

Lecture 3: Sketches for  $L_p$  normsLecturer: *Przemysław Uznański*Scribes: *Wiktoria Garbarek*

## 1 $p$ -stable distributions

**Definition 1.** A distribution  $\mathcal{D}$ , with mean 0, is called *stable*, if for  $X_1, \dots, X_n \sim \mathcal{D}$  which are independent and  $a_1, \dots, a_n \in \mathbb{R}$  there is  $\sum_i a_i X_i = b \cdot Z$  for some  $b \in \mathbb{R}$  and  $Z \sim \mathcal{D}$ .

**Definition 2.** A distribution  $D$  is  $p$ -stable if it is stable and coefficient  $b$  from previous definition satisfies

$$b = \left( \sum_i |x_i|^p \right)^{1/p}$$

**Remark 3.** (Zolotarev, 1986)  $p$ -stable distribution exists if and only if  $0 < p \leq 2$ .

For  $p \in \{\frac{1}{2}, 1, 2\}$  we know closed form formulas, e.g.:

- Normal distribution, that is  $f(x) = \frac{e^{-x^2/2}}{\sqrt{2\pi}}$ , is 2-stable.
- Cauchy distribution, that is  $f(x) = \frac{1}{(1+x^2)\pi}$ , is 1-stable.
- Lévy distribution is  $\frac{1}{2}$ -stable.

**Remark 4.** Except for  $p = 2$ , those distributions are heavy tailed, that is  $\mathbb{E}[|\mathcal{D}|] = \infty$  and  $\mathbb{E}[\mathcal{D}^2] = \infty$  (this has to be, as by Central Limit Theorem distributions with finite moments cannot be stable, unless its normal distribution).

## 2 Sketches for $L_p$ norm [Indyk 2000]

Pick random coefficients  $r_i \sim \mathcal{D}_p$  for  $i = 1..n$ , where  $\mathcal{D}_p$  is a  $p$ -stable distribution. Then

$$Z = \sum_i x_i r_i$$

is a sketch of vector  $\mathbf{x}$ , since  $Z \sim |\mathbf{x}|_p \mathcal{D}_p$ . We will of course run many parallel instances of sketching process (as usual).

**Remark 5.** Our sketches are linear functions, so linear combination of sketches is also a sketch.

Whenever update  $(x_i, c_i)$  comes, to maintain the sketch we compute  $Z := Z + c_i \cdot r_{x_i}$

Challenges:

- How to draw random values from  $p$ -stable distribution?
- How to extract the result?
- How much independence is required?

## 2.1 Drawing from $p$ -stable

- $p = 1$ : If  $U \sim \mathcal{U}(0, 1)$  then  $\tan \pi(U - \frac{1}{2})$  is distributed with the Cauchy distribution
- $p = 2$ : We can use Box-Muller transformation. If  $U, V \sim \mathcal{U}(0, 1)$  and are iid, then  $\sqrt{-2 \ln U} \cdot \cos 2\pi V$  is distributed as a normal distribution.
- $p \in (0, 2)$  and  $p \neq 1$ : In this case  $p$ -stable distribution can be simulated by method derived by Chambers, Mallows and Stuck (1976). If  $U, V \sim \mathcal{U}(0, 1)$  and are iid and let's set  $\Theta(U) = \pi(U - \frac{1}{2})$ . Then

$$\frac{\sin p\Theta(U)}{\cos^{\frac{1}{p}} \Theta(U)} \left( \frac{\cos(\Theta(U) \cdot (1-p))}{-\ln V} \right)^{\frac{1-p}{p}}$$

is distributed as a  $p$ -stable distribution.

## 2.2 Extracting the result via median

Recall  $Z \sim |\mathbf{x}|_p \mathcal{D}_p$  Expected value is useless, since it is infinite, so let's consider median:

$$\text{median}(|Z|) \sim |\mathbf{x}|_p \cdot \text{median}(|\mathcal{D}_p|)$$

How to extract median of a distribution  $X$  (on  $\mathbb{R}_+$ )?

Let  $F$  be CDF of distribution  $X$ . Let's also sample  $k$  values  $x_1, \dots, x_k \sim X$ , and output  $\text{median}(x_1, \dots, x_k)$ . By Chernoff bound, if we have  $k = O(\log(\frac{1}{\delta})/\varepsilon^2)$ , then

$$F^{-1}(\frac{1}{2} - \varepsilon) \leq \text{median}(x_1, \dots, x_k) \leq F^{-1}(\frac{1}{2} + \varepsilon)$$

If  $F'$  is not too flat around  $F^{-1}(\frac{1}{2})$  (i.e. median), then  $F^{-1}(\frac{1}{2} \pm \varepsilon)$  are actually  $1 \pm C \cdot \varepsilon$  approximations of median (required  $F'(x) \geq \frac{1}{C}$  in a given range). This becomes an issue when  $p \rightarrow 0$ , but we don't have to care for constant  $p$ .

$$\text{Estimator: } \frac{\text{median}(|Z|)}{\text{median}(|\mathcal{D}_p|)} = \frac{\text{median}(|Z_1|, \dots, |Z_k|)}{\text{median}(|\mathcal{D}_p|)}$$

## 2.3 Geometric mean estimator [Li 2008]

Use geometric mean as an estimator.

