Computer Aided Geometric Design
–
B-Spline Curves

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B-Spline Curves

- **Most shapes are too complicated to define using a single Bézier curve.**
- A **spline curve is a sequence of Bézier curves** that are connected together
- $\mathcal{C}^1$ continuity is easy to attain with Bézier curves, but $\mathcal{C}^2$ and higher continuity is cumbersome.

B-spline curves

This is a method for representing a sequence of degree n Bézier curves that join automatically with $\mathcal{C}^{n-1}$ continuity, regardless of where the control points are placed.

- An open string of m Bézier curves of degree n involves $nm+1$ distinct control points,
- whereas we will see the same string of Bézier curves can be expressed using only $m+n$ B-spline control points (assuming $\mathcal{C}^{n-1}$ continuity).

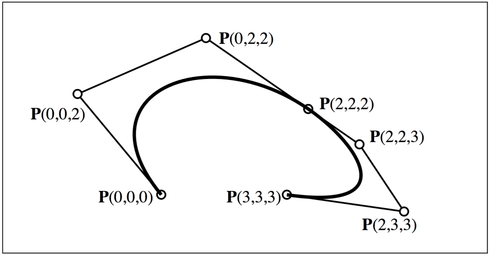
Polar Form

- Introduced by Ramshaw (late 80's), can be thought as another method of labeling the control points; these labels are referred to as **polar values**.
- Next, we show how to derive all important algorithms for Bézier and B-spline curves from the following four rules for polar values.

Rule 1. Control points of a Bézier curve

The control points of a degree n Bézier curve $\mathrm{P}_{[a,b]}(t)$ are relabeled

$$\mathrm{P}_i := \mathrm{P}(u_1, u_2, \ldots, u_n), \;\; u_1 = \cdots = u_{n-i} = a, \;\; u_{n-i+1} = \cdots = u_n = b.$$



For a degree 3 Bézier curve, polar values are

$\mathrm{P}_0 = \mathrm{P}(a, a, a), \; \mathrm{P}_1 = \mathrm{P}(a, a, b),$

$\mathrm{P}_2 = \mathrm{P}(a, b, b), \; \mathrm{P}_3 = \mathrm{P}(b, b, b).$

Polar Form

Rule 2. A polar value is symmetric in its arguments.

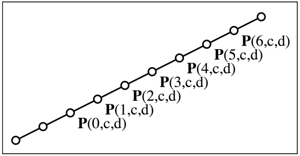
This means that the order of the arguments can be changed without changing the polar value.

For example, $P(1,0,0,2) = P(0,1,0,2) = P(2,1,0,0) = P(0,0,1,2) = \cdots$.

Rule 3. A polar value is multi-affine in its arguments.

This means that a polar value is affine in each of its argument. Given $P(u_1,\ldots,u_{n-1},a)$ and $P(u_1,\ldots,u_{n-1},b)$, for any value c :

$$P(u_1,\ldots,u_{n-1},c) = \frac{(b-c)P(u_1,\ldots,u_{n-1},a)+(c-a)P(u_1,\ldots,u_{n-1},b)}{b-a}.$$



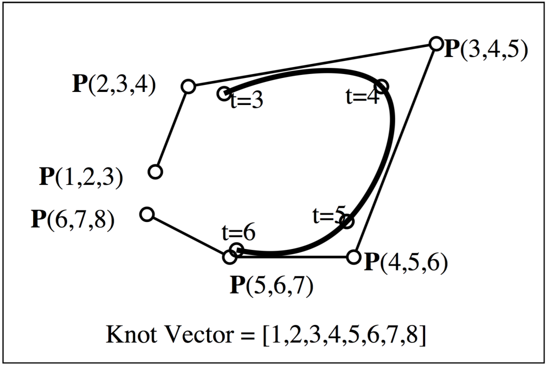
For example,

$$P(0,t,1) = (1-t)P(0,0,1) + tP(0,1,1).$$

Polar Form

Rule 4. B-spline curve.

For a **degree** n B-spline curve with a **knot vector** $[t_1, t_2, t_3, \ldots]$, the arguments of the polar values consist of groups of n **adjacent knots** from the knot vector, with the i^{th} polar value being $\mathrm{P}(t_i, t_{i+1}, \ldots, t_{i+n-1})$.



