

Lecture 5: Dimensionality reduction

Lecturer: *Przemysław Uznański*

Scribe:

1 Dimensionality reduction recap

Two versions of the JL lemma.

Theorem 1 (Distributional JL). *For any integer n , and $0 < \varepsilon, \delta < \frac{1}{2}$, and $m = \mathcal{O}(\log(1/\delta)/\varepsilon^2)$ there is distribution \mathcal{D} over matrices $\mathbb{R}^{m \times n}$ such that for every $x \in \mathbb{R}^n$ such that $\|x\|_2 = 1$:*

$$\Pr_{\Pi \sim \mathcal{D}} (|\Pi \cdot x|_2^2 - 1 > \varepsilon) < \delta$$

note: we fix the distribution, and it works for all the vectors
implies

Theorem 2 (Metric JL). *For X , a set of n points in dimension n , there exists linear $f : X \rightarrow \mathbb{R}^m$ for $m = \mathcal{O}(\log(n)/\varepsilon^2)$ that preserves distances approximately, that is*

$$\forall_{i,j} (1 - \varepsilon) |x_i - x_j|_2 \leq |f(x_i) - f(x_j)|_2 \leq (1 + \varepsilon) |x_i - x_j|_2.$$

follows from DJL by setting $\delta < 1/n^2$ and taking union bound

we already know that: $N(0, 1) \cdot 1/\sqrt{m}$ iid coefficients work scaled rademacher $+1/-1 \cdot 1/\sqrt{m}$ iid coefficients work

JL is a dimensionality reduction in L2 norm. Problem: applying a single projection takes $\mathcal{O}(nm)$ time.

2 Fast JL [AC09]

Main idea is to structurize the matrix, so the time to apply matrix $\mathcal{O}(n + m \log m)$

$$\Pi = \frac{1}{\sqrt{m}} \cdot S \cdot H \cdot D$$

where

- S is $n \times m$ sampling matrix (each row has single 1 in random position, rows are independent)
- H is a $m \times m$ Fourier matrix or Hadamard matrix (we need $HH = I$ and $\max_{i,j} |H_{i,j}| \leq 1/\sqrt{m}$)
- D is a $m \times m$ $\text{diag}(\sigma)$ where σ is a vector of independent Rademachers

D applies in time $\mathcal{O}(m)$, H applies in time $\mathcal{O}(m \log m)$, S applies in time $\mathcal{O}(n)$

Theorem 3. *For $m = \mathcal{O}(\log(1/\delta) \cdot \log(n/\delta) \cdot \varepsilon^{-2})$ and $\|x\|_2 = 1$ we have*

$$\Pr_{\Pi} (|\Pi \cdot x|_2^2 - 1 > \varepsilon) < \delta$$

We will need:

Theorem 4 (Khintchine inequality). *For any $p \geq 1$, $x \in \mathbb{R}^n$ and (σ_i) independent Rademacher,*

$$\left(\mathbb{E} \left| \sum_i x_i \sigma_i \right|^p \right)^{1/p} \leq \mathcal{O}(\sqrt{p}) \|x\|_2$$

Theorem 5 (Chernoff bound). *X_1, \dots, X_n are independent random variables and $X_i \in [0, \tau]$. Let $\mu = \mathbb{E} \sum_i X_i$. Then*

$$\Pr \left[\left| \sum_i X_i - \mu \right| > \varepsilon \mu \right] < 2 \exp \left(-\frac{\varepsilon^2 \mu}{2\tau} \right)$$

Proof of main result. Denote $y = \frac{1}{\sqrt{n}} H D x$ and $z = \sqrt{\frac{n}{m}} \cdot S \cdot y$.

First ur goal is to show bound on $\|y\|_\infty$.

$$y_i = \left(\frac{1}{\sqrt{n}} H D x \right)_i = \sum_{j=1}^n \sigma_j \frac{1}{\sqrt{n}} \gamma_{i,j} x_j = \sigma \odot w^{(i)}$$

where $w^{(i)}$ is a vector $w_j^{(i)} = \frac{1}{\sqrt{n}} \gamma_{i,j} x_j$.

