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Computational method for fuzzy arithmetic operations on triangular fuzzy numbers by extension principle

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A B S T R A C T

Fuzzy arithmetic operations are applied to mathematical equations that include fuzzy numbers, which are commonly used to represent non-probabilistic uncertainty in different applications. Although there are two mathematical approaches available in the literature for implementing fuzzy arithmetic (i.e., the α-cut approach, and the extension principle approach), the existing computational methods are mainly focused on implementing the α-cut approach due to its simplicity. However, this approach causes overestimation of uncertainty in the resulting fuzzy numbers, a phenomenon that reduces the interpretability of the results. This overestimation can be reduced by implementing fuzzy arithmetic using the extension principle; however, existing computational methods for implementing the extension principle approach are limited to the use of min and drastic product t-norms. Using the min t-norm produces the same result as the α-cuts and interval calculations approach, and the drastic product t-norm is criticized for producing resulting fuzzy numbers that are highly sensitive to the changes in the input fuzzy numbers. This paper presents original computational methods for implementing fuzzy arithmetic operations on triangular fuzzy numbers using the extension principle approach with product and Lukasiewicz t-norms. These computational methods contribute to the different applications of fuzzy arithmetic; they reduce the overestimation of uncertainty, as compared to the α-cut approach, and they reduce the sensitivity of the resulting fuzzy numbers to changes in the input fuzzy numbers, as compared to the extension principle approach using drastic product t-norm.

1. Introduction

Fuzzy set theory, developed by Zadeh [1], is a powerful tool for modeling subjective and imprecise information in different contexts. Introduced by Zadeh [2], fuzzy numbers are a specific type of fuzzy sets used for representing values of real world parameters when the exact values are not measurable due to a lack of knowledge or incomplete information [3]. Fuzzy numbers have been applied in many different areas, such as engineering problems [4,5], decision making problems [6,7], and data analysis [8]. In such applications, fuzzy arithmetic is frequently applied to mathematical equations that in-

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clude fuzzy numbers by using one of the two approaches introduced in the literature: (1) the α-cut approach, and (2) the extension principle approach using different t-norms. Hereafter, this paper will refer to the α-cut approach as "standard fuzzy arithmetic" and to the extension principle approach as "extended fuzzy arithmetic". Though the literature contains extensive discussion on the mathematical aspects of the two fuzzy arithmetic implementation approaches (see Dubois and Prade [9] for review), in order to implement fuzzy arithmetic operations on fuzzy numbers in different applications, computational methods are required. In the existing body of knowledge of fuzzy logic, there are few computational methods developed for implementing fuzzy arithmetic, which are mainly focused on standard fuzzy arithmetic due to its simplicity. Accordingly, developing computational methods for the implementation of extended fuzzy arithmetic operations contributes to the more effective use of fuzzy numbers in different applications.

Standard fuzzy arithmetic is the most common approach for implementing fuzzy arithmetic in different applications, due to its simplicity and to the availability of computational methods [6,10]. However, the standard fuzzy arithmetic approach is criticized for accumulation of fuzziness, a phenomenon that causes overestimation of uncertainties in the resulting fuzzy numbers [11–13]. The uncertainty overestimation problem can be reduced by implementing extended fuzzy arithmetic, using any t-norm lower than the min t-norm. While extended fuzzy arithmetic can be implemented using any given t-norm including the four common t-norms (i.e., min, product, Lukasiewicz, and drastic product), existing computational methods are limited to the use of only two of these t-norms: min and drastic product. Klir [14] introduced a computational method for implementing extended fuzzy arithmetic using the min t-norm on trapezoidal fuzzy numbers, and Kolesarova [15], Mesiar [16], and Hong and Do [17] developed a computational method for implementing extended fuzzy arithmetic using the drastic product t-norm on triangular fuzzy numbers. Extended fuzzy arithmetic using the min t-norm produces the same result as standard fuzzy arithmetic [10,18], leading to similar overestimation of uncertainty in the resulting fuzzy numbers. Moreover, as the drastic product is a non-continuous t-norm, implementation of fuzzy operations using this t-norm produces resulting fuzzy numbers that are highly sensitivity to changes in the input fuzzy numbers [3]. Given that min is the highest t-norm (i.e., returns the highest membership value for the results of fuzzy operations), performing extended fuzzy arithmetic using any t-norm other than min reduces the uncertainty overestimation problem [11–13]. Furthermore, implementation of fuzzy arithmetic using any continuous t-norm decreases the sensitivity of the resulting fuzzy numbers to changes in the input fuzzy numbers, as compared to the drastic product t-norm. Although the product and Lukasiewicz t-norms are two common t-norms [3] with the two aforementioned characteristics (i.e., lower than the min t-norm, and continuous t-norm), there is a research gap in development of computational methods for implementing extended fuzzy arithmetic using these two t-norms due to the complexity associated developing such computational methods. This paper introduces an original computational method for implementing extended fuzzy arithmetic using product and Lukasiewicz t-norms on triangular fuzzy numbers, which reduces the uncertainty overestimation problem in the resulting fuzzy numbers. The computational method proposed in this paper is developed for triangular fuzzy numbers, which are commonly used in a wide range of applications [3]. Accordingly, this paper fills a gap in the research for computational methods for implementation of extended fuzzy arithmetic using two common t-norms: the product and Lukasiewicz t-norms. The algorithms for implementation of the proposed computational methods are also presented in this paper. Finally, standard fuzzy arithmetic and extended fuzzy arithmetic using min, product, Lukasiewicz, and drastic product t-norms are compared in terms of the amount of uncertainty included in the resulting fuzzy number, and the sensitivity of the resulting fuzzy number to changes in the input fuzzy numbers.

The proceeding sections of this paper are organized as follows. The second section provides a brief review of fuzzy numbers, arithmetic operations performed on fuzzy numbers, and the uncertainty measures of fuzzy sets. The third section presents the proposed computational method for implementing extended fuzzy arithmetic using product and Lukasiewicz t-norms, and an algorithm for implementing the proposed methods. The fourth section presents a numerical example and compares the two approaches for implementing fuzzy arithmetic. Additionally, in the fourth section, the computational methods presented in this paper are compared to the new methodology for implementing fuzzy arithmetic using horizontal membership functions, and the differences between the two methodologies are discussed.

2. Preliminaries

2.1. Fuzzy sets

Fuzzy set theory, introduced by Zadeh [1], is a powerful method for representing subjective or imprecise information in different contexts. A fuzzy set is a set of elements with memberships in the set that can vary between $[0, 1]$, in contrast to a classical set, where its elements have memberships of either one (fully belong in the set) or zero (do not belong in the set) [1]. Fuzzy sets are represented by their membership functions, which determine the degree of membership of each element (e.g., each point in the universe of discourse) in the fuzzy set. The α-cut of a fuzzy set, denoted as A_α, is a set of all the elements of the universe of discourse whose membership values are equal to or exceed α where $\alpha \in [0, 1]$, as mathematically presented in Eq. (1):

$$A_\alpha = \left\{x \in X \mid A(x) \geq \alpha\right\} \tag{1}$$

Referring to the representation theorem, any fuzzy set can be represented by its α-cuts [3]. The mathematical form of the representation theorem is below shown in Eq. (2):

$$A(x) = \sup_{\alpha \in [0,1]} \left(\alpha A_\alpha(x)\right) \tag{2}$$

where $A(x)$ stands for the membership function of fuzzy set A and A_α represents the α-cut of fuzzy set A at the level of α. The representation theorem implies that any fuzzy operation performed on fuzzy sets (e.g., fuzzy arithmetic) can be implemented on their α-cuts using the classical interval operations [3]. A fuzzy set is convex if for any given point x, where $x_1 \le x \le x_2$, the membership value of x is equal to or greater than the minimum membership values of x_1 and x_2.

2.2. Fuzzy numbers

Fuzzy numbers are a specific type of fuzzy sets, which are represented by the membership functions with the following properties [15]: (1) bounded supports, (2) are normal (i.e., possess at least one point in the universe of discourse, which has a membership value of 1), (3) are convex, and (4) have α-cuts that are closed intervals of real numbers. There are several types of membership functions that meet the above requirements for fuzzy numbers, of which triangular fuzzy numbers are common in different applications of fuzzy numbers [3,20]. Triangular fuzzy numbers are defined by their piecewise linear segments, as presented in Eq. (3):

$$A(x; a_1, a_2, a_3) = \begin{cases} \frac{x-a_1}{a_2-a_1}, & \text{if } a_1 < x \le a_2 \\ \frac{a_3-x}{a_3-a_2}, & \text{if } a_2 < x \le a_3, \qquad a_1 \le a_2 \le a_3 \\ 0, & \text{otherwise} \end{cases} \tag{3}$$

where a_1 and a_3 stand for the lower and upper bounds of the support (respectively) and a_2 is the core of fuzzy number (i.e., $A(a_2; a_1, a_2, a_3) = 1$).

2.3. Fuzzy arithmetic

There are two different approaches for implementing fuzzy arithmetic on fuzzy numbers: (1) the α-cuts and interval calculations approach (i.e., standard fuzzy arithmetic), and (2) the extension principle approach (i.e., extended fuzzy arithmetic) using different t-norms. Standard fuzzy arithmetic can be implemented in the following three steps, as discussed by Alavidoost et al. [6]: (1) the input fuzzy numbers are discretized into a number of α-cuts, (2) interval calculations are implemented on the α-cuts of the input fuzzy numbers to find the α-cut of the resulting fuzzy number, and (3) the resulting fuzzy number is constructed according to the representation theorem (Eq. (1)). Despite the fact that standard fuzzy arithmetic is used in many different applications, this approach causes overestimation of uncertainties in the resulting fuzzy numbers [10,21]. Accordingly, in recent applications, extended fuzzy arithmetic has been preferred to standard fuzzy arithmetic (e.g., [11–13,22]). The extended fuzzy arithmetic approach was first introduced by Nguyen [19] using the min t-norm, as presented in Eq. (4):

$$C(z) = A(x) \circledast B(y) = \sup_{z=x*y} \left(\min\left(A(x), B(y)\right)\right) \tag{4}$$

where $C(z)$ stands for the resulting fuzzy number, $A(x)$ and $B(y)$ represent the two input fuzzy numbers, $\circledast$ stands for any of the four fuzzy arithmetic operations, and $*$ stands for any of the four arithmetic operations implemented on crisp numbers. In the generalized form of extended fuzzy arithmetic, the min t-norm in Eq. (5) can be replaced by any t-norm, as presented in Eq. (5) [23,24]:

$$C(z) = A(x) \circledast B(y) = \sup_{z=x*y} \left(t\left(A(x), B(y)\right)\right) \tag{5}$$

where t represents any t-norm. Thus, the four common t-norms, product, Lukasiewicz, min, and drastic product, can be used to implement extended fuzzy arithmetic in its generalized form. The mathematical representation of the product, Lukasiewicz, min, and drastic product t-norms are presented below in Eq. (6), Eq. (7), Eq. (8), and Eq. (9), respectively. Min is the highest t-norm and drastic product is the lowest t-norm. Additionally, in the literature, the Lukasiewicz t-norm is also referred to as the bounded difference t-norm:

$$t_{product}(x, y) = x \times y \tag{6}$$

$$t_{Lukasiewicz}(x, y) = \max(x + y - 1, 0) \tag{7}$$

$$t_{\min}(x, y) = \min(x, y) \tag{8}$$

$$t_{drastic\ product}(x, y) = \begin{cases} x, & y = 1 \\ y, & x = 1 \\ 0, & \text{otherwise} \end{cases} \tag{9}$$

Triangular fuzzy numbers are closed under extended addition and subtraction using the min t-norm, which means that the result of implementing extended fuzzy addition or subtraction using the min t-norm on two triangular fuzzy numbers is a triangular fuzzy number. An exact mathematical solution for implementing extended fuzzy addition and subtraction was presented by Dubois et a. [25]. Garg [26] showed that sigmoid fuzzy numbers are closed under any extended fuzzy arithmetic operations using the min t-norm, and he presented an exact mathematical solution for implementing extended fuzzy arithmetic operations on sigmoid fuzzy numbers using the min t-norm. Although exact mathematical solutions are available for implementing some extended fuzzy arithmetic operations using the min t-norm (see [25,26]), implementing extended fuzzy arithmetic using the min t-norm causes the same level of uncertainty overestimation as standard fuzzy arithmetic. Conversely, using any t-norm other than min for implementing extended fuzzy arithmetic reduces the uncertainty overestimation compared to standard fuzzy arithmetic. Moreover, depending on the strength of the t-norm used for implementing extended fuzzy arithmetic, the level of uncertainty overestimation will be reduced to different levels, as compared to standard fuzzy arithmetic. In other words, the lower the t-norm is, the less overestimation of uncertainty will occur in the resulting fuzzy numbers. Accordingly, the drastic product t-norm reduces the uncertainty overestimation in the resulting fuzzy numbers to the lowest level, followed by the Lukasiewicz, product, and min t-norms respectively [27]. Heshmaty and Kandel [28] proved that triangular fuzzy numbers are closed under all extended fuzzy arithmetic operations using drastic product t-norm. Thus, the result of implementing any extended fuzzy arithmetic operation using the drastic product t-norm on two triangular fuzzy numbers can be determined by calculating three points of the resulting fuzzy number (i.e., the core and lower and upper bounds of the support). Kolesarova [15], Mesiar [16], and Hong and Do [17] developed the exact mathematical solution for implementing extended fuzzy arithmetic using the drastic product t-norm on triangular fuzzy numbers.

