Due: 11:59 pm on July 4, 2024

Problem 1 (10pts)

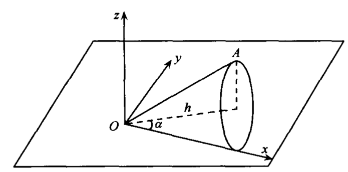
Prove the kinetic energy of a straight rod of uniform mass is

$$T = \frac{1}{6}m(\vec{u}\cdot\vec{u} + \vec{u}\cdot\vec{v} + \vec{v}\cdot\vec{v})$$

where $\vec{u}, \vec{v}$ are the velocities at both ends of the rod.

Problem 2 (10pts)

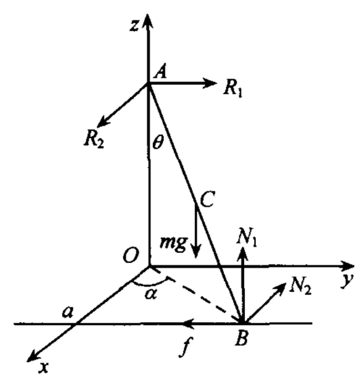
A right circular cone with height h and vertex angle 2α rolls purely around its vertex on a horizontal plane. If its geometric symmetry axis is known to rotate around the vertical axis at a constant angular velocity $\Omega\hat{k}$, prove the velocity and acceleration of the highest point A on the base of the cone are



$$\vec{v_A} = 2\Omega h\cos\alpha\hat{j}, \quad \vec{a_A} = -2\Omega^2 h\cos\alpha\hat{i} - \Omega^2 h\csc\alpha\hat{k}$$

Problem 3 (15pts)

The A end of a uniform heavy rod AB with a length of l can move on a smooth plumb line, and the B end can move along a rough horizontal straight line. The shortest distance between the two straight lines is $a, a < l$, and the static friction coefficient between the B end and the horizontal straight line is μ. If $l > a\sqrt{1+\mu^2}$, and the A end is higher than the B end, prove that the rod is in position



$$\arcsin\frac{a}{l} \le \theta \le \arccos\sqrt{\frac{l^2 - a^2(1+\mu^2)}{l^2(1+4\mu^2)}}$$

Problem 4 (10pts)

Prove the kinetic energy of the rigid body is

$$T = \frac{1}{2}\boldsymbol{v_C}^\top m\boldsymbol{v_C} + \frac{1}{2}\boldsymbol{\omega}^\top \boldsymbol{I}\boldsymbol{\omega}$$

Problem 5 (10pts)

Prove that a system of forces can always be simplified to a force passing through a given point O and the other force acting on any given plane that does not contain point O.

Problem 6 (15pts)

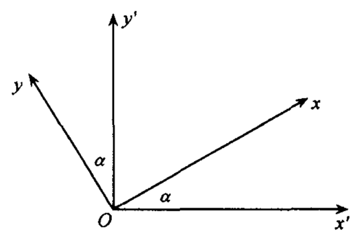
A homogeneous ellipsoid with mass m has an ellipsoid equation of

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

Prove the moment of inertia of the ellipsoid about the three principal axes of inertia of its center of mass are

$$I_a=\frac{1}{5}m(b^2+c^2),\quad I_b=\frac{1}{5}m(c^2+a^2),\quad I_c=\frac{1}{5}m(a^2+b^2)$$

Problem 7 (15pts)



In the above problem, if we do not take the three principal axes of inertia as coordinates, but take the x, y, and z coordinates of the above coordinates rotated around the z-axis by an angle α, prove the inertia tensor about the center of mass is

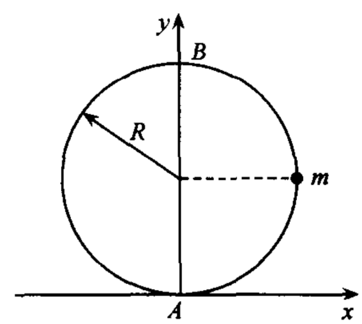
$$I=\frac{1}{5}m\begin{pmatrix} b^2+c^2+(a^2-b^2)\sin^2\alpha & (a^2-b^2)\sin\alpha\cos\alpha & 0\\ (a^2-b^2)\sin\alpha\cos\alpha & c^2+a^2+(b^2-a^2)\sin^2\alpha & 0\\ 0 & 0 & a^2+b^2\end{pmatrix}$$

Problem 8 (15pts)

A homogeneous thin disk with a radius of R and a mass of M, and a particle $m=\frac{5}{4}M$ attached to its edge.

a) Prove the inertia tensor of the system about point A is

$$I_A=\begin{pmatrix}\frac{5}{2} & -\frac{5}{4} & 0\\ -\frac{5}{4} & \frac{3}{2} & 0\\ 0 & 0 & 4\end{pmatrix}MR^2$$



b) Prove the principal axes and principal moments of inertia about point A are

$$I_1=3.346MR^2,\quad \hat{e_1}=(0.828,-0.561,0)$$

$$I_2=0.654MR^2,\quad \hat{e_2}=(0.561,0.828,0)$$

$$I_3=4MR^2,\quad \hat{e_1}=(0,0,1)$$