PHYS2600J Recitation Class 5 (Final)

Week 13

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Outline

- Circuits
- EM Waves
- Optics
- Quantum Physics

RLC Circuit

Differential Equation

$$\frac{\mathrm{d}^2q(t)}{\mathrm{d}t^2}+\frac{R}{L}\frac{\mathrm{d}q(t)}{\mathrm{d}t}+\frac{1}{LC}q(t)=0$$

Solutions

- Underdamped: $R<\sqrt{4L/C}$

$$q(t)=Ae^{-\frac{R}{2L}t}\cos\left(\sqrt{1-\frac{R^2}{4L/C}}\frac{t}{\sqrt{LC}}+\varphi\right)$$

- Overdamped: $R>\sqrt{4L/C}$

$$q(t)=Ae^{-(\frac{R}{\sqrt{4L/C}}+\sqrt{\frac{R^2}{4L/C}-1})\frac{t}{\sqrt{LC}}}+Be^{-(\frac{R}{\sqrt{4L/C}}-\sqrt{\frac{R^2}{4L/C}-1})\frac{t}{\sqrt{LC}}}$$

- Critical: $R=\sqrt{4L/C}$

$$q(t)=(A+Bt)e^{-\frac{R}{2L}t}$$

AC Circuit

AC Current

$$I(t) = I\cos(\omega t + \phi) = I\,\mathrm{Re}\left\{e^{j(\omega t+\phi)}\right\}$$

$$I_{arv} = \frac{1}{T}\int_{t_1}^{t_1+T} |I(t)|\mathrm{d}t = \frac{2}{\pi}I$$

$$I_{rms} = \sqrt{\frac{1}{T}\int_{t_1}^{t_1+T} I^2(t)\mathrm{d}t} = \frac{I}{\sqrt{2}}$$

Resonance

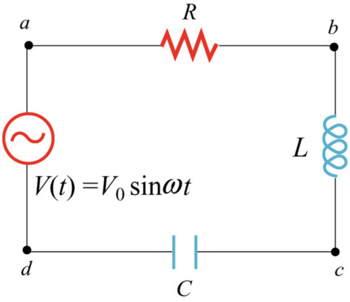
R	R	0
L	$j\omega L$	$\pi/2$
C	$1/(j\omega C)$	$-\pi/2$

$$|Z| = \sqrt{R^2 + (\omega L - \frac{1}{\omega C})^2}, \quad \omega_0 = \frac{1}{\sqrt{LC}}$$

Exercise 1

Suppose an AC generator with $V(t) = (150V)\sin(100t)$ is connected to a series RLC circuit with $R = 40.0\Omega$, $L = 80.0mH$, and $C = 50.0\mu F$.

1. Calculate V_{R0}, V_{L0}, and V_{C0}, the maximum of the voltage drops across each circuit element.
2. Calculate the maximum potential difference across the inductor and the capacitor between points b and d.



Solution 1

1

$$|Z| = \sqrt{R^2 + (\omega L - \frac{1}{\omega C})^2} = 196\Omega$$

$$I_0 = \frac{V_0}{|Z|} = 0.765A$$

$$V_{R0} = I_0 R = 30.6V$$

$$V_{L0} = I_0(\omega L) = 6.12V$$

$$V_{C0} = I_0(\frac{1}{\omega C}) = 153V$$

2

$$|V_{bd} = |V_{L0} - V_{C0}| = 147V$$

Classical Waves

Wave Equation

$$\frac{\partial^2 f}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2} = 0$$

Sinusoidal Wave Function

- $f(x,t) = A\cos(k(x - vt) + \varphi)$
- Complex Notation: $\tilde{f}(x,t) = \tilde{A}e^{i(kx-\omega t)}$
- Parameters:

A	amplitude
k	wave number
$k(x-vt)+\varphi$	phase
φ	phase constant ($0 \le \varphi < 2\pi$)
$\lambda = k/2\pi$	wave length
$T = 2\pi/kv$	period
$\nu = 1/T$	frequency
$\omega = 2\pi\nu$	angular frequency

