

## Lecture 10: Compressed Sensing

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## 1 Overview

A version of sparse recovery: given linear measurements, recover the signal. In general impossible if  $m < n$ , but is possible if we make additional assumptions.

**Definition 1.** Given  $A \in \mathbb{R}^{m \times n}$ , and  $y = Ax \in \mathbb{R}^m$ , recover  $x \in \mathbb{R}^n$  given that  $x$  is  $k$ -sparse.

Rephrasing problem:

$$\begin{aligned} & \text{minimize} \quad \|x'\|_0 \\ & \text{subject to} \quad Ax' = Ax \end{aligned}$$

(this is hard problem, NP-hard?)

The approach: solve the following program

$$\begin{aligned} & \text{minimize} \quad \|x'\|_1 \\ & \text{subject to} \quad Ax' = Ax \end{aligned}$$

The insight is that signals with small  $\|x'\|_1$  should be sparse (the weight of  $x'$  is concentrated on a few “optimal” coordinates). Particularly, optimizing  $\|x'\|_2$  would not work.

(Adding noise: subject to  $\|Ax - Ax'\|_2 \leq \varepsilon$ .)

The program can be expressed as LP, with  $n^{\Theta(1)}$  algorithm.

**Theorem 2.** Let  $m = \Theta(k \log(n/k))$ . Pick  $A \in \mathbb{R}^{m \times n}$  with entries  $\mathcal{N}(0, 1)$ . W.h.p. the following holds for some  $C$ : the output  $x'$  of the LP satisfies

$$\|x' - x\|_2 \leq \frac{C}{\sqrt{k}} \min_{\|x''\|_0 \leq k} \|x - x''\|_1. \quad (1)$$

In fact any JL-type distribution works.

### 1.1 Restricted Isometry Property

**Definition 3.** We say that matrix  $A$  satisfies  $(k, \varepsilon)$ -RIP if for any  $k$ -sparse  $x$  there is

$$\|Ax\|_2 = (1 \pm \varepsilon)\|x\|_2.$$

**Theorem 4.** JL matrix with  $m = \Theta(k \log(n/k))$  satisfies  $(k, 1/3)$ -RIP.

*Proof.* Recall proof of subspace embeddings. Here as well it is enough to show the property for  $\|x\|_2 = 1$ . For any  $K \subseteq [n]$ ,  $|K| = k$ , we take  $B_K = \{x : \|x\|_2 = 1 \wedge \text{support}(x) \subseteq K\}$ , that is vectors spanned on coordinates in  $K$  of length 1.

Pick  $(1/7)$ -net  $N_K$  for  $B_K$  of size  $2^{\mathcal{O}(k)}$ . For any  $x \in B_K$  there is  $x = \sum_{i \geq 0} \alpha_i x_i$  where  $\alpha_i \leq 7^{-i}$  and  $x_i \in B_K$ , and  $\alpha_0 = 1$ .

For  $A$  properly set (JL with constant  $1/7$ ), there is

$$\|Ax\|_2 = \|A \sum_{i \geq 0} \alpha_i x_i\|_2 \leq \sum_i \alpha_i \|Ax_i\|_2 \leq \sum_i \frac{8}{7} \cdot 7^{-i} \leq \frac{4}{3}$$

and

$$\|Ax\|_2 = \|A \sum_{i \geq 0} \alpha_i x_i\|_2 \geq \|Ax_0\|_2 - \sum_{i \geq 1} \alpha_i \|Ax_i\|_2 \geq \frac{6}{7} - \sum_{i \geq 1} \frac{8}{7} 7^{-i} = \frac{6}{7} - \frac{4}{3} \cdot \frac{1}{7} = \frac{2}{3}$$

How large the  $m$  needs to be? It needs to work for union of all  $1/7$ -nets. We have  $|\bigcup N_K| \leq \binom{n}{k} \cdot 2^{\mathcal{O}(k)} = 2^{\mathcal{O}(k \log(n/k))}$ .  $\square$

## 1.2 Basis pursuit

**Theorem 5.**  $(4k, 1/3)$ -RIP implies (1).

*Proof.* Denote  $h = x' - x$ . Let us reorder  $x$  so that  $x_1, \dots, x_k$  are the largest magnitude coefficients, and then so that  $h_{k+1}, \dots$  are sorted (ignoring that  $h_1, \dots, h_k$  might be arbitrary). Denote  $A = \{1, \dots, k\}$  and  $A_j = \{k + 5jk + 1, \dots, 6k + 5jk\}$  for  $j \geq 0$ .