Although the drastic product t-norm, reduces the uncertainty overestimation to the lowest level, it is a non-continuous t-norm. Thus, implementation of fuzzy operations using this t-norm produces resulting fuzzy numbers that are highly sensitive to changes in the input fuzzy numbers [3]. Pedrycz and Gomide [3] suggest that the drastic product t-norm can be ruled out for fuzzy operations in some applications, such as system modeling, decision making, or optimization problems (refer to Klement and Navara [29], Jenei [30], and Jenei [31] for a full review). The product and Lukasiewicz t-norms are appropriate for implementing extended fuzzy arithmetic, as they mitigate the two limitations of implementing extended fuzzy arithmetic using min and drastic product t-norms (i.e., uncertainty overestimation and high sensitivity). Due to the continuity of these two t-norms, using them for implementing extended fuzzy arithmetic can reduce the sensitivity of the resulting fuzzy numbers to changes in the input fuzzy numbers, as compared to the drastic product t-norm. Since these two t-norms are both lower than the min t-norm (i.e., yield smaller membership values for the results of fuzzy operations, as compared to the min t-norm), using these two t-norms for implementing extended fuzzy arithmetic reduces the overestimation of uncertainty, as compared to standard fuzzy arithmetic. Additionally, in some applications of fuzzy numbers, implementing fuzzy arithmetic using one of these two t-norms is preferred to other t-norms based on the characteristics of the problem. As an example, Baldwin et al. [32] discussed that, if fuzzy numbers represent probability values, fuzzy arithmetic operations should be implemented by extended fuzzy arithmetic approach using the product t-norm.

Dubois et al. [25] proved that the implementation of extended fuzzy addition on triangular fuzzy numbers using the Lukasiewicz t-norm produces the same result as the drastic product t-norm (i.e., a triangular fuzzy number). However, triangular fuzzy numbers are not closed under extended fuzzy addition and multiplication using the product t-norm and extended fuzzy multiplication using Lukasiewicz t-norm; thus, there is no exact mathematical solution available for the implementation of these operations on triangular fuzzy numbers. Where triangular fuzzy numbers are not closed under any given extended fuzzy operation, the resulting fuzzy number can be calculated using one of the two following methods: (1) approximate methods, which approximate the resulting fuzzy number as a triangular fuzzy number [33,34]; or (2) computational methods, which discretize the input fuzzy numbers and calculate the resulting fuzzy number using these points [35]. According to Brunelli and Mezei [33], the application of approximate methods for implementing fuzzy arithmetic operations can lead to misleading results. Computational methods can be more accurate than approximate methods, because the computational methods determine the exact membership value of the resulting fuzzy number at discrete points of its support. Dong and Wong [35] explain that discrete methods for implementing extended fuzzy arithmetic are computationally more efficient than using non-linear programing to achieve the exact solution, since unlike the non-linear programing method, the discrete method does not change if the input fuzzy numbers change. This paper introduces computational methods for implementing extended fuzzy addition and multiplication using product t-norm and extended fuzzy multiplication using Lukasiewicz t-norm on triangular fuzzy numbers. The computational methods presented in this paper for the implementation of extended fuzzy arithmetic contribute to the application of fuzzy numbers in different contexts.

2.4. Uncertainty measures of fuzzy sets

Fuzzy sets are used to represent two types of uncertainty in different applications: subjective uncertainty, which refers to the uncertainty associated with defining a variable with linguistic terms [36], and resolutional uncertainty, which refers to the uncertainty due to the inability to specify the exact value of a variable [36]. There are different measures of uncertainty for fuzzy sets, and the appropriate measure of uncertainty for a given fuzzy set is determined based on the type of uncertainty represented by the fuzzy set (i.e., subjective or resolutional uncertainty). The entropy measure (also known as the measure of fuzziness) is appropriate for measuring the uncertainty of the fuzzy set representing subjective uncertainty [37]. The entropy of fuzzy set A, defined on the universe of discourse X, is calculated using Eq. (10):

$$H(A) = \int_X g\big(A(x)\big)dx \tag{10}$$

where g is a function that is monotonically increasing in $[0, 0.5]$ and monotonically decreasing in $[0.5, 1]$. Given two fuzzy sets that represent a given subjective variable, the fuzzy set with the larger value of entropy has a higher level of uncertainty. On the other hand, the cardinality measure is appropriate for measuring the uncertainty of fuzzy set that represents resolutional uncertainty [36]. The cardinality of fuzzy set A, defined on the universe of discourse X, is calculated using Eq. (11):

$$Card(A) = \int_X A(x)dx \tag{11}$$

Given two fuzzy sets that represent the imprecise value of a given variable, the fuzzy set with the larger value of cardinality has a higher level of uncertainty [3]. The specificity measure, introduced by Yager [38], is another measure of uncertainty that is appropriate for measuring the uncertainty of the fuzzy sets that represent resolutional uncertainty. The specificity of a fuzzy set has a trade-off relationship with its cardinality; the higher the cardinality of a set, the lower its specificity [3]. The specificity of fuzzy set A, defined on the universe of discourse X, is calculated using Eq. (12):

$$Spec(A) = \int_0^{\alpha_{\max}} h\big(Card(A_\alpha)\big)d\alpha \tag{12}$$

where h is a function that is monotonically decreasing in $[0, 1]$. Given two fuzzy sets that represent the imprecise value of a given variable, the fuzzy set with the smaller value of specificity has a higher level of uncertainty [3]. In different applications, fuzzy numbers are commonly used to represent the imprecise value of objective variables, which would be represented by crisp numbers if they could be measured precisely. Thus, fuzzy numbers are commonly used to represent resolutional uncertainty, and the appropriate measure of uncertainty for fuzzy numbers is either the cardinality measure or the specificity measure. Since the specificity of a fuzzy set is a function of its cardinality, neither of these two measures is preferred over the other for comparing the uncertainty of fuzzy sets that represent the imprecise value of a given variable. Accordingly, due to the computational simplicity of the cardinality measure, this measure is used to compare the uncertainty of the resulting fuzzy numbers produced by standard fuzzy arithmetic and extended fuzzy arithmetic in the numerical example presented in the section 4 of this paper.

3. Computational method for extended fuzzy arithmetic

This section presents a computational method for implementing extended fuzzy addition and multiplication using product and Lukasiewicz t-norms. Where $A \ominus B = A \oplus (-1) \times B$, and $A \oslash B = A \otimes (1)/B$, extended fuzzy subtraction and extended fuzzy division can be implemented using the computational methods presented in this paper. The computational methods presented in this paper are implemented in the following three-step process: firstly, the resulting fuzzy number support is discretized into a finite number of points; secondly, the exact membership value for each point is calculated; and thirdly, the resulting fuzzy number is constructed using the discrete points (determined in the first step) and their membership values (calculated in the second step).

Due to the fact that min is the highest t-norm, regardless of the t-norm used for implementing extended fuzzy arithmetic, the results of the operations will have a support that is the same length or smaller compared to the results of extended fuzzy arithmetic using the min t-norm [3]. The computational method presented in this paper considers the largest possible support of the extended fuzzy arithmetic results (i.e., results of extended fuzzy arithmetic using the min t-norm); discretizes these results into a finite number of points; and calculates the membership values for each point. Based on the proposal made by Pedrycz and Gomide [3], this computational method implements fuzzy arithmetic on the increasing and the decreasing parts of the two input fuzzy numbers, and then combines the two parts to develop the resulting fuzzy number. Next, the membership value for each point is calculated using the computational methods presented in Section 3.1 and Section 3.2 for implementing extended fuzzy arithmetic operations using product and Lukasiewicz t-norms, respectively. For this purpose, consider two generic triangular fuzzy numbers $A(x; a_1, a_2, a_3)$ and $B(y; b_1, b_2, b_3)$ which are presented in Eq. (13) and Eq. (14) respectively,

$$A(x; a_1, a_2, a_3) = \begin{cases} \overline{\alpha_a}x + \overline{\beta_a}, & \text{if } a_1 \le x < a_2 \\ \underline{\alpha_a}x + \underline{\beta_a}, & \text{if } a_2 \le x \le a_3, \\ 0, & \text{otherwise} \end{cases} \quad \overline{\alpha_a} > 0,\ \underline{\alpha_a} < 0, a_1 \le a_2 \le a_3 \tag{13}$$

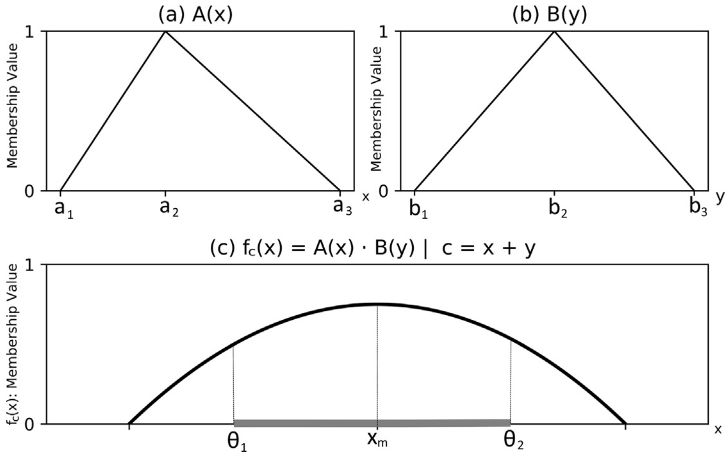

Fig. 1. Extended fuzzy addition using product t-norm.

$$B(y;b_1,b_2,b_3)=\begin{cases}\overline{\alpha_b}y+\overline{\beta_b}, & \text{if } b_1\le y<b_2\\ \underline{\alpha_b}y+\underline{\beta_b}, & \text{if } b_2\le y\le b_3,\\ 0, & \text{otherwise}\end{cases}\quad \overline{\alpha_b}>0,\ \underline{\alpha_b}<0,\ b_1\le b_2\le b_3 \tag{14}$$

3.1. Computational method: extended fuzzy arithmetic using product t-norm

3.1.1. Extended fuzzy addition using product t-norm

The mathematical form of extended fuzzy addition using the product t-norm is presented in Eq. (15):

$$C(z)=A(x)\oplus B(y)=\sup_{z=x+y}\big(\mathrm{A}(x)\times B(y)\big) \tag{15}$$

Proposition 1. *For each discrete point of $z=c$ in the increasing part of the support of the resulting fuzzy number, where $c\in[a_1+b_1,a_2+b_2]$, the value of $\sup_{z=x+y}(A(x)\times B(y))$ is calculated using one of the three cases presented in Eq. (16) to Eq. (18):*

$$\textit{Case 1: if } \theta_1\le x_m^1\le\theta_2: C(c)=A(x_m)\times B(y_m) \tag{16}$$

$$\textit{Case 2: if } \theta_2< x_m^1: C(c)=A(\theta_2)\times B(c-\theta_2) \tag{17}$$

$$\textit{Case 3: if } x_m^1<\theta_1: C(c)=A(\theta_1)\times B(c-\theta_1) \tag{18}$$

where the value of θ_1, θ_2, and x_m^1 and y_m^1 are calculated using Eq. (19) to (21) respectively,

$$\theta_1=\max(a_1,c-b_2) \tag{19}$$

$$\theta_2=\min(a_2,c-b_1) \tag{20}$$

$$x_m^1=\frac{c}{2}-\frac{\overline{\beta_a}\,\overline{\alpha_b}-\overline{\alpha_a}\,\overline{\beta_b}}{2\times\overline{\alpha_a}\,\overline{\alpha_b}},\qquad y_m^1=c-x_m^1=\frac{c}{2}+\frac{\overline{\beta_a}\,\overline{\alpha_b}-\overline{\alpha_a}\,\overline{\beta_b}}{2\overline{\alpha_a}\,\overline{\alpha_b}} \tag{21}$$

