PHYS1600J HW6.

Solution 1.

$$\begin{cases} m_1(\ddot{r} - r\dot{\varphi}^2) = -T \\ m_1(r\ddot{\varphi} + 2\dot{r}\dot{\varphi}) = \frac{m_1}{r}\frac{d}{dt}(r^2\dot{\varphi}) = 0 \\ m_2\ddot{z} = m_2 g - T \end{cases}$$

$$\Rightarrow \begin{cases} r^2\dot{\varphi} = r_0\sqrt{\frac{m_2 g r_0}{m_1}} \\ (m_1+m_2)\ddot{r} - \frac{m_2 g r_0^3}{r^3} = -m_2 g \end{cases}$$

let $r^{-3} = r_0^{3}\left(1 - \frac{3\xi}{r_0}\right)$. $\Rightarrow (m_1+m_2)\ddot{\xi} + \frac{3m_2 g}{r_0}\xi = 0$.

Since $\ddot{\xi} = -\ddot{z}$, $\ddot{z} + \frac{3m_2 g}{(m_1+m_2)r_0} z = \text{Const}$.

$$\omega = \sqrt{\frac{3m_2 g}{(m_1+m_2)r_0}}, \quad T = 2\pi\sqrt{\frac{(m_1+m_2)r_0}{3m_2 g}}$$

Solution 2.

Complex equation: $m\ddot{z} + m\omega_0^2 z + \gamma\dot{z} = Ae^{i\omega t}$

Steady-state solution: $z = z_0 e^{i\omega t}$

$$z_0 = \frac{A}{m(\omega_0^2-\omega^2) + i\gamma\omega} = Be^{-i\phi}.$$

$$B = \frac{A}{\sqrt{m^2(\omega_0^2-\omega^2)^2 + \gamma^2\omega^2}}, \quad \phi = \arctan\frac{\gamma\omega}{m(\omega_0^2-\omega^2)}$$

Method I: work done by force $F = \mathrm{Re}(Ae^{i\omega t})$.

$$P = \mathrm{Re}F \cdot \mathrm{Re}\dot{z} = \frac{1}{4}(F^*\dot{z} + F\dot{z}^*)$$

$$\langle P\rangle = \frac{i\omega AB}{4}(e^{-i\varphi} - e^{i\varphi}) = \frac{\omega AB}{2}\sin\varphi = \frac{\omega AB}{2}\frac{B}{A}\gamma\omega$$

$$= \gamma\frac{\omega^2B^2}{2} = \frac{\gamma\omega^2A^2}{2[m^2(\omega_0^2-\omega^2) + \gamma^2\omega^2]}$$

Method II: work done by the dissipative term,

$$\langle P'\rangle = \gamma(\mathrm{Re}\dot{z})^2 = \gamma\frac{\dot{z}\dot{z}^*}{2} = \gamma\frac{\omega^2B^2}{2} \quad \text{(the same)}$$

Solution 3.

(a) $$\begin{cases} m\ddot{x}_1 = k(x_2 - x_1) \\ M\ddot{x}_2 = k(x_1 - 2x_2 + x_3) \\ m\ddot{x}_3 = -k(x_3 - x_2) \end{cases}$$

let $x_1 = A_1 e^{i\omega t}$, $x_2 = A_2 e^{i\omega t}$, $x_3 = A_3 e^{i\omega t}$

$$\begin{cases} (k - m\omega^2)A_1 - kA_2 = 0. \\ -kA_1 + (2k - M\omega^2)A_2 - kA_3 = 0. \\ -kA_1 + (k - m\omega^2)A_3 = 0. \end{cases}$$

$$\begin{vmatrix} k - m\omega^2 & -k & 0 \\ -k & 2k - M\omega^2 & -k \\ 0 & -k & k - m\omega^2 \end{vmatrix} = 0.$$

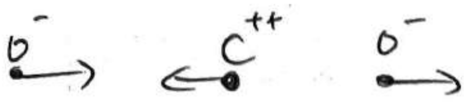
$$\Rightarrow \omega_1 = \sqrt{\frac{k}{m}}, \quad \omega_2 = \sqrt{\frac{(2m+M)k}{mM}}, \quad \omega_3 = 0.$$

(b) Plug in for A_1, A_2, A_3.

