# VP160 Mid Big RC Part II

## Week 7

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June 29, 2024

# Collisions

**Momentum**

$$\sum \vec{p_f} = \sum \vec{p_i} \tag{1}$$

**Kinetic Energy**

$$\sum K_f = e^2 \sum K_i, \quad e \begin{cases} = 0 & \text{perfectly inelastic} \\ \in (0,1) & \text{inelastic} \\ = 1 & \text{perfectly elastic} \\ > 1 & \text{superelastic} \end{cases} \tag{2}$$

# Center of Mass (COM)

**Definition**

$$\vec{R_C} = \frac{\sum m_i \vec{r_i}}{\sum m_i} = \frac{\iiint_\Omega \rho(\vec{r})\vec{r}\mathrm{d}V}{\iiint_\Omega \rho(\vec{r})\mathrm{d}V} \tag{3}$$

**Pappus-Guldinus (Optional)**

- Solid of revolution with uniformly continuous mass.

$$\bar{y_C} = \frac{A}{2\pi L} = \frac{V}{2\pi A} \tag{4}$$

## Angular Momentum

**Definition**

$$\vec{L} = \sum \vec{r_i} \times m_i \vec{v_i} = \sum \vec{r_i} \times \vec{p_i} \tag{5}$$

**Angular Momentum Theorem**

$$\frac{\mathrm{d}\vec{L}}{\mathrm{d}t} = \vec{\tau} = \vec{\tau}_{ext} + \cancelto{0}{\vec{\tau}_{int}} \tag{6}$$

**Conservation of the Angular Momentum Law**

If the net torque of external forces on a system of particles is equal to zero, then the total angular momentum of that system is conserved.

# Rigid Body

**Definition**

A body is called rigid if $|\vec{r_i} - \vec{r_j}| = const$ for any point $i, j$ in the body.

**Total Momentum**

$$\vec{p} = \underbrace{M\vec{v}_{O'}}_{translational} + \underbrace{M\vec{\omega} \times \vec{r'}_{com}}_{rotational} = M\vec{v}_C \tag{7}$$

**Total Angular Momentum**

$$\begin{aligned}\vec{L} &= M\vec{r}_{O'} \times \vec{v}_{O'} + M\vec{r}_{O'} \times (\vec{\omega} \times \vec{r'}_{com}) + M\vec{r'}_{com} \times \vec{v}_{O'} + \sum m_i \vec{r_i'} \times (\vec{\omega} \times \vec{r_i'}) \\ &= M\vec{r}_C \times \vec{v}_C + \sum m_i \vec{r_i'} \times (\vec{\omega} \times \vec{r_i'}) = \vec{L}_C + \vec{L'}\end{aligned} \tag{8}$$

# Moment of Inertia Tensor

**Notes:** The notation refers to Landau's Mechanics.

**Definition**

$$I_{jk} = \sum m(x_l^2\delta_{ik} - x_i x_k) = \int \rho(x_l^2\delta_{ik} - x_i x_k)\mathrm{d}V \tag{9}$$

**Kinetic Energy of A Rigid Body**

$$T = \frac{1}{2}MV^2 + \frac{1}{2}I_{ik}\Omega_i\Omega_k \tag{10}$$

**Angular Momentum of A Rigid Body**

$$L_i' = \Omega_k \sum m(x_l^2\delta_{ik} - x_i x_k) = I_{ik}\Omega_k \tag{11}$$

**Principle Axes & Principle Moments of Inertia**

By choosing a better set of $x_1, x_2, x_3$, $I = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix}$ (diagonal form).

# How to Find Principle Axes

1. Find the mass center $C$ of the rigid body, let $C$ be the origin.
2. Use the current coordinates $x, y, z$ to derive the tensor of inertia $I$.
3. Let $det(I - \lambda I_n) = 0$ to find $\lambda_1, \lambda_2, \lambda_3$.
4. Plug back $\lambda_i$ into the equation $(I - \lambda_i I_n)\vec{u_i} = 0$, find the solution $\vec{u_1}, \vec{u_2}, \vec{u_3}$.
5. Use the direction of $\vec{u_1}, \vec{u_2}, \vec{u_3}$ as axes, calculate the new $I_p$.

