PHYS2600J Recitation Class 3

Week 8

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Basics of Vector Analysis

- Vector Algebra
- Differential Calculus
- Integral Calculus
  - $\int_{\mathbf{a}}^{\mathbf{b}} (\boldsymbol{\nabla} f) \cdot \mathrm{d}\boldsymbol{l} = f(\mathbf{b}) - f(\mathbf{a})$
  - $\int_V (\boldsymbol{\nabla} \cdot \mathbf{v}) \, \mathrm{d}\tau = \oint_{\partial V} \mathbf{v} \cdot \mathrm{d}\boldsymbol{A}$
  - $\int_S (\boldsymbol{\nabla} \times \mathbf{v}) \cdot \mathrm{d}\mathbf{A} = \oint_{\partial S} \mathbf{v} \cdot \mathrm{d}\mathbf{l}$
- Coordinates
- Theory of Vector Fields
  - $\mathbf{f} = -\boldsymbol{\nabla} V + \boldsymbol{\nabla} \times \mathbf{A}$

Basics of Electrostatics

- Coulomb's Law
  - $\vec{F} = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r^2}\hat{\mathbf{r}}$
- Gauss's Law
  - $\oint_S \vec{\mathbf{E}} \cdot \mathrm{d}\vec{\mathbf{A}} = \frac{Q_{enc}}{\epsilon_0}$
  - $\boldsymbol{\nabla} \cdot \vec{\mathbf{E}} = \frac{\rho(\vec{r})}{\varepsilon_0}$
- Electric Potential
  - $V(\vec{\mathbf{r}}) = -\int_O^{\vec{\mathbf{r}}} \vec{\mathbf{E}} \cdot \mathrm{d}\vec{\mathbf{l}}$
  - $\nabla^2 V = -\frac{\rho}{\varepsilon_0}$
- Electrostatic Energy
  - $U(\vec{\mathbf{r}}) = qV(\vec{\mathbf{r}})$
  - $U_{conf} = \frac{1}{8\pi\varepsilon_0}\sum_i \sum_{j,i\neq j} \frac{q_i q_j}{r_{ij}} = \frac{1}{2}\sum_i q_i V(\vec{\mathbf{r_i}})$
  - $U_{conf} = \frac{1}{2}\int_\Omega \rho V \,\mathrm{d}\tau = \frac{\varepsilon_0}{2}\left(\oint_\Sigma V\vec{\mathbf{E}} \cdot \mathrm{d}\vec{\mathbf{A}} + \int_\Omega E^2 \,\mathrm{d}\tau\right)$
  - $U_{conf} = \frac{\varepsilon_0}{2}\int_{\mathbb{R}^3} E^2 \,\mathrm{d}\tau$
- Conductors
  - Basic Properties
  - $C = \frac{Q}{V}$

Basics of Magnetostatics

- The Lorentz Force Law
  - $\vec{\mathbf{F}} = \frac{\mathrm{d}\vec{\mathbf{p}}}{\mathrm{d}t} = q(\vec{\mathbf{E}} + \vec{\mathbf{v}} \times \vec{\mathbf{B}})$
- The Biot-Savart Law
  - $\vec{\boldsymbol{B}}(\vec{\boldsymbol{r}}) = \frac{\mu_0}{4\pi} \int_{\Gamma} \frac{I\mathrm{d}\vec{\mathbf{l}}}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}'}|^2} \times \frac{\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}'}}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}'}|}$
- Ampère's Law
  - $\oint \vec{\boldsymbol{B}} \cdot \mathrm{d}\vec{\boldsymbol{l}} = \mu_0 I_{enc}$
  - $\boldsymbol{\nabla} \times \vec{\boldsymbol{B}} = \mu_0 \vec{\boldsymbol{J}}$
- Magnetic Vector Potential
  - $\vec{\mathbf{B}} = \boldsymbol{\nabla} \times \vec{\mathbf{A}}$
  - Gauge: $\boldsymbol{\nabla} \cdot \vec{\mathbf{A}}$

Basics of Electrodynamics

- Electromotive Force
  - $\vec{\mathbf{J}} = \sigma\vec{\mathbf{E}}$
  - $\varepsilon = \int(\vec{\mathbf{v}} \times \vec{\mathbf{B}}) \cdot \mathrm{d}\vec{\mathbf{l}}$
  - $\varepsilon = -\frac{\partial \Phi_B}{\partial t}$
- Electromagnetic Induction
  - $L = \frac{\Phi}{I}$
  - $M_{12} = M_{21} = \frac{\Phi_2}{I_1} = \frac{\Phi_1}{I_2} = \frac{\mu_0}{4\pi} \oint \oint \frac{\mathrm{d}\mathbf{l}_1 \cdot \mathrm{d}\mathbf{l}_2}{r}$
- Maxwell's Equations
  - $\oint \vec{\mathbf{D}} \cdot \mathrm{d}\vec{\mathbf{a}} = Q_{enc}$ $\quad \boldsymbol{\nabla} \cdot \vec{\mathbf{D}} = \rho$
  - $\oint \vec{\mathbf{E}} \cdot \mathrm{d}\vec{\mathbf{l}} = -\frac{\mathrm{d}\Phi_B}{\mathrm{d}t}$ $\quad \boldsymbol{\nabla} \times \vec{\mathbf{E}} = -\frac{\partial \vec{\mathbf{B}}}{\partial t}$
  - $\oint \vec{\mathbf{B}} \cdot \mathrm{d}\vec{\mathbf{a}} = 0$ $\quad \boldsymbol{\nabla} \cdot \vec{\mathbf{B}} = 0$
  - $\oint \vec{\mathbf{H}} \cdot \mathrm{d}\vec{\mathbf{l}} = I_{enc} + \varepsilon_0 \frac{\mathrm{d}\Phi_E}{\mathrm{d}t}$ $\quad \boldsymbol{\nabla} \times \vec{\mathbf{H}} = \vec{\mathbf{J}} + \varepsilon_0 \frac{\partial \vec{\mathbf{E}}}{\partial t}$

