Tiling and the Extension Theorem

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Plane Tiling A countable family of closed sets that covers the plane without gaps or overlaps.

Cover Family of sets, the union of which covers the plane with no gaps.

Packing Family of sets without overlap.

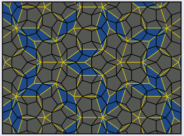


Figure: Plane tiling[1]

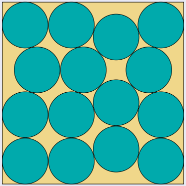


Figure: Packing[2]

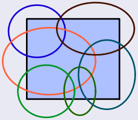


Figure: Cover

[1]Source: https://commons.wikimedia.org/wiki/File:Penrose_Tiling_(P1_over_P3).svg

[2]Source: https://commons.wikimedia.org/wiki/File:Circles_packed_in_square_15.svg

Tiling Family of sets $\mathscr{T} = \{T_1, T_2, \ldots\}$ such that the union of *tiles* T_i is the whole plane, and the interiors of tiles T_i are disjoint.

Patch Finite number of tiles in the tiling, the union of which is a closed topological disc.

Congruent Tiles or tilings are *congruent* if they can be made to coincide by rigid motion of the plane.

Congruence Transformation Mapping of the plane onto itself with preserves all distances.

- Rotation
- Translation
- Reflection
- Glide reflection

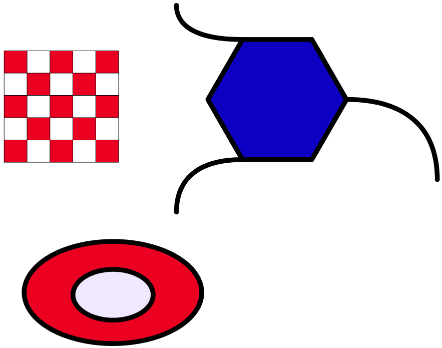
Edge Boundary of a tile, or an arc on the boundary of a tile.

Vertex Isolated point on the boundary of a tile.

Prototile A closed set congruent to a tile.

Well-Behaved Tilings

N1 Every tile of $\mathscr{T}$ is a closed topological disk.



N2 The intersection of every two tiles of $\mathscr{T}$ is a connected set, and does not consist of disjoint parts.

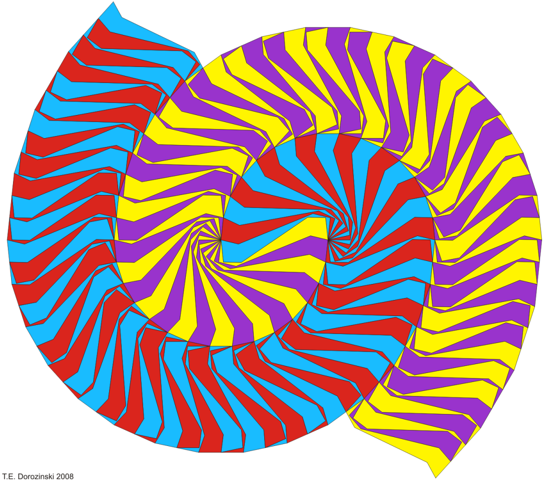


Figure: Voderberg Spiral[3]

[3]Source: https://commons.wikimedia.org/wiki/File:Voderberg-1.png

N3 The tiles of $\mathscr{T}$ are uniformly bounded.

inparamater Every tile in $\mathscr{T}$ may contain a circular disc of radius u.

outparameter Every tile in $\mathscr{T}$ may be contained in a circular disc of radius U.

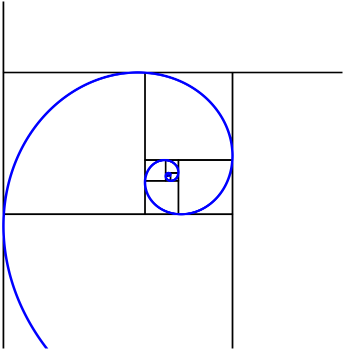


Figure: Golden Spiral[4]