- **Use symmetry to "guess" the principle axes.**

## Theorems

**Notes:** The notation refers to Landau's Mechanics.

**Perpendicular Axis Theorem**

- Laminar body lies in the $x_1x_2$-plane.

$$I_1 + I_2 = I_3 \tag{12}$$

**Parallel Axis Theorem (Huygens–Steiner)**

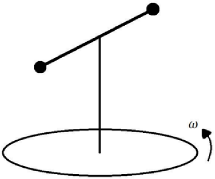
For the displacement $a$ from the center of mass to the new point,

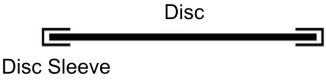
$$I'_{jk} = I_{jk} + M(a^2\delta_{ik} - a_i a_k) \tag{13}$$

# Exercise 1

A uniform thin rod with a mass of $m$ and a length of $2R$. The center of the thin rod is fixed on the upper end of a lightweight vertical bracket, and the angle between the thin rod and the bracket is 60°. A small ball with a mass of M is fixed at each end of the thin rod. The lower end of the bracket is fixed on a uniform disk with a radius of R and a mass of $m/2$. The disk is nested in a fixed disk sleeve, and at the initial moment, the system rotates at an angular velocity $\omega$. It is known that the force between the disk and the disk sleeve only exists at the edge of the disk. Gravity is ignored.

- Calculate the angular momentum of the system at the initial moment.

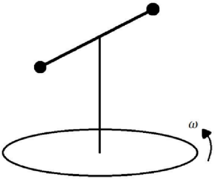


# Exercise 2

A uniform thin rod with a mass of $m$ and a length of $2R$. The center of the thin rod is fixed on the upper end of a lightweight vertical bracket, and the angle between the thin rod and the bracket is 60°. A small ball with a mass of M is fixed at each end of the thin rod. The lower end of the bracket is fixed on a uniform disk with a radius of R and a mass of $m/2$. The disk is nested in a fixed disk sleeve, and at the initial moment, the system rotates at an angular velocity $\omega$. It is known that the force between the disk and the disk sleeve only exists at the edge of the disk. Gravity is ignored.

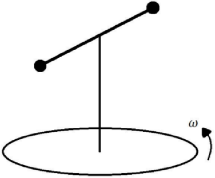
- Without considering friction, calculate the magnitude, direction, and position of the forces acting on the system at the initial moment.

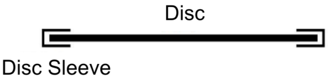



# Exercise 3

A uniform thin rod with a mass of $m$ and a length of $2R$. The center of the thin rod is fixed on the upper end of a lightweight vertical bracket, and the angle between the thin rod and the bracket is 60°. A small ball with a mass of M is fixed at each end of the thin rod. The lower end of the bracket is fixed on a uniform disk with a radius of R and a mass of $m/2$. The disk is nested in a fixed disk sleeve, and at the initial moment, the system rotates at an angular velocity $\omega$. It is known that the force between the disk and the disk sleeve only exists at the edge of the disk. Gravity is ignored.

- If the coefficient of friction between the edge of the disk and the disk sleeve is $\mu = 1.5$, calculate the angular acceleration of the system and the magnitude, direction, and position of the forces at the initial moment.



# Solution 1

Method I (Calculate the inertia tensor):

$$\begin{aligned}
I_{xx} &= m(h^2 + \frac{11R^2}{24}) + M(2h^2 + \frac{19R^2}{12}) \\
I_{xy} &= 0 \\
I_{xz} &= 0 \\
I_{yx} &= 0 \\
I_{yy} &= m(h^2 + \frac{5R^2}{24}) + M(2h^2 + \frac{R^2}{12}) \\
I_{yz} &= -\frac{\sqrt{3}(m+6M)}{12}R^2 \\
I_{zx} &= 0 \\
I_{zy} &= -\frac{\sqrt{3}(m+6M)}{12}R^2 \\
I_{zz} &= \frac{m+3M}{2}R^2
\end{aligned} \tag{14}$$

# Solution 1

$$I = \begin{bmatrix} m(h^2 + \frac{11R^2}{24}) + M(2h^2 + \frac{19R^2}{12}) & 0 & 0 \\ 0 & m(h^2 + \frac{5R^2}{24}) + M(2h^2 + \frac{R^2}{12}) & -\frac{\sqrt{3}(m+6M)}{12}R^2 \\ 0 & -\frac{\sqrt{3}(m+6M)}{12}R^2 & \frac{m+3M}{2}R^2 \end{bmatrix} \tag{15}$$

$$\begin{aligned} L_x &= I_{xz}\omega = 0 \\ L_y &= I_{yz}\omega = -\frac{\sqrt{3}(m+6M)}{12}R^2\omega \\ L_z &= I_{zz}\omega = \frac{m+3M}{2}R^2\omega \end{aligned} \tag{16}$$