Proof. For each constant value of $z=c$, there is a function $f_c(x)$ that determines all the possible values of $A(x)\times B(y)$, such that $c=x+y$, and $c\in[a_1+b_1,a_2+b_2]$. Therefore, the maximum point of $f_c(x)$ is the membership value of the resulting fuzzy number at point $z=c$. Fig. 1(a) and Fig. 1(b) present the two input fuzzy numbers, while Fig. 1(c) presents function $f_c(x)$, which is mathematically defined below:

$$f_c(x)=A(x)\times B(y)=A(x)\times B(c-x)=(\overline{\alpha_a}\,\overline{\alpha_b})\big(cx-x^2\big)+x(\overline{\alpha_a}\,\overline{\beta_b})+(\overline{\beta_a}\,\overline{\alpha_b})(c-x)+(\overline{\beta_a}\,\overline{\beta_b})$$

The domain of function $f_c(x)$ is calculated as the intersection of the two boundary conditions of the input fuzzy numbers, as shown below:

$$\begin{cases}1)\ a_1\le x\le a_2\\ 2)\ b_1\le y\le b_2 \quad\Rightarrow\quad b_1\le c-x\le b_2 \quad\Rightarrow\quad c-b_2\le x\le c-b_1 \quad\Rightarrow\quad \underbrace{\max(a_1,c-b_2)}_{\theta_1}\le x\le\underbrace{\min(a_2,c-b_1)}_{\theta_2}\\ 3)\ a_1+b_1\le c\le a_2+b_2\end{cases}$$

$f_c(x)$ is a quadratic function; thus, it has an extremum point, which is calculated as presented below:

$$\frac{\partial f_c(x)}{\partial x} = (\overline{\alpha_a}\,\overline{\alpha_b})(c-2x) + (\overline{\alpha_a}\,\overline{\beta_b}) - (\overline{\beta_a}\,\overline{\alpha_b}) = 0 \quad \Rightarrow \quad x_m^1 = \frac{c}{2} - \frac{\overline{\beta_a}\,\overline{\alpha_b} - \overline{\alpha_a}\,\overline{\beta_b}}{2 \times \overline{\alpha_a}\,\overline{\alpha_b}}, \qquad y_m^1 = \frac{c}{2} + \frac{\overline{\beta_a}\,\overline{\alpha_b} - \overline{\alpha_a}\,\overline{\beta_b}}{2\overline{\alpha_a}\,\overline{\alpha_b}}$$

The extremum point of function $f_c(x)$ is always a maximum point, as illustrated below:

$$\frac{\partial^2 f_c(x)}{\partial x^2} = -2\overline{\alpha_a}\,\overline{\alpha_b} \begin{cases} \overline{\alpha_a} > 0 \\ \overline{\alpha_b} > 0 \end{cases} \quad \Rightarrow \quad \frac{\partial^2 f_c(x_m)}{\partial x^2} < 0 \quad \Rightarrow \quad f_c\left(x_m^1\right) \text{ is always maximum point of } f_c(x)$$

Thus,

$$\text{if } \theta_1 \le x_m^1 \le \theta_2 : C(c) = \sup_{z=x+y}\big(A(x) \times B(y)\big) = A\left(x_m^1\right) \times B\left(y_m^1\right)$$

$$\text{if } \theta_2 < x_m^1 : \frac{\partial f_c(x < x_m^1)}{\partial x} > 0 \quad \Rightarrow \quad f_c(x < x_m) \text{ is increasing} \quad \Rightarrow \quad \sup_{z=x+y}\big(A(x) \times B(y)\big) = A(\theta_2) \times B(c-\theta_2)$$

$$\text{if } x_m^1 < \theta_1 : \frac{\partial f_c(x > x_m^1)}{\partial x} < 0 \quad \Rightarrow \quad f_c(x < x_m) \text{ is decreasing} \quad \Rightarrow \quad \sup_{z=x+y}\big(A(x) \times B(y)\big) = A(\theta_1) \times B(c-\theta_1) \qquad \square$$

Proposition 2. *For each discrete point of $z = c$ in the decreasing part of the support of the resulting fuzzy number, where $c \in [a_2 + b_2, a_3 + b_3]$, the value of $\sup_{z=x+y}(A(x) \times B(y))$ is calculated using one of the three cases presented in Eq. (22) to Eq. (24):*

$$\textit{Case 1: if } \theta_3 \le x_m^2 \le \theta_4 : C(c) = A\left(x_m^2\right) \times B\left(y_m^2\right) \tag{22}$$

$$\textit{Case 2: if } \theta_4 < x_m^2 : C(c) = A(\theta_4) \times B(c-\theta_4) \tag{23}$$

$$\textit{Case 3: if } x_m^2 < \theta_3 : C(c) = A(\theta_3) \times B(c-\theta_3) \tag{24}$$

where the values of θ_3, θ_4, and x_m^2 and y_m^2 are calculated using Eq. (25) to (27) respectively,

$$\theta_3 = \max(a_2, c-b_3) \tag{25}$$

$$\theta_4 = \min(a_3, c-b_2) \tag{26}$$

$$x_m^2 = \frac{c}{2} - \frac{\underline{\underline{\beta_a}}\,\underline{\alpha_b} - \underline{\alpha_a}\,\underline{\underline{\beta_b}}}{2\underline{\alpha_a}\,\underline{\alpha_b}}, \qquad y_m^2 = c - x_m = \frac{c}{2} + \frac{\underline{\underline{\beta_a}}\,\underline{\alpha_b} - \underline{\alpha_a}\,\underline{\underline{\beta_b}}}{2\underline{\alpha_a}\,\underline{\alpha_b}} \tag{27}$$

Proof. The proof for Proposition 2 is similar to the proof provided for Proposition 1. $\square$

Based on Proposition 1 and Proposition 2, an algorithm is developed for the implementation of extended fuzzy addition using product t-norm in Python programming language, which is presented in Fig. 2.

3.1.2. Extended fuzzy multiplication using product t-norm

The mathematical form of extended fuzzy multiplication using the product t-norm is presented in Eq. (28). For any two triangular fuzzy numbers A and B with positive supports (i.e., $a_1 > 0$ and $b_1 > 0$), extended fuzzy multiplication using the product t-norm is implemented using the following method:

$$C(z) = A(x) \otimes B(y) = \sup_{z=x\times y}\big(A(x) \times B(y)\big) \tag{28}$$

Proposition 3. *For each discrete point of $z = c$ in the increasing part of the support of the resulting fuzzy number, where $c \in [a_1 b_1, a_2 b_2]$, the value of $\sup_{z=x\times y}(A(x) \times B(y))$ is calculated using one of the three cases presented in Eq. (29) to (31):*

$$\textit{Case 1: if } \theta_1 \le x_m^1 \le \theta_2 : C(c) = A\left(x_m^1\right) \times B\left(y_m^1\right) \tag{29}$$

$$\textit{Case 2: if } \theta_2 < x_m^1 : C(c) = A(\theta_2) \times B\left(\frac{c}{\theta_2}\right) \tag{30}$$

$$\textit{Case 3: if } x_m^1 < \theta_1 : C(c) = A(\theta_1) \times B\left(\frac{c}{\theta_1}\right) \tag{31}$$

```
def ExtendedAdditionProduct([a1,a2,a3],[b1,b2,b3]):
    for c in ResultDomain:
        if (a1+b1)<=c<=(a2+b2):
            xm=c/2-(betta_a1*alpha_b1-alpha_a1*betta_b1)/(2*alpha_a1*alpha_b1);ym=c-xm
            tmin=max(a1,c-b2);tmin=min(a2,c-b1)
            if tmin<=xm<=tmax:
                ResultRange.append(A(xm)*B(ym))
            elif tmax<xm:
                ResultRange.append(A(tmax)*B(c-tmax))
            elif xm<tmin:
                ResultRange.append(A(tmin)*B(c-tmin))
            end if
        else if (a2+b2)<=c<=(a3+b3):
            xm=c/2-(betta_a2*alpha_b2-alpha_a2*betta_b2)/(2*alpha_a2*alpha_b2);ym=c-xm
            tmin=max(a2,c-b3);tmin=min(a3,c-b2)
            if tmin<=xm<=tmax:
                ResultRange.append(A(xm)*B(ym))
            elif tmax<xm:
                ResultRange.append(A(tmax)*B(c-tmax))
            elif xm<tmin:
                ResultRange.append(A(tmin)*B(c-tmin))
            end if
        end if
    end for
    return [ResultDomain,ResultRange]
```

Fig. 2. Programming algorithm for extended fuzzy addition using product t-norm.

where the values of θ_1, θ_2, and x_m^1 and y_m^1 are calculated using Eq. (32) to (34) respectively,

$$\theta_1 = \max\left(a_1, \frac{c}{b_2}\right) \tag{32}$$

$$\theta_2 = \min\left(a_2, \frac{c}{b_1}\right) \tag{33}$$

$$x_m^1 = \sqrt{\frac{\overline{\beta_a}\,\overline{\alpha_b}}{\overline{\alpha_a}\,\overline{\beta_b}}} \times c, \qquad y_m^1 = \frac{c}{x_m} = \sqrt{\frac{\overline{\alpha_a}\,\overline{\beta_b}}{\overline{\beta_a}\,\overline{\alpha_b}}} \times c \tag{34}$$

Proof. For each constant value of $z = c$, there is a function $f_c(x)$ that determines all the possible values of $A(x) \times B(y)$, such that $c = x \times y$, and $c \in [a_1 b_1, a_2 b_2]$. Therefore, the maximum point of $f_c(x)$ is the membership value of the resulting fuzzy number at point c. Fig. 3(a) and Fig. 3(b) present the two input fuzzy numbers, while Fig. 3(c) presents function $f_c(x)$, which is mathematically defined below:

$$f_c(x) = A(x) \times B(y) = A(x) \times B\left(\frac{c}{x}\right) = (\overline{\alpha_a}\,\overline{\alpha_b})c + (\overline{\alpha_a}\,\overline{\beta_b})x + (\overline{\beta_a}\,\overline{\alpha_b})\left(\frac{c}{x}\right) + (\overline{\beta_a}\,\overline{\beta_b}), \quad x \neq 0$$

The domain of function $f_c(x)$ is calculated as the intersection of the two boundary conditions of the input fuzzy numbers, as shown below:

$$\begin{cases} 1)\ a_1 \le x \le a_2 \\ 2)\ b_1 \le y \le b_2 \quad \Rightarrow \quad b_1 \le \dfrac{c}{x} \le b_2 \quad \Rightarrow \quad \dfrac{c}{b_2} \le x \le \dfrac{c}{b_1} \quad \Rightarrow \quad \underbrace{\max\left(a_1, \dfrac{c}{b_2}\right)}_{\theta_1} \le x \le \underbrace{\min\left(a_2, \dfrac{c}{b_1}\right)}_{\theta_2} \\ 3)\ a_1 b_1 \le c \le a_2 b_2 \end{cases}$$

Function $f_c(x)$ has two extremum points, which are calculated as presented below:

$$\frac{\partial f_c(x)}{\partial x} = (\overline{\alpha_a}\,\overline{\beta_b}) - \frac{(\overline{\beta_a}\,\overline{\alpha_b})c}{x^2} = 0 \quad \Rightarrow \quad x_m^1 = \pm\sqrt{\frac{\overline{\beta_a}\,\overline{\alpha_b}}{\overline{\alpha_a}\,\overline{\beta_b}} \times c}$$

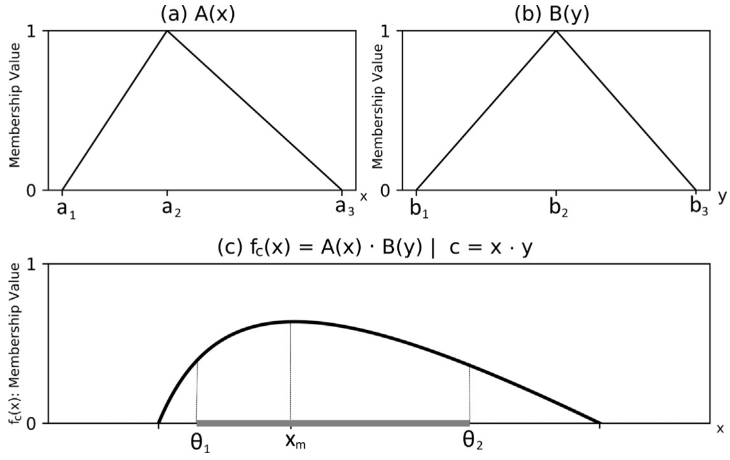

Fig. 3. Extended fuzzy multiplication using product t-norm.

