# VP160 Recitation Class 1

## Week 3

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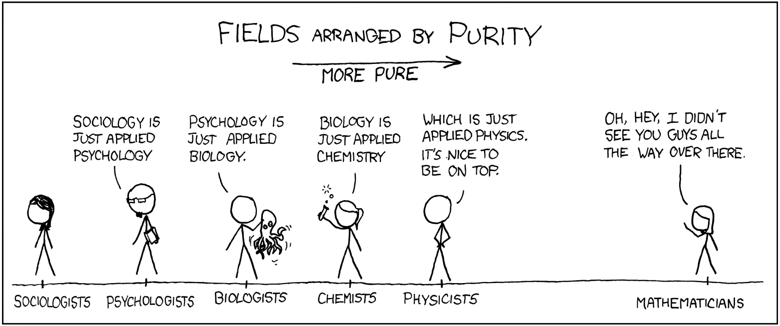
# Before We Start

**How to learn VP160 (Physics) well**

- Good Mathematical Foundation
- Clear Physical Concepts
- General Physical Principles




R. Munroe, Purity. https://xkcd.com/435/.

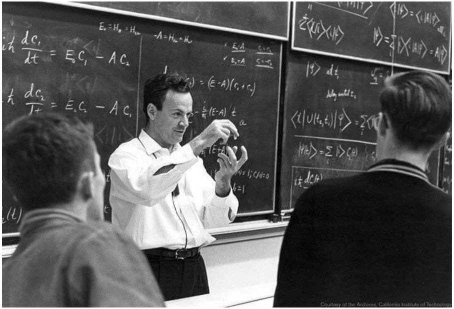
# Before We Start

**What's in RC**

- Concepts/Principles + Exercises

"Understand. Don't memorize. Learn principles, not formulas."

— Richard Feynman




From https://www.economymonitor.com/.

# Units

## Unit Prefix

- $k$(unit prefix)$m$(unit)

| $p$ | $n$ | $\mu$ | $m$ | $c$ | $k$ | $M$ | $G$ |
|---|---|---|---|---|---|---|---|
| Pico | Nano | Micro | Milli | Centi | Kilo | Mega | Giga |
| $10^{-12}$ | $10^{-9}$ | $10^{-6}$ | $10^{-3}$ | $10^{-2}$ | $10^{3}$ | $10^{6}$ | $10^{9}$ |

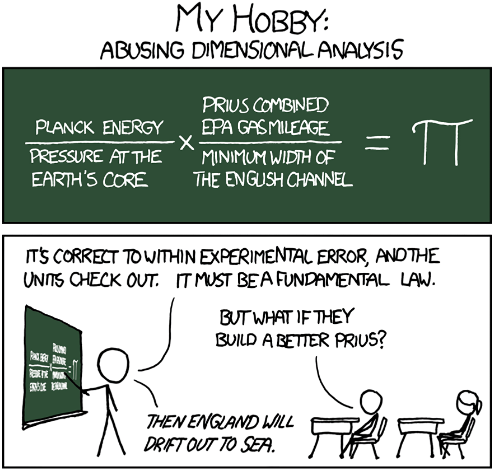
## SI System of Units

| Quantity | Basic Unit | Basic Unit Symbol |
|---|---|---|
| LENGTH | metre | m |
| MASS | kilogram | kg |
| TIME | second | s |
| TEMPERATURE | kelvin | K |
| QUANTITY OF MATTER | mole | mol |
| ELECTRIC CURRENT | ampere | A |
| LUMINOUS INTENSITY | candela | cd |

[a]

[a] From https://www.learnalberta.ca/content/memg/division03/International%20System%20of%20Units/index.html.

# Dimension Analysis




R. Munroe, Dimension Analysis. https://xkcd.com/687/.

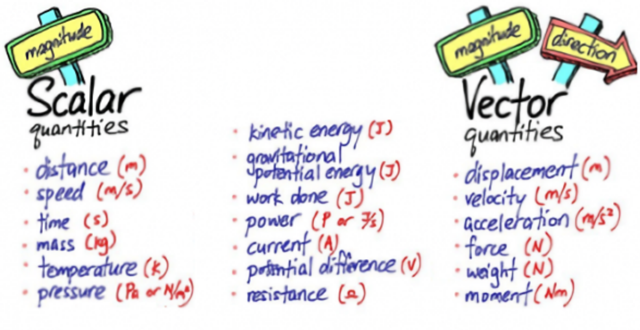
# Scalar and Vector

## Definition of Scalars

Scalars are quantities that only have magnitude.