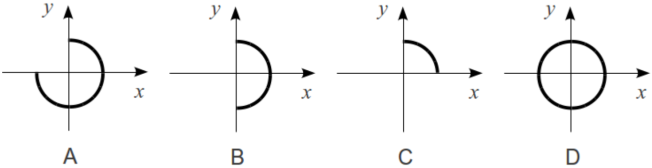
Others

- Circuits
  - KCL: $\sum_k I_k = 0$
  - KVL: $\sum_k V_k = 0$
  - RLC: $\frac{d^2 I(t)}{dt^2} + \frac{R}{L}\frac{dI(t)}{dt} + \frac{1}{LC} I(t) = 0$
- Polarization & Magnetization
  - Electric Dipole
  - Magnetic Moment
  - $\vec{\mathbf{D}} = \varepsilon \vec{\mathbf{E}}, \vec{\mathbf{B}} = \mu \vec{\mathbf{H}}$
- Uniqueness Theorem
- The Method of Images
- Lenz's law

Review lecture notes carefully, especially the unfamiliar concepts!

Exercise 1 | HW1 P1A

Four circular plastic rods are charged uniformly, each with charge $Q < 0$. Rank the four arrangements according to the magnitude of the electric field at the origin. Explain your answer.

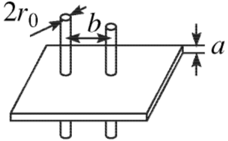


Exercise 2

A conical surface (an empty ice-cream cone) carries a uniform surface charge with density σ. The height of the cone is h, as is the radius of the top. Find the electric potential difference between points $\mathbf{r}_A$ (the vertex) and $\mathbf{r}_B$ (the center of the top).

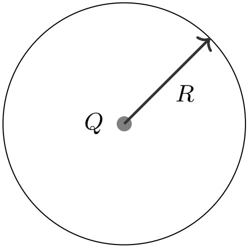
Exercise 3 | HW2 P3A

Two thin conducting wires shaped as cylinders with radii r_0, separated by distance b from each other, are attached to a large conducting slab of thickness a, where $a \ll r_0 \ll b$. Estimate the resistance between the wires, assuming that the conductivity of the wires σ_0 is much larger than the conductivity of the slab σ.



Exercise 4

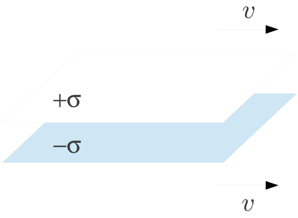
There is a metal ball (R, Q) in space. Calculate the force between the upper and lower half balls.



Exercise 5 | HW3 P3B

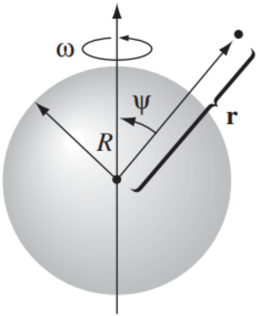
A large parallel-plate capacitor with uniform surface charge of density σ on the upper plate and $-\sigma$ on the lower is moving with a constant speed v, as shown in the figure below.

1. Find the magnetic field between the plates and also above and below them.
2. Find the magnetic force per unit area on the upper plate, including its direction.
3. At what speed v would the magnetic force balance the electric force?



Exercise 6 | RC2 P1

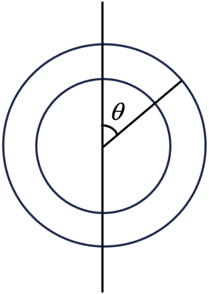
A spherical shell of radius R, carrying a uniform surface charge σ, is set spinning at an angular velocity ω. What is the magnetic field distribution?



Exercise 7 | Add-HW1 P2

A spherical capacitor with inner and outer radii R_1 and R_2 is filled with a dielectric whose permittivity varies with the angle $\varepsilon(\theta) = \varepsilon_0(1 + \varepsilon_r \sin\theta \cos^2\theta)$.

1. Find the capacitance of this capacitor.
2. When the inner and outer conductive spheres are charged with $+Q$ and $-Q$ respectively, find the electric field distribution inside the capacitor.



Exercise 8

Two circular loops of wire share a common axis of symmetry - which we'll denote as x axis - as shown in the figure below. A larger loop of radius R_1, is at the coordinate origin. A smaller loop of radius $R_2 < R_1$ is at the distance x from the coordinate origin, where $x \gg R_1$. A current I_1 runs in the larger loop, with the direction indicated in the figure. Due to a large distance between the loops, we can assume that the magnetic field is nearly uniform through the second, smaller loop. If I_1, changes in time, what is the emf ε_2 induced in the second loop? What is the coefficient of mutual inductance, M, between the two loops?

Thanks for listening!

References

D. J. Griffiths.
Introduction to Electrodynamics.
Cambridge University Press, 5 edition, 2023.