HW 8 Solutions.

Problem 1.

Part A.

1. $\int_0^a \psi^*(x)\psi(x)\,dx = \int_0^a |A|^2 \sin^2\frac{\pi x}{a}\,dx = 1 \Rightarrow |A| = \sqrt{\frac{2}{a}}$.

2. $\int_{-\infty}^{+\infty} |A|^2 \exp(-a^2x^2)\,dx = 1 \Rightarrow |A| = \sqrt[4]{\frac{a^2}{\pi}}$

3. $P(x) = e^{-ikx}e^{ikx} = 1$

   If $\int_{-\infty}^{+\infty} |A|^2 e^{-ikx}e^{ikx}\,dx = 1 \Rightarrow A = 0$.

   $\int_{-\infty}^{+\infty} \frac{1}{2\pi}\exp(-ik'x)\exp(ikx)\,dx = \delta(k-k')$

4. $P(x) = \left|\int_{-\infty}^{+\infty} \delta(x'-x_0)\,\delta(x-x')\,dx'\right|^2 = \delta^2(x-x_0)$.

   $\int_{-\infty}^{+\infty} \delta(x-x_1)\,\delta(x-x_2)\,dx = \delta(x_1-x_2)$

Part B.

1. $P(r)\,dr = \left(\int_0^{\pi}\int_0^{2\pi} |\psi(r,\theta,\varphi)|^2 \sin\theta\,d\theta\,d\varphi\right) r^2\,dr$.

2. $P(\theta,\varphi)\,d\Omega = \left(\int_0^{\infty} |\psi(r,\theta,\varphi)|^2 r^2\,dr\right) d\Omega$.

Problem 2.

Part A.

$\bar{x} = \bar{p} = 0$. $\Delta x \sim a$. $\Delta p \sim p$.

$\Delta x \cdot \Delta p \geq \frac{\hbar}{2}$. $\Rightarrow p \geq \frac{\hbar}{2a}$.

energy is $E_0 = \frac{p^2}{2m} \sim \frac{\hbar^2}{8ma^2}$

Part B.

$\psi(x) = \frac{A}{4}(2\pi\hbar)^{1/2}\left[(2\pi\hbar)^{-1/2}e^{\frac{i}{\hbar}p_1x} + (2\pi\hbar)^{-1/2}e^{\frac{i}{\hbar}p_2x} - (2\pi\hbar)^{-1/2}e^{\frac{i}{\hbar}p_3x} - (2\pi\hbar)^{-1/2}e^{\frac{i}{\hbar}p_4x} + 2(2\pi\hbar)^{-1/2}e^{\frac{i}{\hbar}p_5x}\right]$.

$p_1 = \hbar k$, $w_1 = \frac{\pi\hbar}{8}|A|^2$. $p_2 = -\hbar k$, $w_2 = \frac{\pi\hbar}{8}|A|^2$

$p_3 = 2\hbar k$, $w_3 = \frac{\pi\hbar}{8}|A|^2$ $p_4 = -2\hbar k$, $w_4 = \frac{\pi\hbar}{8}|A|^2$

$p_5 = 0$, $w_5 = \frac{\pi\hbar}{2}|A|^2$

$\Rightarrow \bar{p} = 0$ $\bar{T} = \frac{1}{2m}\overline{p^2} = \frac{5k^2\hbar^2}{8m}$.

From normalization, $\sum_{i=1}^{5} w_i = 1 \Rightarrow A = (\pi\hbar)^{-1/2}$

Part C 1. $\psi(p,0) = \frac{1}{(2\pi\hbar)^{1/2}}\int_{-\infty}^{+\infty} e^{-ipx/\hbar}\psi(x,0)\,dx = \frac{Aa}{\sqrt{2\hbar}}\exp(-\frac{a^2p^2}{4\hbar^2})$.

2. $\psi(p,t) = \frac{Aa}{\sqrt{2\hbar}}\exp(-\frac{a^2p^2}{4\hbar} - \frac{ip^2t}{2m\hbar})$

$\psi(x,t) = \frac{Aa}{\sqrt{a^2+2i\hbar t/m}}\exp[-\frac{x^2}{(a^2+2i\hbar t/m)}]$.

Problem 3.

Part A. 1. $\psi_n = \sqrt{\frac{2}{a}}\sin\frac{n\pi x}{a}$, $E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}$

$\psi(x,0) = \sum_n A_n\psi_n$, $\psi(x,0) = A_1\psi_1 + A_2\psi_2$.

where $A_1 = \sqrt{\frac{4}{5}}$. $A_2 = \sqrt{\frac{1}{5}}$

$\psi(x,t_0) = \sqrt{\frac{4}{5}}\left(\sqrt{\frac{2}{a}}\sin\frac{\pi x}{a}\right)e^{-iE_1t_0/\hbar} + \sqrt{\frac{1}{5}}\left(\sqrt{\frac{2}{a}}\sin\frac{2\pi x}{a}\right)e^{-iE_2t_0/\hbar}$

2. $\langle E\rangle = \frac{4\pi^2\hbar^2}{5ma^2}$

3. $\int_0^{a/2}\psi^*(x,t_0)\psi(x,t_0)\,dx = \frac{1}{2} + \frac{16}{15\pi}\cos\frac{3\pi^2\hbar t_0}{2ma^2}$

Part B.

$\psi_n(x) = \sqrt{\frac{2}{l}}\sin(\frac{\pi x}{l}n)$, $E_n = \frac{\hbar}{2m}(\frac{\pi}{l}n)^2$. $n = 1, 2, 3, \ldots$

$\langle n|\psi(t=0)\rangle = \int_0^l dx\sqrt{\frac{2}{l}}\sin(\frac{\pi x}{l}n)\sqrt{\frac{30}{l^5}}\,x(l-x)$

$= 4\sqrt{15}\left(\frac{1}{n\pi}\right)^3[1-(-1)^n]$

$\psi(x,t) = \sum_{n=1}^{\infty}\langle n|\psi(t=0)\rangle\psi_n(x)\exp(-i\frac{E_n}{\hbar}t)$

$= 8\sum_{n=1}^{\infty}\sqrt{\frac{30}{l}}\frac{1}{(2n+1)^3\pi^3}\sin\left(\frac{2n+1}{l}\pi x\right)e^{-i\frac{\hbar}{2m}\left(\frac{2n+1}{l}\pi\right)^2t}$