Subdivision of Bézier Curves

To illustrate how polar values work, we show how to derive the de Casteljau algorithm using only the first three rules.

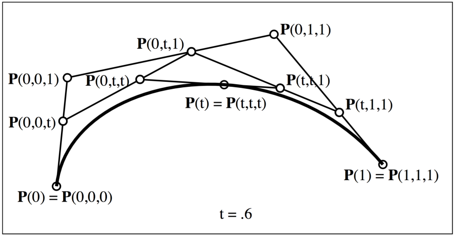
- Given a cubic Bézier curve $P_{[0,1]}(t)$, we wish to split it into $P_{[0,t]}$ and $P_{[t,1]}$.
- The original control points are given by the polar values

$$P(0,0,0),\ P(0,0,1),\ P(0,1,1),\ P(1,1,1).$$

and we need to compute the polar values

$$P(0,0,0),\ P(0,0,t),\ P(0,t,t),\ P(t,t,t)$$

$$P(t,t,t),\ P(t,t,1),\ P(t,1,1),\ P(1,1,1).$$



Corollary

$$\forall t\ :\ P(t,t,t) = P(t)$$

Knot Vectors

Knot Vector

It is a list of parameter values, or *knots*, that specify the parameter intervals for the individual Bézier curves that make up a B-spline. It must be a non-decreasing sequence of real numbers.

- For example, if a cubic B-spline is made of four Bézier curves with parameter intervals $[1, 2]$, $[2, 4]$, $[4, 5]$ and $[5, 8]$, then the knot vector would be

$$[t_1, t_2, 1, 2, 4, 5, 8, t_8, t_9].$$

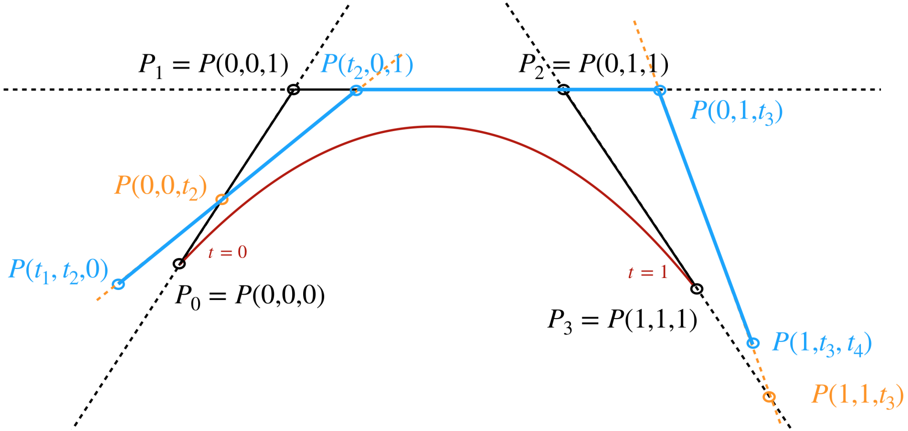
- There are two (one less of the degree) extra knots appended and prepended to the knot vector. They control the *end conditions* of the B-spline (see later).

Terminology:

- If any knot is repeated, it is referred to as a multiple knot (see later).
- A B-spline whose knots are evenly spaced is known as a uniform B-spline.

Cubic Bézier curves as B-Splines

Bézier and B-Spline control polygones of the same cubic Bézier curve



- Knot vector as a Bézier curve : $[0,0,0,1,1,1]$,
- Knot vector as a B-Spline curve : $[t_1, t_2, 0, 1, t_3, t_4]$.

Extracting Bézier Curves from B-Splines

Böhm algorithm

How does one find the control points of the Bézier curves that make up a B-spline?