EM Waves

in Vacuum

$$\nabla^2\left(\vec{\mathbf{E}},\vec{\mathbf{B}}\right)-\mu_0\varepsilon_0\frac{\partial^2(\vec{\mathbf{E}},\vec{\mathbf{B}})}{\partial t^2}=0$$

in Conducting Medium

$$\nabla^2\left(\vec{\mathbf{E}},\vec{\mathbf{H}}\right)-\sigma\mu\frac{\partial(\vec{\mathbf{E}},\vec{\mathbf{H}})}{\partial t}-\mu\varepsilon\frac{\partial^2(\vec{\mathbf{E}},\vec{\mathbf{H}})}{\partial t^2}=0$$

Complex Notation

$$\nabla=ik,\quad \frac{\partial}{\partial t}=-i\omega$$

Poynting Vector

$$\vec{\mathbf{S}}=\vec{\mathbf{E}}\times\vec{\mathbf{H}}$$

$$\boldsymbol{\nabla}\cdot\vec{\mathbf{S}}=-\frac{\partial u}{\partial t}$$

Exercise 2

1. Consider a long straight cylindrical wire of electrical conductivity σ and radius a carrying a uniform axial current of density J. Calculate the magnitude and direction of the Poynting vector at the surface of the wire.
2. Consider a thick conducting slab (conductivity σ) exposed to a plane EM wave with peak amplitudes E_0, B_0. Calculate the Poynting vector within the slab, averaged in time over a wave period. Consider σ large, i.e. $\sigma \gg \omega\varepsilon_0$.
3. In part 2, if σ is infinite, what is the value of the average Poynting vector everywhere in space?

Solution 2

1

$$\vec{\mathbf{E}} = \frac{J}{\sigma}\mathbf{e_z}, \quad \vec{\mathbf{B}} = \frac{\mu_0 J a}{2}\mathbf{e}_\theta$$

$$\vec{\mathbf{S}} = \frac{1}{\mu_0}\vec{\mathbf{E}} \times \vec{\mathbf{B}} = -\frac{J^2 a}{2\sigma}\mathbf{e_r}$$

2

$$\vec{\mathbf{k}} = (\beta + i\alpha)\mathbf{e_z}, \quad \alpha = \beta \approx \sqrt{\frac{\omega\mu_0\sigma}{2}} \quad \text{(from HW Problems)}$$

$$\vec{\mathbf{E}}(\vec{\mathbf{r}}, t) = \vec{\mathbf{E}}_0(\vec{\mathbf{r}})e^{-\alpha z}e^{i(\beta z - \omega t)}$$

$$\vec{\mathbf{S}} = \vec{\mathbf{E}} \times \vec{\mathbf{H}} = \vec{\mathbf{E}} \times \frac{1}{\omega\mu_0}(\vec{\mathbf{k}} \times \vec{\mathbf{E}}) = \sqrt{\frac{\sigma}{\omega\mu_0}}e^{i\frac{\pi}{4}}E^2\mathbf{e_z}$$

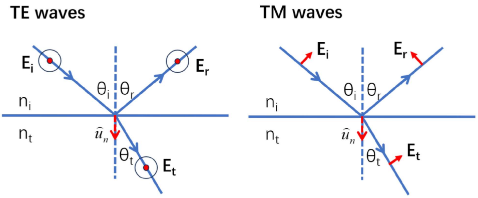
$$\langle\vec{\mathbf{S}}\rangle = \frac{\sqrt{2}}{4}\sqrt{\frac{\sigma}{\omega\mu_0}}E_0^2 e^{-2\alpha z}\mathbf{e_z}$$

3

$$\sigma \to \infty, \alpha \to \infty, \sqrt{\sigma}e^{-2\alpha z} \to 0, \quad \langle\vec{\mathbf{S}}\rangle \to 0$$