[4] Source: https://commons.wikimedia.org/wiki/File:GoldenSpiralLogarithmic.svg

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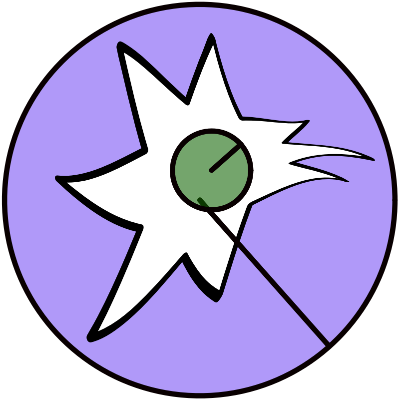
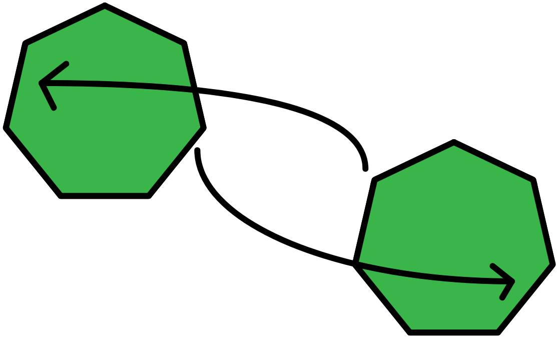


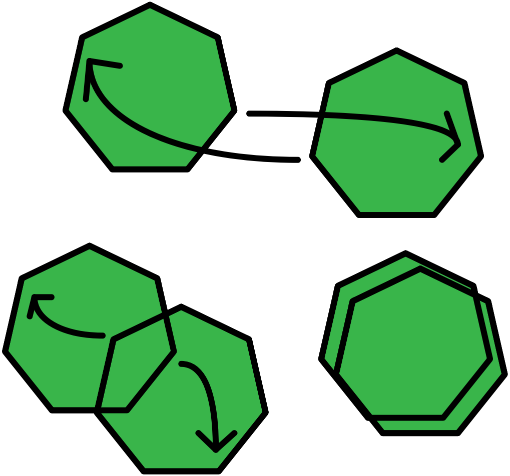
Hausdorff Distance

$$\delta(T_1, T_2) = \max\left\{ \sup_{x_2 \in T_2} \inf_{x_1 \in T_1} |x_1 - x_2|, \sup_{x_1 \in T_1} \inf_{x_2 \in T_2} |x_1 - x_2| \right\} \qquad (1)$$



Convergence

A sequence of tiles (T_n) will *converge* to limit tile T where $\delta(T_n, T) \to 0$ as $n \to \infty$.



Selection Theorem

If all tiles T_i are congruent to a bounded tile T_0, and there is some point P_0 common to all tiles T_i, it is possible to select a convergent subsequence from (T_i) that converges to a limit tile T' that is also congruent to T_0.

Extension Theorem

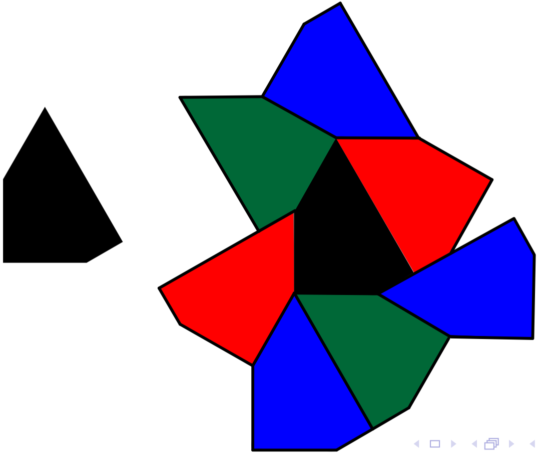
Extentension Theorem

Let $\mathscr{U}$ be a finite set of prototiles, each of which is a closed topological disc. If $\mathscr{U}$ tiles over arbitrarily large circular discs D, then $\mathscr{U}$ admits a tiling of the plane.

Heesch's Problem

Heesch's Problem

For which positive integers r does there exist a prototile T such that T can be surrounded r times, but not $r+1$ times, by tiles congruent to T?



Surround Given a set of prototiles $\mathscr{T}$ that admit a patch $\mathscr{A}_0$, $\mathscr{A}_0$ can be *surrounded* to form patch $\mathscr{A}_1$ if the closure of the plane not covered by $\mathscr{A}_1$ is disjoint from $\mathscr{A}_0$.

