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VP160 Final Big RC Part II
Week 12

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Lagrangian Mechanics

Degrees of Freedom

3D space, N particles, C holonomic constraints:

$$f = 3N - C \tag{1}$$

Lagrangian

$$L = T - V \tag{2}$$

Euler-Lagrange Equation

For $i = 1, 2, ..., f$:

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_i}) - \frac{\partial L}{\partial q_i} = 0 \tag{3}$$

Thermal Physics

Irreversible process	$dW = -P_{ext}dV$
Reversible process	$dW = -PdV$
First Law	$dU = dQ + dW = dQ - PdV$
Enthalpy	$H = U + PV$ $dH = dU + PdV + VdP$
Heat Capacities	$C_V = \left(\dfrac{\partial U}{\partial T}\right)_V$ and $C_P = \left(\dfrac{\partial H}{\partial T}\right)_P$
Entropy	$dS = \dfrac{dQ_{rev}}{T_{surr}}$ and $\Delta S_{surr} = \dfrac{-Q}{T_{\sigma}}$
Second Law	$\Delta S_{univ} = \Delta S_{system} + \Delta S_{surr} \geq 0$
Mass Balance	$\dfrac{dm_{CV}}{dt} = \dot{m}_{in} - \dot{m}_{out}$
Energy Balance	$\dfrac{d(mU)_{CV}}{dt} + \Delta\left[\left(H + \frac{1}{2}u^2 + gz\right)\dot{m}\right] = \dot{Q} + \dot{W}_s$
Entropy Balance	$\dfrac{d(mS)_{CV}}{dt} + \Delta(mS) - \sum_j \dfrac{\dot{Q}_j}{T_{\sigma,j}} = \dot{S}_G \geq 0$
Ideal Gas EOS	$PV = RT$
Virial EOS	$Z = \dfrac{PV}{RT} = 1 + \dfrac{BP}{RT}$ $= 1 + \dfrac{\tilde{B}P_r}{T_r}$ $\tilde{B} = \dfrac{BP_c}{RT_c} = B^0 + \omega B^1$ $B^0 = 0.083 - \dfrac{0.422}{T_r^{1.6}}$ $B^1 = 0.139 - \dfrac{0.172}{T_r^{4.2}}$
van der Waals EOS	$P = \dfrac{RT}{V-b} - \dfrac{a}{V^2}$ $a = \dfrac{27R^2T_c^2}{64P_c}$ $b = \dfrac{RT_c}{8P_c}$ $T_r = \dfrac{T}{T_c}, P_r = \dfrac{P}{P_c}$
Generalised Correlations	Gas: $Z = Z^0 + \omega Z^1$ Liquid: $V^{sat} = V_c Z_c^{(1-T_r)^{2/7}}$

For Multiple Component: $M = (1-x^v)M^l + x^v M^v$
Where M represents V,U,H,S

For Ideal gas:

$\Delta U = \int_{T_1}^{T_2} C_V dT$	$\Delta H = \int_{T_1}^{T_2} C_P dT$
$\Delta S = \int_{T_1}^{T_2} \dfrac{C_V}{T} dT + R \ln\dfrac{V_2}{V_1}$	$\Delta S = \int_{T_1}^{T_2} \dfrac{C_P}{T} dT - R \ln\dfrac{P_2}{P_1}$

For Ideal Gas and Reversible Process:

$dQ = C_V dT + \dfrac{RT}{V} dV$	$dW = -\dfrac{RT}{V} dV$
$dQ = C_P dT - \dfrac{RT}{P} dP$	$dW = -R dT + \dfrac{RT}{P} dP$

Isochoric Process: $dV = 0$	$W = -\int_1^2 PdV = 0$ $Q = \Delta U = \int_1^2 C_V dT$ $\Delta H = \int_1^2 C_P dT$	
Isothermal Process: $dT = 0$	$W = -\int_1^2 \dfrac{RT}{V} dV = RT\ln\dfrac{V_2}{V_1}$ $Q = -W = RT\ln\dfrac{V_2}{V_1} = -RT\ln\dfrac{P_2}{P_1}$ $\Delta U = \int_1^2 C_V dT = 0$ $\Delta H = \int_1^2 C_P dT = 0$	
Isobaric Process: $dP = 0$	$W = -\int_1^2 PdV = -P(V_2 - V_1) = -R(T_2 - T_1)$ $\Delta U = \int_1^2 C_V dT$ $Q = \Delta H = \int_1^2 C_P dT$	
Adiabatic Process: $dQ = 0$	$W = \Delta U = \int_1^2 C_V dT$ $\Delta H = \int_1^2 C_P dT$	$\gamma = C_P/C_V$ $PV^\gamma = constant$ $TV^{\gamma-1} = constant$ $TP^{(1-\gamma)/\gamma} = constant$
Polytropic Process	$W = -\int_1^2 PdV$ $Q = \Delta U - W$	$PV^\delta = constant$ $TV^{\delta-1} = constant$ $TP^{(1-\delta)/\delta} = constant$