Reflection and Refraction

Fresnel Equations



Boundary Condition: $\hat{\boldsymbol{u}}_{\boldsymbol{n}} \times (\boldsymbol{E_i} + \boldsymbol{E_r}) = \hat{\boldsymbol{u}}_{\boldsymbol{n}} \times \boldsymbol{E_t}$

$$r_{TE} = \frac{n_i \cos\theta_i - n_t \cos\theta_t}{n_i \cos\theta_i + n_t \cos\theta_t} \quad r_{TM} = \frac{n_t \cos\theta_i - n_i \cos\theta_t}{n_t \cos\theta_i + n_i \cos\theta_t}$$

$$t_{TE} = \frac{2n_i \cos\theta_i}{n_i \cos\theta_i + n_t \cos\theta_t} \quad t_{TM} = \frac{2n_i \cos\theta_i}{n_t \cos\theta_i + n_i \cos\theta_t}$$

π Phase Shift: $r_{TE}, r_{TM} < 0 (n_i < n_t)$ or $r_{TE} > 0, r_{TM} < 0 (n_i > n_t)$
Brewster's Angle / Polarization Angle: $r_{TM} = 0$, $\theta_p = \arctan(n_t/n_i)$

Interference

Young's Double-Slit Experiment

$$\frac{I}{I_0} = [\cos(\delta/2)]^2, \delta = \frac{2\pi d \sin\theta}{\lambda}$$

Newton's Rings

$$r_N = \begin{cases} \sqrt{\lambda R N} & \text{dark rings} \\ \sqrt{\lambda R (N - \frac{1}{2})} & \text{bright rings} \end{cases}$$

Interference in Thin Films

$$\frac{I_T}{I_0} = \frac{(1-R)^2}{(1-R)^2 + 4R\sin^2(\delta/2)}, \quad \frac{I_R}{I_0} = \frac{4R\sin^2(\delta/2)}{(1-R)^2 + 4R\sin^2(\delta/2)}$$

$$\delta = \frac{4\pi n h \cos i}{\lambda}$$

Diffraction

Fraunhofer Diffraction

Single Slit:

$$\frac{I}{I_0} = [\frac{\sin(\alpha)}{\alpha}]^2, \alpha = \frac{\pi a \sin\theta}{\lambda}$$

Rectangular Aperture:

$$\frac{I}{I_0} = [\frac{\sin(\alpha)}{\alpha}]^2[\frac{\sin(\beta)}{\beta}]^2, \alpha = \frac{\pi a \sin\theta}{\lambda}, \beta = \frac{\pi b \sin\theta}{\lambda}$$

Circular Aperture:

$$\frac{I}{I_0} = [\frac{2J_1(x)}{x}]^2, x = \frac{2\pi a \sin\theta}{\lambda}$$

$$\Delta\theta = 0.61\frac{\lambda}{a} = 1.22\frac{\lambda}{D}$$

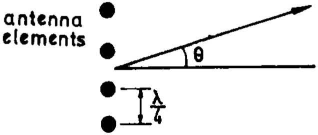
Diffraction Grating

$$\frac{I}{I_0} = [\frac{\sin(\alpha)}{\alpha}]^2[\frac{\sin(N\beta)}{\beta}]^2, \alpha = \frac{\pi a \sin\theta}{\lambda}, \beta = \frac{\pi d \sin\theta}{\lambda} = \frac{\pi(a+b)\sin\theta}{\lambda}$$

Exercise 3

There is a phased array of radar antenna elements spaced at $1/4$ wavelength. The transmitting phase is shifted by steps of $\pi/6$ from element to element, i.e., element 0 has 0 phase, element 1 has $\pi/6$ phase shift, element 2 has $2\pi/6$, element 3 has $3\pi/6$, etc.

