

Lecture 12: Coresets

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Scribe: -

1 Coresets

Setup: given set of $P \subseteq \mathbb{R}^d$, compute $C_P() : \mathbb{R}^d \rightarrow \mathbb{R}$. For example:

- *MEB*, a minimal enclosing ball: $C_P(o)$ is a radius of minimal enclosing ball for set P centered at o .

$|P| = n$ is large, so storing it explicitly is out of the question. Observe that for fixed o , a single $p \in P$ is enough, that is $\forall o \exists p \in P C_P(o) = C_{\{p\}}(o)$. Can we generalize this observation so it works for all $o \in \mathbb{R}^2$ at once?

Definition 1. We say that $S \subseteq P$ is a c -coreset for P if for any o and any $T \subseteq \mathbb{R}^d$ there is:

$$C_{S \cup T}(o) \leq C_{P \cup T}(o) \leq c \cdot C_{S \cup T}(o)$$

Note: this is a stronger definition than just requiring that $f_S(o)$ is a c -approximation to $C_P(o)$ (take $T = \emptyset$).

Observe that in case of *MEB* we have for $A \subseteq B$: $C_A(o) \leq C_B(o)$, so the first inequality is "for free".

Lemma 2 (Merge property). If S is a c -coreset for P , and S' is a c -coreset for P' , then $S \cup S'$ is a c^2 -coreset for $P \cup P'$.

Lemma 3 (Reduce property). If S is a c -coreset for P and P is a c -coreset for Q , then S is a c^2 -coreset for Q .

We sometimes want a stronger property (which for example *MEB* satisfies)

Property 4 (Disjoint merge). If S is a c -coreset for P and S' is a c -coreset for P' and $P \cap P' = \emptyset$, then $S \cup S'$ is a c -coreset for $P \cup P'$.

Exercise: proof for *MEB*.

Theorem 5. Assume that a problem is supported by a $(1 + \alpha)$ -coreset of size $f(\alpha)$, computable in linear space, with disjoint merge property. Then there is a streaming algorithm with $1 + \varepsilon$ guarantee, with space $\mathcal{O}(f(\varepsilon/\log n) \log n)$.

Proof. Sketch of a proof: put stream of n elements into binary tree. Each node stores coreset for the range below it.

For two sibling nodes N_1, N_2 covering sets A_1, A_2 , at level i , and parent node N at level $i + 1$, there is:

- N_1 is $(1 + \alpha)^i$ -coreset for A_1 (and the same for N_2, A_2)

- $N_1 \cup N_2$ is $(1 + \alpha)^i$ -coreset for $A_1 \cup A_2$
- N is constructed as $(1 + \alpha)$ -coreset for $N_1 \cup N_2$
- thus N is $(1 + \alpha)^{i+1}$ -coreset for $A_1 \cup A_2$

As a end-result, we have in the root $(1 + \alpha)^{\log n}$ -coreset for whole input. Selecting $\alpha = \mathcal{O}(\varepsilon/\log n)$ is enough. \square

1.1 Coreset for MEB

Construction goes as follow. Choose dense set of directions $\{v_i\}_{i=1}^m$, such that for any other direction u , there is always some v_i such that $\text{angle}(u, v_i) \leq \alpha$: this is $\sim \alpha$ -net on unit-ball (up to trigonometry). We can choose such set of $m = (1/\alpha)^{\mathcal{O}(d)}$ directions.

Claim 6. *For any direction v_i , pick $p_i \in P$ that is extremal in that direction. Set $S = \{p_i\}_{i=1}^m$ is a $(1 + \mathcal{O}(\alpha^2))$ -coreset for P .*

Proof. Pick arbitrary T ($T = \emptyset$ w.l.o.g. in this proof), and P and constructed set S . Fix o . Pick furthest point $x \in P$, and close direction v_i , and maximal in this direction point $p_i \in S$. The angle $x-o-p_i$ is small (at most α), so the stretch is upperbounded by $\frac{1}{\cos \alpha} = 1 + \mathcal{O}(\alpha^2)$. \square

1.2 Coreset for median

Approximate median: given sequence of numbers A of number a_1, \dots, a_n , return a such that $(1/2 \pm \varepsilon)n$ elements in A are smaller/larger than a .

Alternative formulation: find a that minimizes $C_A(a) = \max(|\{i : a_i \geq a\}|, |\{i : a_i \leq a\}|)$.

Coreset of size $1/\varepsilon$: pick every εn element from sorted A . Easy to see that

$$C_{A \cup T}(a) \leq C_{A \cup S}(a) \leq (1 + \varepsilon)C_{A \cup T}(a)$$

Plugging into the theorem, we obtain streaming median computation in space $\mathcal{O}(\log^2 n/\varepsilon)$. In fact this works for any quantile computation (but the error is additive). Improvement: pre-filter and keep only $1/\varepsilon^2$ elements (randomly), so space becomes $\mathcal{O}(\log^2(1/\varepsilon)/\varepsilon)$.

