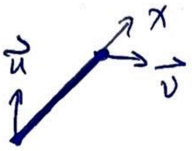
VP160 HW5 Solutions

Solution 1.

Method I: velocity of x: $\vec{u} + \frac{\vec{v}-\vec{u}}{l} x$.



$$T = \frac{1}{2}\int_0^l \|\vec{u} + \frac{\vec{v}-\vec{u}}{l} x\|^2 \frac{m}{l}\,dx$$

$$= \frac{1}{6} m(\vec{u}\cdot\vec{u} + \vec{u}\cdot\vec{v} + \vec{v}\cdot\vec{v}).$$

Method II: $\vec{v_c} = \frac{1}{2}(\vec{u}+\vec{v})$

$$\omega = \frac{1}{l\sin\alpha}|\vec{v}-\vec{u}|, \quad \alpha = \angle \vec{\omega}, \vec{l}$$

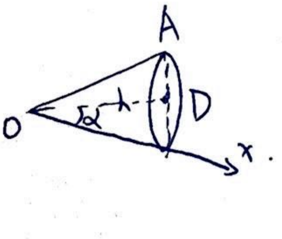
$$I_c = \frac{1}{12} m l^2 \sin^2\alpha.$$

$$T = \frac{1}{2} m \vec{v_c}^2 + \frac{1}{2} I_c \vec{\omega}^2$$

$$= \frac{1}{6} m(\vec{u}\cdot\vec{u} + \vec{u}\cdot\vec{v} + \vec{v}\cdot\vec{v}).$$

Solution 2.

Method I: Assume the angular velocity of the rigid body is ω.



$$\vec{\omega} = -\omega\hat{i}.$$

$$\vec{v_D} = \vec{\omega}\times\vec{OD} = \omega h\sin\alpha\,\hat{j}.$$

Since the angular velocity of OD is $\Omega\hat{k}$.

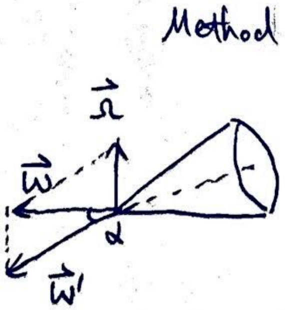
$$\vec{v_D} = \Omega\hat{k}\times\vec{OD} = \Omega h\cos\alpha\,\hat{j} \Rightarrow \omega = \Omega\cot\alpha.$$

Then, $\vec{v_A} = \vec{\omega}\times\vec{OA} = 2\Omega h\cos\alpha\,\hat{j}.$

$$\vec{a_A} = \frac{d\vec{\omega}}{dt}\times\vec{OA} + \vec{\omega}\times(\vec{\omega}\times\vec{OA})$$

$$= -2\Omega^2 h\cos\alpha\,\hat{i} - \Omega^2 h\csc\alpha\,\hat{k}.$$

Method II: Thinking of relative motion,



$$\vec{v_A} = \vec{\omega}'\times\vec{OA} + \vec{\Omega}\times\vec{OA}.$$

$$\vec{a_A} = \vec{\omega}'\times(\vec{\omega}'\times\vec{OA}) + \vec{\Omega}\times(\vec{\Omega}\times\vec{OA}) + 2\vec{\Omega}\times(\vec{\omega}\times\vec{OA})$$

$$\vec{\Omega} = \Omega\hat{k}, \quad \vec{\omega}' = -\Omega\csc\alpha(\cos\alpha\,\hat{i} + \sin\alpha\,\hat{k})$$

$$\vec{OA} = h\sec\alpha(\cos 2\alpha\,\hat{i} + \sin 2\alpha\,\hat{k}).$$

Solution 3.

$AO = l\cos\theta$, $OB = l\sin\theta$, $\cos\alpha = \frac{a}{l\sin\theta}$. $\sin\alpha = \frac{\sqrt{l^2\sin^2\theta - a^2}}{l\sin\theta}$

$A(0, 0, l\cos\theta)$, $B(a, \sqrt{l^2\sin^2\theta - a^2}, 0)$, $C(\frac{a}{2}, \frac{1}{2}\sqrt{l^2\sin^2\theta - a^2}, \frac{1}{2}l\cos\theta)$

Since $\vec{F} = 0$. $\vec{M}_A = 0$.