# Solution 1

Method II (Calculate the angular momentum):

$$\begin{gathered} I(rod + balls) = \frac{m+6M}{3}R^2 \\ I(disc) = \frac{1}{4}mR^2 \end{gathered} \tag{17}$$

$$\begin{aligned} L(rod + balls)_{z'} &= I\omega_{z'} = \frac{\sqrt{3}(m+6M)}{6}R^2\omega \\ L(rod + balls)_y &= -\frac{1}{2}L(rod + balls)_{z'} = -\frac{\sqrt{3}(m+6M)}{12}R^2\omega \\ L(rod + balls)_z &= \frac{\sqrt{3}}{2}L(rod + balls)_{z'} = \frac{(m+6M)}{4}R^2\omega \\ L(disc)_z &= I\omega = \frac{1}{4}mR^2\omega \end{aligned} \tag{18}$$

$$\begin{gathered} L_y = -\frac{\sqrt{3}(m+6M)}{12}R^2\omega \\ L_z = \frac{m+3M}{2}R^2\omega \end{gathered} \tag{19}$$

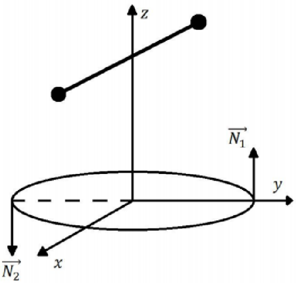
# Solution 2

$$\vec{\tau} = \frac{\mathrm{d}\vec{L}}{\mathrm{d}t} = \vec{\omega} \times \vec{L} = \frac{\sqrt{3}(m+6M)}{12} R^2 \omega^2 \hat{i} \tag{20}$$

$$|\vec{\tau}| = 2NR \tag{21}$$

$$N = \frac{\sqrt{3}(m+6M)}{24} R\omega^2 \tag{22}$$

For $\vec{N}_1$, position: $(0, R, 0)$, direction: $(0, 0, 1)$.
For $\vec{N}_2$, position: $(0, -R, 0)$, direction: $(0, 0, -1)$.



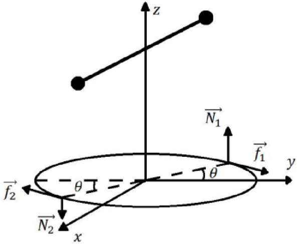
# Solution 3

$$\vec{\tau}=\frac{\mathrm{d}\vec{L}}{\mathrm{d}t}=\vec{\omega}\times\vec{L}+\frac{\mathrm{d}|\vec{L}|}{\mathrm{d}t}\frac{\vec{L}}{|\vec{L}|}=\frac{\sqrt{3}(m+6M)}{12}R^2\omega^2\hat{i}-\frac{\sqrt{3}(m+6M)}{12}R^2\beta\hat{j}+\frac{m+3M}{2}R^2\beta\hat{k} \tag{23}$$

$$\vec{\tau}=2NR\cos\theta\hat{i}+2NR\sin\theta\hat{j}-2\mu NR\hat{k} \tag{24}$$

$$\beta=-\sqrt{\frac{3(m+6M)^2}{13m^2+60mM+36M^2}\omega^2} \tag{25}$$

$$\tan\theta=\sqrt{\frac{3(m+6M)^2}{13m^2+60mM+36M^2}} \tag{26}$$

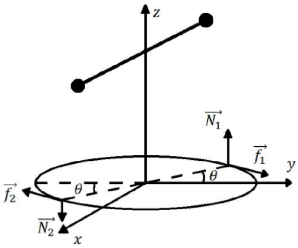


# Solution 3

$$N = \frac{\sqrt{3}(m+3M)(m+6M)}{6\sqrt{13m^2+60mM+36M^2}} R\omega^2 \tag{27}$$

$$f = \frac{\sqrt{3}(m+3M)(m+6M)}{4\sqrt{13m^2+60mM+36M^2}} R\omega^2 \tag{28}$$

For $\vec{F_1}$, position: $R(-\sin\theta, \cos\theta, 0)$, direction: $(\frac{3}{\sqrt{13}}\cos\theta, \frac{3}{\sqrt{13}}\sin\theta, \frac{2}{\sqrt{13}})$.
For $\vec{F_2}$, position: $R(\sin\theta, -\cos\theta, 0)$, direction: $-(\frac{3}{\sqrt{13}}\cos\theta, \frac{3}{\sqrt{13}}\sin\theta, \frac{2}{\sqrt{13}})$.



**Thanks for listening!**

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