VP160 Recitation Class 4

Week 9

Zixiang Lin

University of Michigan - Shanghai Jiao Tong University Joint Institute

July 10, 2024

Harmonic Oscillator

Simple Harmonic Oscillator

$$\ddot{x} + \omega_0^2 x = 0 \tag{1}$$

Damped Harmonic Oscillator

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0 \tag{2}$$

Driven Harmonic Oscillator

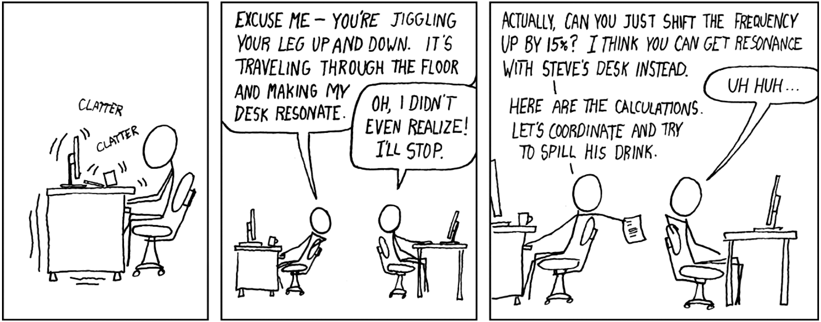
$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = \alpha\cos(\omega_d t) \tag{3}$$

Resonance

Definition

$$\max A = A\Big|_{\omega_d=\sqrt{\omega_0^2-2\beta^2}} = \frac{\alpha}{2\beta\sqrt{\omega_0^2-\beta^2}} \tag{4}$$

$$\lim_{\beta\to 0}\omega_d \to \omega_0, \quad \lim_{\beta\to 0}\max A \to \infty \tag{5}$$



R. Munroe, Resonance. https://xkcd.com/228/.

Wave

Classical Wave Equation

$$\frac{\partial^2 \xi}{\partial x^2} - \frac{\partial^2 \xi}{v_p^2 \partial t^2} = 0 \tag{6}$$

Phase Velocity

$$v_p = \frac{\lambda}{T} = \frac{\omega}{k} = \frac{\omega \lambda}{2\pi} = \sqrt{\frac{T}{\rho}} \text{ (on a string)} \tag{7}$$

Complex Wave Function

$$\tilde{\xi} = \tilde{A} e^{i(kx - \omega t)} \tag{8}$$

Rate of Energy Transmission

$$P_{avg} = \frac{1}{2} \rho v \omega^2 A^2 \tag{9}$$

Normal Mode

Coupled Oscillators

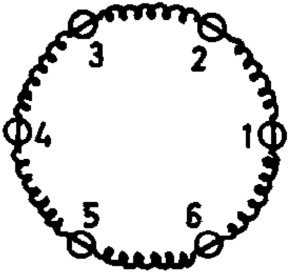
$$\begin{cases} \ddot{x}_1 + a_1\dot{x}_1 + b_1\dot{x}_2 + c_1x_1 + d_1x_2 = 0 \\ \ddot{x}_2 + a_2\dot{x}_1 + b_2\dot{x}_2 + c_2x_1 + d_2x_2 = 0 \end{cases} \tag{10}$$

Standing Waves

$$\xi(x, t) = 2A\sin\left(\frac{2\pi x}{\lambda}\right)\cos(\omega t) \tag{11}$$

Exercise 1

A model of benzene ring useful for some purposes is a wire ring strung with 6 frictionless beads, with springs taut between the beads. The beads each has mass m and the springs all have spring constant K. The ring is fixed in space. Calculate the eigenfrequencies of the normal modes.



