**Due: 11:59 pm on May 28, 2024**

## Problem 1 (10pts)

Dimensional analysis[1] is the analysis of the relationships between different physical quantities by identifying their base quantities (such as length, mass, time, and electric current) and units of measurement (such as meters and grams) and tracking these dimensions as calculations or comparisons are performed. As an example, the dimension of the physical quantity speed $v$ is

$$\dim v = \frac{length}{time} = \frac{L}{T} = T^{-1}L$$

Prove the following statements by the dimensional analysis.
(a) Pythagorean theorem: The three sides of a right triangle satisfy $c^2 = a^2 + b^2$.
(b) The period $T$ of small oscillations in a simple pendulum has the form $T = C\sqrt{\frac{L}{g}}$.

## Problem 2 (10pts)

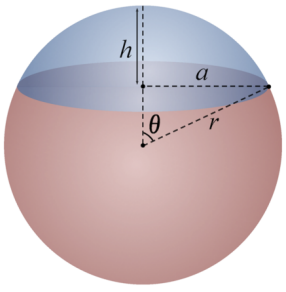
When $t = 0$, a particle starts from the origin and moves along the curve $x^2 + (y - r)^2 = r^2$ in $+x$ direction at a constant speed. When $t = T$, it returns to the origin for the first time.
(a) Find the displacement, the velocity and the acceleration of the particle when $t = T/3$.
(b) Find the average velocity and the average acceleration from $t = 0$ to $t = T/3$.

## Problem 3 (15pts)

The positions of two particles at an instant of time are given by the vectors $\boldsymbol{r_1} = 4\hat{n_x} + 3\hat{n_y} + 8\hat{n_z}$ and $\boldsymbol{r_2} = 2\hat{n_x} + 10\hat{n_y} + 5\hat{n_z}$, respectively (in the units of $m$). Find
(a) the magnitudes of these vectors,
(b) the displacement vector $\boldsymbol{r_{12}}$ from the position of particle 1 to the position of particle 2 and the unit vector in the direction of vector $\boldsymbol{r_{12}}$,
(c) the angles between all pairs of vectors $\boldsymbol{r_1}$, $\boldsymbol{r_2}$, and $\boldsymbol{r_{12}}$ (express the angles both in radians and degrees),
(d) the orthogonal projection (vector) of the vector $\boldsymbol{r_2}$ onto the vector $\boldsymbol{r_1}$,
(e) the cross product $\boldsymbol{r_2} \times \boldsymbol{r_{12}}$,
(f) the cylindrical coordinates of the point defined by the position vector $\boldsymbol{r_1}$ (use a calculator/computer to find the numerical values).

## Problem 4 (10pts)

Using the integration in Spherical Coordinates, calculate the volume of the spherical cap[2] in blue.
*Hint:* $V = \iiint \rho^2 \sin\theta d\rho d\theta d\varphi$.



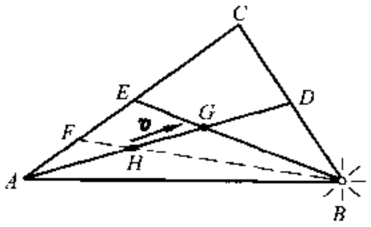

[1] Dimensional analysis. In *Wikipedia*. https://en.wikipedia.org/wiki/Dimensional_analysis.
[2] Spherical cap. In *Wikipedia*. https://en.wikipedia.org/wiki/Spherical_cap.

## Problem 5 (10pts)

Prove[3]
(a) the scalar triple product equation: $\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c})=\boldsymbol{b}\cdot(\boldsymbol{c}\times\boldsymbol{a})=\boldsymbol{c}\cdot(\boldsymbol{a}\times\boldsymbol{b})$,
(b) the vector triple product equation: $\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c})=\boldsymbol{b}(\boldsymbol{a}\cdot\boldsymbol{c})-\boldsymbol{c}(\boldsymbol{a}\cdot\boldsymbol{b})$.

## Problem 6 (15pts)

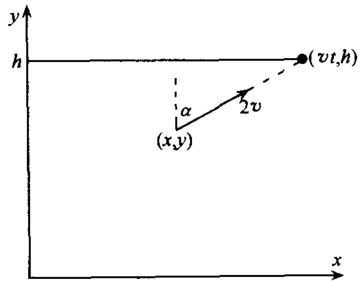
A particle $H$ moves with uniform speed along the midline $AD$ of triangle $ABC$. Suppose there is a light source at $B$, and $BE$ is the center line of the triangle. Assume point $F$ is the projection of particle $H$ on $AC$, prove that the ratio of velocity of point $F$ when passing through three points $A$, $E$, and $C$ is $2^2:3^2:4^2$, and the ratio of acceleration is $2^3:3^3:4^3$.




*Hint:* $\overline{AF}=\dfrac{a\overline{AH}+b}{c\overline{AH}+d}$ [projective geometry[4]].

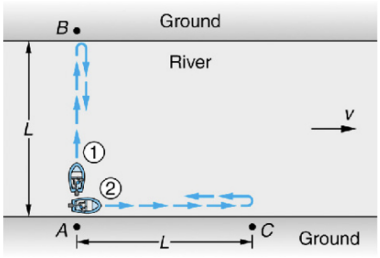
## Problem 7 (15pts)

A target moves along $y=h$ in the $x-y$ plane at a constant speed $v$. When $t=0$, it is at the point $x=0, y=h$. The missile starts from the origin at $t=0$ and moves at a constant speed $2v$. The speed direction always points to the target. Find the trajectory of the missile and the time it takes for the missile to hit the target.




## Problem 8 (15pts)

Two equally matched rowers race each other over courses. Each oarsman rows at speed $c$ in still water, the current in the river moves at speed $v<c$. Boat 1 goes from $A$ to $B$, a distance $L$, and back. Boat 2 goes from $A$ to $C$, also a distance $L$, and back. $A$, $B$, and $C$ are points on the riverbank. Which boat wins the race, or is it a tie?




[3]Triple product. In *Wikipedia*. https://en.wikipedia.org/wiki/Triple_product.
[4]Projective geometry. In *Wikipedia*. https://en.wikipedia.org/wiki/Projective_geometry.