

Algorithms for Big Data

Fall Semester 2019

Exercise Set 2

Pseudorandomness: emulating perfect randomness in a predictable manner. Recall a following measure of *quality*

Definition 1 Consider a family of hash functions $\mathcal{H} = \{h : [u] \rightarrow [m]\}^1$. We say that \mathcal{H} is *k-wise independent* if for any distinct $x_1, \dots, x_k \in [u]$ and any (not necessarily distinct) $y_1, y_2, \dots, y_k \in [m]$ there is

$$\Pr_{h \in \mathcal{H}}(h(x_1) = y_1 \wedge \dots \wedge h(x_k) = y_k) = \Theta(m^{-k}).$$

Informally: those hash-functions are indistinguishable from perfectly random hashing when evaluated simultaneously at k values.

We claim that (i) k -wise independence is good enough to "fool" algorithms into behaving as if provided with perfect randomness and (ii) this type of pseudo-randomness can be stored using small space.

Exercise 1:

Let $p > u$ be prime number. Let $\mathbf{a} = a_0, \dots, a_{k-1}$ be vector of coefficients. Let $h_{\mathbf{a}} : [u] \rightarrow [m]$ be defined as $h_{\mathbf{a}}(x) = [(\sum_{i=0}^{k-1} a_i x^i) \bmod p] \bmod m$. Show that $\mathcal{H} = \{h_{\mathbf{a}} \mid a_0, \dots, a_{k-1} \in [p]\}$ is k -wise independent.

Hint:

Polynomial of degree $k - 1$ in \mathbb{Z}_p is uniquely defined by its value on k distinct points.

Exercise 2:

Show that families of hash-functions from previous exercise are not $(k + 1)$ -wise independent.

Exercise 3:

Show a lower-bound of $\Omega(k \log m)$ bits necessary to represent (store) a hash-function from k -wise independent hash-function family. How much space do we need to represent perfectly random hash-function?

Exercise 4:

Let X_1, X_2, \dots, X_n be pairwise independent random variables. Show that $\text{Var}[\sum_i X_i] = \sum_i \text{Var}[X_i]$.

¹ $[u] = \{0, 1, \dots, u - 1\}$ is called an *universe*.

Exercise 5:

Missing part of Morris' algorithm analysis: show inductively that $\mathbb{E} \left[(2^{X_n})^2 \right] = \frac{3}{2}n^2 + \frac{3}{2}n + 1$.

Exercise 6:

(2 pts)

Consider following idea for concentrating Flajolet-Martin approach. Let $r_1, r_2, \dots, r_n \in [0, 1]$ be picked uniformly and independently at random, and let X_k be k -th smallest value among r_1, \dots, r_n . Find $\mathbb{E}[X_k]$ and $\text{Var}[X_k]$. Use it to derive streaming algorithm for distinct elements (see Bar-Yossef et al. 2002).