Homework #4

Due: 11:59 pm on Oct 31, 2024

Problem 1

Part A

A long, straight, solid cylinder of radius a, oriented with its axis in the z-direction, carries an electric current of density

$$\mathbf{J}(\mathbf{r}) = \begin{cases} \frac{b}{r} \exp\left(\frac{r-a}{\delta}\right) \hat{\mathbf{k}} & \text{for } r \leq a \\ 0 & \text{otherwise,} \end{cases}$$

where r is the radial distance from the axis of the cylinder and $a, b, \delta > 0$ are constants (what are their units?).

1. Let I_0 be the total current passing through the entire cross section of the wire. Obtain an expression for I_0 in terms of a, b, δ.
2. Use Ampère's law to find the magnetic field $\mathbf{B}$ in the region $r > a$. Express your answer in terms of I_0 rather than b.
3. Obtain an expression for the current I through a circular cross section of radius $r \leq a$ and centered at the cylinder axis. Express your answer in terms of I_0.
4. Use Ampère's law to find the magnetic field $\mathbf{B}$ in the region $r \leq a$.

Part B

Is Ampère's law consistent with the general rule that you know from calculus that divergence-of-curl is always zero? Show that Ampère's law *cannot* be valid, in general, outside magnetostatics.

Problem 2

Part A

A metal bar of length L, mass m and resistance R is placed on long frictionless metal rails that are inclined at an angle φ above the horizontal. The rails have negligible resistance. A uniform magnetic field of magnitude B is directed downward. The bar is released from rest and slides down the rails.

1. What is the terminal speed of the bar?
2. Is the direction of the current induced in the bar from A to B or from B to A?
3. What is the induced current in the bar when the terminal speed has been reached?
4. After the terminal speed has been reached, at what rate is electrical energy being converted to thermal energy in the resistance of the bar?
5. After the terminal speed has been reached, at what rate is work being done on the bar by gravity? Compare your answer to that in part (d).

Part B

A square shaped conducting loop lies in the xy-plane. The coordinates of its vertices are: $(0,0,0),(0,a,0),(a,a,0)$, and $(a,0,0)$. A magnetic field $\mathbf{B}(\mathbf{r},t)=(0,0,4t^2y)$ is applied. Find the emf and the direction of the resulting current at any instant $t>0$.

Part C

A rectangular loop of wire of length a, width b, and resistance R is initially ($t=0$) placed next to an infinitely long wire carrying current I, so that the side with length a is a distance d from the wire. The loop moves away from the long wire with velocity $\mathbf{v}$ pointing in the direction lying in the plane of the loop and perpendicular to the wire. Find the magnitude of the magnetic flux through the loop, and the current I_{loop} induced in the loop at any instant of time $t>0$.

Part D

A square loop, side a, resistance R, lies a distance s from an infinite straight wire that carries current I. Now someone cuts the wire, so that I drops to zero. In what direction does the induced current in the square loop flow, and what total charge passes a given point in the loop during the time this current flows?

Problem 3

Part A

A capacitor has two parallel plates with area A separated by a distance d. The space between plates is filled with a material having relative dielectric permittivity ϵ_r. The material is not a perfect insulator, but has resistivity ρ. The capacitor is initially charged with charge of magnitude Q_0 on each plate, which gradually discharges by conduction through the dielectric.

1. Calculate the conduction current density $J_c(t)$ in the dielectric.
2. Show that at any instant the displacement current density in the dielectric is equal in magnitude to the conduction current density but opposite in direction, so the *total* current density is zero at every instant.

Part B

It is impossible to have a uniform electric field that abruptly drops to zero in a region of space in which the magnetic field is constant. Prove this statement.

1. In the bottom half of a piece of paper, draw evenly spaced horizontal lines representing a uniform electric field to your right. Use dashed lines to draw a rectangle $ABCDA$ with horizontal side AB in the electric field region and horizontal side CD in the top half of your paper where $\mathbf{E}=0$.
2. Show the integral along the loop formed by your rectangle contradicts Faraday's law.