The relative amplitudes are

$$\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \text{ for } \omega_1 \quad \text{and} \quad \begin{pmatrix} 1 \\ -\frac{2m}{M} \\ 1 \end{pmatrix} \text{ for } \omega_2.$$

ω_1



ω_2.

Solution 4.

(a) $v = \lambda \nu = 2l \cdot \nu = 2 \times 0.5 \times 200 = 200\ m/s$.

(b)

Before reflection After reflection

(c)

$f_1 = 400 Hz$ $f_2 = 600 Hz$.

Solution 5.

(a) $y = y_0 \sin[\omega(t - \frac{x}{c})]$.

(b) $\dot{y} = \omega y_0 \cos[\omega(t - \frac{x}{c})]$.

$\frac{1}{2}\Delta m \cdot \omega^2 y_0^2 = \frac{1}{2}\rho \omega^2 y_0^2 \Delta x$.

$\bar{E} = \frac{1}{2}\rho \omega^2 y_0^2$.

(c) Power transmitted: $\frac{1}{2}\rho c \omega^2 y_0^2$.

(d) Since $c = \sqrt{\frac{T}{\rho}}$, $T = \rho c^2$.

$F_y(t) = -T\left(\frac{\partial y}{\partial x}\right)_{x=0} = \rho c \omega y_0 \cos(\omega t)$.

Solution 6.

(a) For $n \neq 0$,

$$\ddot{x}_n = -\frac{K}{m}\left[(x_n - x_{n-1}) + (x_n - x_{n+1})\right] = -\frac{K}{m}\left[2x_n - x_{n+1} - x_{n-1}\right).$$

Setting $x_n = Ae^{i(kan-\omega t)}$.

$$\Rightarrow -\omega^2 x_n = -\frac{2K}{m}\left[1-\cos(ka)\right]x_n$$

$$\Rightarrow \omega^2 = \frac{2K}{m}\left[1-\cos(ka)\right].$$

(b) Try the solution .

$$\begin{cases} x_n = \left(Ae^{ikan} + Be^{-ikan}\right)e^{-i\omega t} & n \leq 0. \\ x_n = Ce^{i(kan-\omega t)} & n \geq 0. \end{cases}$$

For $n=0$. $C = A+B$.

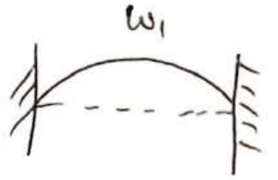
$$\ddot{x}_0 = -\frac{K}{m_0}\left(2x_0 - x_1 - x_{-1}\right)$$

$$\Rightarrow A = \left\{-1 + \frac{im\sin(ka)}{(m-m_0)\left[1-\cos(ka)\right]}\right\}B.$$

$$\left|\frac{B}{A}\right|^2 = \left\{1 + \left(\frac{m}{m-m_0}\right)^2\left[\frac{\sin(ka)}{1-\cos(ka)}\right]^2\right\}^{-1}.$$

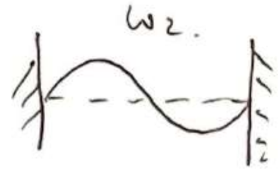
Solution 7

(a) $\frac{\partial^2 y}{\partial x^2} - \frac{\mu}{T}\frac{\partial^2 y}{\partial t^2} = 0.\quad v = \sqrt{\frac{T}{\mu}}.$



$$\omega_1 = \frac{2\pi}{\lambda_1} v = \frac{2\pi}{2L} v = \frac{\pi}{L}\sqrt{\frac{T}{\mu}}.$$

$$y_1 = A_1 \sin\left(\frac{\pi x}{L}\right)\cos(\omega_1 t + \varphi_1).$$

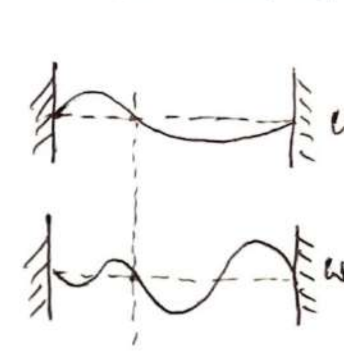


$$\omega_2 = \frac{2\pi}{\lambda_2} v = \frac{2\pi}{L}\sqrt{\frac{T}{\mu}}.$$

$$y_2 = A_2 \sin\left(\frac{2\pi x}{L}\right)\cos(\omega_2 t + \varphi_2).$$

$A_1, A_2, \varphi_1, \varphi_2$ are constants determined from initial condition.