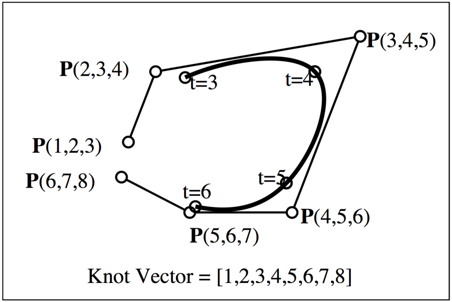
- Consider the B-spline below consisting in three Bézier curves in the domains $[3,4]$, $[4,5]$ and $[5,6]$.
- The control points of these Bézier curves have polar values

$$P(3,3,3), P(3,3,4), P(3,4,4), P(4,4,4),$$

$$P(4,4,4), P(4,4,5), P(4,5,5), P(5,5,5),$$

$$P(5,5,5), P(5,5,6), P(5,6,6), P(6,6,6).$$

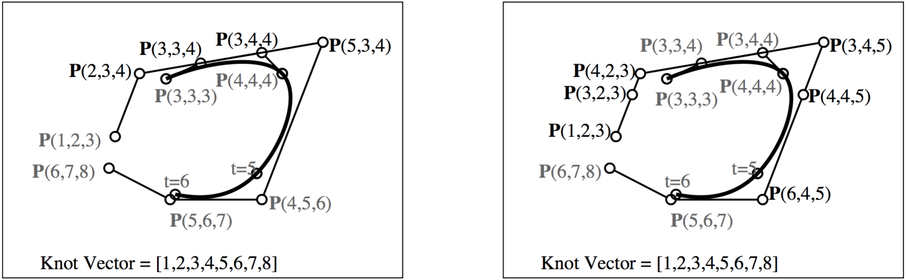
- Goal : compute these polar values.



Extracting Bézier Curves from B-Splines

For the Bézier curve over $[3, 4]$,

- First step: compute $P(3, 3, 4)$ and $P(3, 4, 4)$.
- Second step: compute the auxiliary points $P(3, 2, 3)$ and $P(4, 4, 5)$.
- Finally, compute $P(3, 3, 3)$ and $P(4, 4, 4)$.



(a) First Step.

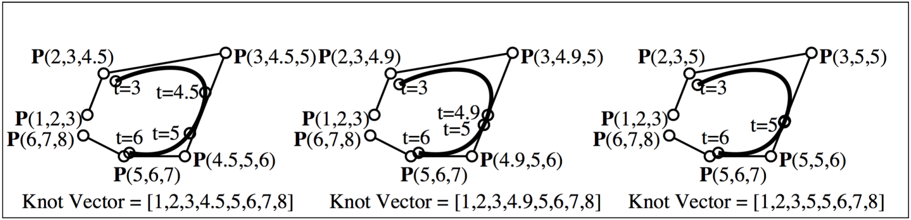
(b) Second Step.

B-splines posses the property of *local control*

The four Bézier control points are derived from four B-spline control points; $P(5, 6, 7)$ and $P(6, 7, 8)$ do not affect this Bézier segment.

Multiple knots

- If a knot vector contains two identical non-end-condition knots $t_i = t_{i+1}$, The B-spline can be thought as containing a zero-length Bézier curve over $[t_i, t_{i+1}]$.
- Below is an illustration of two points which are moved together.



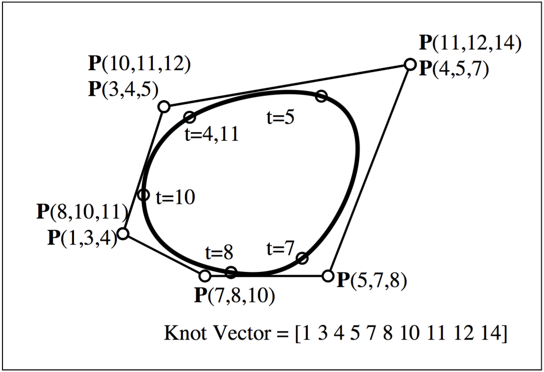
Multiple knot diminishes the continuity

The continuity of two adjacent Bézier curves across a knot of multiplicity k is generally $n - k$.

Periodic B-Splines

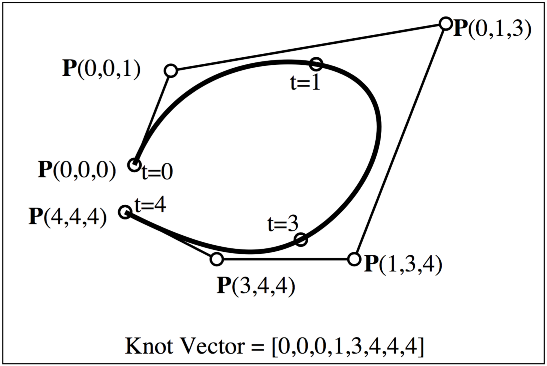
A periodic B-spline is a B-spline which closes on itself.