2 Graph algorithms

2.1 Certificates

Graph-theoretic approach, for decision problems.

Definition 7. *For property \mathcal{P} , and a graph G , we say that G' is a strong certificate for G if: for any H , $G \cup H$ is in \mathcal{P} iff $G' \cup H$ is in \mathcal{P} .*

Examples:

- connectivity: any spanning forest
- non-bipartiteness: any spanning forest + single odd-cycle inducing edge
- edge/vertex connectivity

2.2 Spanners

Subgraph approximately preserving distances:

Definition 8. H , an edge subgraph of G , is a t -spanner of G , if for any u, v there is

$$d_H(u, v) \leq t \cdot d_G(u, v)$$

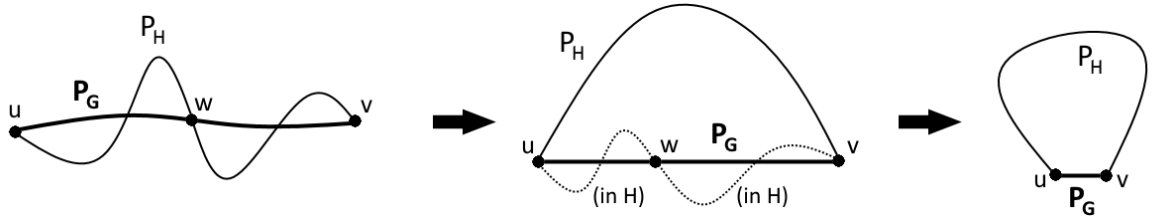
Theorem 9. Any unweighted G contains t -spanner with $\mathcal{O}(n^{1+2/(t-1)})$ edges, and it can be computed in one pass.

Proof. Maintain subgraph. Process edges one-by-one. If an edge causes a cycle of length $\leq t + 1$, do not insert it.

1. This constructs a t -spanner. (Nice easy exercise)

Proof. Let's denote a spanner by H as previously.

- (by contradiction) Assume that H maintained in that way is not a t -spanner. So, there exist a pair of vertices u, v (counterexample) s.t. $d_H(u, v) > td_G(u, v)$. Let's choose some shortest path between u, v in G , P_G and in H , P_H . The proof will have three stages.



- Now we argue that there also exists a counterexample s.t. P_G and P_H don't cross - only common vertices between them are u, v . This follows, because if there is a vertex w s.t. $w \neq u, w \neq v, w \in P_G, w \in P_H$, then we can substitute one of u, v (wlog, u) with w in the counterexample. That's because P_G, P_H are shortest paths between u, v in G, H , so the equalities below are true (if they wouldn't be, there would exist some shorter path):

$$d_H(u, w) + d_H(w, v) = d_H(u, v) > td_G(u, v) = td_G(u, w) + td_G(w, v)$$

so either $d_H(u, w) > td_G(u, w)$ or $d_H(w, v) > td_G(w, v)$ (in other case contradiction by summing sides of opposite (\leq) inequalities). By substituting $u \leftarrow w$ as many times as needed we have $P_G \cap P_H = \{u, v\}$.

- We have that P_G, P_H only have u, v common and now we argue that there also exists a counterexample s.t. $|P_G| = 1$. Similarly as previously, if there exists some w on P_G between u, v , we can substitute one of u, v (wlog, u) with it in the counterexample. This is because from triangle inequality in H we have (please note that there is no longer an equality in the left part because w is not on P_H and we need paths for $d_H(u, w), d_H(w, v)$ that contain some edges from outside of P_H , but the equality in the right part which we need is still true as $w \in P_G$):

$$d_H(u, w) + d_H(w, v) \geq d_H(u, v) > td_G(u, v) = td_G(u, w) + td_G(w, v)$$

then by same argument as previously, either $d_H(u, w) > td_G(u, w)$ or $d_H(w, v) > td_G(w, v)$. By substituting $u \leftarrow w$ as many times as needed, we finally have that u, v are neighbours in G (please note that by such substitution, we can again have intersections between P_G, P_H and we may need to move to the previous step, but $|P_G|$ constantly decreases).

- As we now have a counterexample where $|P_G| = 1$ and $|P_H| > t|P_G| = t$, we arrive at a contradiction: when we were processing the only edge on P_G , we should have added it to H as it didn't cause a cycle of length $\leq t + 1$ - it doesn't cause that now as $d_H(u, v) = |P_H| \geq t + 1$ and in the time of processing of that edge it also didn't as H at that time was a subgraph of the final H .

□

2. The graph is sparse enough:

- Let $d = 2m/n$ be average degree.
- There is subgraph H of G with minimum degree $d' = d/2$: keep removing vertices with degree smaller than d' . We cannot end with no vertices, since then we removed less than m edges.
- H has every vertex of degree at least m/n , and no cycle of length $t + 1$ or less.
- BFS tree of depth $t/2$ has no cycles in H
- $(m/n - 1)^{t/2} \leq |H| \leq n$, which implies bound on m

□