Homework #5

Due: 11:59 pm on Nov 6, 2024

Problem 1

Part A

Prove the equation given in class:

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} = q(\mathbf{E} + \mathbf{V} \times \mathbf{B}) = q[-\nabla V - \frac{\partial \mathbf{A}}{\partial t} + \mathbf{V} \times (\nabla \times \mathbf{A})]$$

Part B

Check that the vector potential $\mathbf{A}$ defined in two different ways as

1. $\mathbf{A}(\mathbf{r}) = \frac{1}{2}(\mathbf{B} \times \mathbf{r})$, with $\mathbf{B} = B\hat{n}_z$;
2. $\mathbf{A}(\mathbf{r}) = xB\hat{n}_y$, called the Landau gauge, with uniform magnetic field of magnitude B in the z-axis direction.

Also check that in both cases div$\mathbf{A} = 0$, *i.e.* in both are some particular realizations of the Coulomb gauge.

Problem 2

Part A | Optional

Consider two light bulbs connected in series around a solenoid producing a sinusoidally varying magnetic field. If we alter the circuit by connecting a thick copper wire across the circuit, we find that the top bulb gets much brighter and the bottom one no longer glows. Why does this happen?

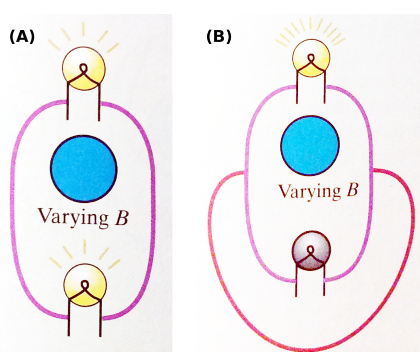

Figure 1: 2A

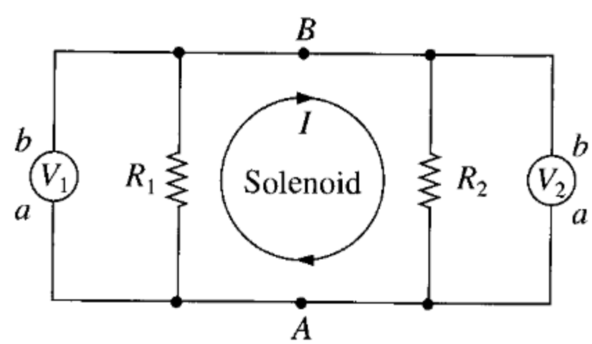

Figure 2: 2B

Part B

The current in a long solenoid piercing the plane of the circuit is increasing linearly with time, so that the magnetic flux through the cross-section of the solenoid $\Phi_B = \alpha t$. Two voltmeters are connected to diametrically opposite points A and B, together with resistors R_1 and R_2.
Show that the readings on voltmeters are $V_1 = \frac{\alpha R_1}{R_1+R_2}$ and $V_2 = -\frac{\alpha R_2}{R_1+R_2}$, respectively, *i.e.* $V_1 \neq V_2$, even though they are connected to the same points.
Assume that these are ideal voltmeters that draw negligible current (huge resistance), and that a voltmeter registers $\int_a^b \mathbf{E} \circ d\mathbf{l}$ between the terminals and through the meter.

Problem 3

Part A

Two coils are wrapped around each other. The current travels in the same sense around each coil. One coil has self-inductance L_1, and the other coil has self-inductance L_2. The mutual inductance of the two coils is M. Show that if the two coils are connected in parallel, the equivalent inductance of the combination is $L = \frac{L_1 L_2 - M^2}{L_1 + L_2 - 2M}$.

Part B

A transformer takes an input AC voltage of amplitude V_1, and delivers an output voltage of amplitude V_2, which is determined by the turns ratio, $V_2/V_1 = N_2/N_1$. If $N_2 > N_1$ the output voltage is greater than the input voltage. Why doesn't this violate conservation of energy? *Answer*: Power is the product of voltage and current; evidently if the voltage goes up, the current must come down.
The purpose of this problem is to see exactly how this works out, in a simplified model.

1. In an ideal transformer the same flux passes through all turns of the primary and of the secondary. Show that in this case $M^2 = L_1 L_2$, where M is the mutual inductance of the coils and L_1, L_2 are their individual self-inductance.

2. Suppose the primary is driven with AC voltage $V_{\text{in}} = V_1 \cos\omega t$, and the secondary connected to a resistor with resistance R. Show that the two currents satisfy the relations
$$L_1 \frac{dI_1}{dt} + M \frac{dI_2}{dt} = V_1 \cos\omega t, \quad L_2 \frac{dI_2}{dt} + M \frac{dI_1}{dt} = -I_2 R.$$

3. Using the result in (a) solve the equations for I_1 and I_2. (Assume that I_1 has no DC component.)

4. Show that the output voltage $V_{\text{out}} = I_2 R$ divided by the input voltage V_{in} is equal to the turns ratio, *i.e.* $V_{\text{out}}/V_{\text{in}} = N_2/N_1$.

5. Calculate the input power $P_{\text{in}} = V_{\text{in}} I_1$ and the output power $P_{\text{out}} = V_{\text{out}} I_2$, and show that their averages over a full cycle are equal.

Part C

One application of LRC series circuits is to high-pass or low-pass filters, which filter out either the low- or high-frequency components of a signal.

1. In a high-pass filter the output voltage is taken across the LR combination. Derive an expression for V_{hi}/V_s, the ratio of the output and source amplitudes as a function of the angular frequency ω of the source. Show that when ω is small, this ratio is proportional to ω and thus is small, and show that the ratio approaches unity in the limit of large frequency.

2. In a low-pass filter the output voltage is taken across the capacitor in an LRC circuit. Derive an expression for V_{lo}/V_s, the ratio of the output and source amplitudes as a function of the angular frequency ω of the source. Show that when ω is large this ratio is proportional to ω^{-2} and thus is small, and show that the ratio approaches unity in the limit of small frequency.

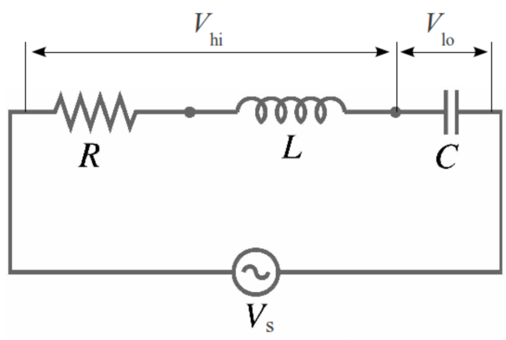

Figure 3: 3C