One of the extremum points of function $f_c(x)$ is always located on the negative side of the universe of discourse, and the other one is always located on the positive side. However, the extremum point that is on the negative side of the universe of discourse is not included in the domain of the function. Thus, $x_m^1 = \sqrt{\frac{\overline{\beta_a}\overline{\alpha_b}}{\overline{\alpha_a}\overline{\beta_b}}c}$ is the only extremum point of function $f_c(x)$. The extremum point of function $f_c(x)$ is always a maximum point, as illustrated below:

$$\frac{\partial^2 f_c(x)}{\partial x^2} = 2 \times \frac{(\overline{\beta_a}\overline{\alpha_b})c}{x^3}, \qquad \begin{cases} \overline{\alpha_b} > 0 \\ c > 0 \\ \overline{\beta_a} < 0 \\ x^3 > 0 \end{cases} \quad \Rightarrow \quad \frac{\partial^2 f_c(x)}{\partial x^2} < 0 \quad \Rightarrow \quad f_c\left(x_m^1\right) \text{ is always maximum point of } f_c(x)$$

Thus,

$$\text{if } \theta_1 \le x_m^1 \le \theta_2 \quad \Rightarrow \quad \max\left(f_c(x)\right) = f_c\left(x_m^1\right) = A\left(x_m^1\right) \times B\left(y_m^1\right)$$
$$\text{if } \theta_2 < x_m^1 : \frac{\partial f_c(x < x_m^1)}{\partial x} > 0 \quad \Rightarrow \quad \max\left(f_c(x)\right) = f_c(\theta_2) = A(\theta_2) \times B\left(\frac{c}{\theta_2}\right)$$
$$\text{if } x_m^1 < \theta_1 : \frac{\partial f_c(x_m^1 < x)}{\partial x} < 0 \quad \Rightarrow \quad \max\left(f_c(x)\right) = f_c(\theta_1) = A(\theta_1) \times B\left(\frac{c}{\theta_1}\right) \qquad \square$$

Proposition 4. *For each discrete point of $z = c$ in the decreasing part of the support of the resulting fuzzy number, where $c \in [a_2b_2, a_3b_3]$, the value of $\sup_{z=x\times y}(A(x) \times B(y))$ is calculated using one of the three cases presented in Eq. (35) to Eq. (37):*

$$\textit{Case 1: if } \theta_3 \le x_m^2 \le \theta_4 : C(c) = A\left(x_m^2\right) \times B\left(y_m^2\right) \tag{35}$$

$$\textit{Case 2: if } \theta_4 < x_m^2 : C(c) = A(\theta_4) \times B\left(\frac{c}{\theta_4}\right) \tag{36}$$

$$\textit{Case 3: if } x_m^2 < \theta_3 : C(c) = A(\theta_3) \times B\left(\frac{c}{\theta_3}\right) \tag{37}$$

where the values of θ_3, θ_4, and x_m^2 and y_m^2 are calculated using Eq. (38) to (40) respectively,

$$\theta_3 = \max\left(a_2, \frac{c}{b_3}\right) \tag{38}$$

$$\theta_4 = \min\left(a_3, \frac{c}{b_2}\right) \tag{39}$$

$$x_m^2 = \sqrt{\frac{\underline{\beta_a}\,\underline{\alpha_b}}{\underline{\alpha_a}\,\underline{\beta_b}} \times c}, \qquad y_m^2 = \frac{c}{x} = \sqrt{\frac{\underline{\alpha_a}\,\underline{\beta_b}}{\underline{\beta_a}\,\underline{\alpha_b}} \times c} \tag{40}$$

Proof. The proof for Proposition 4 is similar to the proof provided for Proposition 3. $\square$

```
def ExtendedMultiplicationProduct([a1,a2,a3],[b1,b2,b3]):
    for c in ResultDomain:
        if (a1*b1)<=c<=(a2*b2):
            xm=sqrt((betta_a1*alpha_b1)/(alpha_a1*betta_b1)*c);ym=c/xm
            tmin=max(a1,c/b2);tmin=min(a2,c/b1)
            if tmin<=xm<=tmax:
                ResultRange.append(A(xm)*B(ym))
            elif tmax<xm:
                ResultRange.append(A(tmax)*B(c/tmax))
            elif xm<tmin:
                ResultRange.append(A(tmin)*B(c/tmin))
            end if
        else if (a2+b2)<=c<=(a3+b3):
            xm=sqrt((betta_a2*alpha_b2)/(alpha_a2*betta_b2)*c);ym=c/xm
            tmin=max(a2,c/b3);tmin=min(a3,c/b2)
            if tmin<=xm<=tmax:
                ResultRange.append(A(xm)*B(ym))
            elif tmax<xm:
                ResultRange.append(A(tmax)*B(c/tmax))
            elif xm<tmin:
                ResultRange.append(A(tmin)*B(c/tmin))
            end if
        end if
    end for
    return [ResultDomain,ResultRange]
```

Fig. 4. Programming algorithm for extended fuzzy multiplication using product t-norm.

Based on Proposition 3 and Proposition 4, an algorithm is developed for the implementation of extended fuzzy addition using product t-norm in Python programming language, which is presented in Fig. 4.

If the two input fuzzy numbers (i.e., $A(x)$ and $B(x)$) have negative supports (i.e., $a_3 < 0$ and $b_3 < 0$), then there exists an $A'(x) = -1 \otimes A(x)$ and a $B'(x) = -1 \otimes B(y)$ where $A'(x)$ and $B'(y)$ are two triangular fuzzy numbers with positive supports (i.e., $a'_1 > 0$ and $b'_1 > 0$). Due to the associativity and commutativity of extended fuzzy multiplication [3,39], the resulting fuzzy number can be computed using $A'(x)$ and $B'(y)$ through the method presented in this section; where

$$A(x) \otimes B(y) = \left(-1 \otimes A'(x)\right) \otimes \left(-1 \otimes B'(y)\right)=\left((-1) \otimes (-1)\right) \otimes \left(A'(x) \otimes B'(y)\right) = A'(x) \otimes B'(y)$$

Moreover, if one of the two input triangular fuzzy numbers has a negative support, and the other one has a positive support (i.e., $a_3 < 0$ and $b_1 > 0$), then there exists some $A'(x) = -1 \otimes A(x)$ such that $A'(x)$ is a triangular fuzzy number that has a positive support (i.e., $a'_3 < 0$). Due to the associativity of extended fuzzy multiplication [39], the resulting fuzzy number can be computed using $A'(x)$ and $B(y)$ through the method presented in this section, where

$$A(x) \otimes B(y) = \left((-1) \otimes A'(x)\right) \otimes B(y)=(-1) \otimes \left(A'(x) \otimes B(y)\right)$$

3.2. Computational method: extended fuzzy arithmetic using Lukasiewicz t-norm

3.2.1. Extended fuzzy addition using Lukasiewicz t-norm

Dubois et al. [25] proved that the implementation of extended fuzzy addition on triangular fuzzy numbers using the Lukasiewicz t-norm produces the same result as the drastic product t-norm (i.e., a triangular fuzzy number). Accordingly, the result of extended fuzzy addition using Lukasiewicz t-norm on two triangular fuzzy number, A and B (refer to Eq. (9) and Eq. (10)) is calculated using the mathematical solution provided by Kolesárová [15] as presented in Eq. (41):

$$A(x) \oplus B(y) = C\left(z; a_2 + b_2 - \max(a_2 - a_1, b_2 - b_1), a_2 + b_2, a_2 + b_2 + \max(a_3 - a_2, b_3 - b_2)\right) \tag{41}$$

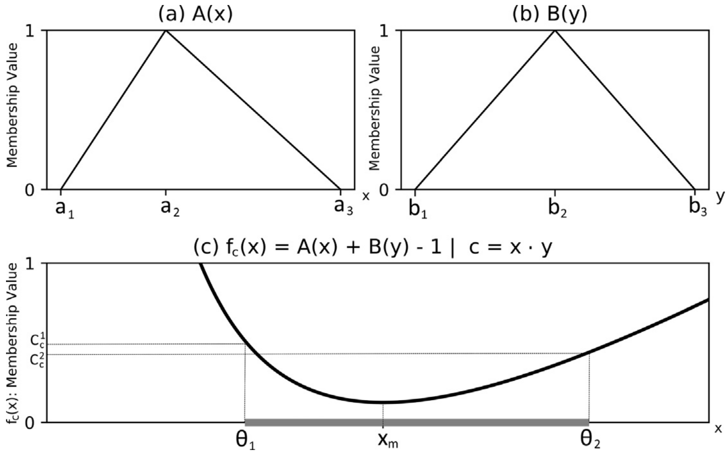

Fig. 5. Extended fuzzy multiplication using Lukasiewicz t-norm (increasing part).

3.2.2. Extended fuzzy multiplication using Lukasiewicz t-norm

The mathematical form of extended fuzzy multiplication using Lukasiewicz t-norm is presented in Eq. (42). For any two triangular fuzzy numbers A and B with positive supports (i.e., $a_1 > 0$ and $b_1 > 0$) extended fuzzy multiplication using the Lukasiewicz t-norm can be implemented using the following method:

$$C(z) = A(x) \otimes B(y) = \sup_{z=x\times y}\big(A(x)tB(y)\big) = \sup_{z=x\times y}\big(\max\big(A(x) + B(y) - 1, 0\big)\big) \tag{42}$$

Proposition 5. *For each discrete point of $z = c$ in the increasing part of the support of the resulting fuzzy number, where $c \in [a_1b_1, a_2b_2]$, the value of $\sup_{z=x\times y}(A(x)t_L B(y))$ is calculated using Eq. (43):*

$$C(c) = \max\left(A(\theta_1) + B\left(\frac{c}{\theta_1}\right) - 1, A(\theta_2) + B\left(\frac{c}{\theta_2}\right) - 1, 0\right) \tag{43}$$

where the values of θ_1 and θ_2 are calculated using Eq. (44) and (45) respectively,

$$\theta_1 = \max\left(a_1, \frac{c}{b_2}\right) \tag{44}$$

$$\theta_2 = \min\left(a_2, \frac{c}{b_1}\right) \tag{45}$$

Proof. For each constant value of $z = c$, there is a function $f_c(x)$ that determines all the possible values of $A(x) + B(y) - 1$, such that $c = x \times y$, and $c \in [a_1b_1, a_2b_2]$. Therefore, the maximum point of $f_c(x)$ is the membership value for of the resulting fuzzy number at point c. Fig. 5(a) and Fig. 5(b) present the two input fuzzy numbers, while Fig. 5(c) presents function $f_c(x)$, which is mathematically defined below:

$$f_c(x) = A(x) + B(y) - 1 = A(x) + B\left(\frac{c}{x}\right) - 1 = (\overline{\alpha_a}x + \overline{\beta_a}) + \left(\overline{\alpha_b}\left(\frac{c}{x}\right) + \overline{\beta_b}\right) - 1$$

The domain of the function $f_c(x)$ is calculated as the intersection of the two boundary conditions of the input fuzzy numbers, as shown below:

$$\begin{cases} 1)\ a_1 \le x \le a_2 \\ 2)\ b_1 \le y \le b_2 \quad \Rightarrow \quad b_1 \le \dfrac{c}{x} \le b_2 \quad \Rightarrow \quad \dfrac{c}{b_2} \le x \le \dfrac{c}{b_1} \quad \Rightarrow \quad \underbrace{\max\left(a_1, \dfrac{c}{b_2}\right)}_{\theta_1} \le x \le \underbrace{\min\left(a_2, \dfrac{c}{b_1}\right)}_{\theta_2} \\ 3)\ a_1b_1 \le c \le a_2b_2 \end{cases}$$

Function $f_c(x)$ has two extremum points, which are calculated as shown below:

$$\frac{\partial f_c(x)}{\partial x} = \overline{\alpha_a} - \frac{\overline{\alpha_b}c}{x^2} = 0 \quad \Rightarrow \quad x = \pm\sqrt{\frac{\overline{\alpha_b}c}{\overline{\alpha_a}}}$$

One of the extremum points of function $f_c(x)$ is always located on the negative side of the universe of discourse, while the other one is always located on the positive side. However, the extremum point that is on the negative side of the universe of discourse is not included in the domain of the function. Thus, $x_m^1 = \sqrt{\frac{\overline{\alpha_b}c}{\overline{\alpha_a}}}$ is the only extremum point for function $f_c(x)$. The extremum point of function $f_c(x)$ is always a minimum point, as proven below:

$$\frac{\partial^2 f_c(x)}{\partial x^2} = 2 \times \frac{\overline{\alpha_b}c}{x^3}, \qquad \begin{cases} \overline{\alpha_b} > 0 \\ c > 0 \\ x^3 > 0 \end{cases} \quad \Rightarrow \quad \frac{\partial^2 f_c(x)}{\partial x^2} > 0 \quad \Rightarrow \quad f_c\left(x_m^1\right) \text{ is always minimum point of function } f_c(x)$$

