Solutions for HW 5.

Problem 1.

Part A. We have $\vec{\mathcal{E}} = -\nabla V - \frac{\partial \vec{A}}{\partial t}$. $\vec{B} = \nabla \times \vec{A}$.

Just plug in, then we can get the final expression.

Part B. 1. $\vec{B} = \nabla \times \vec{A} = \nabla \times \frac{1}{2}(\vec{B} \times \vec{r}) = \frac{1}{2}(\partial_x B_x + \partial_y B_y)\hat{k} = B\hat{k}$.

$\nabla \cdot \vec{A} = 0$

2. $\vec{B} = \nabla \times \vec{A} = B\hat{k}$. $\nabla \cdot \vec{A} = 0$.

Problem 2.

Part A. Upper bulb includes the circuit with the varying B.

while lower bulb's circuit excludes the B.

Part B. $V = -\frac{d\phi_B}{dt} = -\alpha$. $V_1 = \alpha \frac{R_1}{R_1+R_2}$. $V_2 = -\alpha \frac{R_2}{R_1+R_2}$.

Problem 3.

Part A. $V = L_1 \dot{I}_1 + M\dot{I}_2 = L_2\dot{I}_2 + M\dot{I}_1$

$\Rightarrow (M - L_1)\dot{I}_1 = (M - L_2)\dot{I}_2$

$$L = \frac{V}{\dot{I}_1 + \dot{I}_2} = \frac{L_1L_2 - M^2}{L_1 + L_2 - 2M}.$$

Part B. 1. $M = L_2 \frac{N_1}{N_2} = L_1 \frac{N_2}{N_1} \Rightarrow M^2 = L_1 L_2$.

2. $V_{in} = -\mathcal{E}_1 = \frac{d\phi_{B1}}{dt} \Rightarrow V_1 \cos\omega t = L_1 \frac{d}{dt} I_1 + M \frac{d}{dt} I_2$

$\mathcal{E}_2 = I_2 R = -\frac{d\phi_{B2}}{dt} \Rightarrow L_2 \frac{dI_2}{dt} + M\frac{dI_1}{dt} = -I_2 R$.

3. $\begin{cases} L_1\dot{I}_1 + \sqrt{L_1L_2}\,\dot{I}_2 = V_1\cos\omega t \\ L_2\dot{I}_2 + \sqrt{L_1L_2}\,\dot{I}_1 = -I_2R \end{cases} \Rightarrow \begin{cases} I_1 = \frac{V_1}{L_1\omega}\sin\omega t + \frac{L_2V_1}{L_1R}\cos\omega t \\ I_2 = -\frac{MV_1}{L_1R}\cos\omega t. \end{cases}$

4. $V_{out}/V_{in} = \frac{M}{L_1} = \frac{\sqrt{L_2}}{\sqrt{L_1}} = \frac{N_2}{N_1}$

5. $P_{in} = V_{in}I_1$, $P_{out} = V_{out}I_2$. $\Rightarrow \overline{P_{in}} = \overline{P_{out}} = \frac{1}{2}\frac{V_1^2L_2}{L_1R}$.

Part C. 1. $\tilde{V}_{hi}/\tilde{V}_s = \dfrac{R + j\omega L}{R + \frac{1}{j\omega C} + j\omega L} \Rightarrow V_{hi}/V_s = \sqrt{\dfrac{\omega^2C^2(R^2 + \omega^2L^2)}{\omega^2R^2C^2 + (1 + \omega^2LC)^2}}$

$\omega \to$ small : $V_{hi}/V_s = RC\omega \propto \omega$. $\omega \to$ big $V_{hi}/V_s = 1$.

2. $\tilde{V}_{hi}/\tilde{V}_s = \dfrac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C} + j\omega L} \Rightarrow V_{hi}/V_s = 1/\sqrt{\omega^2R^2C^2 + (1 + \omega^2LC)^2}$

$\omega \to$ small: $V_{hi}/V_s = 1$. $\omega \to$ big. $V_{hi}/V_s = \frac{1}{\omega^2LC} \propto \omega^{-2}$