VP160 HW4 Solutions

Solution 1.

Method I. (Momentum theorem)

$$(F - g\,dm)\,dt = dm\left(0 - (-\sqrt{2gh})\right) \Rightarrow F = \sqrt{2gh}\,\lambda.$$

$$N = F + Mg + \lambda g t = (\sqrt{2gh} + gt)\lambda + Mg$$

Method II. (Mass changing)

$$m\frac{dv}{dt} + (\cancel{v}^{\,0} - \sqrt{2gh})\frac{dm}{dt} = F = \left(M + \frac{dm}{dt}t\right)g - N$$

$$N = (M + \lambda t)g + \sqrt{2gh}\,\lambda.$$

Solution 2.

$$m_A v_A + m_B v_B \cos\beta = I\cos\alpha, \quad m_B v_B \sin\beta = I\sin\alpha.$$

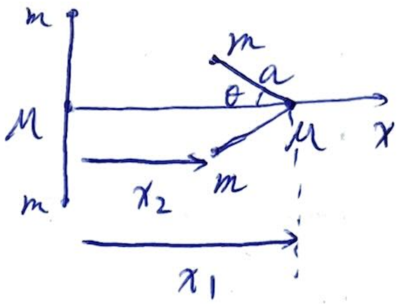
$$v_A = v_B\cos\beta \quad \text{(inextensible rope)}.$$

$$\Rightarrow \quad \beta = \arctan\left(\frac{m_A + m_B}{m_B}\tan\alpha\right)$$

$$= \arcsin\left(\frac{(m_A + m_B)\tan\alpha}{\sqrt{m_B^2 + (m_A + m_B)^2\tan^2\alpha}}\right) = \arccos\left(\frac{m_B}{\sqrt{m_B^2 + (m_A + m_B)^2\tan^2\alpha}}\right)$$

$$I = \frac{m_B(m_A + m_B)v_B}{\sqrt{(m_A + m_B)^2\tan^2\alpha + m_B^2}}$$

Solution 3.



$$(2m+M)x_c = Mx_1 + 2mx_2.$$

$$v_c = \frac{M}{M+2m}v.$$

When $t=\tau$, $x_1 = b$. $x_2 = b-a$.

$$\Rightarrow b = \frac{Mv\tau + 2ma}{M+2m}$$

Kinemic Energy: $\frac{1}{2}M\dot{x}_1^2 + 2\times\frac{1}{2}m(\dot{x}_2^2+\dot{y}_2^2) = \frac{1}{2}Mv^2.$

When $t=\tau$, $\dot{x}_1 = \dot{x}_2 = v_c = \frac{Mv}{M+2m}$

$$\Rightarrow \dot{y}_2^2 = a^2\cos^2\theta\,\dot{\theta}^2 = \frac{M}{M+2m}v^2 \Rightarrow \dot{\theta}^2 = \frac{Mv^2}{(M+2m)a^2}$$

Since $M\dot{x}_1 + 2m\dot{x}_2 = Mv$, $x_2 = x_1 - a\cos\theta$

$$(M+2m)\dot{x}_1 + 2ma\sin\theta\,\dot{\theta} = Mv.$$

$$(M+2m)\ddot{x}_1 + 2ma\sin\theta\,\ddot{\theta} + 2ma\cos\theta\,\dot{\theta}^2 = 0.$$

When $t=\tau$. $\ddot{x}_1 = -\frac{2Mm}{(M+2m)^2}\frac{v^2}{a}$

$$T = -\frac{1}{2}M\ddot{x}_1 = \frac{mM^2v^2}{(M+2m)^2a}$$

Solution 4.

$$m\frac{dv}{dt} + u\frac{dm}{dt} = -mg$$

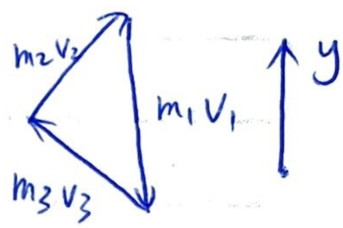
when $t=0$, $m=m_0$, $\frac{dv}{dt}=0$. $\Rightarrow$ $\frac{dm}{dt} = -\frac{m_0}{u}g$.

Integral: $m(t) = m_0 - \frac{m_0}{u}gt$

Then, $a(t) = \frac{dv}{dt} = \frac{g^2 t}{u-gt}$

$$v(t) = \int_0^t a(t)dt = -gt + u\ln\frac{u}{u-gt}$$

$$h(t) = \int_0^t v(t)dt = -(ut+\frac{1}{2}gt^2) + \frac{u}{g}(u+gt)\ln\frac{u}{u-gt}.$$



Solution 5.

