if there is an open neighborhood $\mathscr{U} \subset \mathscr{B}$ of x_0 such that $\mathscr{U} \cap \operatorname{Crit}\mathscr{E}$ is a relatively open smooth submanifold of $\mathscr{B}$ and (3.6) holds at x_0.

This definition generalizes the classical concept of Morse-Bott functionals on finite-dimensional spaces to functionals on Banach spaces. The Morse-Bott property is closely linked to the optimal Łojasiewicz-Simon inequality, as demonstrated in the following theorem.

Theorem 3.6 (Optimal Łojasiewicz-Simon Inequality for C^2-Morse-Bott Functionals; Feehan [11], Theorem 1.2)**.** *Let $\mathscr{X}, \mathscr{Y}, \mathscr{G}, \mathscr{H}$ be Banach spaces with continuous embeddings $\mathscr{X} \subset \mathscr{G}$ and $\mathscr{Y} \subset \mathscr{H} \subset \mathscr{G}^* \subset \mathscr{X}^*$. Let $\mathscr{U} \subset \mathscr{X}$ be an open subset, $\mathscr{E} : \mathscr{U} \to \mathbb{R}$ be a C^2-function, and $x_\infty \in \mathscr{U}$ be a critical point of $\mathscr{E}$, so $\mathscr{E}'(x_\infty) = 0$. Let $\mathscr{M} : \mathscr{U} \to \mathscr{Y}$ be a C^1 gradient map for $\mathscr{E}$, and require that $\mathscr{E}$ be Morse-Bott at x_∞, so $\mathscr{U} \cap \operatorname{Crit}\mathscr{E}$ is relatively open smooth submanifold of $\mathscr{X}$ and $K := \operatorname{Ker}\mathscr{E}''(x_\infty) = T_{x_\infty}\operatorname{Crit}\mathscr{E}$. Suppose that,* *for each $x \in \mathscr{U}$, the bounded linear operator $\mathscr{M}'(x) : \mathscr{X} \to \mathscr{Y}$ has an extension $\mathscr{M}_1(x) : \mathscr{G} \to \mathscr{H}$ such that the following map is continuous:*

$$\mathscr{U} \ni x \mapsto \mathscr{M}_1(x) \in \mathscr{L}(\mathscr{G}, \mathscr{H}). \tag{3.7}$$

Assume that $K \subset \mathscr{X}$ has a closed complement $\mathscr{X}_0 \subset \mathscr{X}$, that $\mathscr{K} := \operatorname{Ker} \mathscr{M}_1(x_\infty) \subset \mathscr{G}$ has a closed complement $\mathscr{G}_0 \subset \mathscr{G}$ and that $\operatorname{Ran} \mathscr{M}_1(x_\infty) \subset \mathscr{H}$ is a closed subspace. Then there exists constants $c, \sigma > 0$ such that for every $x \in \mathscr{U}$ which satisfies $\|x - x_\infty\|_{\mathscr{X}} < \sigma$,

$$c \cdot |\mathscr{E}(x) - \mathscr{E}(x_\infty)|^{1/2} \leq \|\mathscr{M}(x)\|_{\mathscr{H}}. \tag{3.8}$$

See also [16], where Feehan established the converse of the above theorem.

Now, we can apply the above theorem to establish the optimal Łojasiewicz-Simon inequality for the $L - \lambda A$ functional.

Corollary 3.7 (Łojasiewicz-Simon Inequality for the $L - \lambda A$ functional)**.** *Let $\lambda > 0$ be any fixed constant. For $L - \lambda A : L^2(\mathbb{S}^1) \to \mathbb{R}$ and its gradient map $d(L - \lambda A) : L^2(\mathbb{S}^1) \to H^2(\mathbb{S}^1)$, there exists constants $c, \sigma > 0$ depending on λ such that*

$$c \cdot |(L - \lambda A)(f) - \pi\lambda^{-1}|^{1/2} \leq \|d(L - \lambda A)_f\|_{H^2} \qquad (\forall f \in L^2(\mathbb{S}^1) \text{ s.t. } \|f - \lambda^{-1}\|_{L^2} < \sigma). \tag{3.9}$$

Proof. Let $\mathscr{X} = \mathscr{G} = L^2(\mathbb{S}^1)$ and $\mathscr{Y} = \mathscr{H} = H^2(\mathbb{S}^1)$. These spaces are well-known to be Hilbert spaces and, consequently, are also Banach spaces. There exists a canonical embedding $H^2(\mathbb{S}^1) \subset (L^2(\mathbb{S}^1))^* \cong L^2(\mathbb{S}^1)$. The functional $\mathscr{E} = (L - \lambda A) : L^2(\mathbb{S}^1) \to \mathbb{R}$ is C^2-functional defined on $\mathscr{U} = \mathscr{X} = L^2(\mathbb{S}^1)$.

