VP160 Recitation Class 3

Week 5

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Kinetic Energy

Definition

The kinetic energy of an object is the form of energy that it possesses due to its motion.

$$E_k = \frac{1}{2}mv^2 \tag{1}$$

The work is the energy transferred to or from an object via the application of force along a displacement.

$$W = \int \vec{F} \cdot \mathrm{d}\vec{s} \tag{2}$$

Kinetic Energy Theorem

$$W = \Delta E_k = E_k - E_{k0} \tag{3}$$

Derivation & Integration

Derivation

$$\frac{\mathrm{d}}{\mathrm{d}y}F(x) = \frac{\mathrm{d}}{\mathrm{d}x}F(x) \cdot \frac{\mathrm{d}x}{\mathrm{d}y} \tag{4}$$

Integration

$$\int F(x)\mathrm{d}x = xF(x) - \int x\mathrm{d}F(x) \tag{5}$$

Euler Relation

Formula

$$e^{ix} = \cos x + i \sin x \tag{6}$$

Coriolis Force

$$\vec{F}_{cor} = -2m(\vec{\omega} \times \vec{v}) \tag{7}$$

$$\vec{v} = v_x + iv_y, \quad \vec{\omega} \times \vec{v} = -i\omega\vec{v} \tag{8}$$

The Trigonometric Functions Slide 476 JOINT INSTITUTE 交大密西根学院

The Trigonometric Functions

We first note that we define

$$e^z := \exp z \qquad \text{for } z \in \mathbb{C}.$$

We then introduce the well-known trigonometric cosine and sine functions $\cos, \sin \colon \mathbb{R} \to \mathbb{R}$ by

$$\cos(x) := \operatorname{Re} e^{ix} = \frac{e^{ix} + e^{-ix}}{2} = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!},$$

$$\sin(x) := \operatorname{Im} e^{ix} = \frac{e^{ix} - e^{-ix}}{2i} = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}.$$

The equation

$$e^{ix} = \cos(x) + i \sin(x)$$

is sometimes called the ***Euler relation***.

Hohberger, Horst. MATH1860J Slides. Fall 2022.

Approximation

Taylor Series

$$F(x) = F(x_0) + \sum F^{(n)}(x_0)(x - x_0)^n/n! \tag{9}$$

Expansion ($x_0 = 0$)

$$F(x) = \Sigma A_n x^n \tag{10}$$

First Order Linear Ordinary Differential Equation

Integrating Factor $M(x)$

$$y' + P(x)y = Q(x) \tag{11}$$

$$M(x)y' + M(x)P(x)y = M(x)y' + M'(x)y \Rightarrow M(x) = C_0 e^{\int P(x)\mathrm{d}x} \tag{12}$$

$$e^{\int P(x)\mathrm{d}x} y = \int e^{\int P(x)\mathrm{d}x} Q(x)\mathrm{d}x + C \tag{13}$$

$$y = e^{-\int P(x)\mathrm{d}x} \int Q(x) e^{\int P(x)\mathrm{d}x}\mathrm{d}x + Ce^{-\int P(x)\mathrm{d}x} \tag{14}$$

General Integral Curves of First Order ODEs Slide 271 JOINT INSTITUTE 交大密西根学院

Integrating factors (Euler Multipliers)

13.7. Definition. Let g, h be continuous functions on an open set $D \subset \mathbb{R}^2$. A function M with $M(x, y) \neq 0$ defined on D is said to be an ***integrating factor*** or ***Euler multiplier*** for the differential equation

$$h(x, y)y' + g(x, y) = 0 \tag{13.8}$$

if the vector field

$$F^{\perp}(x, y) = \begin{pmatrix} M(x, y)g(x, y) \\ M(x, y)h(x, y) \end{pmatrix}$$

has a potential function.

Of course, the main difficulty now is ***finding the correct integrating factor $M(x, y)$.***

Hohberger, Horst. MATH2860J Slides. Fall 2023.

Dynamics

Newton's Second Law of Motion

$$F(x,\dot{x},t)=m\ddot{x}=\frac{\mathrm{d}}{\mathrm{d}t}(m\dot{x})=m\dot{x}\frac{\mathrm{d}\dot{x}}{\mathrm{d}x}=\frac{\mathrm{d}}{\mathrm{d}x}(\frac{1}{2}m\dot{x}^2) \tag{15}$$

Expressions in Different Coordinates

$$\begin{cases}F_x=m\ddot{x}\\F_y=m\ddot{y}\end{cases} \tag{16}$$

$$\begin{cases}F_r=m(\ddot{r}-r\dot{\theta}^2)\\F_\theta=m(r\ddot{\theta}+2\dot{r}\dot{\theta})\end{cases} \tag{17}$$

$$\begin{cases}F_n=m\frac{v^2}{\rho}=mv\dot{\phi}\\F_\tau=m\dot{v}\end{cases} \tag{18}$$

Exercise 1

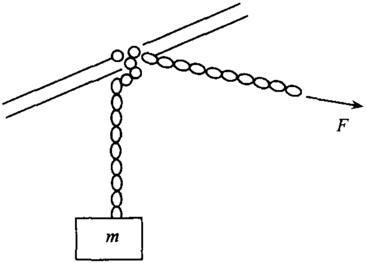
A small ball of mass m is tied to one end of an inextensible light rope. It passes through a small hole on the table and moves on a smooth horizontal table. The ball moves in a circle with a radius of r_1 and an angular velocity of ω_1 around the hole. There is no friction between the rope and the hole. Find the work that the pulling force need to do to reduce the radius of the ball's circular motion from $r = r_1$ to $r = r_2$?