Thus,

$$\text{if } \theta_1 \le x_m^1 \le \theta_2 \quad \Rightarrow \quad \max(f_c(x) = \max\left(f_c(\theta_1), f_c(\theta_2)\right) = \max\left(A(\theta_1) + B\left(\frac{c}{\theta_1}\right) - 1, A(\theta_2) + B\left(\frac{c}{\theta_2}\right) - 1, 0\right)$$

$$\text{if } \theta_2 < x_m^1 : \frac{\partial f_c(x < x_m^1)}{\partial x} < 0 \quad \Rightarrow \quad \max\left(f_c(x)\right) = \max\left(A(\theta_1) + B\left(\frac{c}{\theta_1}\right) - 1, 0\right)$$

$$\text{if } x_m^1 < \theta_1 : \frac{\partial f_c(x > x_m^1)}{\partial x} > 0 \quad \Rightarrow \quad \max\left(f_c(x)\right) = \max\left(A(\theta_2) + B\left(\frac{c}{\theta_2}\right) - 1, 0\right) \qquad \square$$

Proposition 6. *For each discrete point of $z = c$ in the decreasing part of the support of the resulting fuzzy number, where $c \in [a_2 b_2, a_3 b_3]$, the value of $\sup_{z=x\times y}(A(x) t_L B(y))$ is calculated using one of the three cases presented in Eq. (46) to Eq. (48).*

$$\textit{Case 1: if } \theta_3 \le x_m^2 \le \theta_4 : C(c) = \max\left(A\left(x_m^2\right) + B\left(y_m^2\right) - 1, 0\right) \tag{46}$$

$$\textit{Case 2: if } \theta_4 < x_m^2 : C(c) = \max\left(A(\theta_4) + B\left(\frac{c}{\theta_4}\right) - 1, 0\right) \tag{47}$$

$$\textit{Case 3: if } x_m^2 < \theta_3 : C(c) = \max\left(A(\theta_3) + B\left(\frac{c}{\theta_3}\right) - 1, 0\right) \tag{48}$$

where the values of θ_3, θ_4, and x_m^2 and y_m^2 are calculated using Eq. (49) to (51) respectively,

$$\theta_3 = \max\left(a_2, \frac{c}{b_3}\right) \tag{49}$$

$$\theta_4 = \min\left(a_3, \frac{c}{b_2}\right) \tag{50}$$

$$x_m^2 = \sqrt{\frac{\underline{\alpha_b}c}{\underline{\alpha_a}}}, \qquad y_m^2 = \frac{c}{x_m} = \sqrt{\frac{\underline{\alpha_a}c}{\underline{\alpha_b}}} \tag{51}$$

Proof. For each constant value of $z = c$, there is a function $f_c(x)$ that determines all the possible values of $A(x) + B(y) - 1$, such that $c = x \times y$, and $c \in [a_2 b_2, a_3 b_3]$. Therefore, the maximum point of $f_c(x)$ is the membership value of the resulting fuzzy number at point c. Fig. 6(a) and Fig. 6(b) present the two input fuzzy numbers, while Fig. 6(c) presents function $f_c(x)$, which is mathematically defined below:

$$f_c(x) = A(x) + B(y) - 1 = A(x) + B\left(\frac{c}{x}\right) - 1 = (\underline{\alpha_a}x + \underline{\beta_a}) + \left(\underline{\alpha_b}\left(\frac{c}{x}\right) + \underline{\beta_b}\right) - 1$$

The domain of function $f_c(x)$ is calculated as the intersection of the two boundary conditions of the input fuzzy numbers, as shown below:

$$\begin{cases} 1)\ a_2 \le x \le a_3 \\ 2)\ b_2 \le y \le b_3 \quad \Rightarrow \quad b_2 \le \dfrac{c}{x} \le b_3 \quad \Rightarrow \quad \dfrac{c}{b_3} \le x \le \dfrac{c}{b_3} \quad \Rightarrow \quad \underbrace{\max\left(a_2, \dfrac{c}{b_3}\right)}_{\theta_3} \le x \le \underbrace{\min\left(a_3, \dfrac{c}{b_2}\right)}_{\theta_4} \\ 3)\ a_2 b_2 \le c \le a_3 b_3 \end{cases}$$

Function $f_c(x)$ has two extremum points, which are calculated as presented below:

$$\frac{\partial f_c(x)}{\partial x} = \underline{\alpha_a} - \frac{\underline{\alpha_b}c}{x^2} = 0 \quad \Rightarrow \quad x = \pm\sqrt{\frac{\underline{\alpha_b}c}{\underline{\alpha_a}}}$$

One of the extremum points of function $f_c(x)$ is always located on the negative side of the universe of discourse, while the other is always located on the positive side. However, the extremum point that is on the negative side of the universe of

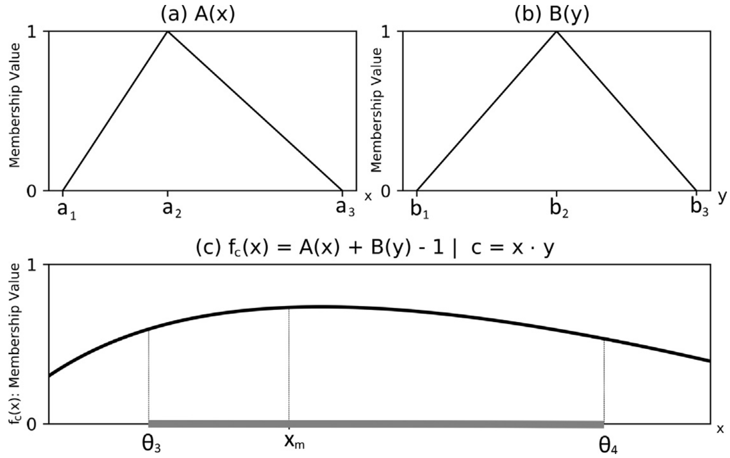

Fig. 6. Extended fuzzy multiplication using Lukasiewicz t-norm (decreasing part).

```
def ExtendedMultiplicationLukasiewicz([a1,a2,a3],[b1,b2,b3]):
    for c in ResultDomain:
        if (a1*b1)<=c<=(a2*b2):
            tmin=max(a1,c/b2);tmin=min(a2,c/b1)
                ResultRange.append(max(A(tmin)+B(c/tmin)-1,A(tmax)+B(c/tmax)-1,0))
        else if (a2+b2)<=c<=(a3+b3):
            xm=sqrt((alpha_b2*c)/alpha_a2);ym=c/xm
            tmin=max(a2,c/b3);tmin=min(a3,c/b2)
            if tmin<=xm<=tmax:
                ResultRange.append(A(xm)*B(ym))
            elif tmax<xm:
                ResultRange.append(A(tmax)*B(c/tmax))
            elif xm<tmin:
                ResultRange.append(A(tmin)*B(c/tmin))
            end if
        end if
    end for
    return [ResultDomain,ResultRange]
```

Fig. 7. Programming algorithm for extended fuzzy multiplication using Lukasiewicz t-norm.

discourse is not included in the domain of the function. Thus, $x_m^2 = \sqrt{\frac{\alpha_b c}{\alpha_a}}$ is the only extremum point of function $f_c(x)$. The extremum point of function $f_c(x)$ is always a maximum point, as proven below:

$$\frac{\partial^2 f_c(x)}{\partial x^2} = 2 \times \frac{\alpha_b c}{x^3}, \qquad \begin{cases} \alpha_b < 0 \\ c > 0 \\ x^3 > 0 \end{cases} \quad \Rightarrow \quad \frac{\partial^2 f_c(x)}{\partial x^2} < 0 \quad \Rightarrow \quad f_c\left(x_m^1\right) \text{ is always maximum point of function } f_c(x)$$

Thus,

$$\begin{aligned}
&\text{if } \theta_3 \le x_m^2 \le \theta_4 \quad \Rightarrow \quad \max\left(f_c(x)\right) = f_c\left(x_m^2\right) = \max\left(A\left(x_m^2\right) + B\left(y_m^2\right) - 1, 0\right) \\
&\text{if } \theta_4 < x_m^2 : \frac{\partial f_c(x < x_m^2)}{\partial x} > 0 \quad \Rightarrow \quad \max\left(f_c(x)\right) = \max\left(A(\theta_4) + B\left(\frac{c}{\theta_4}\right) - 1, 0\right) \\
&\text{if } x_m^2 < \theta_3 \quad \Rightarrow \quad \frac{\partial f_c(x > x_m^2)}{\partial x} < 0 \quad \Rightarrow \quad \max\left(f_c(x)\right) = \max\left(A(\theta_3) + B\left(\frac{c}{\theta_3}\right) - 1, 0\right)
\end{aligned}$$

Based on Proposition 5 and Proposition 6, an algorithm is developed for the implementation of extended fuzzy addition using product t-norm in Python programming language, which is presented in Fig. 7. □

3.3. Discussion

The mathematical proof presented in this section shows that for the increasing part of the resulting fuzzy number, implementing extended fuzzy multiplication using the Lukasiewicz t-norm on triangular fuzzy numbers produces the same result as implementing extended fuzzy multiplication using the drastic product t-norm. The mathematical form of extended fuzzy multiplication using the drastic product t-norm is presented below:

$$C(z) = A(x) \otimes B(y) = \sup_{z=x\times y} \big(A(x) t_d B(y)\big)$$

Membership values for discrete points in the increasing part of the resulting fuzzy number support are calculated using the following method. For each constant value of $z = c$, there is a function $g_c(x)$ that determines all the possible values of $(A(x)t_d B(y))$, such that $c = xy$, and $c \in [a_1b_1, a_2b_2]$ and t_d stands for the drastic product t-norm. Therefore, the maximum point for function $g_c(x)$ occurs in one of the two boundary points of its domain. Function $g_c(x)$ is mathematically defined below:

$$g_c(x) = \begin{cases} A(\frac{c}{b_2}), & \text{if } a_1 \le \frac{c}{b_2} \\ B(\frac{c}{a_2}), & \text{if } b_1 \le \frac{c}{a_2} \\ 0, & \text{otherwise} \end{cases} \quad \Rightarrow \quad C(c) = \sup_{z=x\times y} \big(A(x) t B(y)\big) = \max\big(g_c(x)\big)$$

For the Lukasiewicz t-norm, the value of $\sup_{z=x\times y}(A(x)t_L B(y))$ is calculated using the following equation:

$$\sup_{z=x\times y} \big(A(x) t_d B(y)\big) = \max\left(A(\theta_1) + B\left(\frac{c}{\theta_1}\right) - 1, A(\theta_2) + B\left(\frac{c}{\theta_2}\right) - 1, 0\right)$$

where $\theta_1 = \max(a_1, \frac{c}{b_2})$ and $\theta_2 = \min(a_2, \frac{c}{b_1})$ (see Section 3.2.2). Let us assume that $\theta_2 = \min(a_2, \frac{c}{b_1}) = \frac{c}{b_1}$; then,

$$\max\left(A(\theta_2) + B\left(\frac{c}{\theta_2}\right) - 1, 0\right) = \max\left(A\left(\frac{c}{b_1}\right) + B(b_1) - 1, 0\right) = \max\left(A\left(\frac{c}{b_1}\right) + 0 - 1, 0\right) = 0$$

Similarly, if $\theta_1 = \max(a_1, \frac{c}{b_2}) = a_1$, then; $\max(A(\theta_1) + B(\frac{c}{\theta_1}) - 1, 0) = 0$. Let us assume that $\theta_2 = \min(a_2, \frac{c}{b_1}) = a_2$; then,

$$\max\left(A(\theta_2) + B\left(\frac{c}{\theta_2}\right) - 1, 0\right) = \max\left(A(a_2) + B\left(\frac{c}{a_2}\right) - 1, 0\right) = \max\left(1 + B\left(\frac{c}{a_2}\right) - 1, 0\right) = B\left(\frac{c}{a_2}\right)$$

Similarly, if $\theta_1 = \max(a_1, \frac{c}{b_2}) = \frac{c}{b_2}$, then, $\max(A(\theta_1) + B(\frac{c}{\theta_1}) - 1, 0) = A(\frac{c}{b_2})$.