By Theorem 3.3, The critical set of $\mathscr{E}$ is given by $\operatorname{Crit} \mathscr{E} = \lambda^{-1} + \operatorname{span}\{\sin, \cos\}$, which is a translation of a 2-dimensional linear subspace. Let x_∞ denote the constant function λ^{-1}. Since $K = \operatorname{Ker} \mathscr{E}''(x_\infty) = \operatorname{span}\{\sin, \cos\}$, it follows immediately that $K = T_{x_\infty} \operatorname{Crit} \mathscr{E} = \operatorname{span}\{\sin, \cos\}$. In other words, $\mathscr{E}$ is a Morse-Bott functional.

The gradient map $\mathscr{G}$ corresponds to $d(L - \lambda A)$, considered as a map $L^2(\mathbb{S}^1) \to H^2(\mathbb{S}^1) \subset (L^2(\mathbb{S}^1))^*$. Furthermore, since $\mathscr{X} = \mathscr{G}$ and $\mathscr{Y} = \mathscr{H}$, the derivative $\mathscr{M}'(x) : \mathscr{X} \to \mathscr{Y}$ trivially extends to a continuous map $\mathscr{M}_1(x) = \mathscr{M}'(x)$.

As any finite-dimensional subspace of a Banach space has closed complement, the complement $\mathscr{X}_0 = \mathscr{G}_0 = L^2(\mathbb{S}^1)/\operatorname{span}\{\sin, \cos\}$ can be realized as a closed subspace of $L^2(\mathbb{S}^1)$. In Theorem 3.3, we have explicitly demonstrated that $\operatorname{Ran} \mathscr{M}_1(x_\infty)$ is a closed subspace of codimension 2 in $H^2(\mathbb{S}^1)$. Thus, the result follows directly from Theorem 3.6. □

4 Łojasiewicz-Simon Inequality of APCSF

We are now prepared to apply the Łojasiewicz-Simon inequality to establish the exponential decay of the length along the APCSF. Since APCSF is not precisely the gradient flow of the $L - \lambda A$ functional on the space $L^2(\mathbb{S}^1)$, additional estimates are required to control the decay of the length; in particular, establishing both upper and lower bounds for the curvature is crucial.

Theorem 4.1 (Łojasiewicz-Simon Inequality of APCSF)**.** *There exists a positive real number $\delta > 0$, depending on constants $\lambda > 0$ and $c, \sigma > 0$ described in Corollary 3.7, such that the APCSF, starting from any C^2-initial curve with enclosed area $A_0 = \pi\lambda^{-2}$ and whose reciprocal of curvature $f_0 = f(0, -)$ satisfies $\|f_0(-) - \lambda^{-1}\|_{H^1} < \delta$, fulfills the following conditions:*

(i) *It remains within the L^2-neighborhood $\|f(t, -) - \lambda^{-1}\|_{L^2} < \sigma$ described in Corollary 3.7, where the optimal Łojasiewicz-Simon inequality for the $L - \lambda A$ functional holds.*
(ii) *It has positive uniform upper and lower bounds $0 < m \leq f(t, \theta) \leq M$.*
(iii) *Its length $L(t) = L(f(t, -))$ converges exponentially to $2\pi\lambda^{-1}$.*

Proof. Fix a C^2-initial curve $f(0, -)$ satisfying $\|f(0, -) - \lambda^{-1}\|_{H^1} < \frac{1}{10}\lambda^{-1}$. Define T as the supremum of real numbers such that

$$\|f(t, -) - \lambda^{-1}\|_{H^1} \leq \tfrac{1}{8}\lambda^{-1} \qquad (\forall t \in [0, T)). \tag{4.1}$$

Then $T > 0$ by the time-continuity of the solution $f(t, \theta)$, and T depends on $f(0, -)$.