Exercise 2

$$\text{Let } \begin{cases} f_1(x,t) = Ae^{-k(x-vt)^2} \\ f_2(x,t) = A\sin[k(x-vt)] \\ f_3(x,t) = \frac{A}{k(x-vt)^2+1} \\ f_4(x,t) = Ae^{-k(kx^2+vt)} \\ f_5(x,t) = A\sin(kx)\cos(kvt)^3 \end{cases} .$$

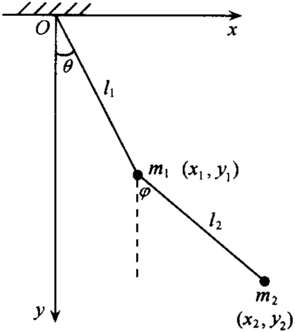
Check whether these functions satisfy the 1D classical wave equation.

Exercise 3

Rewrite the classical wave equation $\frac{\partial^2 \xi(x,t)}{\partial x^2} - \frac{1}{v^2} \frac{\partial^2 \xi(x,t)}{\partial t^2} = 0$ using new variables $\alpha = x + vt, \beta = x - vt$ and show that any solution of this equation may be expressed as a sum of a wave traveling to the left and a wave traveling to the right, i.e. $\xi(x,t) = \xi_1(x+vt) + \xi_2(x-vt)$.

Exercise 4

A particle with mass m_1 is tied to a fixed point O by a light, inextensible rope of length l_1, and another particle with mass m_2 is tied to the first particle by a light, inextensible rope of length l_2. $\theta, \varphi \ll 1$. Find the normal mode of this system and the corresponding angular frequency.



Solution 1

$$-m\omega^2 A_n = K(A_{n-1} + A_{n+1} - 2A_n) \tag{12}$$

Or

$$A_{n-1} + (\frac{m\omega^2}{K} - 2)A_n + A_{n+1} = 0, \quad n = 1, 2, ..., 6 \tag{13}$$

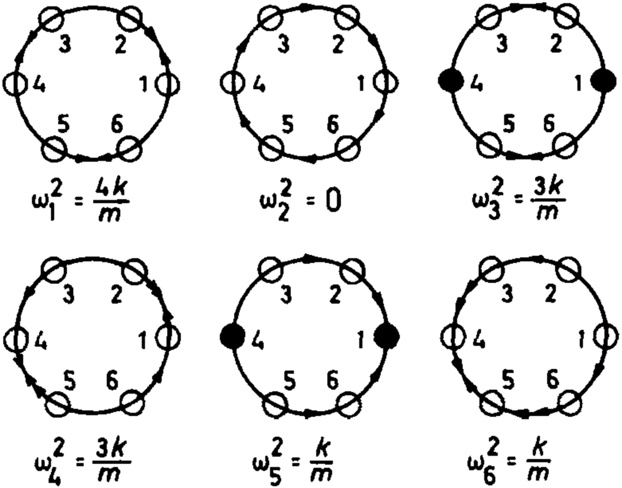
The determinant must vanish

$$\begin{vmatrix} \varepsilon & 1 & 0 & 0 & 0 & 1 \\ 1 & \varepsilon & 1 & 0 & 0 & 0 \\ 0 & 1 & \varepsilon & 1 & 0 & 0 \\ 0 & 0 & 1 & \varepsilon & 1 & 0 \\ 0 & 0 & 0 & 1 & \varepsilon & 1 \\ 1 & 0 & 0 & 0 & 1 & \varepsilon \end{vmatrix} = 0, \quad \varepsilon = \frac{m\omega^2}{K} - 2 \tag{14}$$

$$\varepsilon^6 - 6\varepsilon^4 + 9\varepsilon^2 - 4 = (\varepsilon + 1)^2(\varepsilon - 1)^2(\varepsilon + 2)(\varepsilon - 2) = 0 \tag{15}$$

$$\varepsilon = 2, -2, 1, 1, -1, -1 \Rightarrow \omega^2 = \frac{4K}{m}, 0, \frac{3K}{m}, \frac{3K}{m}, \frac{K}{m}, \frac{K}{m} \tag{16}$$

Solution 1

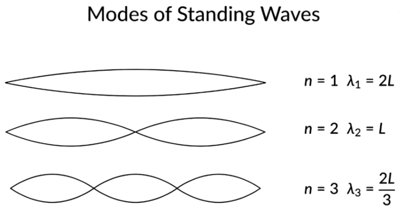


3 2 4 1 5 6
$\omega_1^2 = \frac{4k}{m}$
$\omega_2^2 = 0$
$\omega_3^2 = \frac{3k}{m}$
$\omega_4^2 = \frac{3k}{m}$
$\omega_5^2 = \frac{k}{m}$
$\omega_6^2 = \frac{k}{m}$

Solution 1

How about the situation with more beads?
- Calculating the $n \times n$ determinant is not easy.

