Bézier Surfaces and NURBS Surfaces

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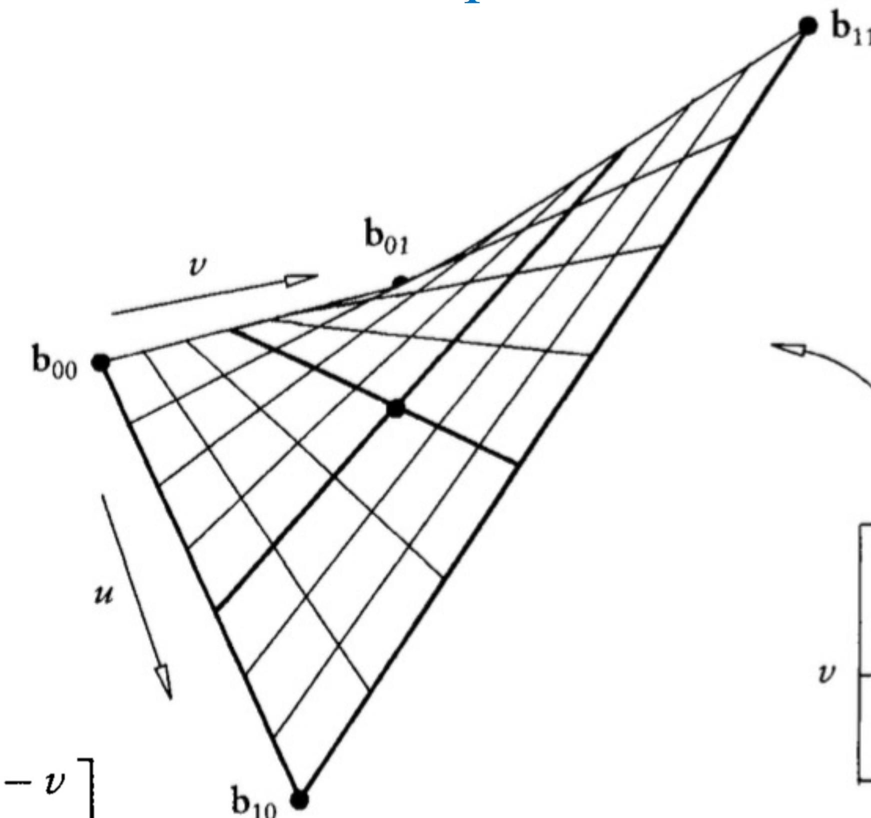
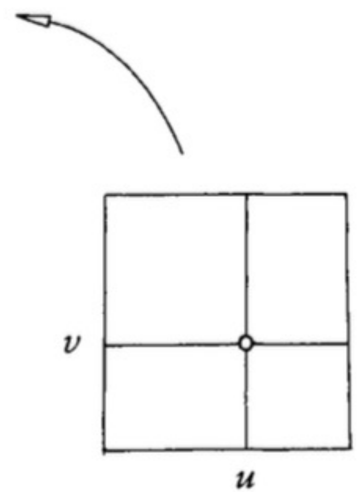
Bilinear interpolation

As linear interpolation fits the « simplest » curve between two points, bilinear interpolation fits the « simplest » surface between four points:



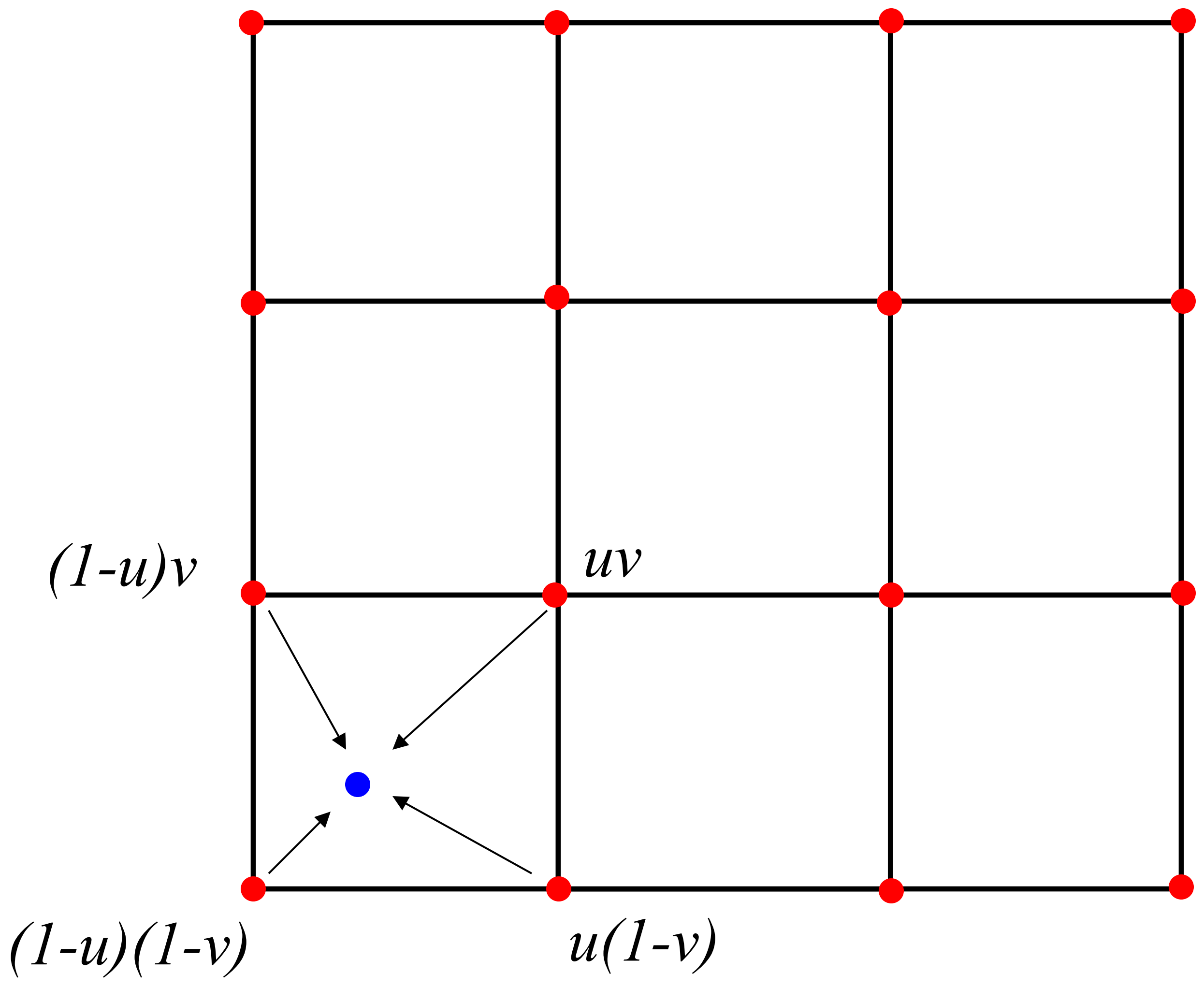


$$\mathbf{x}(u,v) = \sum_{i=0}^{1}\sum_{j=0}^{1} \mathbf{b}_{i,j} B_i^1(u) B_j^1(v)$$