Using Sobolev inequalities, we estimate

$$\|f(t,-)-\lambda^{-1}\|_{L^\infty} \le \left(\sqrt{2\pi}+\frac{1}{\sqrt{2\pi}}\right)\|f(t,-)-\lambda^{-1}\|_{H^1} < \frac{1}{2}\lambda^{-1}. \tag{4.2}$$

This ensures uniform upper and lower bounds for f and its inverse k:

$$\frac{1}{2}\lambda^{-1} \le f(t,\theta) \le \frac{3}{2}\lambda^{-1} \quad , \quad m := \frac{2}{3}\lambda \le k(t,\theta) \le 2\lambda =: M \qquad (\forall t \in [0,T)\ \forall \theta \in \mathbb{S}^1). \tag{4.3}$$

We will estimate several values and norms at time $t \in [0,T)$ using initial values.
Step I. Estimate of $L(t)-2\pi\lambda^{-1}$. The length difference satisfies

$$\begin{aligned} L(t)-2\pi\lambda^{-1} \le L_0 - 2\pi\lambda^{-1} &= \int (f(0,\theta)-\lambda^{-1})\, d\theta \\ &\le \sqrt{2\pi}\left[\int (f(0,\theta)-\lambda^{-1})^2\, d\theta\right]^{1/2} \\ &= \sqrt{2\pi}\|f(0,-)-\lambda^{-1}\|_{L^2}. \end{aligned}$$

Additionally,

$$L(0)-L(t) \le L_0 - 2\pi\lambda^{-1} \le \sqrt{2\pi}\|f(0,-)-\lambda^{-1}\|_{L^2}.$$

Step II. Estimate of $-L_t$. The time derivative of $L(t)$ satisfies

$$\begin{aligned} -L_t(t) = -\left.\frac{dL}{dt}\right|_t &= \int \frac{1}{f(t,\theta)} - \frac{2\pi}{L(t)}\, d\theta \\ &= -\int \frac{\lambda}{f}(f-\lambda^{-1})\, d\theta + \frac{2\pi}{L\lambda^{-1}}(L-2\pi\lambda^{-1}) \\ &= \int \frac{\lambda^2}{f}(f-\lambda^{-1})^2 - \lambda^2(f-\lambda^{-1})\, d\theta + \frac{2\pi}{L\lambda^{-1}}(L-2\pi\lambda^{-1}) \\ &= \int \frac{\lambda^2}{f}(f-\lambda^{-1})^2\, d\theta - 2\pi\frac{\lambda^2}{L}(L-2\pi\lambda^{-1})^2. \end{aligned}$$

Therefore,

$$m\lambda^2\|f(t,-)-\lambda^{-1}\|_{L^2}^2 - 2\pi\frac{\lambda^2}{L_0}(L(t)-2\pi\lambda^{-1})^2 \le -L_t(t) \le M\lambda^2\|f(t,-)-\lambda^{-1}\|_{L^2}^2.$$

Step III. Estimate of $\|f(t,-)-\lambda^{-1}\|_{L^2}^2$. Differentiating $\|f(t,-)-\lambda^{-1}\|_{L^2}^2$, we obtain

$$\begin{aligned} \frac{d}{dt}\|f(t,-)-\lambda^{-1}\|_{L^2}^2 &= 2\int (f-\lambda^{-1})f_t\, d\theta \\ &= 2\int \frac{ff_{\theta\theta}-2f_\theta^2}{f^2} + \cancelto{0}{(-1+2\pi L^{-1}f)} - \cancelto{0}{\lambda^{-1}(f^{-1})_{\theta\theta}} + \lambda^{-1}\left(f^{-1}-\frac{2\pi}{L}\right) d\theta \\ &= 2\int \cancelto{0}{(f^{-1}f_\theta)_\theta} - \frac{f_\theta^2}{f^2} + \lambda^{-1}\left(f^{-1}-\frac{2\pi}{L}\right) d\theta \\ &= -2\lambda^{-1}L_t - 2\int \frac{f_\theta^2}{f^2}\, d\theta \\ &\le -2\lambda^{-1}L_t. \end{aligned}$$

Integration yields

$$\begin{aligned} \|f(t,-)-\lambda^{-1}\|_{L^2}^2 &\le 2\lambda^{-1}(L_0-L(t)) + \|f(0,-)-\lambda^{-1}\|_{L^2}^2 \\ &\le 2\lambda^{-1}(L_0-2\pi\lambda^{-1}) + \|f(0,-)-\lambda^{-1}\|_{L^2}^2 \\ &\le \|f(0,-)-\lambda^{-1}\|_{L^2}(2\sqrt{2\pi}\lambda^{-1} + \|f(0,-)-\lambda^{-1}\|_{L^2}) \\ &\le 6\lambda^{-1}\|f(0,-)-\lambda^{-1}\|_{L^2}. \end{aligned}$$