Thus, for each discrete point of $z = c$ in the increasing part of the support of the resulting fuzzy number, the value of $\sup_{z=x\times y}(A(x)t_L B(y))$ can be calculated using the following function $h_c(x)$ as presented below:

$$h_c(x) = \begin{cases} A(\frac{c}{b_2}), & \text{if } a_1 \le \frac{c}{b_2} \\ B(\frac{c}{a_2}), & \text{if } b_1 \le \frac{c}{a_2} \\ 0, & \text{otherwise} \end{cases} \quad \Rightarrow \quad \forall c \in [a_1b_1, a_2b_2]: \sup_{z=x\times y} \big(A(x) t_L B(y)\big) = \sup_{z=x\times y} \big(A(x) t_d B(y)\big), \quad z = c$$

Thus, if any point in the increasing part of the resulting fuzzy number support has a membership value greater than zero, its membership value is equal for both Lukasiewicz and drastic product t-norms. However, the membership values in the decreasing parts of the resulting fuzzy numbers are not necessarily equal for Lukasiewicz and drastic product t-norms. Finally, if one of the two input triangular fuzzy numbers has a negative support and the other one has positive support, it can be proven that the decreasing parts of the fuzzy numbers resulting from extended fuzzy multiplication using Lukasiewicz and drastic product t-norms overlap.

4. Numerical example

This section presents numerical examples of extended fuzzy arithmetic implemented on triangular fuzzy numbers. The numerical examples use the computational methods proposed in Section 3 to illustrate the results of implementing extended fuzzy addition and multiplication using product and Lukasiewicz t-norms on triangular fuzzy numbers. Moreover, extended fuzzy arithmetic using the min t-norm is performed using the computational method proposed by Klir [14], while a computational method proposed by Lin et al. [13] is used to implement extended fuzzy arithmetic using the drastic product t-norm. This section presents numerical example for implementing extended fuzzy arithmetic on triangular fuzzy numbers.

Example. Consider two triangular fuzzy numbers A and Bthat both have positive supports. The mathematical forms of the two fuzzy numbers are presented in Eq. (52) and Eq. (53) respectively. Fig. 8(a) and Fig. 8(b) show the graphical

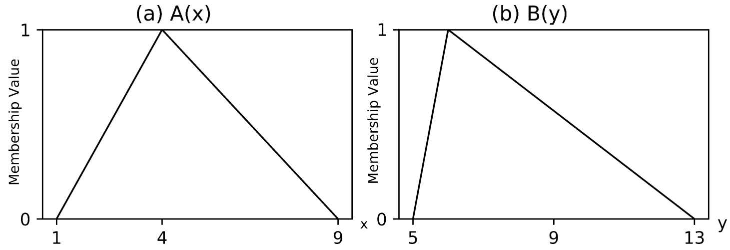

Fig. 8. Triangular input fuzzy numbers $A(x, 1, 4, 9)$ and $B(y, 5, 6, 13)$.

Table 1
Extended fuzzy addition results for $A(x) \oplus B(y)$ using product t-norm.

$c = x + y$	x_m	θ_{min}	θ_{max}	x	$A(x)$	y	$B(y)$	Membership value $C(c) = t(A(x), B(y))$
6.00	1.00	1.00	1.00	1.00	0.00	5.00	0.00	0.00
7.78	1.89	1.78	2.78	1.89	0.30	5.89	0.89	0.26
9.56	2.78	3.56	4.00	3.56	0.85	6.00	1.00	0.85
11.33	3.67	4.00	5.33	4.00	1.00	7.33	0.81	0.81
13.11	4.56	4.00	7.11	4.56	0.89	8.56	0.63	0.56
14.89	5.45	4.00	8.89	5.45	0.71	9.45	0.51	0.36
16.67	6.34	4.00	9.00	6.34	0.53	10.34	0.38	0.20
18.44	7.22	5.44	9.00	7.22	0.36	11.22	0.25	0.09
20.22	8.11	7.22	9.00	8.11	0.18	12.11	0.13	0.02
22.00	9.00	9.00	9.00	9.00	0.00	13.00	0.00	0.00

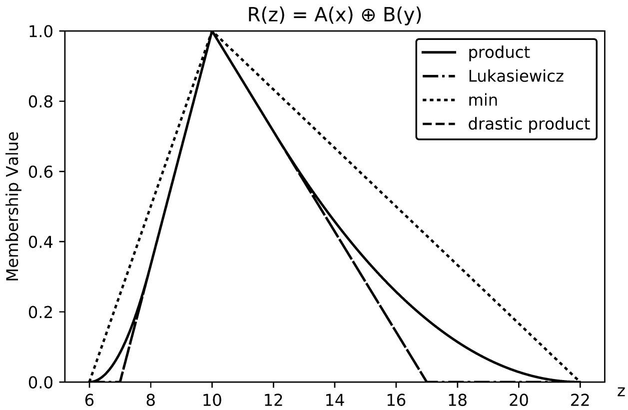

Fig. 9. Extended fuzzy addition $A(x) \oplus B(y)$ using different t-norms.

representations of A and B respectively,

$$A(x; 1, 4, 9) = \begin{cases} \frac{1}{3}x - \frac{1}{3}, & \text{if } 1 \le x < 4 \\ -\frac{1}{5}x + \frac{9}{5}, & \text{if } 4 \le x \le 9 \\ 0, & \text{otherwise} \end{cases} \tag{52}$$

$$B(y; 5, 6, 13) = \begin{cases} y - 5, & \text{if } 5 \le y < 6 \\ -\frac{1}{7}y + \frac{13}{7}, & \text{if } 6 \le y \le 13 \\ 0, & \text{otherwise} \end{cases} \tag{53}$$

To perform extended fuzzy addition using the product t-norm on the two fuzzy numbers, the membership values for 10 points of the resulting fuzzy number support are calculated. Table 1 presents the 10 points, the membership value for each point, and the values of x, y, $A(x; 1, 4, 9)$ and $B(y; 5, 6, 13)$.

Fig. 9 presents the results of implementing extended fuzzy addition using the four common t-norms (i.e., product, Lukasiewicz, min, and drastic product t-norms) on the two triangular fuzzy numbers.

The min t-norm is the highest t-norm, which implies that the resulting fuzzy number has the largest membership value for each point of the support if extended fuzzy addition is implemented using the min t-norm (refer to Fig. 9). Conversely, the drastic product t-norm is the lowest t-norm, thus the resulting fuzzy number has the smallest membership value for

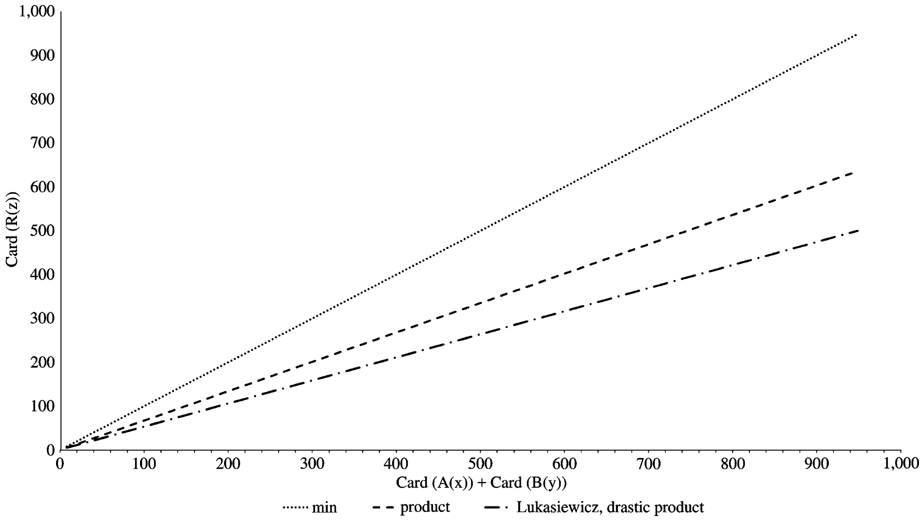

Fig. 10. Changes in the cardinality of extended fuzzy addition results with respect to changes in the cardinality of input fuzzy numbers.

each point of the support if extended fuzzy addition is implemented using the drastic product t-norm. Finally, as presented in Fig. 9, the fuzzy numbers resulting from the implementation of extended fuzzy addition using the Lukasiewicz and drastic product t-norms are overlapping for all points of the support. In order to compare the overestimation of uncertainty in the resulting fuzzy numbers presented in Fig. 9, the cardinality of each of the four fuzzy numbers is calculated. Pedrycz and Gomide [3] discussed that a higher value of cardinality is associated with a higher level of uncertainty, and a lower value of cardinality is associated with a lower level of uncertainty (or a higher level of specificity).

The cardinality of the resulting fuzzy numbers produced by extended fuzzy addition using min, product, Lukasiewicz, and drastic product t-norms is equal to 8.00, 5.65, 5.00, and 5.00 respectively. The cardinality of the resulting fuzzy number produced by the min t-norm is 60% larger than the cardinality of the resulting fuzzy number produced by the drastic product and Lukasiewicz t-norms. Moreover, the cardinality of the resulting fuzzy number produced by the min t-norm is 42% larger than the cardinality of the resulting fuzzy number produced by the product t-norm. Finally, the cardinality of the resulting fuzzy number produced by the product t-norm is 13% larger than the cardinality of the resulting fuzzy number produced by the Lukasiewicz and drastic product t-norms. In conclusion, (1) the min t-norm results in the highest level of uncertainty overestimation, (2) although the support of the resulting fuzzy number produced by the min and product t-norms is equal, the cardinality of the resulting fuzzy number produced by the min t-norm is 42% larger than the cardinality of the resulting fuzzy number produced by the product t-norm, and (3) the difference between the cardinality of the resulting fuzzy number produced by the min and product t-norms is more significant than the difference between the cardinality of the resulting fuzzy number produced by the product, Lukasiewicz, and drastic product t-norms. For further analysis of the uncertainty overestimation caused by extended fuzzy addition, changes in the cardinality of the resulting fuzzy number as a result of increases in the cardinality of the input fuzzy numbers is calculated. For this purpose, the length of the support of each input fuzzy number is increased, and the cardinality of the resulting fuzzy number for extended fuzzy addition using min, product, Lukasiewicz, and drastic product t-norm is calculated. Fig. 10 shows the changes in the cardinality of the resulting fuzzy number with respect to changes in the cardinality of the input fuzzy numbers.

Referring to Fig. 10, the min t-norm produces the resulting fuzzy number with the largest value of cardinality (i.e., the highest level of uncertainty); the uncertainty overestimation caused by this t-norm becomes more significant as the cardinality of input fuzzy numbers increases.

In order to implement extended fuzzy multiplication using the product t-norm on the two fuzzy numbers, the membership values for 10 points of the resulting fuzzy number support are calculated. Table 2 presents the 10 points, the membership value for each point, and values of x, y, A and B.

Similarly, membership values for the 10 points of the resulting fuzzy number support are calculated in order to implement extended fuzzy multiplication using the Lukasiewicz t-norm on the two fuzzy numbers. Table 3 presents the 10 points, the membership value for each point, and the values of x, y, A and B. Fig. 11 presents the results of implementing extended fuzzy multiplication using the four common t-norms (i.e., product, Lukasiewicz, min, and drastic product t-norms) on the two triangular fuzzy numbers.

Fig. 12 shows that the increasing parts of the resulting fuzzy numbers overlap for the Lukasiewicz and drastic product t-norms. However, there is no overlap on the decreasing parts of the resulting fuzzy numbers. The cardinality of the resulting fuzzy numbers produced by extended fuzzy multiplication using min, product, Lukasiewicz, and drastic product t-norms is equal to 50.67, 34.19, 26.45, and 24.00 respectively. The cardinality of the resulting fuzzy number produced by the min t-norm is 111% larger than the cardinality of the fuzzy number produced by the drastic product t-norm, 92% larger than the Lukasiewicz t-norm, and 48% larger than the product t-norm. Moreover, the cardinality of the resulting fuzzy number

Table 2
Extended fuzzy multiplication results for $A(x) \otimes B(y)$ using product t-norm.