1. At what angle θ is the zero order constructive interference transmission lobe?
2. Are there any secondary lobes?



Solution 3

$$\delta = \frac{2\pi}{\lambda}\frac{\lambda}{4}\sin\theta + \frac{\pi}{6} = \frac{\pi}{2}(\sin\theta + \frac{1}{3})$$

$$\tilde{A}_{tot} = \sum_{k=0}^{N-1} \tilde{A}e^{ik\delta}, \quad I \sim [\frac{\sin(N\delta/2)}{\sin(\delta/2)}]^2$$

1

$$\delta/2 = 0 \rightarrow \theta = -\arcsin\frac{1}{3}$$

2

$$\delta/2 = \pm\pi \rightarrow \sin\theta = -\frac{1}{3} \pm 4$$

which means no secondary lobe can occur.

Introduction to Quantum Physics

Wave Function

- Normalization: $\iiint_{\mathbb{R}^3} |\Psi(\vec{\mathbf{r}},t)|^2 \mathrm{d}^3 r = 1$
- Probability Density: $\Pr(\vec{\mathbf{r}},t) = |\Psi(\vec{\mathbf{r}},t)|^2$
- Stationary State: $|\Psi(\vec{\mathbf{r}},t)|^2 = \Phi(\vec{\mathbf{r}})$
- Inner Product: $\langle \cdot,\cdot \rangle \in \mathcal{L}^2(\mathbb{R}^3), \langle \Psi, \Phi \rangle = \iiint_{\mathbb{R}^3} \Psi^*(\vec{\mathbf{r}},t)\Phi(\vec{\mathbf{r}},t)\,\mathrm{d}^3 r$

Operators

- Position Operator: $\hat{x} = x$
- Momentum Operator: $\hat{p} = -i\hbar\frac{\partial}{\partial x}$
- Hamiltonian Operator: $\hat{H} = \hat{K} + \hat{V} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)$
- Energy Operator: $\hat{E} = i\hbar\frac{\partial}{\partial t}$

Hermitian

- Hermitian Conjugate: $\langle \hat{A}^\dagger \Psi, \Phi \rangle = \langle \Psi, \hat{A}\Phi \rangle$
- Hermitian Operator: $\hat{A}^\dagger = \hat{A}$

Introduction to Quantum Physics

Physical Quantity

- Average: $\langle A\rangle_\Psi = \langle \Psi, \hat{A}\Psi\rangle = \int_{-\infty}^{\infty} A|\Psi(x,t)|^2 dx$
- Variance: $(\Delta_A^{(\Psi)})^2 = \langle (A - \langle A\rangle_\Psi)^2\rangle_\Psi = \langle A^2\rangle_\Psi - \langle A\rangle_\Psi^2$
- Standard Deviation: $\Delta_A^{(\Psi)} = \sqrt{\langle A^2\rangle_\Psi - \langle A\rangle_\Psi^2}$

Commutator

$$[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$$

1. $[\hat{A}, \hat{B}] = -[\hat{B}, \hat{A}]$
2. $[\hat{A}, \hat{B}\hat{C}] = \hat{B}[\hat{A}, \hat{C}] + [\hat{A}, \hat{B}]\hat{C}$
3. $[\hat{A}, \hat{B} + \hat{C}] = [\hat{A}, \hat{B}] + [\hat{A}, \hat{C}]$
4. $[\hat{A}, \gamma] = 0$
5. $[\hat{A}, f(\hat{A})] = 0$

Introduction to Quantum Physics

Schrödinger–Robertson Uncertainty Principle

$$\Delta_A^{(\Psi)}\Delta_B^{(\Psi)} \geq \frac{1}{2}|\langle\Psi, i[\hat{A},\hat{B}]\Psi\rangle|$$

Heisenberg Uncertainty Principle

$$\Delta_x^{(\Psi)}\Delta_p^{(\Psi)} \geq \frac{\hbar}{2}$$

1D Schrödinger Equation

$$\hat{H}\Psi = \hat{E}\Psi : -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi(x,t) + V(x)\Psi(x,t) = i\hbar\frac{\partial}{\partial t}\Psi(x,t)$$