Step IV. Estimate of $\|k(t,-)-\bar{k}(t)\|_{L^2}^2$.

$$\begin{aligned}
\|k(t,-)-\bar{k}(t)\|_{L^2}^2 &= \int \left(\frac{1}{f(t,\theta)}-\frac{2\pi}{L(t)}\right)^2 d\theta = \int f^{-2}L^{-2}\left(L-2\pi f\right)^2 d\theta \\
&\le \frac{M^2}{L^2}\int \left[(L-2\pi\lambda^{-1})-2\pi(f-\lambda^{-1})\right]^2 d\theta \\
&\le 2\frac{M^2}{L^2}\left[2\pi(L-2\pi\lambda^{-1})^2+4\pi^2\int (f-\lambda^{-1})^2\, d\theta\right] \\
&\le 2M^2\lambda^2\left[\frac{(L_0-2\pi\lambda^{-1})^2}{2\pi}+\|f(t,-)-\lambda^{-1}\|_{L^2}^2\right] \\
&\le 2M^2\lambda^2\left[\|f(0,-)-\lambda^{-1}\|_{L^2}^2+6\lambda^{-1}\|f(0,-)-\lambda^{-1}\|_{L^2}\right] \\
&\le 13M^2\lambda\|f(0,-)-\lambda^{-1}\|_{L^2}.
\end{aligned}$$

Step V. Estimate of $\|k_\theta(0,-)\|_{L^2}^2$.

$$\|k_\theta(0,-)\|_{L^2}^2 = \int [(f^{-1})_\theta]^2\, d\theta = \int \frac{f_\theta^2}{f^4}\, d\theta \le M^4\int f_\theta^2\, d\theta = M^4\|f_\theta(0,-)\|_{L^2}^2$$

Step VI. Estimate of $\|k_\theta(t,-)\|_{L^2}^2$.

$$\begin{aligned}
\frac{d}{dt}(\|k(t,-)-\bar{k}(t)\|_{L^2}^2-\|k_\theta(t,-)\|_{L^2}^2) &= 2\int (k-\bar{k})(k_t-\bar{k}_t)\, d\theta - 2\int k_\theta k_{\theta t}\, d\theta \\
&= 2\int (k_{\theta\theta}+k-\bar{k})k_t\, d\theta - 2\bar{k}_t\int (k-\bar{k})\, d\theta \\
&= 2\int k^2(k_{\theta\theta}+k-\bar{k})^2\, d\theta - 4\pi\frac{L_t^2}{L^2} \\
&\ge -4\pi\frac{L_t^2}{L^2} \ge -\pi^{-1}M\lambda^4\|f(t,-)-\lambda^{-1}\|_{L^2}^2\cdot(-L_t) \\
&\ge -2M\lambda^3\|f(0,-)-\lambda^{-1}\|_{L^2}\cdot(-L_t)
\end{aligned}$$

Integration yields

$$\begin{aligned}
\|k(t,-)-\bar{k}(t)\|_{L^2}^2-\|k_\theta(t,-)\|_{L^2}^2 \ge{}& \|k(0,-)-\bar{k}(0)\|_{L^2}^2-\|k_\theta(0,-)\|_{L^2}^2 \\
&- 2M\lambda^3\|f(0,-)-\lambda^{-1}\|_{L^2}\cdot(L_0-L(t)).
\end{aligned}$$

In particular, this implies

$$\begin{aligned}
\|k_\theta(t,-)\|_{L^2}^2 &\le \|k(t,-)-\bar{k}(t)\|_{L^2}^2+\|k_\theta(0,-)\|_{L^2}^2+2M\lambda^3\|f(0,-)-\lambda^{-1}\|_{L^2}\cdot(L_0-L(t)) \\
&\le 13M^2\lambda\|f(0,-)-\lambda^{-1}\|_{L^2}+M^4\|f_\theta(0,-)\|_{L^2}^2+6M\lambda^3\|f(0,-)-\lambda^{-1}\|_{L^2}^2 \\
&\le 52\lambda^3\|f(0,-)-\lambda^{-1}\|_{L^2}+16\lambda^4\|f(0,-)-\lambda^{-1}\|_{H^1}^2 \\
&\le 54\lambda^3\|f(0,-)-\lambda^{-1}\|_{H^1}.
\end{aligned}$$

Step VII. Estimate of $\|f_\theta(t,-)\|_{L^2}^2$.

$$\|f_\theta(t,-)\|_{L^2}^2 = \int f^4(f^{-1})_\theta^2\, d\theta \le m^{-4}\int k_\theta^2\, d\theta = m^{-4}\|k_\theta(t,-)\|_{L^2}^2 \le 274\lambda^{-1}\|f(0,-)-\lambda^{-1}\|_{H^1}$$

This demonstrates that f remains within the specified H^1-neighborhood (where the Łojasiewicz inequality holds) throughout the flow, provided that f is initially contained in a sufficiently small H^1-neighborhood of λ^{-1}.