Another method is inspired by the standing wave structure.



We can think of it as equivalent to having a standing wave on the ring.

Solution 1

Begin with

$$A_{n-1} + (\frac{m\omega^2}{K} - 2)A_n + A_{n+1} = 0, \quad n = 1, 2, ..., 6 \tag{17}$$

Let

$$A_n = Ae^{i(\omega t - \frac{2\pi}{\lambda} x_n)} = Ae^{i(\omega t - \frac{2\pi}{\lambda} \frac{nL}{6})} \tag{18}$$

Plug in

$$\omega^2 = 2(1 - \cos\left(\frac{\pi L}{3\lambda}\right))\frac{K}{m} \tag{19}$$

$$\lambda = \frac{L}{6}, \frac{2L}{6}, \frac{3L}{6}, \frac{4L}{6}, \frac{5L}{6}, \frac{6L}{6} \Rightarrow \omega^2 = \frac{4K}{m}, 0, \frac{3K}{m}, \frac{3K}{m}, \frac{K}{m}, \frac{K}{m} \tag{20}$$

Solution 2

$$\frac{\partial^2 f_i}{\partial x^2} = \begin{cases} 2(2k(x-vt)^2-1)kAe^{-k(x-vt)^2}, i=1 \\ -k^2 A\sin(kx-kvt), i=2 \\ \frac{2kA(3k(x-vt)^2-1)}{(k(x-vt)^2+1)^3}, i=3 \\ 2k^2A(2k^2x^2-1)e^{-k(kx^2+vt)}, i=4 \\ -k^2A\sin(kx)\cos^3(kvt), i=5 \end{cases}$$

$$\frac{\partial^2 f_i}{\partial t^2} = \begin{cases} 2(2k(x-vt)^2-1)kv^2Ae^{-k(x-vt)^2}, i=1 \\ -k^2v^2 A\sin(kx-kvt), i=2 \\ \frac{2kv^2A(3k(x-vt)^2-1)}{(k(x-vt)^2+1)^3}, i=3 \\ k^2v^2Ae^{-k(kx^2+vt)}, i=4 \\ 3k^2v^2A\sin(kx)\cos(kvt)(1-3\sin^2(kvt)), i=5 \end{cases} \tag{21}$$

1D classical wave equation

$$\frac{\partial^2 f_i}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 f_i}{\partial t^2} = 0 \tag{22}$$

Obviously, f_1, f_2, f_3 satisfy the 1D classical wave equation, f_4, f_5 does not.

Solution 3

$$x = \frac{\alpha + \beta}{2}, y = \frac{\alpha - \beta}{2} \tag{23}$$

$$\frac{\partial \xi}{\partial x} = \frac{\partial \xi}{\partial \alpha} + \frac{\partial \xi}{\partial \beta}, \frac{\partial \xi}{\partial t} = v(\frac{\partial \xi}{\partial \alpha} - \frac{\partial \xi}{\partial \beta}) \tag{24}$$

$$\frac{\partial^2 \xi}{\partial x^2} = \frac{\partial^2 \xi}{\partial \alpha^2} + \frac{\partial^2 \xi}{\partial \beta^2} + 2\frac{\partial^2 \xi}{\partial \alpha \partial \beta}, \frac{\partial^2 \xi}{\partial t^2} = v^2(\frac{\partial^2 \xi}{\partial \alpha^2} + \frac{\partial^2 \xi}{\partial \beta^2} - 2\frac{\partial^2 \xi}{\partial \alpha \partial \beta}) \tag{25}$$

Therefore,

$$\frac{\partial^2 \xi}{\partial \alpha \partial \beta} = 0 \tag{26}$$

Let $\xi(\alpha, \beta) = \xi_1(\alpha) + \xi_2(\beta)$, $\frac{\partial^2 \xi}{\partial \alpha \partial \beta} = 0$ since $\frac{\partial^2 \xi_1(\alpha)}{\partial \alpha \partial \beta} = \frac{\partial^2 \xi_2(\beta)}{\partial \alpha \partial \beta} = 0$.