1D Stationary Schrödinger Equation

$$\hat{H}\psi = E\psi : -\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\psi(x) + V(x)\psi(x) = E\psi(x)$$

Introduction to Quantum Physics

Solution of 1D Schrödinger Equation

$$\Psi(x,t) = \sum_n c_n e^{-\frac{i}{\hbar}E_n t}\psi_n(x)$$

Infinite Potential Well

$$E_n = \frac{n^2\pi^2\hbar^2}{2mL^2}$$

$$\psi_n(x) = \begin{cases} \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi}{L}x\right) & 0 \le x \le L \\ 0 & \text{otherwise} \end{cases}$$

Measurement Postulate

$$\psi = \sum_n c_n\psi_n, \quad \Pr(A = a_n) = |c_n|^2$$

Quantum Harmonic Oscillator

$$E_n = (n + \frac{1}{2})\hbar\omega$$

Exercise 4

A particle with mass m is confined in a one-dimensional box of length l

$$\begin{cases} V = \infty, & x < 0 \\ V = 0, & 0 < x < l \\ V = \infty, & x > l \end{cases}$$

At time $t = 0$, the particle wave function is

$$\begin{cases} \psi = \sqrt{\frac{30}{l^5}} x(l - x), & 0 < x < l \\ \psi = 0, & x > l \text{ or } x < 0 \end{cases}$$

Find the series representation and expression for the series coefficients of $\psi(x, t > 0)$.

Solution 4

$$\psi_n = \sqrt{\frac{2}{l}} \sin\Big(\frac{n\pi}{l}x\Big), \quad E_n = \frac{n^2\pi^2\hbar^2}{2ml^2}$$

$$c_n = \int_0^l \sqrt{\frac{2}{l}} \sin\Big(\frac{n\pi}{l}x\Big) \sqrt{\frac{30}{l^5}} x(l-x)\mathrm{d}x = 4\sqrt{15}\frac{1-(-1)^n}{n^3\pi^3}$$

$$\Psi(x,t) = \sum_n c_n e^{-\frac{i}{\hbar}E_n t}\psi_n = 8\sum_{n=1}^{\infty} \sqrt{\frac{30}{l^5}} \frac{\sin\big(\frac{2n+1}{l}\pi x\big) e^{-i\frac{\hbar}{2m}(\frac{2n+1}{l}\pi)^2 t}}{(2n+1)^3\pi^3}$$

Exercise 5

At some instant of time the state of a particle moving in a harmonic potential well $V(x) = m\omega^2 x^2/2$ is described by the (normalized) wave function

$$\psi = \sqrt{\frac{1}{3}}\psi_1 + \sqrt{\frac{2}{3}}\psi_3$$

where ψ_n is the n-th eigenstate of the Hamiltonian.
The total energy of the particle is measured.

1. What are possible outcomes of the measurement?
2. What is the probability that the measurement yields $3\hbar\omega/2, 5\hbar\omega/2, 7\hbar\omega/2$?
3. If the first measurement yields $3\hbar\omega/2$, what is the probability that the subsequent measurement yields $3\hbar\omega/2$?

Solution 5

1

$$E_1 = \hbar\omega(1 + \frac{1}{2}) = \frac{3}{2}\hbar\omega$$

$$E_3 = \hbar\omega(3 + \frac{1}{2}) = \frac{7}{2}\hbar\omega$$

2

$$\Pr(E = E_1) = |c_1|^2 = \frac{1}{3}$$

$$\Pr(E = E_2) = |c_2|^2 = 0$$

$$\Pr(E = E_3) = |c_3|^2 = \frac{2}{3}$$

3 Wave function collapse

$$\Pr(E = E_1) = 1$$

Tips for the Final

- Check whether the dimensions of the answers are correct.
- Check whether the answers are as expected and make physical sense.
- List relative equations if you don't know how to solve the problem.
- Review HW problems and check whether you can solve them easily.
- Review the lecture notes carefully, especially the unfamiliar concepts!

Thanks for listening!

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