Step VIII. Estimate of $\|f(t,-)-\lambda^{-1}\|_{H^1}^2$.

$$\begin{aligned}
\|f(t,-)-\lambda^{-1}\|_{H^1}^2 &= (\|f(t,-)-\lambda^{-1}\|_{L^2}+\|f_\theta(t,-)\|_{L^2})^2 \\
&\le 2(\|f(t,-)-\lambda^{-1}\|_{L^2}^2+\|f_\theta(t,-)\|_{L^2}^2) \\
&\le 2(6\lambda^{-1}\|f(0,-)-\lambda^{-1}\|_{L^2}+274\lambda^{-1}\|f(0,-)-\lambda^{-1}\|_{H^1}) \\
&\le 560\lambda^{-1}\|f(0,-)-\lambda^{-1}\|_{H^1}.
\end{aligned}$$

Step IX. Application of the Łojasiewicz-Simon inequality (Corollary 3.7).
Let

$$\sigma' := \min\left\{\sigma, \frac{c}{225}, \frac{\lambda^{-1}}{10}\right\},$$

where c, σ are the constants in the Łojasiewicz-Simon inequality (Corollary 3.7). Define

$$\delta := \frac{1}{560}\lambda(\sigma')^2.$$

This ensures that for any C^2-initial curve $f(0,-)$ satisfying $\|f(0,-) - \lambda^{-1}\|_{H^1} < \delta$, we have $\|f(t,-) - \lambda^{-1}\|_H^1 < \sigma'$ for all $t \in [0,T)$, where $T > 0$ depends on $f(0,-)$. Recall that T is defined as the supremum of real numbers such that

$$\|f(t,-) - \lambda^{-1}\|_{H^1} \leq \lambda^{-1}/8 \qquad (\forall t \in [0,T)).$$

Assume, for contradiction, that $T < \infty$. Then, there exists a time $t' > T$, arbitrarily close to T, such that

$$\|f(t',-) - \lambda^{-1}\|_{H^1} > \lambda^{-1}/8.$$

However, this contradicts the continuity of $f(t,-)$, as $\|f(t,-) - \lambda^{-1}\|_H^1 < \sigma' \leq \lambda^{-1}/10$ holds for all $t < T$. Thus, $T = \infty$ for any C^2-initial curve $f(0,-)$ satisfying $\|f(0,-) - \lambda^{-1}\|_{H^1} < \delta$.

Consequently, for any C^2-initial curve $f(0,-)$ with $\|f(0,-) - \lambda^{-1}\|_{H^1} < \delta$, the inequality $\|f(t,-) - \lambda^{-1}\|_H^1 < \sigma'$ holds for all $t \in [0,\infty)$.
Step X. Conclusion.
For the solution $f(t,\theta)$, we apply the Łojasiewicz-Simon inequality (Corollary 3.7), which gives

$$\begin{aligned} c(L(t) - 2\pi\lambda^{-1}) &\leq \|d(L - \lambda A)_{f(t,-)}\|_{H^2}^2 \leq 25\lambda^2\|f(t,-) - \lambda^{-1}\|_{L^2}^2 \\ &\leq 25m^{-1}(-L_t + 2\pi\lambda^2 L_0^{-1}(L(t) - 2\pi\lambda^{-1})^2). \end{aligned}$$

Next, note that

$$L(t) - 2\pi\lambda^{-1} \leq \sqrt{2\pi}\|f(0,-) - \lambda^{-1}\|_{L^2} \leq \sqrt{2\pi}\sigma' \leq \frac{\sqrt{2\pi}c}{225} \leq \frac{c}{75} = \frac{cm(2\pi\lambda^{-1})}{100\pi} \leq \frac{cmL_0}{100\pi}.$$