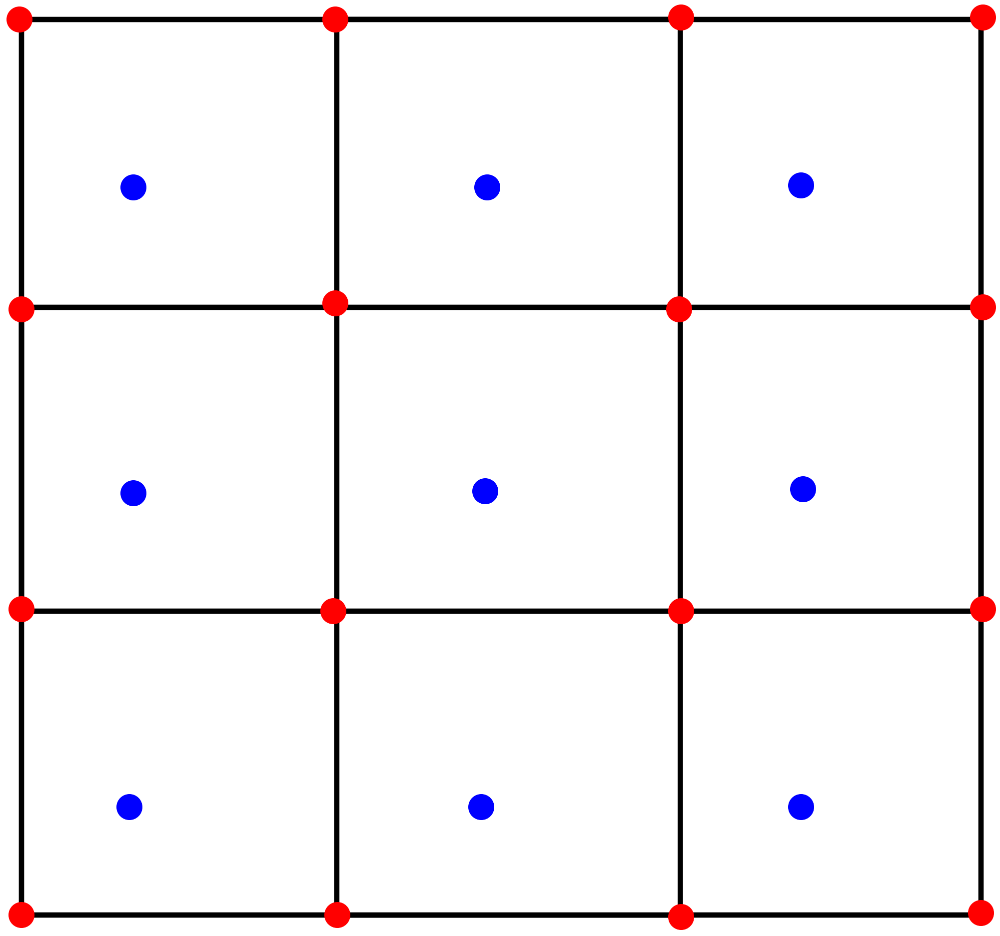
Matrix formulation:

$$\mathbf{x}(u,v) = \begin{bmatrix} 1-u & u \end{bmatrix} \begin{bmatrix} \mathbf{b}_{00} & \mathbf{b}_{01} \\ \mathbf{b}_{10} & \mathbf{b}_{11} \end{bmatrix} \begin{bmatrix} 1-v \\ v \end{bmatrix}$$

