Homework #6

Due: 11:59 pm on Nov 20, 2024

Problem 1

Part A

Electromagnetic waves propagate much differently in conductors than they do in dielectrics or in vacuum. If the resistivity of the conductor is sufficiently low (that is, if it is a sufficiently good conductor), the oscillating electric field of the wave gives rise to an oscillating conduction current that is much larger than the displacement current. In this case, the wave equation for an electric field $\mathbf{E}(x,t) = (0, E_y(x,t), 0)$ propagating in the $+x$ direction within a conductor is

$$\frac{\partial^2 E_y(x,t)}{\partial x^2} = \frac{\mu}{\rho}\frac{\partial E_y(x,t)}{\partial t},$$

where μ is the permeability of the conductor and ρ is its resistivity.

1. Check that
$$E_y(x,t) = E_0 e^{-k_C x} \sin(k_C x - \omega t),$$
where $k_C = \sqrt{\omega\mu/2\rho}$, is a solution of the above wave equation.

2. The exponential term shows that the electric field decreases in amplitude as it propagates. Explain why this happens.
Hint. The field does work to move charges within the conductor. The current of these moving charges causes heating within conductor. Where does this energy come from?

3. Note that the electric-field amplitude decreases by a factor of $1/e$ in a distance $1/k_C$ and calculate this distance for a radio wave with frequency $\nu = 1\text{MHz}$ in copper (resistivity $1.7210^{-8}\Omega \cdot \text{m}$, relative permeability $\mu_r = 1$). Since the distance is so short, electromagnetic waves of this frequency can hardly propagate at all into copper. Instead they are reflected at the surface of the metal. This is why radio waves cannot penetrate through copper or other metals.

Problem 2

Part A

An electromagnetic wave with an angular frequency ω passes through a medium with a free electron density of ρ_e.

1. Find the current density caused by the electric field $\vec{\boldsymbol{E}}$, assuming that the interaction between electrons is ignored;

2. Write the differential equations of electromagnetic waves in the medium;

3. Write the necessary and sufficient conditions for electromagnetic waves to pass through an infinite medium from the equations in 2.

Part B (Hard)

A plane-polarized electromagnetic wave $E = E_{y0}e^{i(kz-\omega t)}$ is incident vertically on a semi-infinite medium with a magnetic permeability of μ, a dielectric constant of ε, and a conductivity of σ.

1. For metals with large σ, the electric field in the medium at low frequencies is derived from the Maxwell equations;
2. The same deduction is made for rarefied plasma. Here, the conductivity is limited by inertia and is not constrained by electron scattering, i.e., $\sigma = i\frac{ne^2}{m\omega}$;
3. Based on the above results, discuss the optical properties of metals in the ultraviolet band.

Problem 3

Part A

A cylindrical conductor with radius a and resistivity ρ carries a constant current I.

1. What are the magnitude and direction of the electric field vector $\mathbf{E}$ and the magnetic field vector $\mathbf{B}$ at a point just outside the wire and at a distance a from the axis of the cylinder?
2. What are the magnitude and direction of the Poynting vector $\mathbf{S}$ at the same point?
3. Find the rate of flow of energy into the volume occupied by a length l of the conductor. *Hint.* What is the physical meaning of the Poynting vector integrated over a closed surface?
4. How does your result compare to the rate of generation of thermal energy in the same volume? Why does the Poynting vector point in the direction found in 2?