From this, we derive

$$\tfrac{1}{75}\lambda c(L(t) - 2\pi\lambda^{-1}) = \tfrac{1}{50}mc(L(t) - 2\pi\lambda^{-1}) \leq \tfrac{1}{25}mc(L(t) - 2\pi\lambda^{-1}) - 2\pi\lambda^2 L_0^{-1}(L(t) - 2\pi\lambda^{-1})^2 \leq -L_t.$$

Integrating this inequality over time yields the exponential decay of $L(t) - 2\pi\lambda^{-1}$:

$$L(t) - 2\pi\lambda^{-1} \leq (L_0 - 2\pi\lambda^{-1})\exp(-\tfrac{1}{75}\lambda ct). \qquad \square$$

5 Łojasiewicz Convergence Theorem for APCSF

Recall that for closed plane curves γ_1 and γ_2, the **Hausdorff distance** between them is defined as

$$\mathrm{dist}(\gamma_1,\gamma_2) = \max\left(\sup_{x\in\gamma_1}\inf_{y\in\gamma_2}|x-y| + \sup_{x\in\gamma_2}\inf_{y\in\gamma_1}|x-y|\right).$$

(Note that we slightly abuse notation by letting γ_1 and γ_2 represents their images, $\mathrm{Im}(\gamma_1)$ and $\mathrm{Im}(\gamma_2)$, respectively.)

Theorem 5.1 (Łojasiewicz Convergence Theorem for APCSF)**.** *The APCSF $\{\gamma_t\}$, starting from an initial curve satisfying the assumptions of Theorem 4.1, converges exponentially to a circle γ_∞ with radius λ^{-1} in both the uniform distance and the Hausdorff distance.*

Proof. This proof adapts the argument from Theorem 2.3, where the finite-dimensional optimal Łojasiewicz inequality was employed to bound the length of the trajectory by an exponential function. However, modifications are necessary because the APCSF is not exactly the gradient flow of the $L - \lambda A$ functional on the space $L^2(\mathbb{S}^1)$.

Let $u \in \mathbb{S}^1$ represents the original parameter of the flow $X(t,u)$, so that $\gamma_t(u) = X(t,u)$ satisfies

$$X_t = (k - \bar{k})N.$$

Consider the trajectory $x(-) = X(-,u)$, which initiates at $x_1 = X(t_1,u) = \gamma_{t_1}(u)$. The trajectory satisfies

$$x(t) = x_1 + \int_{t_1}^{t} X_t(t', x(t'))\, dt'$$

$$|\gamma_{t_2}(u) - \gamma_{t_1}(u)| = |x(t_2) - x(t_1)| \le \int_{t_1}^{t_2} |X_t|(t', x(t'))\, dt' \le \int_{t_1}^{t_2} \|X_t(t', -)\|_{L^\infty}\, dt'.$$

Therefore,

$$\|\gamma_{t_2} - \gamma_{t_1}\|_{L^\infty} \le \int_{t_1}^{t_2} \|X_t(t', -)\|_{L^\infty}\, dt'.$$

Suppose we can bound $\int_{t_1}^{t_2} \|X_t(t', -)\|_{L^\infty}\, dt'$ by an exponential function $C_1 \exp(-C_2 t_1)$. In this case, $\{\gamma_t\}$ forms a uniformly Cauchy sequence, and thus it converges uniformly and exponentially to a continuous curve γ_∞:

$$\|\gamma_t - \gamma_\infty\|_{L^\infty} \le C_1 \exp(-C_2 t).$$

The uniform limit γ_∞ must be a circle of radius λ^{-1} by the equality condition of the isoperimetric inequality, as its length satisfies

$$L(\gamma_\infty) = \lim_{t\to\infty} L(t) = 2\pi\lambda^{-1}.$$

Additionally, for any continuous curves γ_1, γ_2 defined on the parameter domain $u \in \mathbb{S}^1$, we have

$$\sup_{x\in\gamma_1} \inf_{y\in\gamma_2} |x - y| \le \|\gamma_1 - \gamma_2\|_{L^\infty},$$

since for any $x = \gamma_1(u) \in \gamma_1$, we can choose $y := \gamma_2(u)$, which satisfies

$$|x - y| = |\gamma_1(u) - \gamma_2(u)| \le \|\gamma_1 - \gamma_2\|_{L^\infty}.$$

Interchanging the roles of γ_1, γ_2, and then taking the maximum, we obtain

$$\operatorname{dist}(\gamma_1, \gamma_2) \le \|\gamma_1 - \gamma_2\|_{L^\infty}.$$