For a degree n B-spline curve, this requires that
- the first n control points are identical to the last n,
- the first n parameter intervals in the knot vector are identical to the last n.



Bézier end conditions

- We already noted that a knot vector always has $n-1$ extra knots at the beginning and end which do not specify Bézier parameter limits (except in the periodic case), but which influence the shape of the curve at its end.
- In the case of an open B-spline, one usually chooses an n-fold knot at each end to get a Bézier behavior : the curve interpolates the end control points and is tangent to the control polygon at its endpoints (Böhm algorithm).

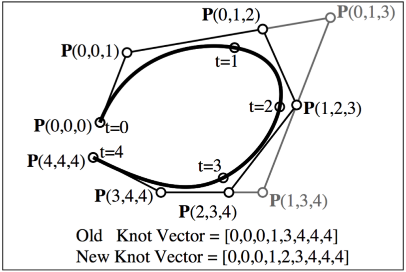


Knot insertion

- It is a standard design tool for B-splines consisting in adding a knot in the knot vector.
- It results in an additional control point and a modification of a few existing ones. The curve itself is unchanged.
- Several applications: evaluation, add local details, *splitting in Bézier segments*, etc.

In the example below a knot at $t = 2$ is inserted.

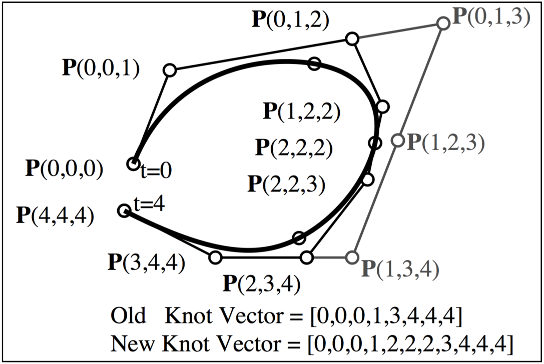
	Initial	After Knot Insertion
Knot Vector:	[(0,0,0,1,3,4,4,4)]	[(0,0,0,1,2,3,4,4,4)]
Control Points:	**P**(0,0,0)	**P**(0,0,0)
	P(0,0,1)	**P**(0,0,1)
		P(0,1,2)
	P(0,1,3)	
		P(1,2,3)
	P(1,3,4)	
		P(2,3,4)
	P(3,4,4)	**P**(3,4,4)
	P(4,4,4)	**P**(4,4,4)



The de Boor algorithm

The de Boor algorithm provides a method for evaluating a B-spline curve.

- Given a parameter value t, find the corresponding point on the B-spline.
- This point $P(t)$ has the polar value $P(t, t, \ldots, t)$.
- It can be computed by inserting n times the knot t.
- Using polar forms, the algorithm is easy to figure out.



B-spline hodographs

- The first derivative (or hodograph) of a B-spline is obtained in a manner similar to that of Bézier curves.
- The hodograph has the same knot vector as the given B-spline, except that the first and last knots are discarded.
- The control points are given by the equation

$$H_i := \frac{n}{t_{i+n} - t_i} \left(\mathrm{P}_{i+1} - \mathrm{P}_i \right)$$

where n is the degree.

Polar forms and the blossoming principle

Polar forms are not simply a labeling scheme

A multivariate polynomial $f(t_1, \ldots, t_n)$ is called

- symmetric if it keeps its values under any permutation of its arguments.
- multi-affine if it is affine in each of its arguments, i.e.

$$f\left(\ldots, \sum_j \alpha_j t_j, \ldots\right) = \sum_j \alpha_j f(\ldots, t_j, \ldots)$$

for all scalars α_j such that $\sum_j \alpha_j = 1$.

Blossoming principle

For every degree m polynomial $p(t)$ there exists a **unique symmetric, multi-affine and multivariate polynomial** $p(t_1, \ldots, t_n)$, called the **polar form**, such that $p(t, \ldots, t) = p(t)$.

Polar forms and the blossoming principle

Suppose given a polynomial $p(t)$ of degree n and denote by $p(t_1, \ldots, t_n)$ its polar form. Then, one can prove the following results.