$c = x \times y$	x_m	θ_{min}	θ_{max}	x	$A(x)$	y	$B(y)$	Membership value $C(c) = t(A(x), B(y))$
5.00	1.00	1.00	1.00	1.00	0.00	5.00	0.00	0.00
17.44	1.87	2.91	3.49	2.91	0.64	6.00	1.00	0.64
29.89	4.55	4.00	4.98	4.55	0.89	6.57	0.92	0.82
42.33	5.41	4.00	7.06	5.41	0.72	7.82	0.74	0.53
54.78	6.16	4.21	9.00	6.16	0.57	8.90	0.59	0.33
67.22	6.82	5.17	9.00	6.82	0.44	9.85	0.45	0.20
79.67	7.43	6.13	9.00	7.43	0.31	10.73	0.32	0.10
92.11	7.99	7.09	9.00	7.99	0.20	11.53	0.21	0.04
104.56	8.51	8.04	9.00	8.51	0.10	12.29	0.10	0.01
117.00	9.00	9.00	9.00	9.00	0.00	13.00	0.00	0.00

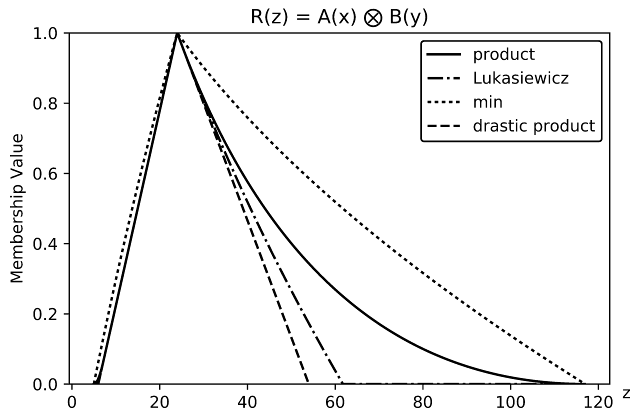

Fig. 11. Extended fuzzy multiplication $A(x) \otimes B(y)$ using different t-norms.

Table 3
Extended fuzzy multiplication results for $A(x) \otimes B(y)$ using Lukasiewicz t-norm.

$c = x \times y$	x_m	θ_{min}	θ_{max}	x	$A(x)$	y	$B(y)$	Membership value $C(c) = t(A(x), B(y))$
5.00	3.87	1.00	1.00	1.00	0.00	5.00	0.00	0.00
12.47	6.12	2.08	2.49	2.08	0.36	6.00	1.00	0.36
19.93	3.77	4.00	3.32	3.32	0.77	6.00	1.00	0.77
24.00	4.14	4.00	4.00	4.00	1.00	6.00	1.00	1.00
27.40	4.42	4.00	4.57	4.42	0.92	6.19	0.97	0.89
34.87	4.99	4.00	5.81	4.99	0.80	6.99	0.86	0.66
42.33	5.50	4.00	7.06	5.50	0.70	7.70	0.76	0.46
49.80	5.96	4.00	8.30	5.96	0.61	8.35	0.66	0.27
57.27	6.40	4.41	9.00	6.40	0.52	8.95	0.58	0.10
64.73	6.80	4.98	9.00	6.80	0.44	9.52	0.50	0.00

produced by the product t-norm is 42% larger than the cardinality of the resulting fuzzy number produced by the drastic product t-norm and 29% larger than the Lukasiewicz t-norm. Finally, the cardinality of the resulting fuzzy number produced by the Lukasiewicz t-norm is 10% larger than the cardinality of the resulting fuzzy number produced by the drastic product t-norm. In conclusion, (1) the min t-norm results in the highest level of uncertainty overestimation, (2) although the support of the resulting fuzzy number produced by the min and product t-norms is equal, the cardinality of the resulting fuzzy number produced by the min t-norm is 92% larger than the cardinality of the fuzzy number produced by the product t-norm, and (3) the difference between the cardinality of the resulting fuzzy number produced by the min and product t-norms is more significant than the difference between the cardinality of the fuzzy number produced by the product and Lukasiewicz t-norms, as well as the difference between the cardinality of the fuzzy number produced by the Lukasiewicz and drastic product t-norms. For further analysis of the uncertainty overestimation caused by extended fuzzy multiplication, changes in the cardinality of the resulting fuzzy number as a result of increases in the cardinality of the input fuzzy numbers is calculated. For this purpose, the length of the support of each input fuzzy number is increased, and the cardinality of the resulting fuzzy number for extended fuzzy multiplication using min, product, Lukasiewicz, and drastic product t-norm is calculated. Fig. 12 shows the changes in the cardinality of the resulting fuzzy number with respect to changes in the cardinality of the input fuzzy numbers.

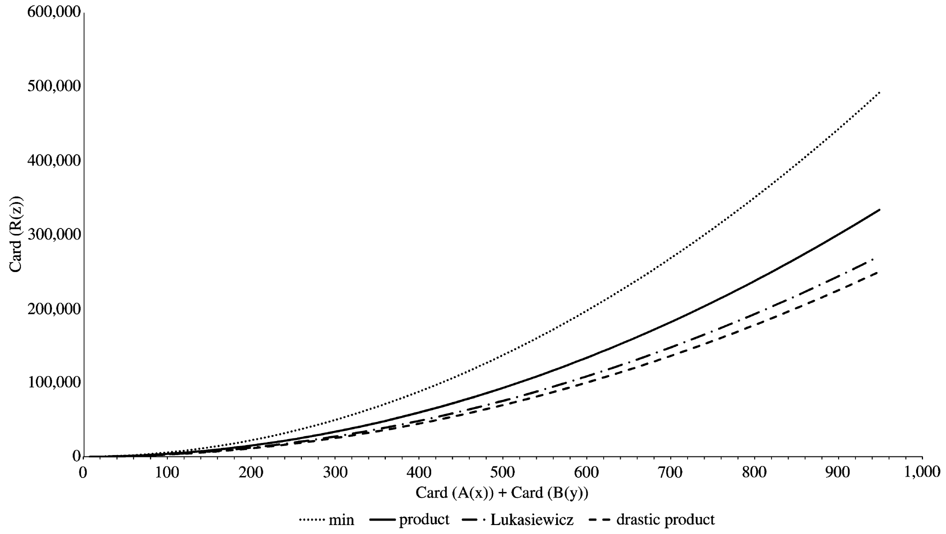

Fig. 12. Changes in the cardinality of extended fuzzy multiplication results with respect to changes in the cardinality of input fuzzy numbers.

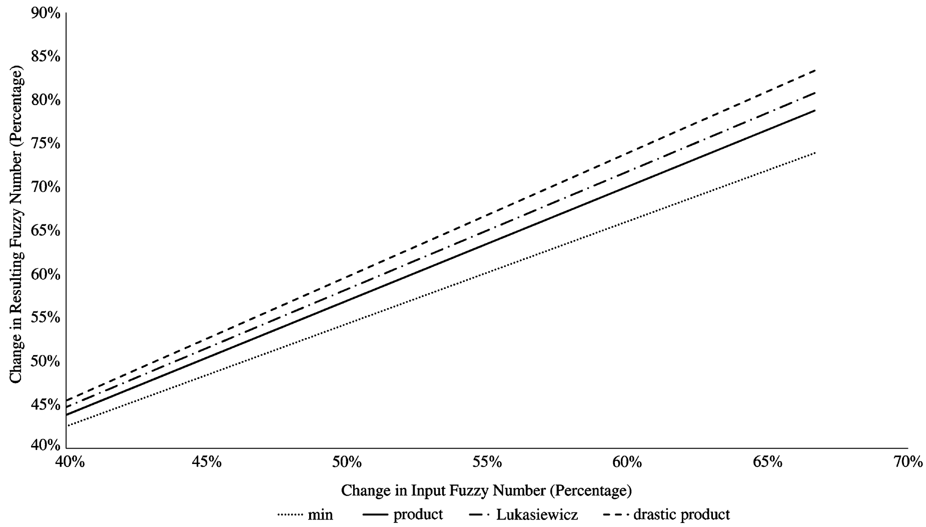

Fig. 13. Sensitivity analysis on the results of the extended fuzzy multiplication.

Referring to Fig. 12, the min t-norm produces the resulting fuzzy number with the largest value of cardinality (i.e., the highest level of uncertainty), and the uncertainty overestimation caused by this t-norm grows quadratically as the cardinality of the input fuzzy numbers increases. Accordingly, as the cardinality of the input fuzzy numbers increases, the significance of uncertainty overestimation caused by implementing extended fuzzy multiplication using the min t-norm increases rapidly.

Next the sensitivity of the resulting fuzzy number produced by the four t-norms is evaluated to the changes in the input fuzzy numbers. For this purpose, the length of the support of one of the input fuzzy numbers is increased by increasing the value of its core and the upper bound of its support, and the result of extended fuzzy multiplication using min, product, Lukasiewicz, and drastic product t-norm is calculated and then defuzzified using center of area (COA) method. Next, the changes (i.e., increment or decrement) of the defuzzified values of the input and resulting fuzzy number are calculated. Fig. 13 shows the results of the sensitivity analysis on the results of extended fuzzy multiplication.

Referring to Fig. 13, the min t-norm produces the resulting fuzzy number with the lowest level of sensitivity to changes in the input fuzzy number; followed by the product, Lukasiewicz and drastic product t-norms respectively. Accordingly, using drastic product for implementing extended fuzzy multiplication produces the resulting fuzzy number with the highest level of sensitivity to changes in the input fuzzy number, and the sensitivity of the resulting fuzzy number increases as changes in the input fuzzy numbers are larger.

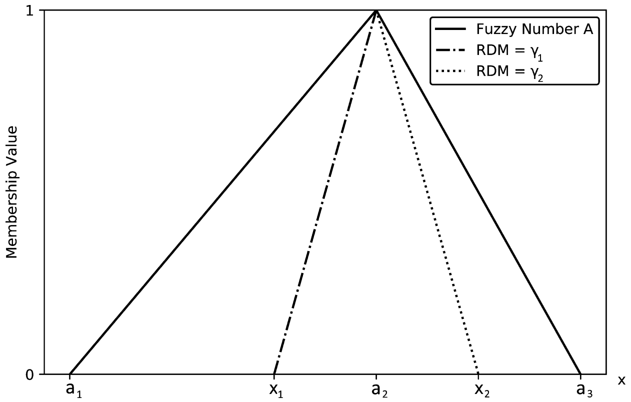

Fig. 14. Representation of triangular fuzzy numbers using horizontal membership functions.

4.1. Proposed computational methods versus fuzzy arithmetic using horizontal membership functions

Due to the complexity associated with the implementation of extended fuzzy arithmetic operations, Tomaszewska [40] introduced a new methodology for implementing fuzzy arithmetic operations on fuzzy numbers using horizontal membership functions of fuzzy numbers. Horizontal membership functions is an alternative to the traditional fuzzy membership functions (refer to Section 2.2) for representing fuzzy numbers using the relative distance measure (RDM) [41]. The value of RDM for a given value of x can be calculated as shown in Eq. (54):

$$\forall x \in [\underline{x}, \overline{x}] : \gamma_x = \frac{x - \underline{x}}{\overline{x} - \underline{x}} \tag{54}$$

where $\underline{x}$ and $\overline{x}$ stand for the lower and upper bounds of the interval respectively and γ_x stands for the value of RDM at point x. Using the RDM, any given fuzzy membership function defined on the universe of discourse X, can be represented by its horizontal membership function, which determines the value of $x \in X$ as a function of membership value (μ) and the RDM value (γ_x). The generic form of the horizontal membership function for the triangular fuzzy number $A(x; a_1, a_2, a_3)$ is shown in Eq. (55) [40]:

$$x = \big(a_1 + (a_2 - a_1)\mu\big) + \big((a_3 - a_1)(1 - \mu)\big)\gamma_x \tag{55}$$

where a_1 and a_3 stand for the lower and upper bounds of the support respectively, and a_2 is the core of fuzzy number (i.e., $A(a_2; a_1, a_2, a_3) = 1$). Fig. 14 shows the representation of the triangular fuzzy number $A(x; a_1, a_2, a_3)$ using horizontal membership functions and two RDM values of γ_1 and γ_2, where $x_1 = (a_3 - a_1)\gamma_1$ and $x_2 = (a_3 - a_1)\gamma_2$; and $\gamma_1 < \gamma_2$.