Thus, uniform exponential convergence implies Hausdorff exponential convergence:

$$\operatorname{dist}(\gamma_t, \gamma_\infty) \le \|\gamma_t - \gamma_\infty\|_{L^\infty} \le C_1 \exp(-C_2 t).$$

Therefore, to establish both Hausdorff and uniform exponential convergence, it is sufficient to show that the L^∞-length of the trajectory is bounded by an exponential function:

$$\int_{t_1}^{t_2} \|X_t(t', -)\|_{L^\infty}\, dt' \le C_1 \exp(-C_2 t_1).$$

We have $|X_t|(u,t) = |k(\theta(u), t) - \bar{k}(t)|$. Changing the parametrization from u to θ does not affect the L^∞-norm. By Sobolev inequalities, we obtain

$$\|k(t,-) - \bar{k}(t)\|_{L^\infty} \le \left(\sqrt{2\pi} + \frac{1}{\sqrt{2\pi}}\right) \max(\|k(t,-) - \bar{k}(t)\|_{L^2}, \|k_\theta(t,-)\|_{L^2}) \le 3\|k(t,-) - \bar{k}(t)\|_{H^1}.$$

For the L^2-norm of $k(t,-) - \bar{k}(t)$, we have

$$\begin{aligned}
\|k(t,-) - \bar{k}(t)\|_{L^2}^2 &\leq \pi^{-1} M^2 \lambda^2 (L(t) - 2\pi\lambda^{-1})^2 + 2M^2\lambda^2 \|f - \lambda^{-1}\|_{L^2}^2(t) \\
&\leq (\pi^{-1} M^2 \lambda^2 + 4\pi M^2 m^{-1} \lambda^2 L_0^{-1})(L(t) - 2\pi\lambda^{-1})^2 + 2M^2 m^{-1}(-L_t) \\
&\leq (\pi^{-1} M^2 \lambda^2 + 4\pi M^2 m^{-1} \lambda^2 L_0^{-1})(L_0 - 2\pi\lambda^{-1})^2 \exp(-\tfrac{2}{75}\lambda c t) + 2M^2 m^{-1}(-L_t) \\
&\leq 14\lambda^4 (L_0 - 2\pi\lambda^{-1})^2 \exp(-\tfrac{2}{75}\lambda c t) + 12\lambda(-L_t) \\
&\leq 100\lambda^4 \|f(0,-) - \lambda^{-1}\|_{L^2}^2 \exp(-\tfrac{2}{75}\lambda c t) + 12\lambda(-L_t) \\
&\leq \lambda^2 \exp(-\tfrac{2}{75}\lambda c t) + 12\lambda(-L_t).
\end{aligned}$$

For the L^2-norm of $k_\theta(t,-)$, we have

$$\begin{aligned}
\|k_\theta(t,-)\|_{L^2}^2 &= \int k_\theta^2 \, d\theta \leq m^{-1} \int k k_\theta^2 \, d\theta = -\frac{1}{2m} \int k^2 k_{\theta\theta} \, d\theta = -\frac{1}{2m} \int k^2 (k^{-2} k_t - (k - \bar{k})) \, d\theta \\
&\leq \frac{M^2}{2m} \left| \int (k^{-1})_t \, d\theta \right| + \frac{1}{2m} \int k^3 (k - \bar{k}) \, ds = \frac{M^2}{2m} \left| \frac{d}{dt} \int \frac{1}{k} \, d\theta \right| + \frac{1}{2m} \int (k^3 - \bar{k}^3)(k - \bar{k}) \, ds \\
&= -\frac{M^2}{2m} \frac{dL}{dt} + \frac{1}{2m} \int (k^2 + k\bar{k} + \bar{k}^2)(k - \bar{k})^2 \, ds \leq \left(\frac{M^2}{2m} + \frac{3M^2}{2m} \right) (-L_t) \\
&= 2M^2 m^{-1} (-L_t) = 12\lambda(-L_t).
\end{aligned}$$

Combining these results, we estimate the H^1-norm of $k(t,-) - \bar{k}(t)$ as follows:

$$\begin{aligned}
\|k(t,-) - \bar{k}(t)\|_{H^1}^2 &= (\|k(t,-) - \bar{k}(t)\|_{L^2} + \|k_\theta(t,-)\|_{L^2})^2 \\
&\leq 2\|k(t,-) - \bar{k}(t)\|_{L^2}^2 + 2\|k_\theta(t,-)\|_{L^2}^2 \\
&\leq 48\lambda(-L_t) + 2\lambda^2 \exp(-\tfrac{2}{75}\lambda c t).
\end{aligned}$$