Denote  $\eta = \min_{\|x''\|_0 \leq k} \|x - x''\|_1 = \|x_{A^c}\|_1$ .

First, bound:

$$\|x\|_1 \geq \|x'\|_1 = \|x + h\|_1 = \|x_A + h_A\|_1 + \|x_{A^c} + h_{A^c}\|_1 \geq \|x_A\|_1 - \|h_A\|_1 - \|x_{A^c}\|_1 + \|h_{A^c}\|_1$$

so

$$\|h_{A^c}\|_1 \leq \|h_A\|_1 + 2\|x_{A^c}\|_1 = \|h_A\|_1 + 2\eta \quad (2)$$

Then for any vector  $y$  with sorted coordinates,  $|y_i| \leq \|y\|_1/i$ . So

$$\|h_{(AA_0)^c}\|_2^2 \leq \|h_{A^c}\|_1^2 \sum_{j=5k}^{\infty} \frac{1}{j^2} \leq \|h_{A^c}\|_1^2 \frac{1}{\mathcal{O}(k)}$$

so, using (2)

$$\|h_{(AA_0)^c}\|_2 = \mathcal{O}\left(\frac{\|h_A\|_1 + \eta}{\sqrt{k}}\right) \quad (3)$$

Then

$$\begin{aligned} 0 &= \|A(x' - x)\|_2 = \|Ah\|_2 \geq \|Ah_{AA_0}\|_2 - \left\| \sum_{j \geq 1} Ah_{A_j} \right\|_2 \\ &\geq \|Ah_{AA_0}\|_2 - \sum_{j \geq 1} \|Ah_{A_j}\|_2 \\ &\geq \frac{2}{3} \|h_{AA_0}\|_2 - \frac{4}{3} \sum_{j \geq 1} \|h_{A_j}\|_2 \end{aligned}$$

Each term in  $h_{A_{j+1}}$  is smaller than average term in  $h_{A_j}$ , so  $\|h_{A_{j+1}}\|_2^2 \leq (5k) \frac{\|h_{A_j}\|_1^2}{(5k)^2} = \frac{\|h_{A_j}\|_1^2}{5k}$ . So

$$\sum_{j \geq 1} \|h_{A_j}\|_2 \leq \sum_{j \geq 0} \frac{\|h_{A_j}\|_1}{\sqrt{5k}} = \frac{\|h_{A^c}\|_1}{\sqrt{5k}} = \frac{\|h_A\|_1 + 2\eta}{\sqrt{5k}} \leq \sqrt{\frac{k}{5k}} \|h_A\|_2 + \frac{2}{\sqrt{5k}} \eta \quad (4)$$

Combining we have

$$\frac{2}{3} \|h_{AA_0}\|_2 \leq \frac{4}{3\sqrt{5}} \|h_A\|_2 + \mathcal{O}\left(\frac{\eta}{\sqrt{k}}\right)$$

so since  $\|h_A\|_2 \leq \|h_{AA_0}\|_2$  and  $\frac{2}{3} < \frac{4}{3\sqrt{5}}$ , there is

$$\|h_{AA_0}\|_2 = \mathcal{O}\left(\frac{\eta}{\sqrt{k}}\right)$$

and this bound also applies in the form of

$$\frac{\|h_A\|_1}{\sqrt{k}} = \mathcal{O}\left(\frac{\eta}{\sqrt{k}}\right).$$

By triangle inequality

$$\|h\|_2 \leq \|h_{AA_0}\|_2 + \|h_{(AA_0)^c}\|_2 = \mathcal{O}\left(\frac{\eta}{\sqrt{k}}\right)$$

□

### 1.3 Iterative Hard Thresholding

Assume  $y = Ax + e$ , where  $e$  is some noise, and  $x$  is  $k$ -sparse.

The algorithm works as follow.

- $x^0 = 0$ .
- $a^{i+1} = x^i + A^T(y - Ax^i)$
- $x^{i+1} = H_k[a^{i+1}]$  where  $H_k$  is projection onto top- $k$  coefficients.