## Definition of Vectors

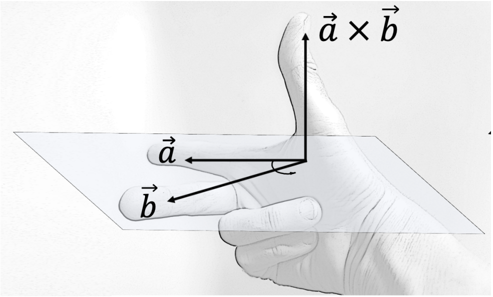
Vectors are quantities that have both magnitude and direction.




From https://www.tutormyself.com/1-13-understand-how-vector-quantities-differ-from-scalar-quantities/.

# Cross Product

## Right Hand Rule



[a]

[a] Sepitropova, Right-hand rule for cross product. In *Wikipedia*.

## Einstein Summation Convention (Optional | OH)

$$\vec{c} = \vec{a} \times \vec{b} = a_i b_j \varepsilon_{ijk} e_k$$

# Common Coordinate Systems

## Cartesian Coordinate System

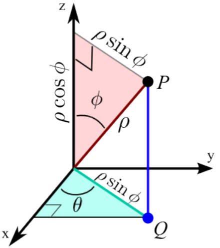
- $(x, y) \mid (x, y, z)$

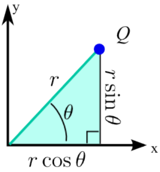
## Polar | Cylindrical Coordinate System

- $(r, \theta) \mid (\rho, \theta, z)$

## Spherical Coordinate System

- $(\rho, \theta, \phi)$




From `https://mathinsight.org/spherical_coordinates`.

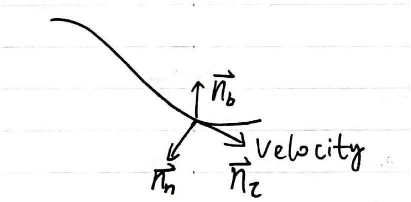
# Natural Coordinate System

## Definition

- $(\hat{n_\tau}, \hat{n_n}, \hat{n_b})$
- $\vec{v} = v\hat{n_\tau}$
- $\vec{a} = \dot{v}\hat{n_\tau} + \frac{v^2}{\rho}\hat{n_n}$

## Radius of Curvature

$$\rho = \frac{(1+y'^2)^{3/2}}{|y''|}(Cartesian) \mid \frac{(r^2+r'^2)^{3/2}}{|r^2+2r'^2-rr''|}(Polar)$$



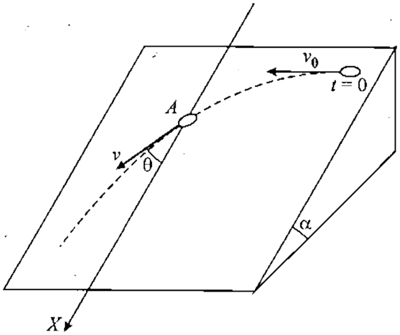

Zijie Qu, *Lecture note_week2*, p11.

# Exercise 1

The trajectory of a particle is a logarithmic spiral $r = be^{k\varphi}$, where $b$ and $k$ are both positive constants and $\dot{r} = c$ ($c$ is a positive constant). When $t = 0$, it is located at $r = b$ and $\varphi = 0$. Find the velocity and acceleration of the particle and how the radius of curvature changes with time.

# Exercise 2

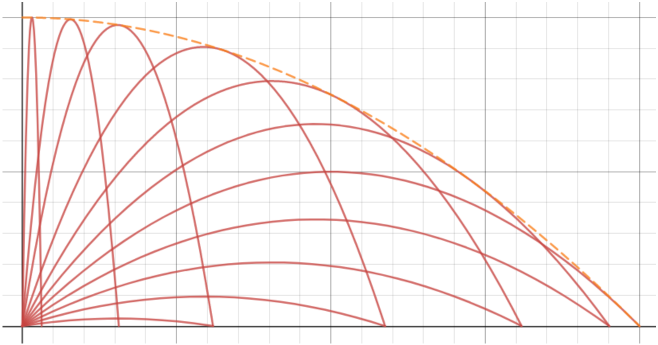
A small disc is placed on an inclined plane forming an angle $\alpha$ with the horizontal and is imparted an initial velocity $v_0$. Find how the velocity of the disc depends on the angle $\theta$ if the friction coefficient is $\mu$.
*Hint:* $\int \csc x dx = -\ln|\csc x + \cot x| + C$




From https://www.sarthaks.com/1781789/.