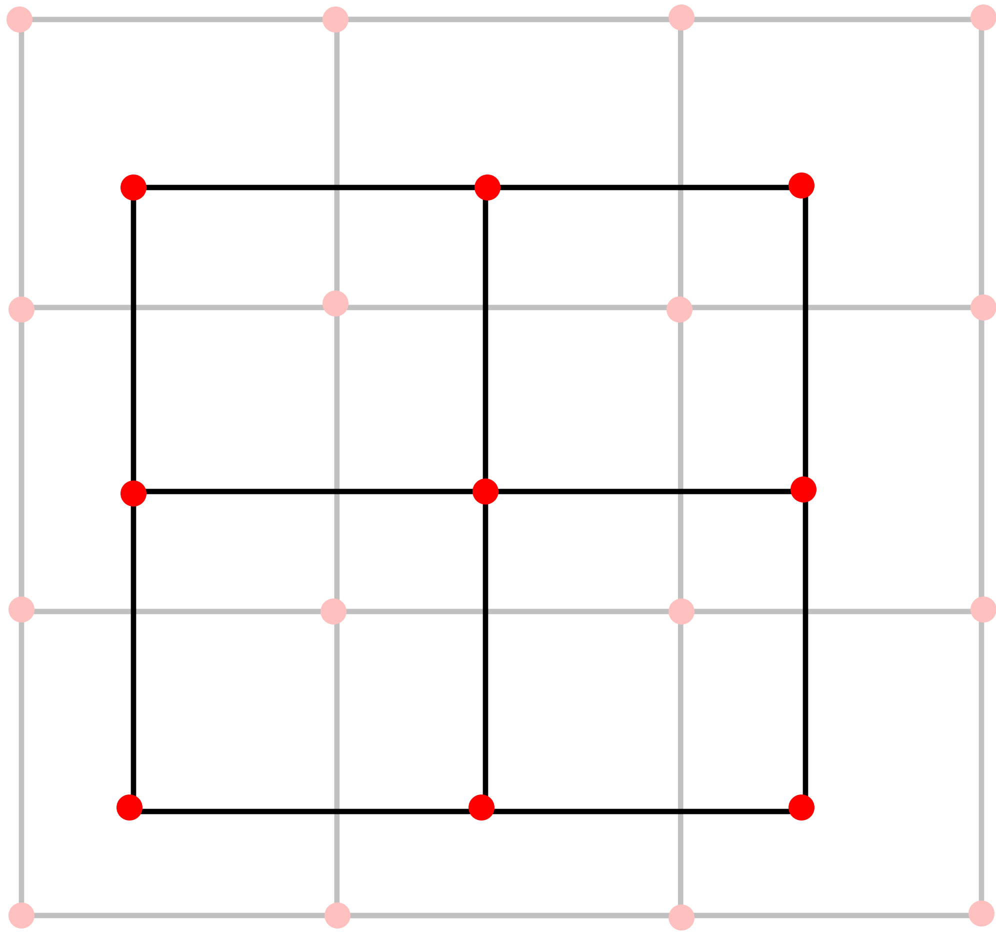
Repeated bilinear interpolation



Repeated bilinear interpolation



Repeated bilinear interpolation



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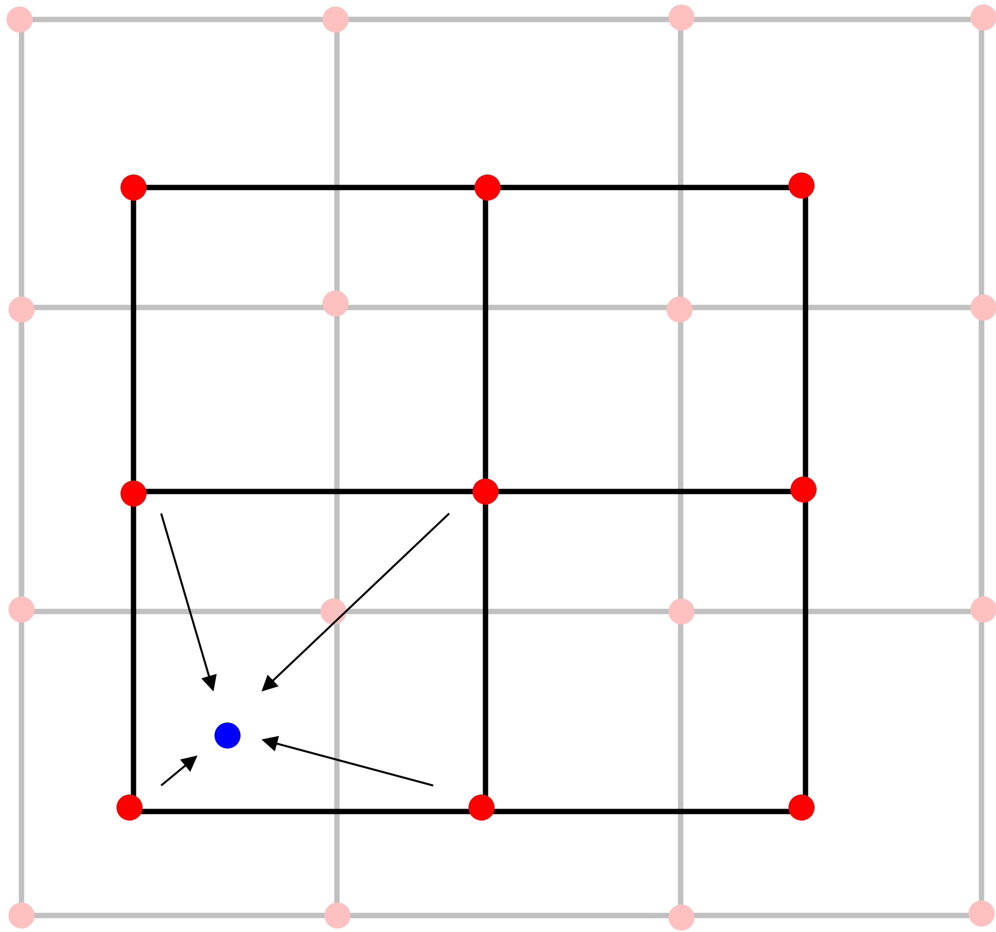
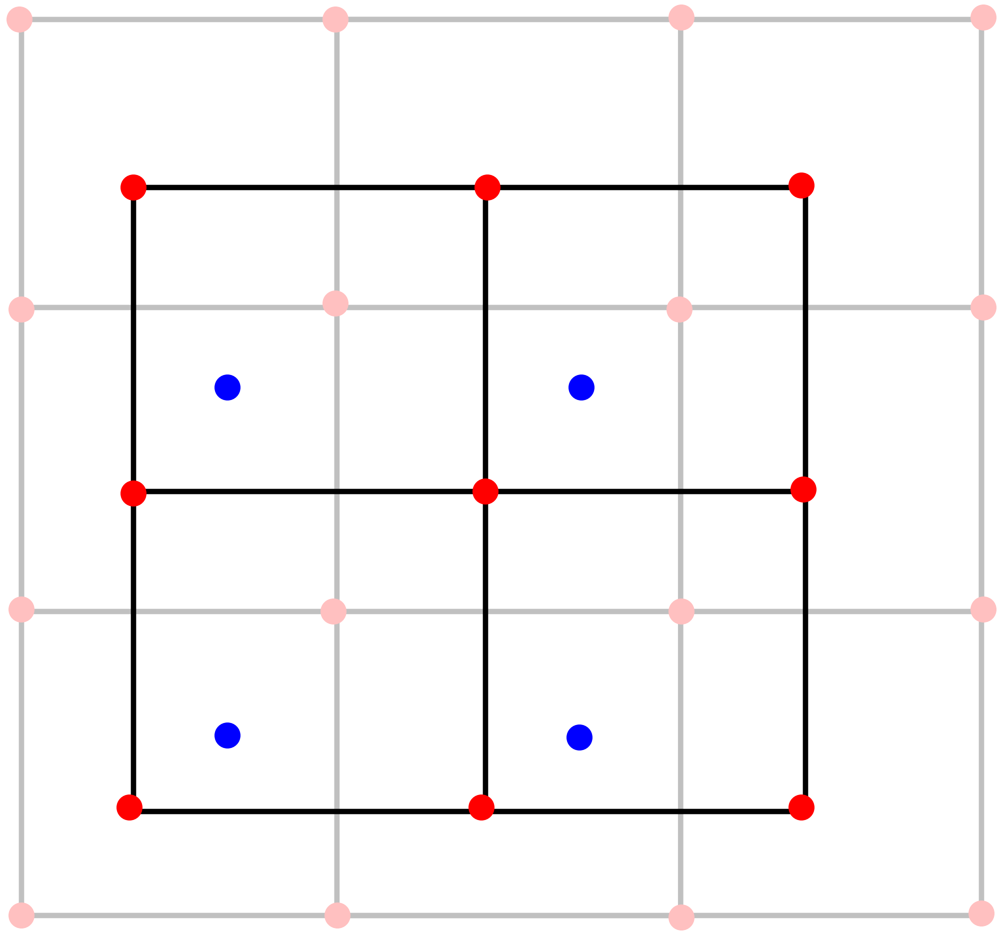