First, $\|y\|_2 = \|x\|_2 = 1$ since (normalized) Hadamard transform preserves L_2 norms, but one can also prove that $\mathbb{E}|y_i|^2 = \mathbb{E}|\sigma \odot w^{(i)}|^2 = \|w^{(i)}\|_2^2 = \sum_j \left(\frac{1}{\sqrt{n}} \gamma_{i,j} x_j \right)^2 = \frac{1}{n} \|x\|_2^2 = \frac{1}{n}$ where we only used that H has $-1/+1$ coefficients and that σ is Rademacher.

By Khintchine, and using length of x and fact that $\gamma_{i,j} \in \{-1, +1\}$, for some absolute constant C :¹

$$\mathbb{E}|y_i|^p \leq \left(C \sqrt{p} \|w^{(i)}\|_2 \right)^p = \left(\sqrt{\frac{\mathcal{O}(p)}{n}} \right)^p$$

by Markov's inequality, for any β

$$\Pr \left[|y_i| \geq \sqrt{\frac{\mathcal{O}(p)}{n}} \cdot \left(\frac{2n}{\delta} \right)^{1/p} \right] = \Pr \left[|y_i|^p \geq \left(C \cdot \sqrt{\frac{p}{n}} \right)^p \cdot \frac{2n}{\delta} \right] \leq \frac{\delta}{2n}$$

Optimizing, the term

$$\varphi(p) = \sqrt{\frac{\mathcal{O}(p)}{n}} \cdot \left(\frac{2n}{\delta} \right)^{1/p} \sim \exp \left(\mathcal{O}(1) + \frac{1}{2} \ln p + \frac{1}{p} \ln \left(\frac{2n}{\delta} \right) \right)$$

minimizes when $p \approx 2 \ln \left(\frac{2n}{\delta} \right)$, so then $\varphi_{\min}(p) = \sqrt{\frac{\mathcal{O}(p)}{n}} \cdot \sqrt{e}$ and so

$$\Pr \left[|y_i| \geq \mathcal{O} \left(\sqrt{\frac{\ln(2n/\delta)}{n}} \right) \right] \leq \frac{\delta}{2n},$$

and taking the union bound

$$\Pr \left[\|y\|_\infty \geq \mathcal{O} \left(\sqrt{\frac{\ln(2n/\delta)}{n}} \right) \right] \leq \frac{\delta}{2}.$$

¹We will denote with C, C', C'', \dots an absolute constants.

So in the following we condition on the event that $\|y\|_\infty = \mathcal{O}\left(\sqrt{\frac{\ln(2n/\delta)}{n}}\right)$.

Now consider $X_i = (z_i)^2$ as a random variable. We have the following:

$$\mathbb{E}X_i = \mathbb{E}(z_i)^2 = \frac{n}{m} \cdot \frac{\|y\|_2^2}{n} = \frac{1}{m}.$$

And from bound on $\|y\|_\infty$ there is $X_i \leq \frac{n}{m} \cdot \mathcal{O}\left(\frac{\ln(2n/\delta)}{n}\right) = \mathcal{O}\left(\frac{\ln(2n/\delta)}{m}\right) = \tau$.

We now apply Chernoff bound, with $\mu = 1$ (keep in mind that $\sum_i X_i = \|z\|_2^2$)

$$\Pr\left[\left|\sum_i X_i - 1\right| > \varepsilon\right] < 2\exp\left(-\frac{\varepsilon^2}{2\tau}\right) = 2\exp\left(-\frac{\varepsilon^2 m}{\mathcal{O}(\ln(2n/\delta))}\right)$$

Now, usually when applying Chernoff bound, $\tau = 1$ and then it is enough to set $m = \mathcal{O}(\varepsilon^{-2} \log(1/\delta))$ to have the probability in the bound be δ . Unfortunately in our case, $\tau \gg 1$ so we have to offset for the log term in the denominator. So we need to set $m = \mathcal{O}(\varepsilon^{-2} \log(1/\delta) \log(n/\delta))$ so we can bound the probability in the Chernoff by $\delta/2$.

Taking union bound over both $\delta/2$ failure probability finishes the proof. \square

Using this approach, this dependency is roughly optimal (that is, we are losing one log vs. optimal JL). However, one trick to reduce ε to optimal at the cost of slower runtime is to apply $m' \times m$ "naive" JL at the end of the chain of matrices, for $m' = \mathcal{O}(\log(1/\delta)\varepsilon^{-2})$. This adds $\mathcal{O}(m' \cdot m) = \mathcal{O}(\varepsilon^{-4} \log^2(1/\delta) \log(n/\delta))$ to the running time though.