Blossoming of Bézier curves

Let $\Delta := [a, b]$ be an arbitrary interval. The polynomial $p(t)$ can be represented as a Bézier polynomial of degree n w.r.t. Δ and its control points are given as

$$\mathrm{P}_i = p(\underbrace{a, \ldots, a}_{n-i}, \underbrace{b, \ldots, b}_{i}).$$

Blossoming of B-spline curves

The polynomial $p(t)$ can be represented as a B-spline segment over a non-decreasing knot sequence $r_n \leq \ldots \leq r_1 < s_1 \leq \ldots \leq s_n$. Its control points are given as

$$\mathrm{P}_i = p(r_1, \ldots, r_{n-i}, s_1, \ldots, s_i).$$

B-spline Curves from Basis Functions

Definition

A degree d B-spline curve with $n+1$ control points can be expressed as

$$\mathrm{P}(t) = \sum_{i=0}^{n} \mathrm{P}_i N_i^d(t).$$

- The knot vector for this curve contains $n+d$ knots, which we will denote

$$[t_1, \ldots, t_{n+d}].$$

- The control point P_i has polar value $\mathrm{P}(t_{i+1}, \ldots, t_{i+d})$.
- The functions $N_i^d(t)$ are called the *B-spline basis functions*.
- The domain of the B-Spline curve is the interval $[t_d, t_{n+1}]$ (eliminate the end-condition knots).
- This curve is composed of $n-d+1$ Bézier curves corresponding to the parameter intervals $[t_{d+j-1}, t_{d+j}]$, $j = 1, \ldots, n-d+1$.

B-Spline Basis Functions

Definition

The functions $N_i^0(t)$ are the step functions; they are defined as

$$N_i^0(t) = \begin{cases} 1 & \text{if } t \in [t_i, t_{i+1}] \\ 0 & \text{otherwise} \end{cases}$$

Then, B-Spline basis functions of degree k are defined using the recurrence relation

$$N_i^k(t) = w_i^k(t) N_i^{k-1}(t) + (1 - w_{i+1}^k) N_{i+1}^{k-1}$$

where

$$w_i^k(t) = \begin{cases} \frac{t-t_i}{t_{i+k}-t_i} & \text{if } t_i < t_{i+k} \\ 0 & \text{otherwise .} \end{cases}$$

N.B.: It is necessary to add two extra knots at the ends : $[\mathrm{t_0}, \mathrm{t_1}, \ldots, \mathrm{t_{n+d}}, \mathrm{t_{n+d+1}}]$

Remark: If for some i, one has $t_i = t_{i+k+1}$ (t_i is a knot of multiplicity $\geq k+2$), then $N_i^k \equiv 0$.

Properties of B-spline Basis Functions

Piecewise polynomials

$N_i^k(t)$ is a piecewise polynomial, each polynomial piece being of degree k

Support

- $N_i^k(t) = 0$ if $t \notin [t_i, t_{i+k+1}]$
- $N_i^k(t) > 0$ if $t \in]t_i, t_{i+k+1}[$.
  $N_i^k(t_i) = 0$, except if $t_i = \cdots = t_{i+k} < t_{i+k+1}$ (knot of multiplicity $k+1$), in which case $N_i^k(t_i) = 1$.

Partition of unity

For any integer m, the B-spline functions form a partition of unity on the interval $[t_k, t_{m-k}]$:

$$\sum_{i=0}^{m-k-1} N_i^k(t) \equiv 1 \quad \forall t \in [t_k, t_{m-k}].$$

$\Rightarrow$ From here, one can develop an extensive theory of B-Spline curves.

Non-uniform rational B-spline (NURBS) curves

Definition

A degree d NURBS curve with $n+1$ control points is expressed as

$$\mathrm{P}(t) = \frac{\sum_{i=0}^{n} w_i \mathrm{P}_i N_i^d(t)}{\sum_{i=0}^{n} w_i N_i^d(t)}.$$

- A weight needs to be introduced for each control point.
- The NURBS curve does not change if all the weights are multiplied by the same nonzero constant.
- It is the central projection of a B-spline curve in one more dimension.
- Most algorithms for B-spline curves extend to NURBS curves (similarly to the Bézier case).