Fig. 14 shows that the increasing part of the support of the fuzzy number has the RDM value of 0 ($\gamma_1 = 0$) and the decreasing part of the support of fuzzy number has the RDM value of 1 ($\gamma_2 = 1$). The horizontal membership functions of fuzzy numbers can be used for implementing arithmetic operations on fuzzy numbers [40,42]. For example the result of fuzzy addition $C(z) = A(x; a_1, a_2, a_3) \oplus B(y; b_1, b_2, b_3)$ can be calculated by Eq. (56) [40,42]:

$$\begin{aligned} z = {} & \big(a_1 + (a_2 - a_1)\mu\big) + \big((a_3 - a_1)(1 - \mu)\big)\gamma_x + \big(b_1 + (b_2 - b_1)\mu\big) + \big((b_3 - b_1)(1 - \mu)\big)\gamma_y, \\ & \mu, \gamma_x, \gamma_y \in [0, 1] \end{aligned} \tag{56}$$

Accordingly, the horizontal membership function of the resulting fuzzy number of fuzzy addition determines the value of z as a function of three variables (i.e., $z = f(\mu, \gamma_x, \gamma_y)$): the membership value of the resulting fuzzy number (μ) and the value of RDM for the two input fuzzy numbers (i.e., γ_x, and γ_y). Thus, the resulting fuzzy number is defined in the 4-dimensional space. In order to transform the results of horizontal membership functions into a Type 1 fuzzy number—which is required in many applications of fuzzy arithmetic for further operations on the results—the resulting fuzzy number must be projected into the 2-dimensional space. For this purpose, two constant values for the RDM of the input fuzzy numbers ($\gamma_x = \gamma_y = \gamma_1$ and $\gamma_x = \gamma_y = \gamma_2$) are considered, where $\gamma_1 \in [0, 0.5]$ and $\gamma_2 \in [0.5, 1]$. Considering two constant values of $\gamma_1 = 0$ and $\gamma_2 = 1$, horizontal membership functions returns the same results as standard fuzzy addition as illustrated next. First, the resulting fuzzy number is calculated using horizontal membership functions as presented in Eq. (57):

$$\begin{aligned} \gamma_x = \gamma_y = 0; \quad & z = \big(a_1 + (a_2 - a_1)\mu\big) + \big(b_1 + (b_2 - b_1)\mu\big) = a_1 + b_1 + (a_2 + b_2 - a_1 - b_1)\mu \\ \gamma_x = \gamma_y = 1; \quad & z = \big(a_1 + (a_2 - a_1)\mu\big) + \big((a_3 - a_1)(1 - \mu)\big) + \big(b_1 + (b_2 - b_1)\mu\big) \\ & \quad + \big((b_3 - b_1)(1 - \mu)\big) = a_3 + b_3 + (a_2 + b_2 - a_3 - b_3)\mu \end{aligned} \tag{57}$$

Next, the result of standard fuzzy addition on two triangular fuzzy numbers is calculated. The result of standard fuzzy addition on two triangular fuzzy numbers is a triangular fuzzy number, which is calculated by Eq. (58) [3]:

$$C(z; c_1, c_2, c_3) = A(x; a_1, a_2, a_3) \oplus B(y; b_1, b_2, b_3), \qquad c_1 = a_1 + b_1, c_2 = a_2 + b_2, c_3 = a_3 + b_3 \tag{58}$$

Thus, the horizontal membership function of the resulting fuzzy number can be determined using Eq. (55); this horizontal membership function is presented in Eq. (59):

$$\begin{aligned} z &= \big(c_1 + (c_2 - c_1)\mu\big) + \big((c_3 - c_1)(1-\mu)\big)\gamma_z \\ &= \big(a_1 + b_1 + (a_2 + b_2 - a_1 - b_1)\mu\big) + \big((a_3 + b_3 - a_1 - b_1)(1-\mu)\big)\gamma_z \end{aligned} \tag{59}$$

The increasing part of the support of the resulting fuzzy number can be determined using Eq. (59), where the value of RDM is equal to zero ($\gamma_z = 0$), and the decreasing part of the support of the resulting fuzzy number can be determined using Eq. (59), where the value of RDM is equal to one ($\gamma_z = 1$) as shown in Eq. (60):

$$\begin{aligned} &\gamma_z = 0 : z = a_1 + b_1 + (a_2 + b_2 - a_1 - b_1)\mu \\ &\gamma_z = 1 : z = a_3 + b_3 + (a_2 + b_2 - a_3 - b_3)\mu \end{aligned} \tag{60}$$

A comparison of Eq. (57) and Eq. (60), shows that implementing fuzzy addition using horizontal membership functions on two triangular fuzzy number returns the same results as the standard fuzzy addition and extended fuzzy addition using the min t-norm (refer to Section 2.3). Thus, the implementation of horizontal membership functions causes the same level of uncertainty overestimation as the standard fuzzy addition and extended fuzzy addition using the min t-norm (see Fig. 9). In the case of fuzzy addition using horizontal membership functions, the overestimation of uncertainty can be reduced by changing the value of RDM for the input fuzzy numbers (i.e., referring to γ_x and γ_x in Eq. (55)). However, by considering any value for RDM except 0 and 1, the shape of the input fuzzy numbers will change. It should be noted that fuzzy arithmetic using horizontal membership functions is an extension of the interval arithmetic approach introduced by Moore [43]. Thus, the reduction of the overestimation of uncertainty by changing the value of RDM of the input fuzzy numbers is associated with changes in the α-cuts of the input fuzzy numbers, which changes the shape of the input fuzzy numbers as well. In contrast, the computational methods presented in this paper reduce the overestimation of uncertainty in the resulting fuzzy number, as compared to the standard fuzzy arithmetic operations, without changing the shape of the input fuzzy numbers; instead, these computational methods reduce the overestimation of uncertainty by implementing extended fuzzy arithmetic operations using the product and Lukasiewicz t-norms. For more information regarding the fuzzy arithmetic using horizontal membership functions, refer to Piegat and Pluciński [42] and Tomaszewska [40].

4.2. Discussion

Implementing extended fuzzy arithmetic using min t-norm produces a resulting fuzzy number with the highest level of uncertainty and the lowest level of sensitivity to the changes in the input fuzzy numbers. Conversely, using drastic product t-norm for implementing extended fuzzy arithmetic yields the lowest level of uncertainty in the resulting fuzzy number, while the resulting fuzzy number has the highest level of sensitivity to changes in the input fuzzy numbers. Accordingly, there is a trade-off between the uncertainty overestimation and the sensitivity of the resulting fuzzy number to changes in the input fuzzy numbers. Referring to the results of cardinality analysis (see Fig. 10 and Fig. 12) and sensitivity analysis (see Fig. 13), there are only two extreme conditions, in which the use of min and drastic product t-norms for extended fuzzy arithmetic is recommended; first, if the cardinality of the input fuzzy numbers is extremely large and the changes in the input fuzzy numbers are extremely small, the use of drastic product t-norm is recommended for extended fuzzy arithmetic; and second, if the cardinality of the input fuzzy numbers is extremely small and the changes in the input fuzzy numbers are extremely large, the use of min t-norm is recommended for extended fuzzy arithmetic. However, in most applications of fuzzy numbers, the use of product and Lukasiewicz t-norms for extended fuzzy arithmetic are recommended, since they simultaneously reduce the uncertainty overestimation (as compared to the min t-norm) and reduce the sensitivity of the resulting fuzzy number to changes in the input fuzzy numbers (as compared to the drastic product t-norm). In the other two extreme conditions, where the cardinality of the input fuzzy numbers and the changes in the input fuzzy numbers are both extremely large or both extremely small, the use of product and Lukasiewicz t-norms for extended fuzzy arithmetic are recommended. According to the results presented in Fig. 10, Fig. 12, and Fig. 13, using the product t-norm for extended fuzzy arithmetic produces a resulting fuzzy number with a higher level of uncertainty and a lower level of sensitivity, as compared to the Lukasiewicz t-norm. Accordingly, in applications of fuzzy numbers, where the sensitivity of the resulting fuzzy number is more important than its level of uncertainty (e.g., in the case of fuzzy control systems), the use of the product t-norm is recommended for implementing extended fuzzy arithmetic. Conversely, if the level of uncertainty of the resulting fuzzy number is more important than its sensitivity to changes in the input fuzzy numbers (e.g., in the case of fuzzy decision making), the use of the Lukasiewicz t-norm is recommended for implementing extended fuzzy arithmetic. In some applications of fuzzy numbers, such as engineering applications [5], fuzzy arithmetic operations are implemented on fuzzy numbers consecutively, where a fuzzy arithmetic operation is implemented on the results of another fuzzy arithmetic operation. In such applications, the overestimation of uncertainty becomes more significant. Accordingly, implementation of extended fuzzy arithmetic using the Lukasiewicz t-norm is suggested in these applications since it reduces the overestimation of uncertainty significantly, as compared to the use of standard fuzzy arithmetic (refer to Fig. 12). Moreover, the implementation of extended fuzzy arithmetic using the Lukasiewicz t-norm reduces the sensitivity of the resulting fuzzy numbers to changes in the input fuzzy numbers, as compared to the use of the drastic product t-norm (refer to Fig. 13).

In many applications of fuzzy numbers, the resulting fuzzy numbers are interpreted using their α-cuts, where each α-cut represents an interval of the resulting fuzzy number with a given level of confidence $\lambda = 1 - \alpha$ [43]. In the case of extended fuzzy arithmetic, at the confidence level of $\lambda \cong 1$, the resulting fuzzy number produced by the product t-norm has almost similar α-cuts to those produced by the min t-norm. Referring to Fig. 11, the α-cuts of the resulting fuzzy numbers for the confidence level of $\lambda = 99\%$ for the min, product, Lukasiewicz and drastic product t-norms are equal to $[5.16, 115.72]$, $[5.85, 104.44]$, $[6.23, 61.66]$, and $[6.23, 53.70]$ respectively. The uncertainty of these α-cuts is equal to the length of the α-cuts. The length of the each of the above α-cuts is equal to 109.84, 98.59, 55.43, and 47.47 for the min, product, Lukasiewicz and drastic product t-norms respectively. The length of the α-cut of the resulting fuzzy number produced by the product t-norm is only 10% smaller than that produced by the min t-norm; on the other hand, the length of the α-cut of the resulting fuzzy number produced by the Lukasiewicz t-norm is 50% smaller than that produced by the min t-norm and 44% smaller than that produced by the product t-norm. Accordingly, in different applications of fuzzy numbers, if the results are presented using their α-cuts at high confidence levels (i.e., $\lambda \cong 1$), the uncertainty of the α-cuts produced by the min and product t-norms is significantly larger than the uncertainty produced by the Lukasiewicz t-norm. Therefore, in these applications, implementation of extended fuzzy arithmetic using Lukasiewicz t-norm may be preferable to the use of the min and product t-norms.

5. Conclusion and future work

In the different applications for fuzzy numbers, fuzzy arithmetic operations are frequently performed in order to solve mathematical equations that use fuzzy numbers, which are used to represent the non-probabilistic uncertainties of real-world variables. There are two different approaches proposed in the literature for implementing fuzzy arithmetic operations: standard fuzzy arithmetic and extended fuzzy arithmetic. The mathematical aspects of the two approaches of fuzzy arithmetic operations are discussed extensively in the literature; however, the existing computational methods for implementing fuzzy arithmetic are mainly focused on the standard fuzzy arithmetic approach due to its simplicity. Despite the fact that standard fuzzy arithmetic is commonly used, this approach causes an overestimation of uncertainties in the resulting fuzzy numbers. The overestimation of uncertainty issue can be reduced by implementing extended fuzzy arithmetic using any t-norm lower than the min t-norm. However, due to the complexity, there are only a few computational methods available in the literature for implementing extended fuzzy arithmetic using the min and drastic product t-norms. Although implementing extended fuzzy arithmetic using the drastic product t-norm reduces the overestimation of uncertainty, this t-norm causes the resulting fuzzy number to be highly sensitive to changes in the input fuzzy numbers. Moreover, extended fuzzy arithmetic using the min t-norm produces the same results as the standard fuzzy arithmetic approach. This paper contributes the application of fuzzy numbers by presenting an original computational method for implementing extended fuzzy arithmetic operations using the product and Lukasiewicz t-norms on triangular fuzzy numbers. Implementation of extended fuzzy arithmetic using the product and Lukasiewicz t-norms reduces the uncertainty overestimation, as compared to the min t-norm, and simultaneously reduces the sensitivity of the resulting fuzzy number, as compared to the drastic product t-norm. Thus, the proposed computational method for implementing extended fuzzy arithmetic using the product and Lukasiewicz t-norms contributes to the applicability of fuzzy numbers in different applications.

The proposed computational method is an exact discrete method that calculates the exact membership values of a finite number of points in the resulting fuzzy number support; this method then constructs the resulting fuzzy number using the discrete points of its support and their membership values. This paper illustrates that performing extended fuzzy addition using the Lukasiewicz t-norm on triangular fuzzy numbers produces the same result as the drastic product t-norm. Moreover, for any two input triangular fuzzy numbers that have positive supports, the fuzzy numbers resulting from extended fuzzy multiplication using the Lukasiewicz and drastic product t-norms overlap in the increasing part of the fuzzy numbers. In the future, the computational method presented in this paper will be further developed to implement extended fuzzy arithmetic using the product and Lukasiewicz t-norms on trapezoidal and Gaussian fuzzy numbers. The current computational method will also be further developed for implementation of extended fuzzy arithmetic using the family of Yager t-norms.

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