(b) Equations for two sections are:



$$\begin{cases} \frac{\partial^2 y}{\partial x^2} - \frac{4\mu}{T}\frac{\partial^2 y}{\partial t^2} = 0 & x \in [0, \frac{L}{3}] \\ \frac{\partial^2 y}{\partial x^2} - \frac{\mu}{T}\frac{\partial^2 y}{\partial t^2} = 0. & x \in [\frac{L}{3}, L] \end{cases}$$

$$\Rightarrow y(x,t) = \begin{cases} (A_1 \cos\omega t + B_1 \sin\omega t)\sin\left(\sqrt{\frac{4\mu}{T}}\,\omega x\right) & x \in [0, \frac{L}{3}] \\ (A_2 \cos\omega t + B_2 \sin\omega t)\sin\left[\sqrt{\frac{\mu}{T}}\,\omega(L-x)\right] & x \in [\frac{L}{3}, L] \end{cases}$$

Consider y and y' are continuous at $x = \frac{L}{3}$.

$$\Rightarrow \begin{vmatrix} \sin\left(\frac{L\omega}{3v_1}\right) & -\sin\left(\frac{L\omega}{3v_1}\right) \\ \frac{\omega}{v_1}\cos\left(\frac{L\omega}{3v_1}\right) & \frac{\omega}{2v_1}\cos\left(\frac{L\omega}{3v_1}\right) \end{vmatrix} = 0.\quad \left(v_1 = \sqrt{\frac{T}{4\mu}}\right).$$

$$\Rightarrow \frac{2L\omega}{3v_1} = \frac{4L\omega}{3}\sqrt{\frac{\mu}{T}} = n\pi.\quad n = 1, 2, 3, \ldots$$

$$\omega_1' = \frac{3}{4}\omega_1 = \frac{3\pi}{4L}\sqrt{\frac{T}{\mu}}.\quad \omega_2' = \frac{3\pi}{2L}\sqrt{\frac{T}{\mu}} = \frac{3}{2}\omega_1.$$

for ω_1', $A_2 = A_1$, $B_2 = B_1$.

for ω_2', $A_2 = -2A_1$, $B_2 = -2B_1$

Solution 8.

(a)
$$\begin{cases} m\ddot{x}' = T_{x'} + 2m\omega\dot{y}'\sin\varphi \\ m\ddot{y}' = T_{y'} - 2m\omega(\dot{x}'\sin\varphi + \dot{z}\cos\varphi) \\ m\ddot{z}' = T_{z'} - mg + 2m\omega\dot{y}'\cos\varphi \end{cases}$$

Since $z' \approx 0$. $T \approx mg$.

$$\begin{cases} \ddot{x}' - 2\omega\dot{y}'\sin\varphi + \frac{g}{l}x' = 0. \\ \ddot{y}' + 2\omega\dot{x}'\sin\varphi + \frac{g}{l}y' = 0. \end{cases}$$

(b)
$$\ddot{\tilde{\xi}} + 2i\omega\dot{\tilde{\xi}}\sin\varphi + \frac{g}{l}\tilde{\xi} = 0.$$

$$\Rightarrow \tilde{\xi} = e^{-i\omega t\sin\varphi}\left(Ae^{i\omega_0 t} + Be^{-i\omega_0 t}\right), \quad \omega_0 = \sqrt{\frac{g}{l}}.$$

(c)
$$\begin{cases} x' = Re\{\tilde{\xi}\} = (A-B)\cos(\omega_0 t)\sin(\omega t\sin\varphi) \\ y' = Im\{\tilde{\xi}\} = (A-B)\cos(\omega_0 t)\cos(\omega t\sin\varphi) \end{cases}$$

$$\Rightarrow \vec{r}' = \left(\sin(\omega t\sin\varphi), \cos(\omega t\sin\varphi)\right)(A-B)\cos(\omega_0 t)$$
$$= \vec{k}R\cos(\omega_0 t).$$

(d)
$$\tilde{T} = \frac{2\pi}{\omega\sin\varphi}$$

$\varphi_{Mohe} = 53°$. $\varphi_{Haikou} = 20°$.

Thus,

$$\tilde{T}_{Mohe} = 1.08\times10^5\,s = 30h.$$

$$\tilde{T}_{Haikou} = 2.52\times10^5\,s = 70h.$$