$$N_1 = mg,\quad \vec{AB} \times (-N_2\hat{i} - f\hat{j} + N_1\hat{k}) + \vec{AC} \times (-mg)\hat{k} = 0.$$

$$\Rightarrow \begin{cases} \frac{1}{2}mg\sqrt{l^2\sin^2\theta - a^2} - f\,l\cos\theta = 0. \\ N_2\, l\cos\theta - \frac{1}{2}mga = 0 \\ -af + N_2\sqrt{l^2\sin^2\theta - a^2} = 0. \end{cases}$$

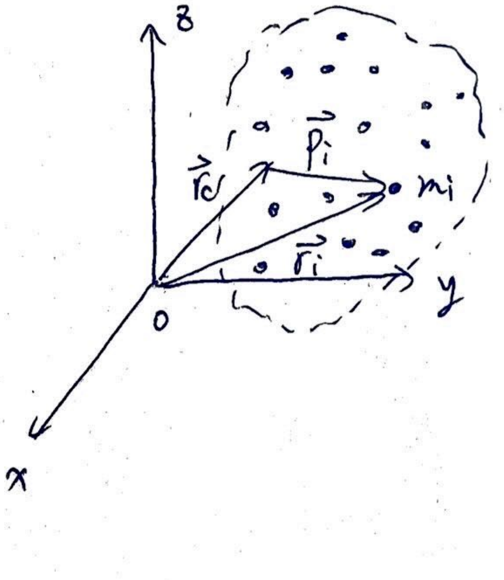
$$\Rightarrow \begin{cases} f = \frac{mg}{2l\cos\theta}\sqrt{l^2\sin^2\theta - a^2}. \\ N_2 = \frac{mga}{2l\cos\theta}. \end{cases}$$

$$\mu\sqrt{N_1^2 + N_2^2} = f \Rightarrow \theta_{max} = \arccos\sqrt{\frac{l^2 - (1+\mu^2)a^2}{(1+4\mu^2)l^2}}.$$

When $B(a, 0, 0) \Rightarrow \theta_{min} = \arcsin\frac{a}{l}$.

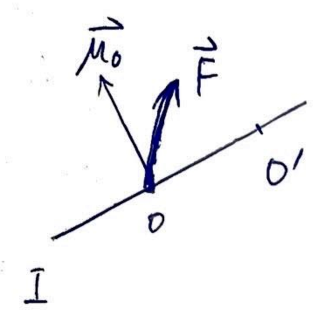
Therefore, $\arcsin\frac{a}{l} \le \theta \le \arccos\sqrt{\frac{l^2 - (1+\mu^2)a^2}{(1+4\mu^2)l^2}}$.

Solution 4.



$$\begin{aligned} T &= \frac{1}{2}\sum_i m_i \|\dot{\vec{r_i}}\|^2 \\ &= \frac{1}{2}\sum_i m_i\left(\|\dot{\vec{r_c}}\|^2 + 2\dot{\vec{r_c}}\cdot\dot{\vec{p_i}} + \|\dot{\vec{p_i}}\|^2\right) \\ &= \frac{1}{2}m\|\dot{\vec{r_c}}\|^2 + \frac{1}{2}\sum_i m_i\|\dot{\vec{p_i}}\|^2. \\ &= \frac{1}{2}v_c^T m v_c + \frac{1}{2}\sum_i m_i\vec{\omega}\cdot(\vec{p_i}\times(\vec{\omega}\times\vec{p_i})) \\ &= \frac{1}{2}v_c^T m v_c + \frac{1}{2}\sum_i m_i\vec{\omega}\cdot(\|\vec{p_i}\|^2\vec{\omega} - (\vec{p_i}\cdot\vec{\omega})\vec{p_i}) \\ &= \frac{1}{2}v_c^T m v_c + \frac{1}{2}\omega^T I\omega. \end{aligned}$$

Solution 5.



A system of forces can be simplified as a force $\vec{F}$ and a torque $\vec{M_o}$.

Draw a perpendicular plane I through O to $\vec{M_o}$.

The lines of action of ~~[illegible]~~ $\vec{M_o}$ and point O are on the paper.

the line of action of $\vec{F}$ is generally not on the paper.

$\vec{OO'} \times \vec{f} = \vec{M_o}$ Assume $\angle \vec{OO'}, \vec{f} = \alpha$.