Repeated bilinear interpolation



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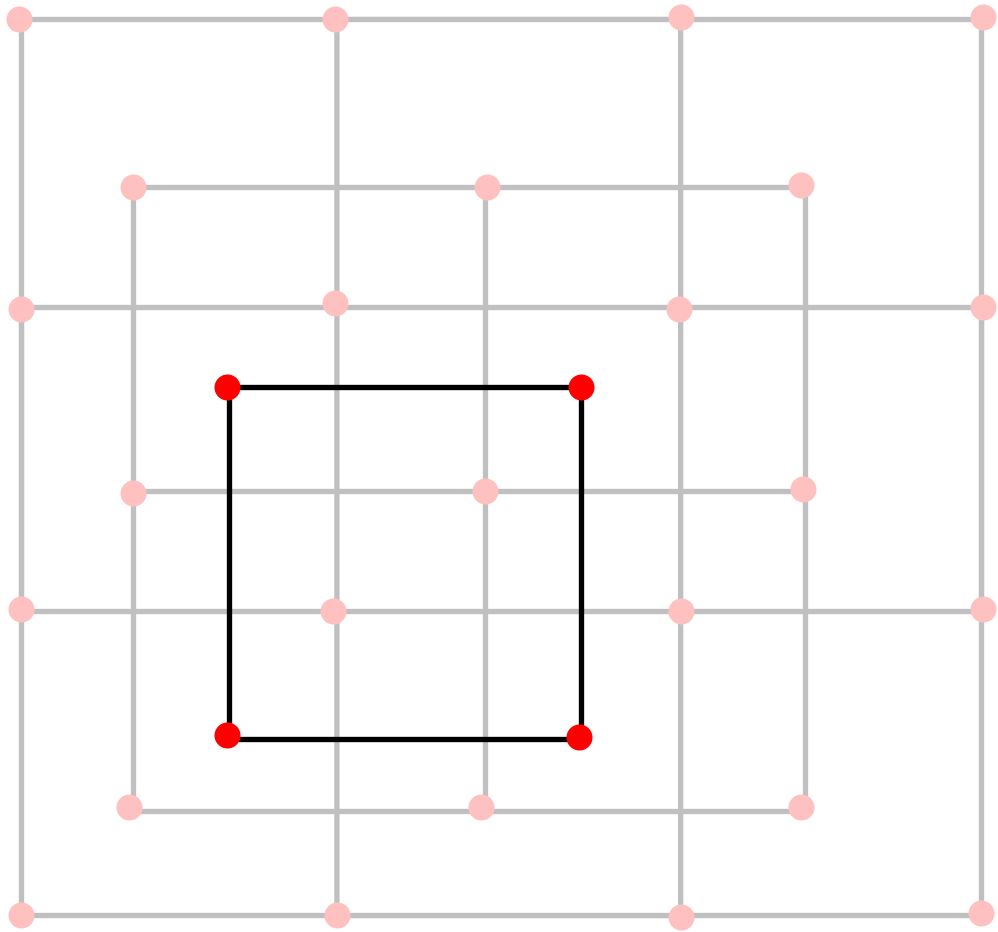
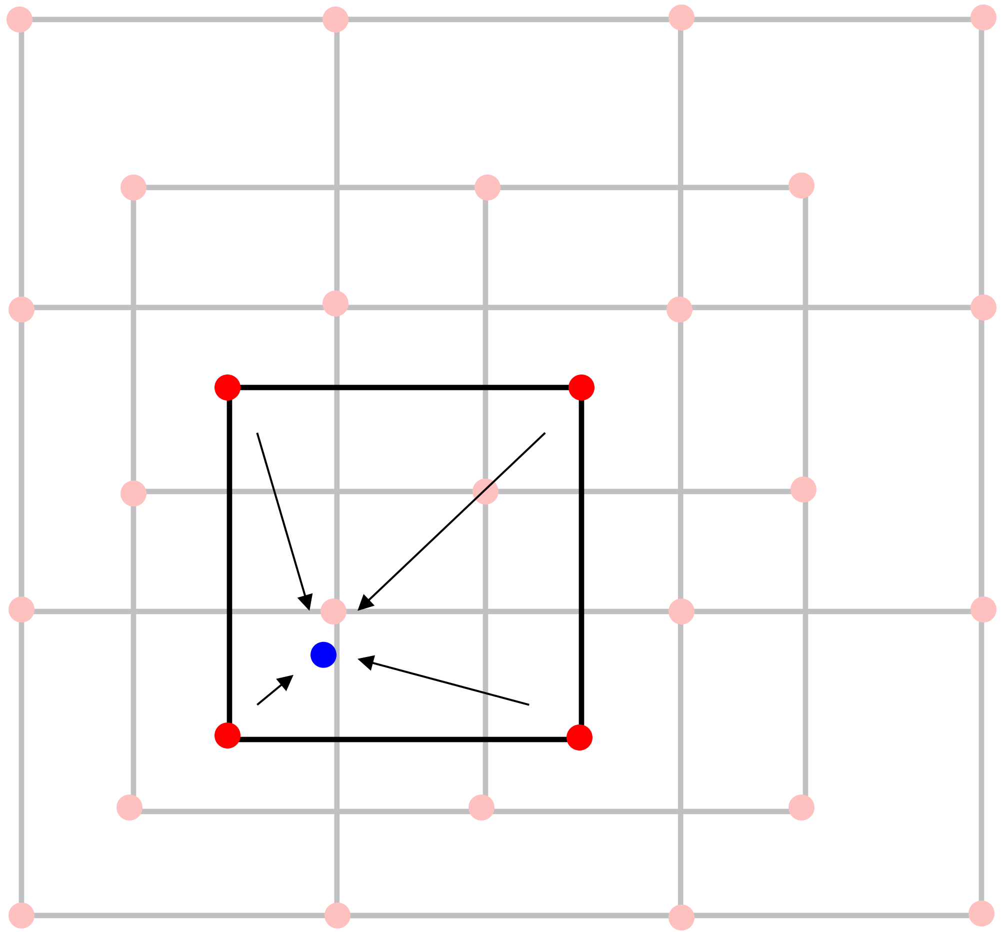
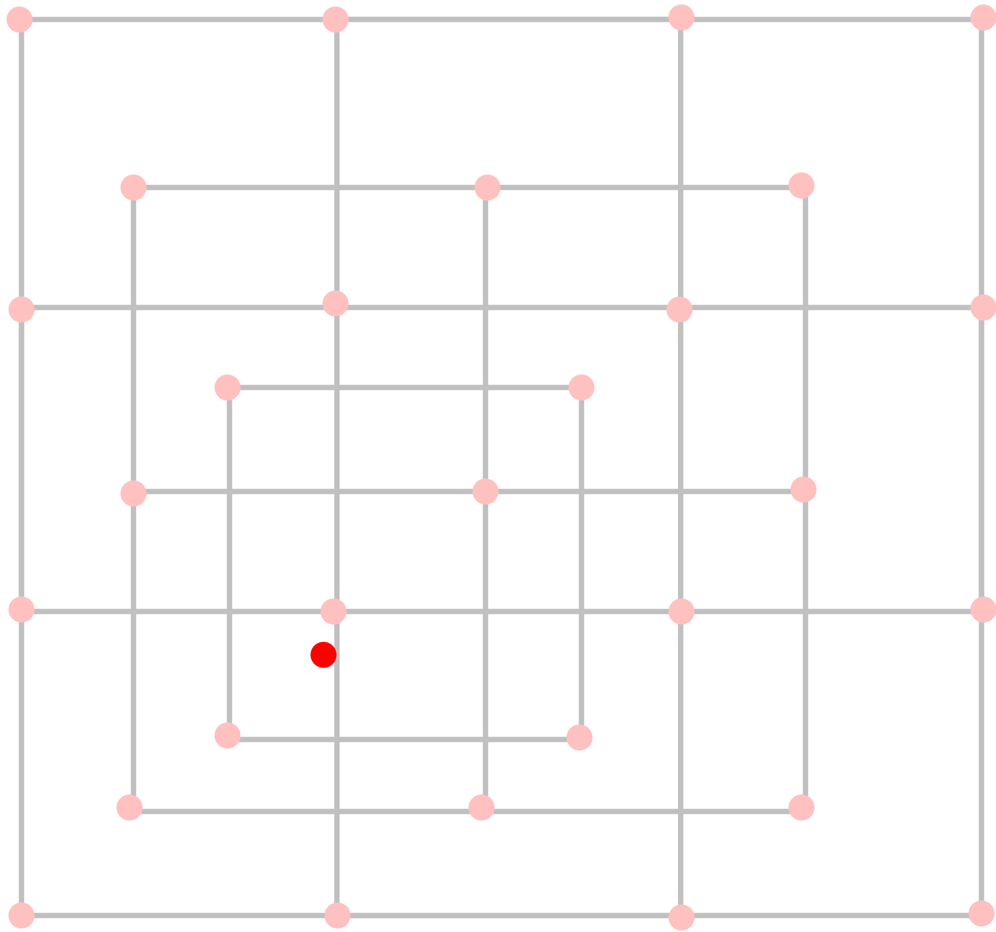