For Residual Properties:

$dU = TdS - PdV$	$dH = TdS + VdP$
$dA = -SdT - PdV$	$dG = -SdT + VdP$
$\left(\dfrac{\partial T}{\partial V}\right)_S = -\left(\dfrac{\partial P}{\partial S}\right)_V$	$\left(\dfrac{\partial T}{\partial P}\right)_S = \left(\dfrac{\partial V}{\partial S}\right)_P$
$\left(\dfrac{\partial P}{\partial T}\right)_V = \left(\dfrac{\partial S}{\partial V}\right)_T$	$\left(\dfrac{\partial V}{\partial T}\right)_P = -\left(\dfrac{\partial S}{\partial P}\right)_T$
$dU = C_V dT + \left[T\left(\dfrac{\partial P}{\partial T}\right)_V - P\right]dV$	$dH = C_P dT + \left[V - T\left(\dfrac{\partial V}{\partial T}\right)_P\right]dP$
$dS = \dfrac{C_V}{T} dT + \left(\dfrac{\partial P}{\partial T}\right)_V dV$	$dS = \dfrac{C_P}{T} dT - \left(\dfrac{\partial V}{\partial T}\right)_P dP$
$H = H^{ig} + H^R$ $\Delta H = \int_{T_1}^{T_2} C_P^{ig} dT + \Delta H_R$	$S = S^{ig} + S^R$ $\Delta S = \int_{T_1}^{T_2} \dfrac{C_P^{ig}}{T} dT - R\ln\dfrac{P_2}{P_1} + \Delta S_R$
$C_P = C_V + T\left(\dfrac{\partial P}{\partial T}\right)_V\left(\dfrac{\partial V}{\partial T}\right)_P$	$d\left(\dfrac{G^R}{RT}\right) = \dfrac{V^R}{RT} dP - \dfrac{H^R}{RT^2} dT$
$V^R = \dfrac{RT}{P}(Z-1)$	$\dfrac{S^R}{R} = -T\left(\dfrac{\partial Z}{\partial T}\right)_P \dfrac{dP}{P} - \int (Z-1)\dfrac{dP}{P}$
$\dfrac{U^R}{RT} = -T\int_0^P \left(\dfrac{\partial Z}{\partial T}\right)_P \dfrac{dP}{P}$	$\dfrac{G^R}{RT} = \int_0^P (Z-1)\dfrac{dP}{P}$

Virial EOS	$Z = \dfrac{PV}{RT} = 1 + \dfrac{BP}{RT}$ $\dfrac{H^R}{RT} = \dfrac{P}{R}\left(\dfrac{B}{T} - \dfrac{dB}{dT}\right)$ $\dfrac{S^R}{R} = -\dfrac{P}{R}\dfrac{dB}{dT}$ $\dfrac{G^R}{RT} = \dfrac{BP}{RT}$	$\dfrac{dB}{dT} = \dfrac{R}{P_c}\dfrac{dB}{dT_r}$ $\dfrac{dB}{dT_r} = \dfrac{dB^0}{dT_r} + \omega\dfrac{dB^1}{dT_r}$ $\dfrac{dB^0}{dT_r} = \dfrac{0.675}{T_r^{2.6}}$ $\dfrac{dB^1}{dT_r} = \dfrac{0.722}{T_r^{5.2}}$
van der Waals EOS	$Z = \dfrac{1}{1-\rho b} - \dfrac{\rho a}{RT}$ $\dfrac{H^R}{RT} = \dfrac{\rho b}{1-\rho b} - \dfrac{2\rho a}{RT}$ $\dfrac{S^R}{R} = \ln\left(1 - \dfrac{\rho b}{RT} + \dfrac{V^R}{RT}\right)$ $\dfrac{G^R}{RT} = \dfrac{\rho b}{1-\rho b} - \dfrac{2\rho a}{RT} - \ln\left(1 - \dfrac{\rho b}{RT} + \dfrac{\rho^2 ab}{RT}\right)$	
Generalized Correlations for Gases	$\dfrac{H^R}{RT_c} = \dfrac{(H^R)^0}{RT_c} + \omega\dfrac{(H^R)^1}{RT_c}$ $\dfrac{S^R}{R} = \dfrac{(S^R)^0}{R} + \omega\dfrac{(S^R)^1}{R}$	

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Exercise 1

A block of mass M is rigidly connected to a massless circular track of radius a on a frictionless horizontal table as shown in Fig. 2.7. A particle of mass m is confined to move without friction on the circular track which is vertical.