3 Sparse JL [DKS10]

Motivation: if x is sparse (that is, $\|x\|_0$ is small), we expect time proportional to $\|x\|_0$.

We consider distributions \mathcal{D} of matrices such that:

- Each column has only s non-zero elements, for some s . (Either deterministically or in expectation.)
- They still provide good dimensionality reduction.

The time to compute Πx is then $s \cdot \|x\|_0$, since each column determines "where" each x_i contributes and can be processed in $\mathcal{O}(s)$ time.

3.1 Dasgupta et al. construction

$s = \mathcal{O}(\varepsilon^{-1} \log(1/\delta) \log^2(m/\delta))$,

$h : [sn] \rightarrow [m]$ be a random hash function, and let $H \in \{-1, 0, 1\}^{m \times sn}$ be such that $H_{ij} = \delta_{i, h(j)} r_j$. (all r are indep. Rademacher). $P \in \{0, 1\}^{sn \times n}$ be such that

$$P_{i,j} = \begin{cases} 1 & \text{for } (j-1)s + 1 \leq i \leq js \\ 0 & \end{cases}$$

Intuition: P creates s copies of each element of input, H hashes each element (after duplication) into $[m]$ together with $+1/-1$ coef. This can be evaluated implicitly without expanding the matrices.

Theorem 6. $\Pi = \frac{1}{\sqrt{s}} H P$ has JL guarantees. Πx can be evaluated in time $\mathcal{O}(s\|x\|_0)$.

We skip the proof.

3.2 Kane, Nelson construction

$s = \mathcal{O}(\varepsilon m) = \mathcal{O}(\varepsilon^{-1} \log(1/\delta))$

Construction 1: matrix $m \times n$, where in each column we place s random $-1/+1$ (sample without replacement), normalized with $\frac{1}{\sqrt{s}}$ coef.

Construction 2: group each column into s blocks, each of size m/s . Pick in each block one $-1/+1$, normalize with $\frac{1}{\sqrt{s}}$ coef.

Construction 2 is effectively the same as **CountSketch**. The proof that it works shows that analysis using more than just 2-independence can show a very good concentration (CountSketch uses median, here we can use average).

4 Missing proofs

Proof of Khintchine Inequality. For any variable X , $\mathbb{E}[|X|^p]^{1/p}$ is increasing with p (see: generalized average inequality), so we can round-up p to even integer. Consider $g_i \sim \mathcal{N}(0, 1)$. Expand $\mathbb{E}[(\sum_i \sigma_i x_i)^p]$ into sum of monomials. Any monomial with odd-exponents vanishes. Similarly in $\mathbb{E}[(\sum_i g_i x_i)^p]$. For any even-exponents α_1, \dots , $\mathbb{E} \prod_i \sigma_i^{\alpha_i} = 1$ while $\mathbb{E} \prod_i g_i^{\alpha_i} \geq 1$, so the gaussian case dominates the rademacher case.²

But $\sum_i g_i x_i$ is itself normal variable $\mathcal{N}(0, \|x\|_2^2)$, so

$$\mathbb{E}[(\sum_i \sigma_i x_i)^p] \leq \mathbb{E}[(\sum_i g_i x_i)^p] = (p-1)!! \|x\|_2^p$$

Asymptotically, $((p-1)!!)^{1/p} \approx (2^{p/2} \cdot (p/2)!)^{1/p} \approx \sqrt{2} \cdot \left(\frac{(p/2)^{p/2}}{e^{p/2}}\right)^{1/p} = \sqrt{\frac{p}{e}}$, so the hidden constant in Khintchine inequality is $\mathcal{O}(\sqrt{p})$. □

References

- [AC09] Nir Ailon and Bernard Chazelle. The fast johnson–lindenstrauss transform and approximate nearest neighbors. *SIAM J. Comput.*, 39(1):302–322, 2009.
- [DKS10] Anirban Dasgupta, Ravi Kumar, and Tamás Sarlós. A sparse johnson: Lindenstrauss transform. In Leonard J. Schulman, editor, *Proceedings of the 42nd ACM Symposium on Theory of Computing, STOC 2010, Cambridge, Massachusetts, USA, 5-8 June 2010*, pages 341–350. ACM, 2010.

²Useful fact is that for $g \sim \mathcal{N}(0, 1)$ and even p , $\mathbb{E}[g^p] = (p-1)!!$.