$OO' \cdot f \sin\alpha = M_o \Rightarrow f = \frac{M_o}{OO' \sin\alpha}$.

Therefore, the system of forces can be $\vec{F} - \vec{f}$ acting on O and a force $\vec{f}$ on a given plane (does not contain O).

Solution 6.

$$m = \rho \int_{-a}^{a} dx \int_{-b\sqrt{1-\frac{x^2}{a^2}}}^{b\sqrt{1-\frac{x^2}{a^2}}} dy \int_{-c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}}^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dz = \frac{4}{3}\rho\pi abc.$$

$$I_1 = \int (y^2+z^2)\, dm = \int (y^2+z^2)\rho\, dx\, dy\, dz$$

$$= 8\rho \int_0^a dx \int_0^{b\cdot\sqrt{1-\frac{x^2}{a^2}}} dy \int_0^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} (y^2+z^2)\, dz = \frac{1}{5} m(b^2+c^2)$$

Similarly, $I_2 = \frac{1}{5} m(c^2+a^2)$, $I_3 = \frac{1}{5} m(a^2+b^2)$.

Solution 7.

$$I = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix}.$$

$$I_{xx} = (\cos\alpha, \sin\alpha, 0)\, I \begin{pmatrix} \cos\alpha \\ \sin\alpha \\ 0 \end{pmatrix} = \frac{1}{5} m(b^2+c^2+(a^2-b^2)\sin^2\alpha)$$

$$I_{yy} = (-\sin\alpha, \cos\alpha, 0)\, I \begin{pmatrix} -\sin\alpha \\ \cos\alpha \\ 0 \end{pmatrix} = \frac{1}{5} m(c^2+a^2+(b^2-a^2)\sin^2\alpha).$$

$$I_{xy} = (\cos\alpha\ \sin\alpha\ 0)\, I \begin{pmatrix} -\sin\alpha \\ \cos\alpha \\ 0 \end{pmatrix} = \frac{1}{5} m(a^2-b^2)\sin\alpha\cos\alpha.$$

$$I_{zz} = \frac{1}{5} m(a^2+b^2)$$

$$I_{xz} = I_{yz} = 0.$$

Solution 8.

$= \frac{1}{4} MR^2 + MR^2$ (parallel axis theorem)

a) $I_{xx} = \frac{5}{4} MR^2 + mR^2 = \frac{5}{2} MR^2$.

$I_{yy} = \frac{1}{4} MR^2 + mR^2 = \frac{3}{2} MR^2$.

$I_{zz} = \frac{5}{2} MR^2 + \frac{3}{2} MR^2 = 4 MR^2$ (perpendicular axis theorem).

$I_{xy} = I_{yx} = -\frac{5}{4} MR^2$.

$I_{xz} = I_{yz} = 0$.

$$\Rightarrow I_A = \begin{pmatrix} \frac{5}{2} & -\frac{5}{4} & 0 \\ -\frac{5}{4} & \frac{3}{2} & 0 \\ 0 & 0 & 4 \end{pmatrix} MR^2 .$$

b) Since $I_{xz} = I_{yz} = 0 \Rightarrow z$ is a principle axis.

$I_3 = I_{zz} = 4MR^2 \quad \vec{e_3} = (0, 0, 1)$.

For the other two,

$$\begin{vmatrix} \frac{5}{2} MR^2 - I & -\frac{5}{4} MR^2 \\ -\frac{5}{4} MR^2 & \frac{3}{2} MR^2 - I \end{vmatrix} = 0 .$$

$\Rightarrow I_1 = 3.346\, MR^2, \quad I_2 = 0.654\, MR^2$.

$$\begin{pmatrix} \frac{5}{2} - I_1/MR^2 & -\frac{5}{4} \\ -\frac{5}{4} & \frac{3}{2} - I_1/MR^2 \end{pmatrix} \begin{pmatrix} e_{1x} \\ e_{1y} \end{pmatrix} = 0 \Rightarrow \vec{e_1} = (0.828, -0.561, 0) .$$

$$\begin{pmatrix} \frac{5}{2} - I_2/MR^2 & -\frac{5}{4} \\ -\frac{5}{4} & \frac{3}{2} - I_2/MR^2 \end{pmatrix} \begin{pmatrix} e_{2x} \\ e_{2y} \end{pmatrix} = 0 \Rightarrow \vec{e_2} = (0.561, 0.828, 0) .$$