(a) Set up the Lagrangian, using θ as one coordinate.
(b) Find the equations of motion.

(*Wisconsin*)

Fig. 2.7.
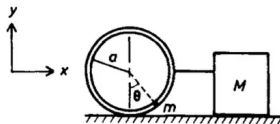
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Exercise 2

Compute the theoretical minimum amount of mechanical work needed to freeze 1 kilogram of water, if the water and surroundings are initially at a temperature $T_0 = 25°C$. The surroundings comprise the only large heat reservoir available.

$$(L_{\text{ice}} = 80 \text{ cal}/g, \quad C_p = 1 \text{ cal}/g \cdot °\text{ C}) .$$

$(UC, Berkeley)$

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Exercise 3

A thermally insulated box is separated into two compartments (volumes V_1 and V_2) by a membrane. One of the compartments contains an ideal gas at temperature T; the other is empty (vacuum). The membrane is suddenly removed, and the gas fills up the two comparments and reaches equilibrium.

(a) What is the final temperature of the gas?

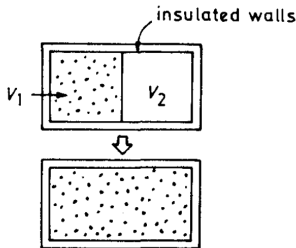(b) Show that the gas expansion process is irreversible.

(*MIT*)

insulated walls

V_1 → V_2

Fig. 1.20.

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Solution 1

Solution:

(a) As the motion of the system is confined to a vertical plane, use a fixed coordinate frame x, y and choose the x coordinate of the center of the circular track and the angle θ giving the location of m on the circular track as the generalized coordinates as shown in Fig. 2.7. The coordinates of the mass m are then $(x + a\sin\theta, -a\cos\theta)$. As M is rigidly connected to the circular track its velocity is $(\dot{x}, 0)$. Hence the Lagrangian is

$$L = T - V = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m[(\dot{x} + a\dot{\theta}\cos\theta)^2 + a^2\dot{\theta}^2\sin^2\theta] + mga\cos\theta$$

$$= \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m[\dot{x}^2 + a^2\dot{\theta}^2 + 2a\dot{x}\dot{\theta}\cos\theta] + mga\cos\theta \ .$$

(b) As

$$\frac{\partial L}{\partial x} = 0 \ ,$$

$$\frac{\partial L}{\partial \dot{x}} = M\dot{x} + m\dot{x} + ma\dot{\theta}\cos\theta \ ,$$

$$\frac{\partial L}{\partial \theta} = -ma\dot{x}\dot{\theta}\sin\theta - mga\sin\theta \ ,$$

$$\frac{\partial L}{\partial \dot{\theta}} = ma^2\dot{\theta} + ma\dot{x}\cos\theta \ ,$$

Lagrange's equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q_i}}\right) - \frac{\partial L}{\partial q_i} = 0$$

give

$$(M + m)\ddot{x} + ma\ddot{\theta}\cos\theta - ma\dot{\theta}^2\sin\theta = 0 \ ,$$

$$a\ddot{\theta} + \ddot{x}\cos\theta + g\sin\theta = 0 \ .$$

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Solution 2

Solution:
The minimum work can be divided into two parts W_1 and W_2: W_1 is used to lower the water temperature from $25°$C to $0°$C, and W_2 to transform water to ice. We find

$$
\begin{aligned}
W_1 &= -\int_{T_0}^{T_f} (T_0 - T) MC_p dT/T \\
&= MC_p T_0 \ln(T_0/T_f) - MC_p(T_0 - T_f) \\
&= 1.1 \times 10^3 \text{ cal} , \\
W_2 &= (T_0 - T_f) LM/T_f = 7.3 \times 10^3 \text{ cal} , \\
W &= W_1 + W_2 = 8.4 \times 10^4 \text{ cal} = 3.5 \times 10^4 \text{ J} .
\end{aligned}
$$

Solution 3

Solution:

(a) Freely expanding gas does no external work and does not absorb heat. So the internal energy does not change, i.e., $dU = 0$. The internal energy of an ideal gas is only a function of temperature; as the temperature does not change in the process, $T_f = T$.

(b) Assuming a quasi-static process of isothermal expansion, we can calculate the change in entropy resulting from the free expansion. In the process, we have $dS = pdV/T, pV = NkT$. Hence,

$$S_f - S = \int dS = \int_{V_1}^{V_1+V_2} \frac{p}{T}dV = Nk \ln \frac{V_1 + V_2}{V_1} > 0 \ .$$

Thus the freely expanding process of the gas is irreversible.

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Information about Final Exam

- 1 concept problem $+$ 6 calculation problems
- It will be more difficult than the midterm exam.
- Lecture notes are very important.
- Good luck!

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Thanks for listening!

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