Output:  $(\prod_{i=1}^k |Z_i|)^{1/k} / \alpha$ , where  $\alpha = e^{\mathbb{E}(\ln |\mathcal{D}_p|)}$ , and  $k = O(\log(\frac{1}{\delta})/\varepsilon^2)$ .

## 2.4 Independence

To use  $p$ -stability we assumed full  $n$ -wise independence - this means in theory huge space in streaming applications, and there is no easy workaround. We present two workarounds that are theoretic in nature and require very sophisticated machinery.

### 2.4.1 First workaround [Indyk 2000]

Apply Nisan's pRNG which is able to:

- Store its seed/local state in a  $\text{polylog}(n)$  space
- Extract its next random number in a small working space and small time
- "Fool" small space (arbitrary) computations as if they were given perfect randomness

### 2.4.2 Second workaround [Kane, Nelson and Woodruff 2008]

Prove that  $k$ -wise independence is enough, for

$$k = \mathcal{O}(\log \frac{1}{\varepsilon} \log \log \log \frac{1}{\varepsilon})$$

## 3 $L_p$ sketching for $p > 2$

### 3.1 2-pass Algorithm

Algorithm (only for sequence of values, not for turnstile):

1. Pick uniformly and at random  $i \in [n]$
2. In the first pass: Select  $x = x_i$
3. In the second pass: Compute  $f_x = |\{j : x_j = x\}|$
4. return output  $Y = n(f_x)^{p-1}$

$$E[Y] = \frac{1}{n} \sum_i n(f_{x_i})^{p-1} = \sum_x (f_x)^{p-1} f_x = \sum_x (f_x)^p = F_p$$

$$E[Y^2] = \frac{1}{n} \sum_i n^2 (f_{x_i})^{2p-2} = n \sum_x (f_x)^{2p-2} f_x = nF_{2p-1}$$

**Claim 6.** *Following inequality holds*

$$nF_{2p-1} \leq m^{1-1/p} (F_p)^2$$

*Proof.*

$$nF_{2p-1} = n\|f\|_{2p-1}^{2p-1} \leq n\|f\|_p^{2p-1} = \|f\|_1 \|f\|_p^{2p-1} \leq m^{1-1/p} \|f\|_p \|f\|_p^{2p-1} = m^{1-1/p} \|f\|_p^{2p} = m^{1-1/p} F_p^2$$

For  $\|f\|_1 \leq m^{1-1/p} \|f\|_p$  and  $\|f\|_{2p-1} \leq \|f\|_p$  see Wikipedia<sup>1</sup>. □

### 3.2 1-pass Algorithm

How to pick random  $i$  in a stream of unknown length?

1. Initialize  $r \leftarrow 0$ .
2. At  $i$ -th step of algorithm: with probability  $\frac{1}{i}$  we set  $x \leftarrow x_i$  and  $r \leftarrow 0$
3. If  $x_i = x$  then  $r \leftarrow r + 1$
4. Return output  $Y' = n(r^p - r^{p-1})$

$$E[Y'] = \sum_x \sum_{r \leq f_x} (r^p - r^{p-1}) = \sum_x (f_x)^p = F_p Y' \leq n p r^{p-1} \leq n p (f_x)^{p-1} = p Y$$

$$E[(Y')^2] \leq p^2 E[Y^2] = p^2 m^{1-1/p} (F_p)^2$$

And thus the number of parallel runs should be  $\mathcal{O}\left(\frac{p^2 m^{1-\frac{1}{p}}}{\varepsilon^2}\right)$

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<sup>1</sup>[https://en.wikipedia.org/wiki/Lp\\_space#Relations\\_between\\_p-norms](https://en.wikipedia.org/wiki/Lp_space#Relations_between_p-norms)

**Fact 7.** *There is a lowerbound for size of sketches for  $L_p$ ,  $p > 2$ :  $\Omega(m^{1-\frac{2}{p}}/\varepsilon^2)$  [see Bar-Yossef, Jayram, Kumar, and Sivakumar 2004]*

**Fact 8.** *Best complexity of sketching is  $\mathcal{O}(m^{1-\frac{2}{p}}/\varepsilon^2)$  – achieved in turnstile model [Indyk and Woodruff 2005].*