Repeated bilinear interpolation



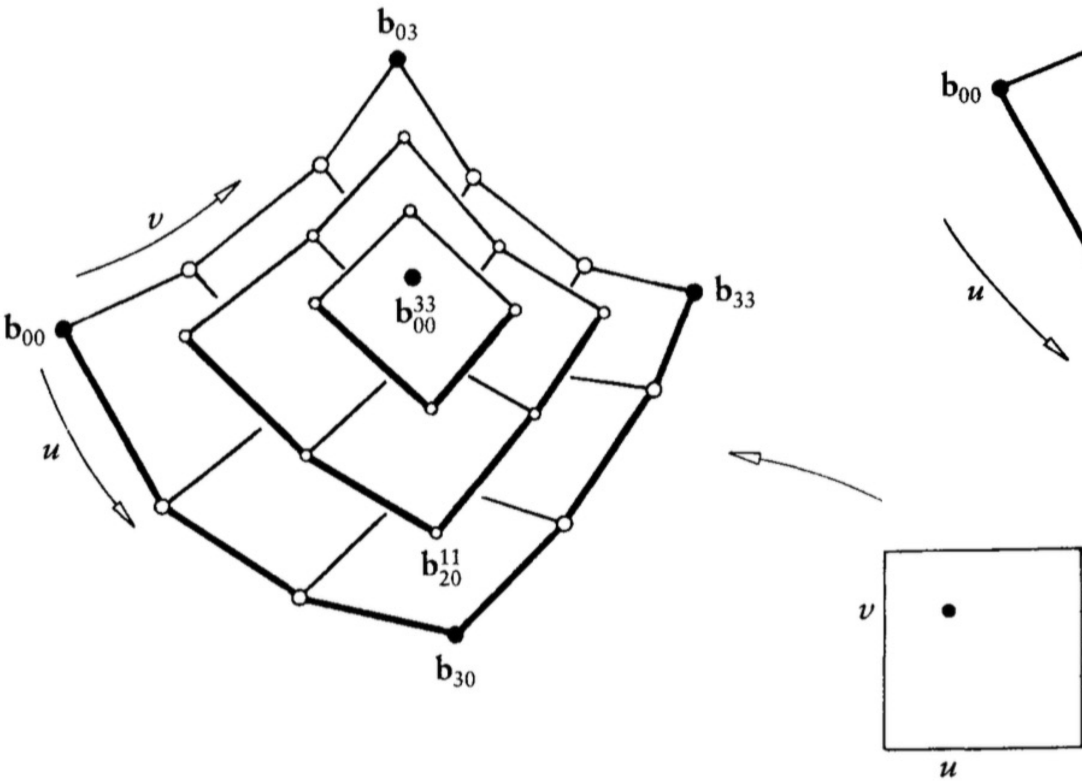
Repeated bilinear interpolation

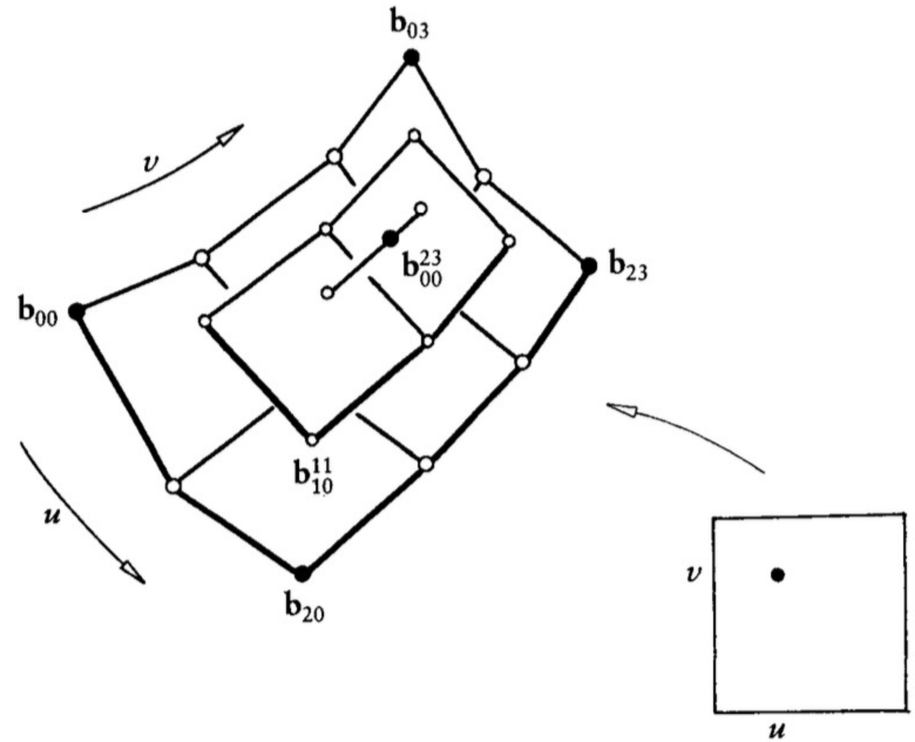


Direct de Casteljau algorithm

- Surfaces are obtained from repeated bilinear interpolation

Bicubic patch





The tensor product approach

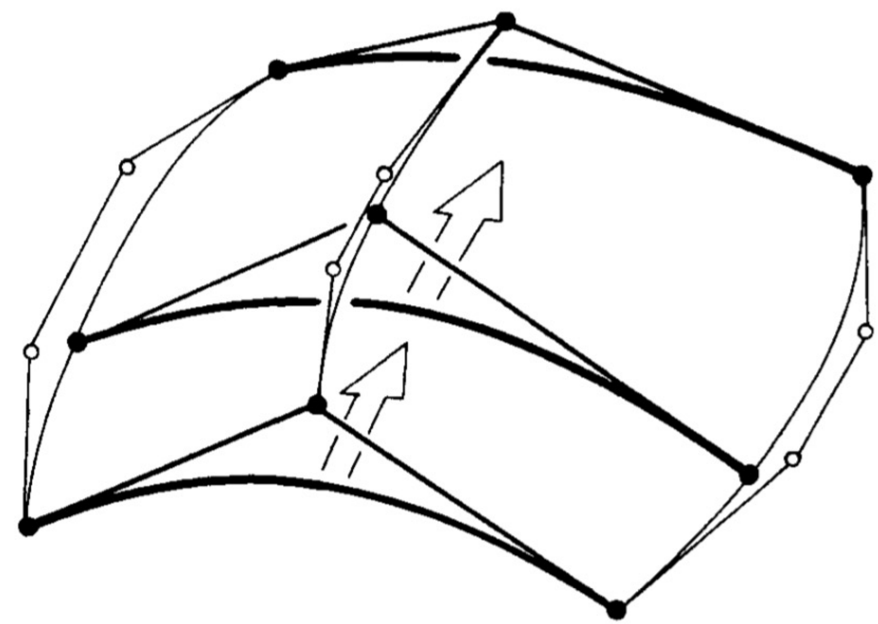
A surface is the locus of a curve that is moving through space

- Initial Bézier curve of degree m :

$$\mathbf{b}^m(u) = \sum_{i=0}^{m} \mathbf{b}_i B_i^m(u).$$

- Each control point draws a Bézier curve of degree n

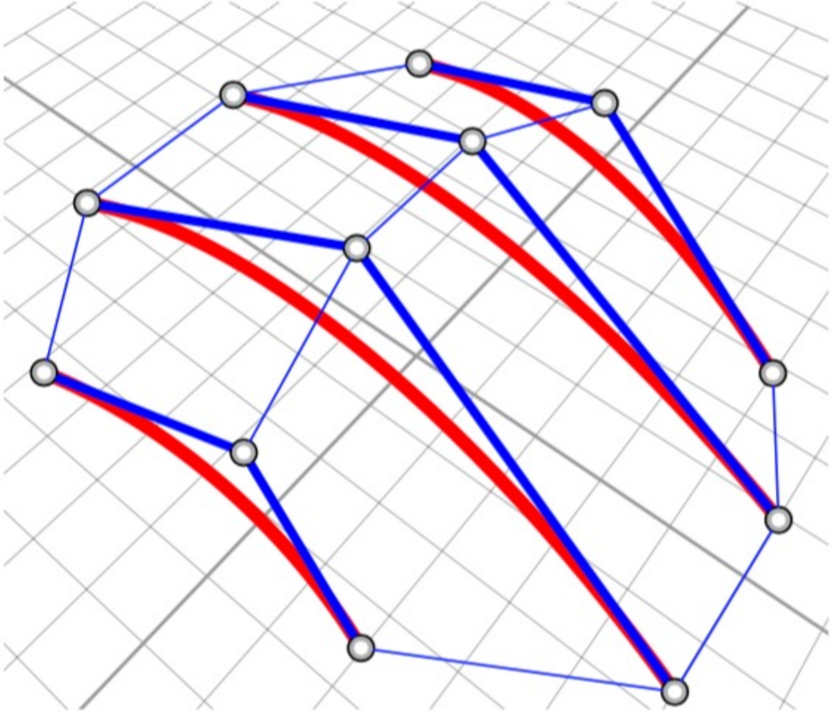
$$\mathbf{b}_i = \mathbf{b}_i(v) = \sum_{j=0}^{n} \mathbf{b}_{i,j} B_j^n(v).$$



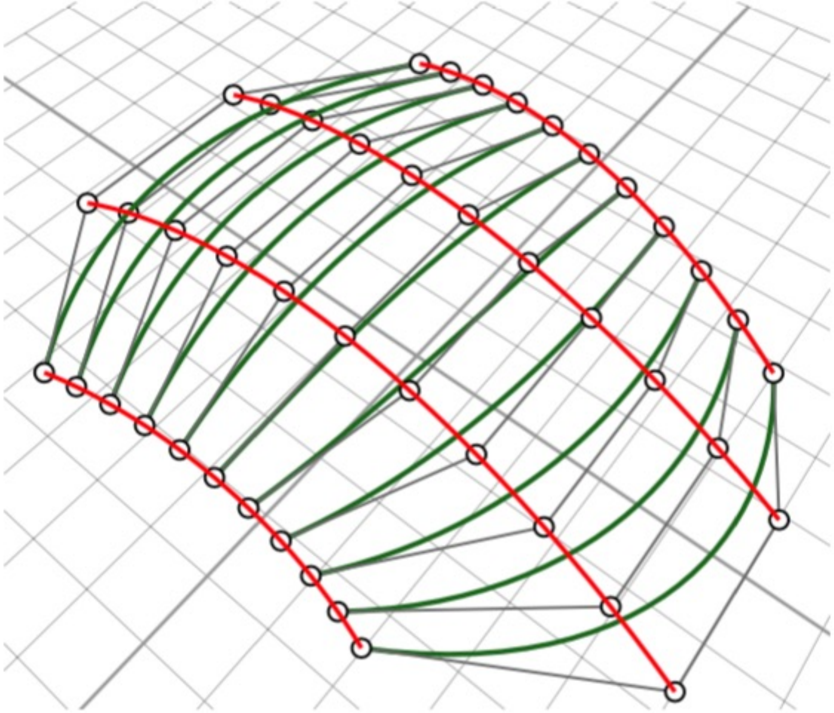
$$\boxed{\mathbf{b}^{m,n}(u,v) = \sum_{i=0}^{m} \sum_{j=0}^{n} \mathbf{b}_{i,j} B_i^m(u) B_j^n(v).}$$

Some special curves

- Four boundary Bézier curves.
- The control curves are the Bézier curves given by a « row » or a « column » of control points.