# Exercise 3

Find the envelope of the projectile trajectories.



From https://math.stackexchange.com/questions/1495086/.

# Solution 1

$$r = be^{k\varphi} = b + ct, \dot{r} = kbe^{k\varphi}\dot{\varphi} = c, \ddot{r} = 0 \tag{1}$$

$$\dot{\varphi} = \frac{c}{kb}e^{-k\varphi}, \ddot{\varphi} = -\frac{c}{kb^2}e^{-2k\varphi}, \varphi = \frac{1}{k}\ln\left(1 + \frac{c}{b}t\right) \tag{2}$$

$$v = \sqrt{\dot{r}^2 + (r\dot{\varphi})^2} = \frac{c}{k}\sqrt{1 + k^2} \tag{3}$$

$$a = \sqrt{(\ddot{r} - r\dot{\varphi}^2)^2 + (r\ddot{\varphi} + 2\dot{r}\dot{\varphi})^2} = \frac{c^2\sqrt{1 + k^2}}{k^2(b + ct)} \tag{4}$$

Since $\vec{a} \perp \vec{v}$

$$a_n = a = \frac{c^2\sqrt{1 + k^2}}{k^2(b + ct)} \tag{5}$$

$$\rho = \frac{v^2}{a_n} = \sqrt{1 + k^2}(b + ct) \tag{6}$$

Or using the formula

$$\rho = \frac{(r^2 + {r'}^2)^{3/2}}{|r^2 + 2{r'}^2 - rr''|} = \sqrt{1 + k^2}(b + ct) \tag{7}$$

# Solution 2

$$\hat{n_\tau} : \dot{v} = g(\sin\alpha\cos\theta - \mu\cos\alpha) \tag{8}$$

$$\hat{n_n} : v\dot{\theta} = -g\sin\alpha\sin\theta \tag{9}$$

$$v = \frac{v_0}{1+\cos\theta}\left(\frac{\sin\theta}{1+\cos\theta}\right)^{\mu\cot\alpha - 1} \tag{10}$$

When $\mu = \tan\alpha$

$$v_f = \lim_{\theta\to 0} v = \frac{v_0}{2} \tag{11}$$

When $\mu > \tan\alpha$

$$v_f = \lim_{\theta\to 0} v = 0 \tag{12}$$

When $\mu < \tan\alpha$

$$v_f = \lim_{\theta\to 0} v \to \infty \tag{13}$$

# Solution 3

$$\begin{cases} x = v\cos\theta t \\ y = v\sin\theta t - \frac{g}{2}t^2 \end{cases} \Rightarrow y = x\tan\theta - \frac{gx^2}{2v^2}\sec^2\theta \tag{14}$$

$$\frac{\partial y}{\partial\theta} = \frac{gx^2}{v^2}\sec^2\theta\left(\frac{v^2}{gx} - \tan\theta\right) = 0 \Rightarrow \tan\theta = \frac{v^2}{gx} \tag{15}$$

Or

$$\frac{gx^2}{2v^2}\tan^2\theta - x\tan\theta + \frac{gx^2}{2v^2} + y = 0, \Delta = x^2 - 4\frac{gx^2}{2v^2}\left(\frac{gx^2}{2v^2} + y\right) = 0 \tag{16}$$

$$y = -\frac{g}{2v^2}x^2 + \frac{v^2}{2g} \tag{17}$$

**Thanks for listening!**

# References


Zijie Qu.
Lecture notes.
2024.

Zeyi Ren.
Recitation class slides.
2021.

Jin Wu.
Recitation class slides.
2023.

# Einstein Notation

**Kronecker Delta**

$$\delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

**Levi-Civita**

$$\varepsilon_{ijk} = \begin{cases} +1 & (i,j,k) = (1,2,3) \mid (2,3,1) \mid (3,1,2)[\textit{even permutation}] \\ -1 & (i,j,k) = (3,2,1) \mid (1,3,2) \mid (2,1,3)[\textit{odd permutation}] \\ 0 & i = j \mid j = k \mid k = i \end{cases}$$

**Basic Formulas**

$$\delta_{ij} a_j = a_i$$

$$\varepsilon_{lij}\varepsilon_{lmn} = \delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}$$

$$\vec{A} = e_i a_i, \vec{A} \cdot \vec{B} = a_i b_i, \vec{A} \times \vec{B} = a_i b_j \varepsilon_{ijk} e_k$$