Exercise 2

The mass of the train is m, and the power it produces is a constant P.
Find $t(v)$ if the resistance on the train is a constant f.
Find $t(v)$ if the resistance on the train is $f \propto v$.

Exercise 3

An object with a mass of m is wound around a horizontal rod through a rope of negligible mass for 5/4 turns, and a horizontal force F is applied to the other end. If the friction factor between the rope and the rod is μ, how much horizontal tension should be applied to keep the object stationary?



Exercise 4

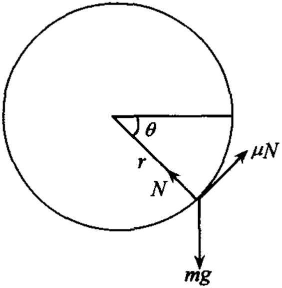
Consider the projectile motion with air resistance. If the projectile motion has the parameters (v_0, α), and the air resistance is $\vec{f} = -mkv\vec{v}$, prove

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = -\frac{ge^{2ks}}{v_0^2 \cos^2 \alpha}$$

where s is the moving distance.

Exercise 5

A particle moves along a rough vertical circle under the action of gravity. It starts from a rest state at one end of the horizontal diameter and comes to rest at the lowest point of the circle. Find the relationship that the friction coefficient μ must satisfy.



Solution 1

$$F_\theta = m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = \frac{m}{r}\frac{\mathrm{d}}{\mathrm{d}t}(r^2\dot{\varphi}) = 0 \tag{19}$$

Thus

$$r_2^2\omega_2 = r_1^2\omega_1 \Rightarrow \omega_2 = \omega_1(\frac{r_1}{r_2})^2 \tag{20}$$

$$W = \frac{1}{2}m(r_2\omega_2)^2 - \frac{1}{2}m(r_1\omega_1)^2 = \frac{1}{2}mr_1^2\omega_1^2(\frac{r_1^2}{r_2^2} - 1) \tag{21}$$

Solution 2

$$\frac{\mathrm{d}}{\mathrm{d}t}(\frac{1}{2}mv^2) = P - fv \tag{22}$$

If $f = Const.$

$$\mathrm{d}t = \frac{mv}{P - fv}\mathrm{d}v \tag{23}$$

$$t(v) = \frac{mP}{f^2} \ln \frac{P}{P - fv} - \frac{mv}{f} \tag{24}$$

If $f \propto v$, assume $f = cv$

$$\mathrm{d}t = \frac{m}{2(P - cv^2)}\mathrm{d}v^2 \tag{25}$$

$$t(v) = \frac{mv}{2f} \ln \frac{P}{P - fv} \tag{26}$$

Solution 3

$$T + \mathrm{d}T \pm \mu \mathrm{d}N = T \tag{27}$$

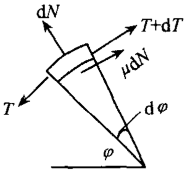
$$\mathrm{d}N = 2T \sin \frac{\mathrm{d}\varphi}{2} = T\mathrm{d}\varphi \tag{28}$$

Then we get

$$F = mge^{\mp \frac{5}{2}\mu\pi} \tag{29}$$

Therefore

$$mge^{-\frac{5}{2}\mu\pi} < F < mge^{\frac{5}{2}\mu\pi} \tag{30}$$



Solution 4

$$m\ddot{x} = -mk\dot{s}\dot{x} \Rightarrow \ddot{x} = -k\dot{s}\dot{x} \tag{31}$$

$$m\ddot{y} = -mk\dot{s}\dot{y} - mg \Rightarrow \ddot{y} = -k\dot{s}\dot{y} - g \tag{32}$$

Since

$$\ddot{y} = \frac{\mathrm{d}}{\mathrm{d}t}(\dot{x}\frac{\mathrm{d}y}{\mathrm{d}x}) = \ddot{x}\frac{\mathrm{d}y}{\mathrm{d}x} + \dot{x}^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} \tag{33}$$

we can get

$$\dot{x}^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -g \tag{34}$$

Since

$$\frac{\mathrm{d}\dot{x}^2}{\dot{x}^2} = -2k\mathrm{d}s \tag{35}$$

we have

$$\dot{x}^2 = v_0^2\cos^2\alpha e^{-2ks} \tag{36}$$

Then

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -\frac{ge^{2ks}}{v_0^2\cos^2\alpha} \tag{37}$$

Solution 5

$$m\dot{v} = mg\cos\theta - \mu N \tag{38}$$

$$m\frac{v^2}{r} = N - mg\sin\theta \tag{39}$$

Then, we get

$$\frac{\mathrm{d}v^2}{\mathrm{d}\theta} + 2(\mu v^2 - gr\cos\theta + \mu gr\sin\theta) = 0 \tag{40}$$

Integrating factor

$$M(\theta) : \frac{\mathrm{d}M}{\mathrm{d}\theta} = 2\mu\theta \Rightarrow M(\theta) = e^{2\mu\theta} \tag{41}$$

Then

$$\mathrm{d}(v^2 e^{2\mu\theta}) + 2(-gr\cos\theta + \mu gr\sin\theta)\mathrm{d}\theta = 0 \tag{42}$$

$$2\mu^2 - 1 + 3\mu e^{-\mu\pi} = 0 \tag{43}$$

Thanks for listening!

References

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