Solution 4

$$m_1 l_1 \ddot{\theta} = -m_1 g\theta + m_2 g(\varphi - \theta) \tag{27}$$

$$m_2 l_2 \ddot{\varphi} = -m_2 g\varphi - m_2 l_1 \ddot{\theta} \tag{28}$$

$$\begin{cases} \ddot{\theta} + \frac{m_1+m_2}{m_1}\frac{g}{l_1}\theta - \frac{m_2}{m_1}\frac{g}{l_1}\varphi = 0 \\ \frac{l_1}{l_2}\ddot{\theta} + \ddot{\varphi} + \frac{g}{l_2}\varphi = 0 \end{cases} \tag{29}$$

Consider the form $\theta = A_\theta e^{i\omega t}, \varphi = A_\varphi e^{i\omega t}$,

$$\begin{cases} (-\omega^2 + \frac{m_1+m_2}{m_1}\frac{g}{l_1})A_\theta - \frac{m_2}{m_1}\frac{g}{l_1}A_\varphi = 0 \\ -\omega^2\frac{l_1}{l_2}A_\theta + (-\omega^2 + \frac{g}{l_2})A_\varphi = 0 \end{cases} \tag{30}$$

$$\begin{vmatrix} -\omega^2 + \frac{m_1+m_2}{m_1}\frac{g}{l_1} & -\frac{m_2}{m_1}\frac{g}{l_1} \\ -\omega^2\frac{l_1}{l_2} & -\omega^2 + \frac{g}{l_2} \end{vmatrix} = 0 \tag{31}$$

$$\omega^4 - \frac{(m_1+m_2)(l_1+l_2)g}{m_1 l_1 l_2}\omega^2 + \frac{(m_1+m_2)g^2}{m_1 l_1 l_2} = 0 \tag{32}$$

$$\omega = \{\frac{g}{2m_1 l_1 l_2}[(m_1+m_2)(l_1+l_2) \pm \sqrt{(m_1+m_2)[m_2(l_1+l_2)^2 + m_1(l_1-l_2)^2]}]\}^{1/2} \tag{33}$$

Overview

- Dynamic Problem-Solving
- 3 Days: Problem $\Longrightarrow$ Paper
- Assumptions + Physical Model + Numerical Results + Conclusion
- Figures are important
- Search articles though Google Scholar with keywords
- Working together with a clear division of labor
- Trust your model and explain the results
- **Honor Code**

Example

Table Tennis Ball Sizes[a]

[a] The Problem B in the 2012 University Physics Competition.

In the year 2000, the International Table Tennis Federation changed the official ball diameters from 38mm to 40mm. The purpose of this was to increase the effects of air resistance, and slow down the game, in order to make it more fun to watch as a televised spectator sport. If the diameter was increased further, would this make the game an even better spectator sport? What would be the best ball diameter, in order to make the game as fun to watch as possible?

Example

Thinking Route

Fun $\Longrightarrow$ Best Time $\Longrightarrow$ Best Velocity $\Longrightarrow$ Best Diameter

Dynamic Analysis

$$\begin{aligned}\vec{F} &= \vec{F}_{gravity} + \vec{F}_{drag} + \vec{F}_{Magnus} \\ &= m\vec{g} - \frac{C_d \rho A}{2}|\vec{v}|\vec{v} + 4\pi C_M r^3 \rho \vec{\omega} \times \vec{v}\end{aligned} \tag{34}$$

Check `http://www.uphysicsc.com/2012-GM-B-414.PDF` for details.

Thanks for listening!

References

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