- Iso-parameter curves are Bézier curves *on the surface* that are obtained by fixing the value of one of the two parameters.

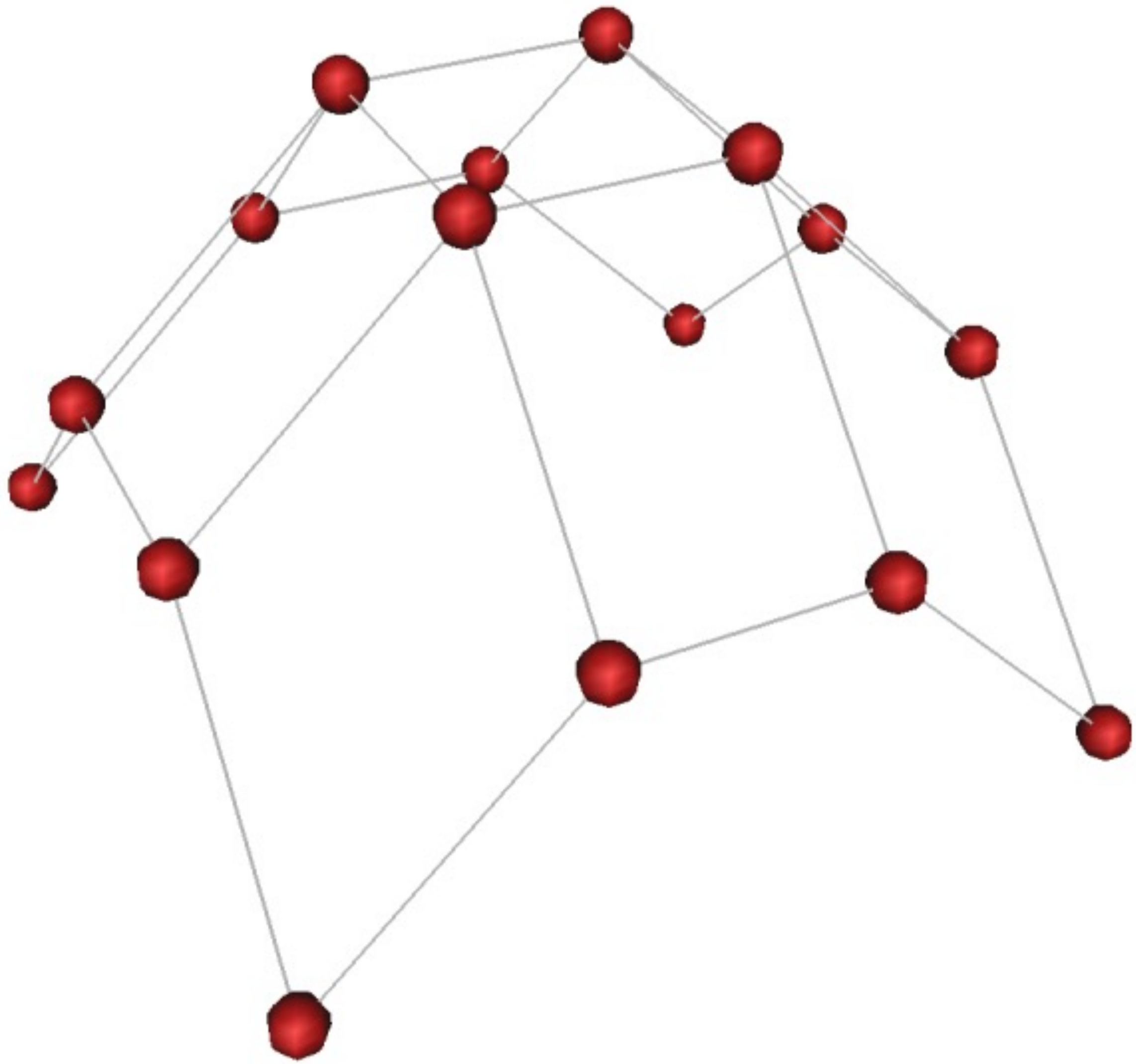


The de Casteljau algorithm

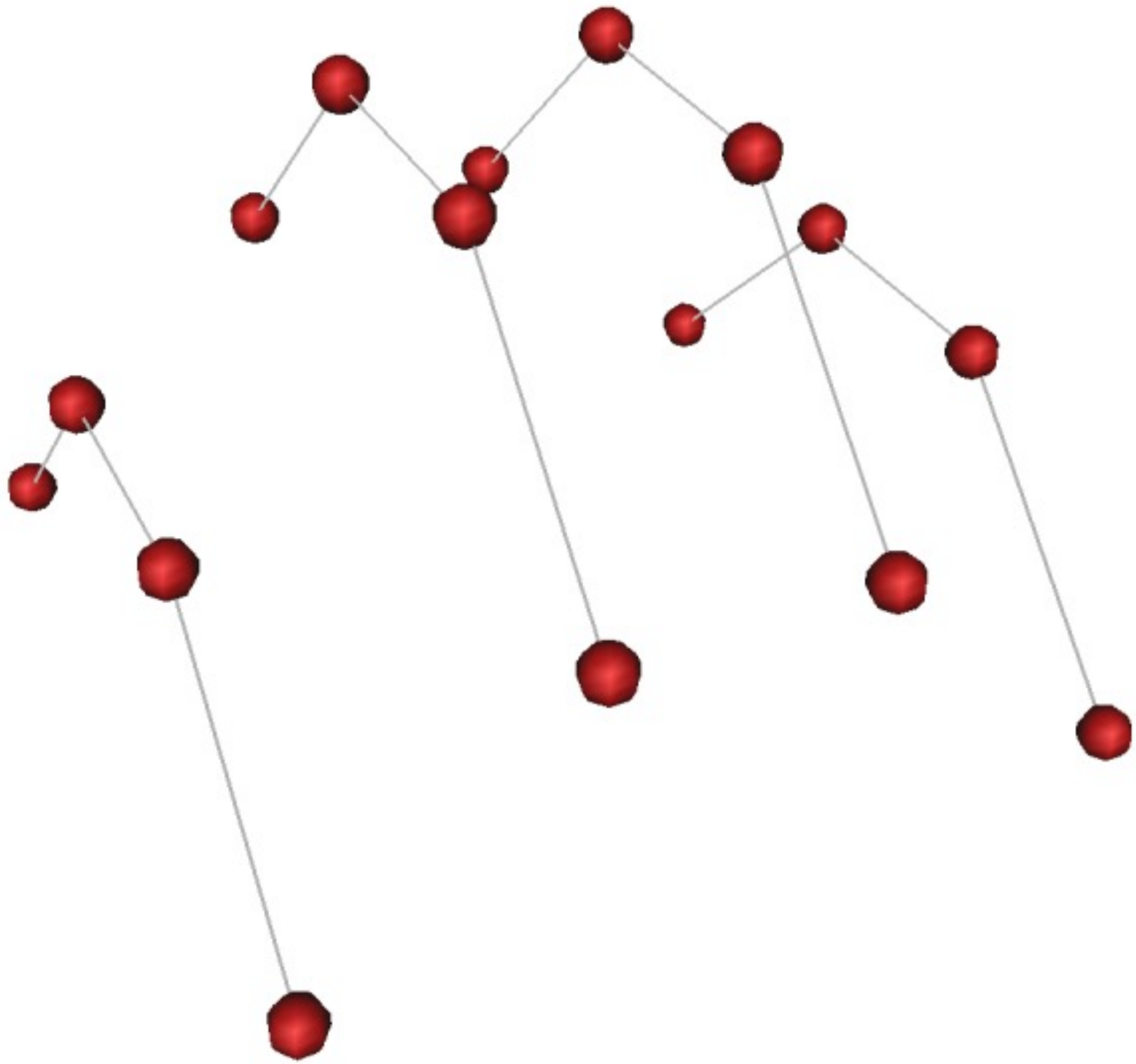
- To evaluate a point on a tensor product Bézier patch corresponding to a given parameter *(s,t),* one can:
  - Evaluate the *v*-control curves at $v=t$,
  - Then evaluate this iso-parameter curve at $u=s$.
  - Alternatively, one could proceed with the u-control curves first.