We expect  $x^t$  to converge onto  $x \pm \mathcal{O}(e)$ , in logarithmic number of iterations. Proof: denote  $r^i = x - x^i$ . Our goal is to show that  $r^i$  decreases by a constant factor at each step.

Denote  $S = \text{support}(x)$  and  $S^i = \text{support}(x^i)$ , and  $B^i = S \cup S^i$ .

We assume we have  $(3k, \varepsilon)$ -RIP for sufficiently small  $\varepsilon$  (like, small constant).

Properties: ( $A_S$  denotes matrix with only columns in  $S$  kept, and all others zeroed out).

**Lemma 6.** For  $S$  such that  $|S| \leq 3k$ :

$$\|(I - A_S^T A_S)x_S\|_2 \leq \varepsilon \|x_S\|_2$$

*Proof.* Follows from  $A_S$  being subspace embedding for  $3k$ -sparse vectors. □

**Lemma 7.** for any disjoint  $S, S'$  with  $|S| + |S'| \leq 3k$

$$\|A_S^T A_{S'} x_{S'}\|_2 \leq \varepsilon \|x_{S'}\|_2$$

*Proof.*

$$\begin{aligned}\|A_S^T A_{S'}\|_2 &= \sup_{\|y\|=1} \|A_S^T A_{S'} y\|_2 = \sup_{\|x\|=1, \|y\|=1} x^T A_S^T A_{S'} y = \sup_{\|x\|=1, \|y\|=1} \langle A x_S, A y_{S'} \rangle \\ &= \sup_{\|x\|=1, \|y\|=1} (\langle x_S, y_{S'} \rangle \pm \varepsilon) \leq \varepsilon\end{aligned}$$

where we used that  $\langle x_S, y_{S'} \rangle = 0$  and that linear combinations of  $x_S, y_{S'}$  are also sparse.  $\square$

**Lemma 8.**

$$\|A_S^T y\|_2^2 \leq (1 + \varepsilon) \|y\|_2^2$$

*Proof.* Follows from  $\|A_S^T\|_2 = \|A_S\|_2 \leq (1 + \varepsilon)$ , which follows from:

$$\|A_S\|_2 = \sum_{\|x\|_2=1} \|A_S x_S\|_2 \leq \sqrt{1 + \varepsilon} \|x_S\| = \sqrt{1 + \varepsilon}$$

$\square$

Now we bound  $\|r^{i+1}\|_2$ :

$$\|r^{i+1}\|_2 = \|x - x^{i+1}\|_2 = \|x_{B^{i+1}} - x_{B^{i+1}}^{i+1}\|_2 \leq \|x_{B^{i+1}} - a_{B^{i+1}}^{i+1}\|_2 + \|x_{B^{i+1}}^{i+1} - a_{B^{i+1}}^{i+1}\|_2$$

since  $x_{B^{i+1}}^{i+1}$  is the best possible  $k$ -sparse approximation to  $a_{B^{i+1}}^{i+1}$ , and  $x_{B^{i+1}}$  is  $k$ -sparse, we have

$$\begin{aligned}\|r^{i+1}\|_2 &\leq 2\|x_{B^{i+1}} - a_{B^{i+1}}^{i+1}\|_2 \\ &\leq 2\|x_{B^{i+1}} - x_{B^{i+1}}^i - A_{B^{i+1}}^T A r^i - A_{B^{i+1}}^T e\|_2 \\ &\leq 2\|r_{B^{i+1}}^i - A_{B^{i+1}}^T A r^i\|_2 + 2\|A_{B^{i+1}}^T e\|_2 \\ &\leq 2\|(I - A_{B^{i+1}}^T A_{B^{i+1}}) r_{B^{i+1}}^i\|_2 + 2\|A_{B^{i+1}}^T A_{B^i - B^{i+1}} r_{B^i - B^{i+1}}^i\|_2 + 2\sqrt{1 + \varepsilon} \|e\|_2 \\ &\leq 2\varepsilon \|r_{B^{i+1}}^i\|_2 + 2\varepsilon \|r_{B^i - B^{i+1}}^i\|_2 + 2\sqrt{1 + \varepsilon} \|e\|_2 \\ &\leq 4\varepsilon \|r^i\|_2 + 2\sqrt{1 + \varepsilon} \|e\|_2\end{aligned}$$

set  $\varepsilon = 1/8$

$$\leq \frac{1}{2} \|r^i\|_2 + 3\|e\|_2$$