$m_1v_1 = m_2v_{2y} + m_3v_{3y}$ (different y here)

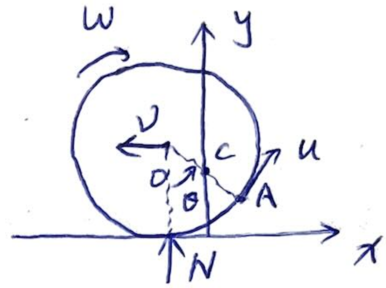
$$h = v_1t_1 + \frac{1}{2}gt_1^2.$$

$$h = -v_{2y}t_2 + \frac{1}{2}gt_2^2$$

$$h = -v_{3y}t_3 + \frac{1}{2}gt_3^2$$

$$\Rightarrow h = \frac{1}{2}g\frac{\Sigma_i m_i t_i}{\Sigma_i \frac{m_i}{t_i}} \quad (i=1,2,3)$$

Solution 6.



(a) $-mv + m(-v - R\omega_0 + u) = 0$.

$-mR^2\omega_0 + m(-v - R\omega_0 + u)R = 0$.

$\Rightarrow \omega_0 = \frac{u}{3R}$.

(b) $\dot\theta = \frac{u}{R} - \omega$, $\dot\theta_0 = \frac{2u}{3R}$.

$y_c = R - \frac{1}{2}R\cos\theta$, $\ddot y_c = \frac{1}{2}R(\dot\theta^2\cos\theta + \ddot\theta\sin\theta)$

$N - 2mg = 2m\ddot y_c$, $N\sin\theta = -3mR\ddot\theta$

$\Rightarrow R\dot\theta^2\sin\theta\cos\theta + R\ddot\theta(3+\sin^2\theta) + 2g\sin\theta = 0$.

Integral: $R\dot\theta^2(3+\sin^2\theta) - \frac{4u^2}{3R} = 4g(\cos\theta - 1)$

When $\theta = \frac{\pi}{2}$, $\dot\theta^2 = \frac{u^2}{3R^2} - \frac{g}{R} \geq 0$. $\Rightarrow u \geq \sqrt{3Rg}$

and $4R\ddot\theta + 2g = 0 \Rightarrow N = \frac{3}{2}mg$.

Solution 7

(a) $\dot x_{cm} = \frac{M\dot x_{car} + Nm\dot x_{man}}{M+Nm} = 0$

$\dot x_{car} = V_{car}$, $\dot x_{man} = V_{car} - V_r \Rightarrow V_{car} = \frac{Nm}{M+Nm}V_r$.

(b) $P_n = MV_n + nmV_n$, $P_{n-1} = MV_{n-1} + (n-1)mV_{n-1} + m(V_{n-1} - V_r)$

Since $P_n = P_{n-1} \Rightarrow V_{n-1} = V_n + \frac{m}{M+nm}V_r$

Thus, $V_0 = \sum_{n=1}^{N} \frac{mV_r}{M+nm}$

(c) $\sum_{n=1}^{N} \frac{1}{M+nm} > \frac{N}{M+Nm} \Rightarrow$ (b) attains a greater velocity.

Solution 8

Suppose the spherical dust particle initially has mass M_0 and radius R_0. Then,

$$M(t) = M_0 + \frac{4}{3}\pi(R^3 - R_0^3)\rho.$$ ρ is the density of water mist.

giving $$\frac{dM}{dt} = \rho \cdot 4\pi R^2 \frac{dR}{dt}.$$

The droplet has a cross section πR^2 and sweeps out a cylinder of volume $\pi R^2 \dot{x}$ in unit time.

$$\frac{dM}{dt} = \alpha \cdot \pi R^2 \dot{x} \propto (\pi R^2 \dot{x})$$ α is a positive constant.

Hence, $$\dot{x} = \frac{4\rho}{\alpha}\dot{R}$$

Momentum theorem: $$\dot{x}\frac{dM}{dt} + M\ddot{x} = Mg.$$

For large t, $M(t) \approx \frac{4}{3}\pi R^3 \rho$, $\frac{dM}{dt} \approx 3M\frac{\dot{R}}{R}$.

then $$\ddot{R} + \frac{3\dot{R}^2}{R} = \frac{\alpha g}{4\rho}$$

setting $R(t) = at^2$, where a is a constant.

then $$a = \frac{\alpha g}{56\rho}.$$

Thus, $$\dot{x} = \frac{4\rho}{\alpha}\cdot 2at = \frac{g}{7}t$$

Hence, the acceleration for large times is $\frac{g}{7}$.