- This algorithm is also used to cut a tensor product Bézier patch into two pieces, either in the *u* or *v* parameter

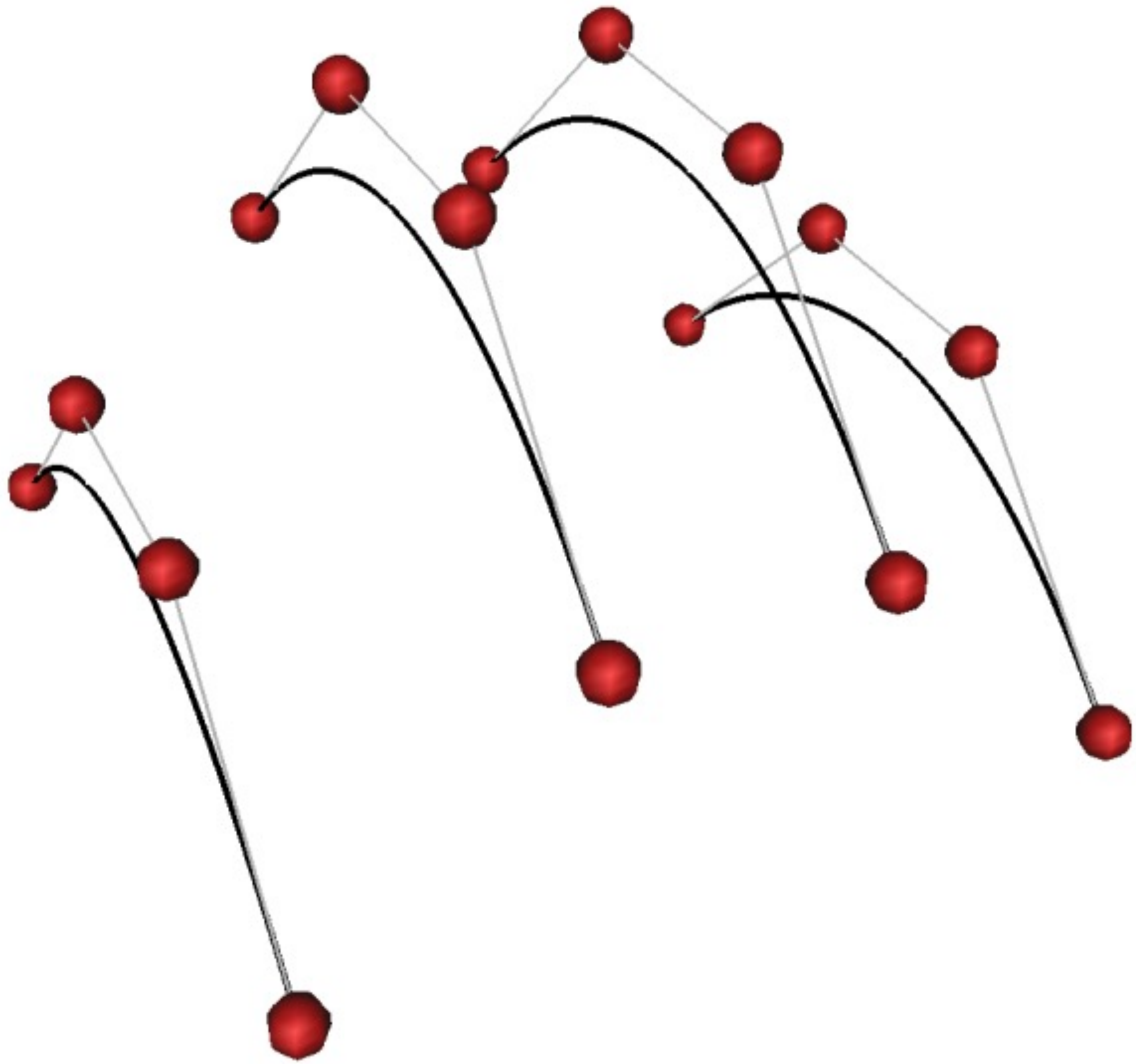
A « bicubic » grid of control points