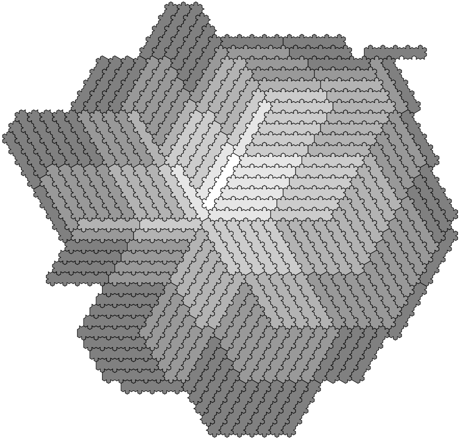


Figure: Tiling of Heesch number 5[5]

[5] Source: `https://commons.wikimedia.org/wiki/File:Heesch-5.svg`

Conjecture

Does there exist a function $f(n, m)$ such that for all prototile sets $\mathscr{P}$ of n prototiles, if every patch of m tiles can be surrounded $f(n, m)$ times, then $\mathscr{P}$ admits a tiling of the plane.

Polyminoes

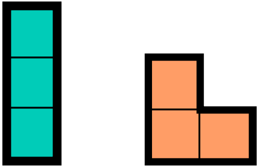


Figure: Trominoes[6]

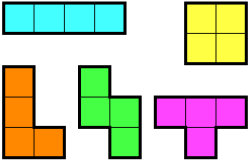


Figure: Tetrominoes[7]

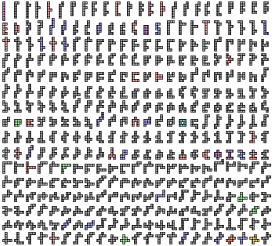


Figure: Octominoes[8]

[1]Source: https://commons.wikimedia.org/wiki/File:Trominoes.svg
[2]Source: https://en.wikipedia.org/wiki/Tetromino
[3]Source:
https://commons.wikimedia.org/wiki/File:The_369_Free_Octominoes.svg

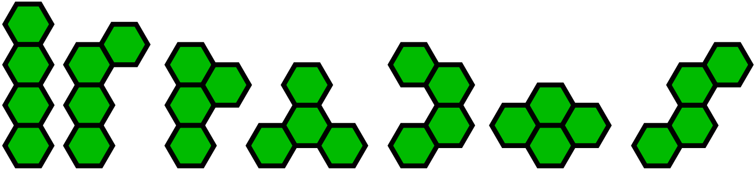


Figure: Tetrahex[9]

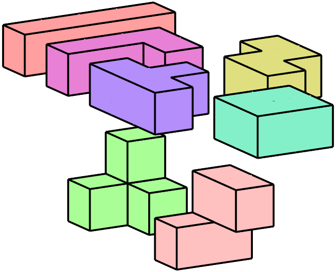


Figure: Tetracubes[10]

[1]Source: https://commons.wikimedia.org/wiki/File:Tetracomb.svg

[2]Source: https://commons.wikimedia.org/wiki/File:Tetracubes.svg

Penrose Tiles

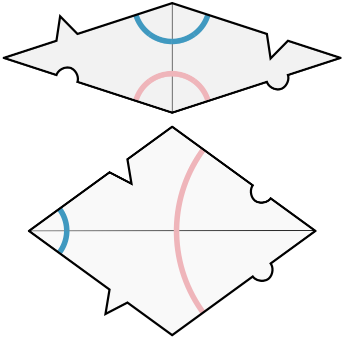


Figure: Penrose P3[11]

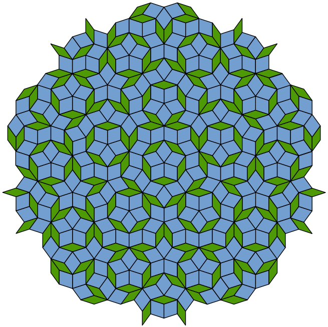


Figure: Penrose P3 Tiling[12]

[1]Source: https://commons.wikimedia.org/wiki/File:Kite_Dart.svg

[2]Source: https://commons.wikimedia.org/wiki/File:Penrose_Tiling_(Rhombi).svg