Integration yields

$$\begin{aligned}
\int_{t_1}^{t_2} \|k(t,-) - \bar{k}(t)\|_{L^\infty} \, dt &\leq 3 \int_{t_1}^{t_2} \|k(t,-) - \bar{k}(t)\|_{H^1} \, dt \\
&\leq 3 \left(\int_{t_1}^{\infty} \|k(t,-) - \bar{k}(t)\|_{H^1}^2 (t + \varepsilon^{-1/\varepsilon})^{1+\varepsilon} \, dt \right)^{1/2} \left(\int_{t_1}^{\infty} (t + \varepsilon^{-1/\varepsilon})^{-1-\varepsilon} \, dt \right)^{1/2} \\
&\leq 3 \left(\int_{t_1}^{\infty} \|k(t,-) - \bar{k}(t)\|_{H^1}^2 (t + \varepsilon^{-1/\varepsilon})^{1+\varepsilon} \, dt \right)^{1/2} \left(\varepsilon^{-1} (t_1 + \varepsilon^{-1/\varepsilon})^{-\varepsilon} \right)^{1/2} \\
&\leq 3 \left(\int_{t_1}^{\infty} 48\lambda(-L_t)(t + \varepsilon^{-1/\varepsilon})^{1+\varepsilon} + 2\lambda^2 (t + \varepsilon^{-1/\varepsilon})^{1+\varepsilon} \exp(-\tfrac{2}{75}\lambda c t) \, dt \right)^{1/2},
\end{aligned}$$

where $\varepsilon > 0$ is fixed. Integrals of the terms involving $(-L_t)$ can be estimated:

$$\begin{aligned}
\int_{t_1}^{\infty} (-L_t)(t + \varepsilon^{-1/\varepsilon})^{1+\varepsilon} \, dt &= -\left[(L(t) - 2\pi\lambda^{-1})(t + \varepsilon^{-1/\varepsilon})^{1+\varepsilon} \right]_{t_1}^{\infty} \\
&\quad + (1+\varepsilon) \int_{t_1}^{\infty} (L(t) - 2\pi\lambda^{-1})(t + \varepsilon^{-1/\varepsilon})^{\varepsilon} \, dt \\
&\leq (L_0 - 2\pi\lambda^{-1})(t_1 + \varepsilon^{-1/\varepsilon})^{1+\varepsilon} \exp(-\tfrac{1}{75}\lambda c t_1) \\
&\quad + (1+\varepsilon)(L_0 - 2\pi\lambda^{-1}) \int_{t_1}^{\infty} (t + \varepsilon^{-1/\varepsilon})^{\varepsilon} \exp(-\tfrac{1}{75}\lambda c t) \, dt
\end{aligned}$$

There exist constants $C_0, C_2 > 0$ such that the following inequalities hold:

$$\begin{aligned}
2\lambda^2 (t + \varepsilon^{-1/\varepsilon})^{1+\varepsilon} \exp(-\tfrac{2}{75}\lambda c t) &\leq C_0 e^{-2C_2 t} \\
48\lambda(1+\varepsilon)(L_0 - 2\pi\lambda^{-1})(t + \varepsilon^{-1/\varepsilon})^{\varepsilon} \exp(-\tfrac{1}{75}\lambda c t) &\leq C_0 e^{-2C_2 t} \\
48\lambda(L_0 - 2\pi\lambda^{-1})(t + \varepsilon^{-1/\varepsilon})^{1+\varepsilon} \exp(-\tfrac{1}{75}\lambda c t) &\leq C_0 e^{-2C_2 t}.
\end{aligned}$$

As a result, there exists a constant $C_1 > 0$, depending on C_0, C_2, λ, such that

$$3\left(\int_{t_1}^{\infty} 2M^2m^{-1}(-L_t)(t+\varepsilon^{-1/\varepsilon})^{1+\varepsilon} + C(t+\varepsilon^{-1/\varepsilon})^{1+\varepsilon}\exp(-\tfrac{1}{3}m\lambda^2 ct)\,dt\right)^{1/2} \le C_1 e^{-C_2 t}.$$

Thus, for any $0 \le t_1 \le t_2$,

$$\int_{t_1}^{t_2} \|k-\bar{k}\|_{L^\infty}\,dt \le C_1\exp(-C_2 t_1).$$

This completes the proof. □

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