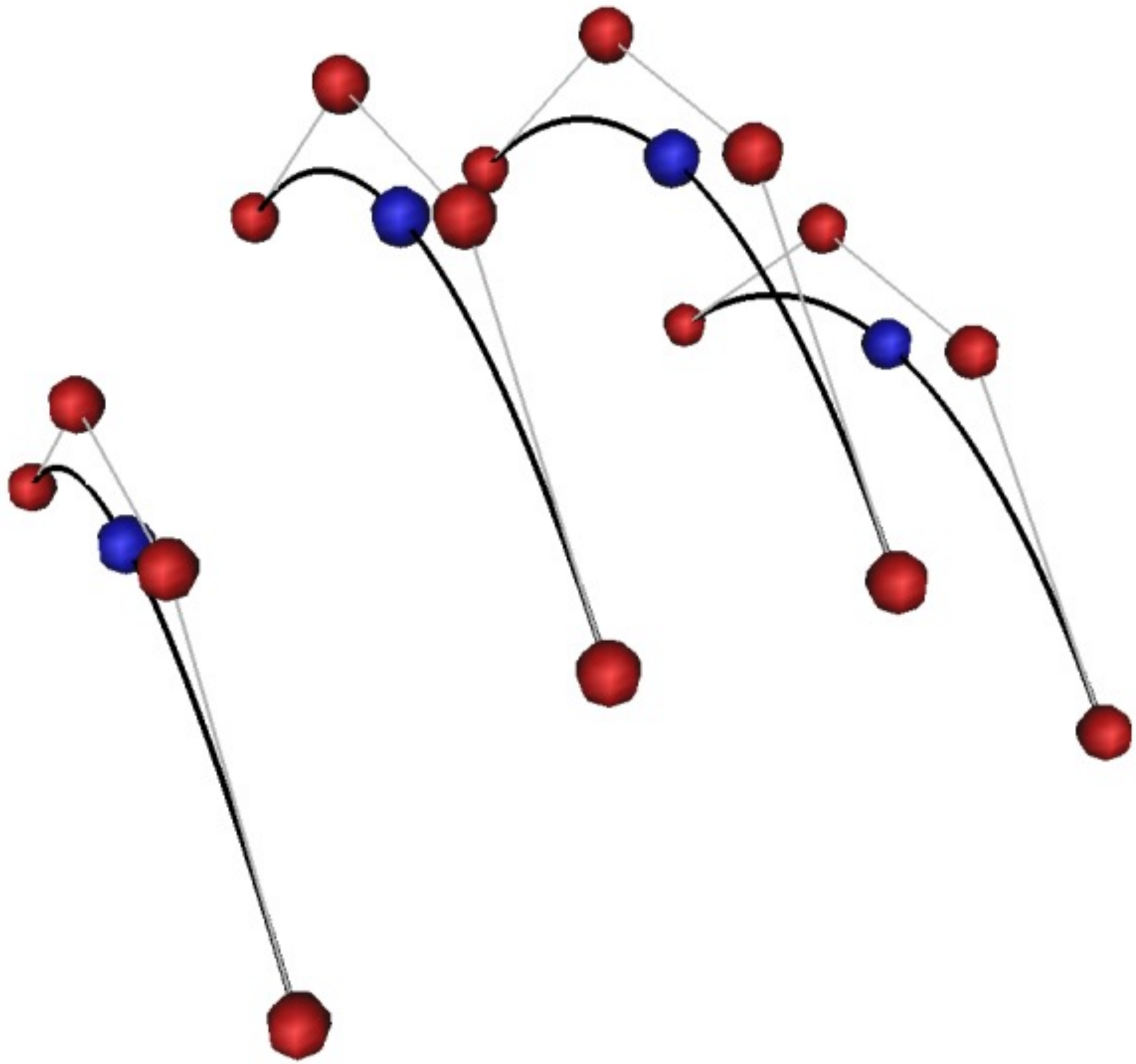
The de Casteljau algorithm



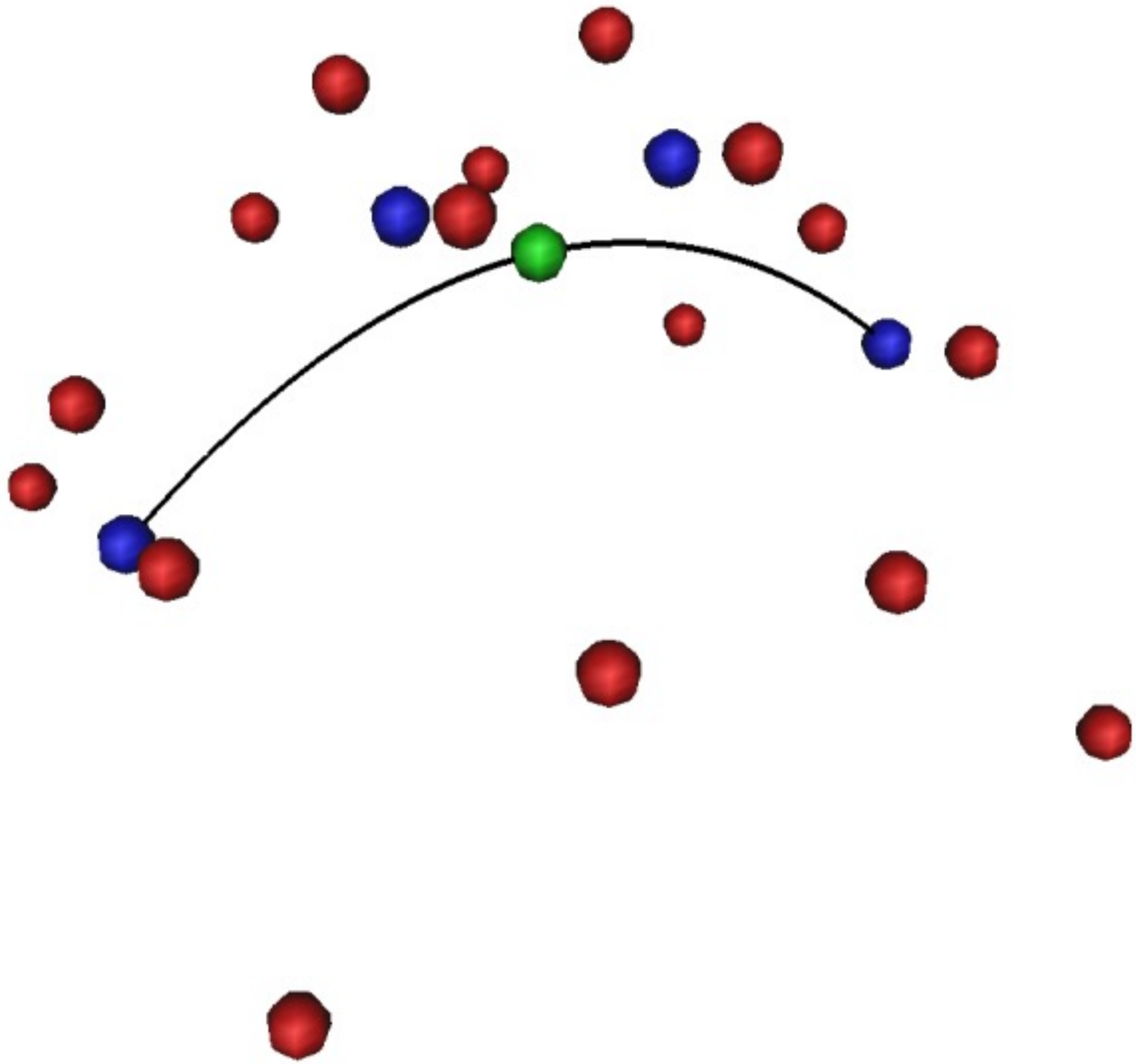
The de Casteljau algorithm



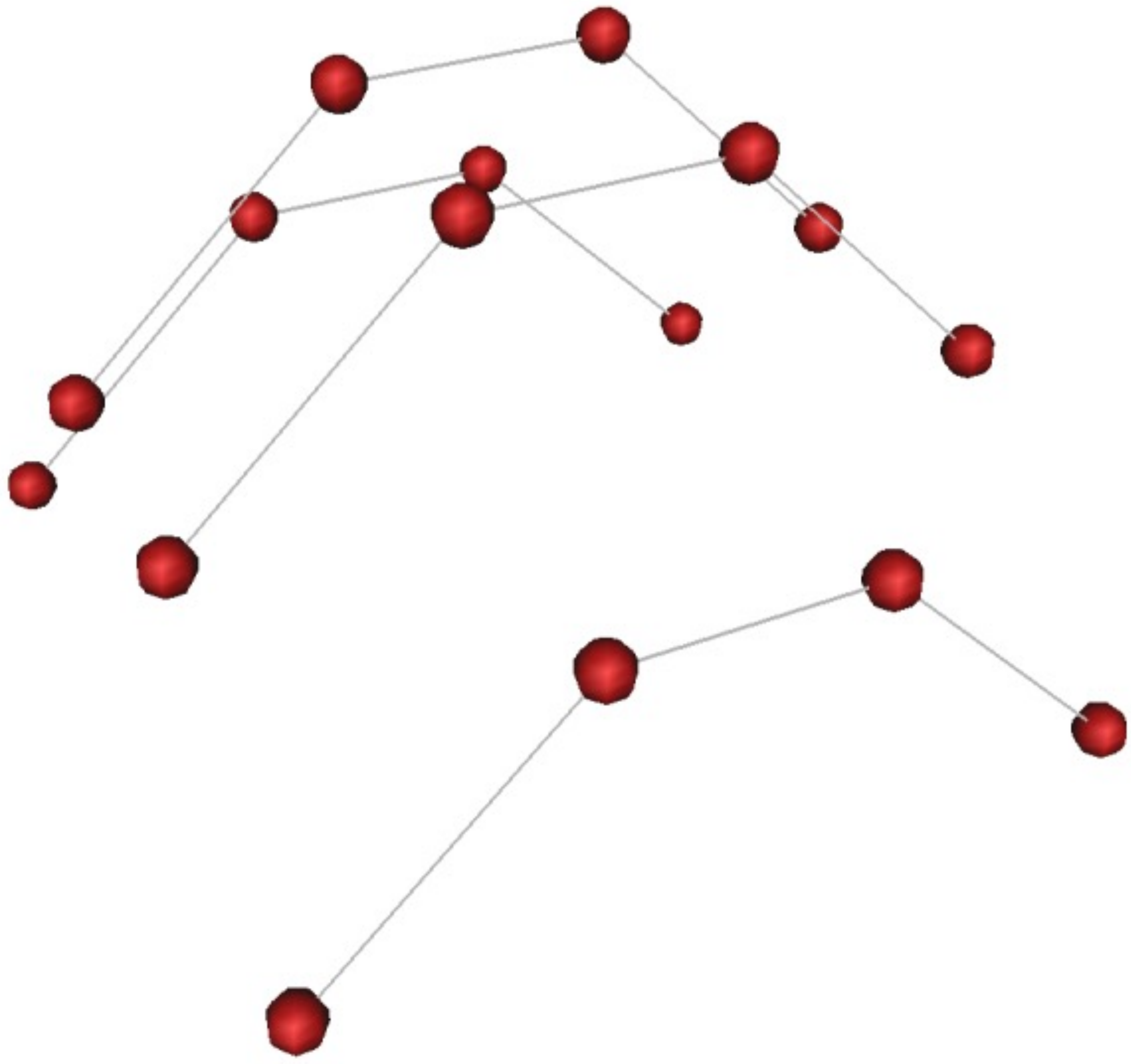
The de Casteljau algorithm



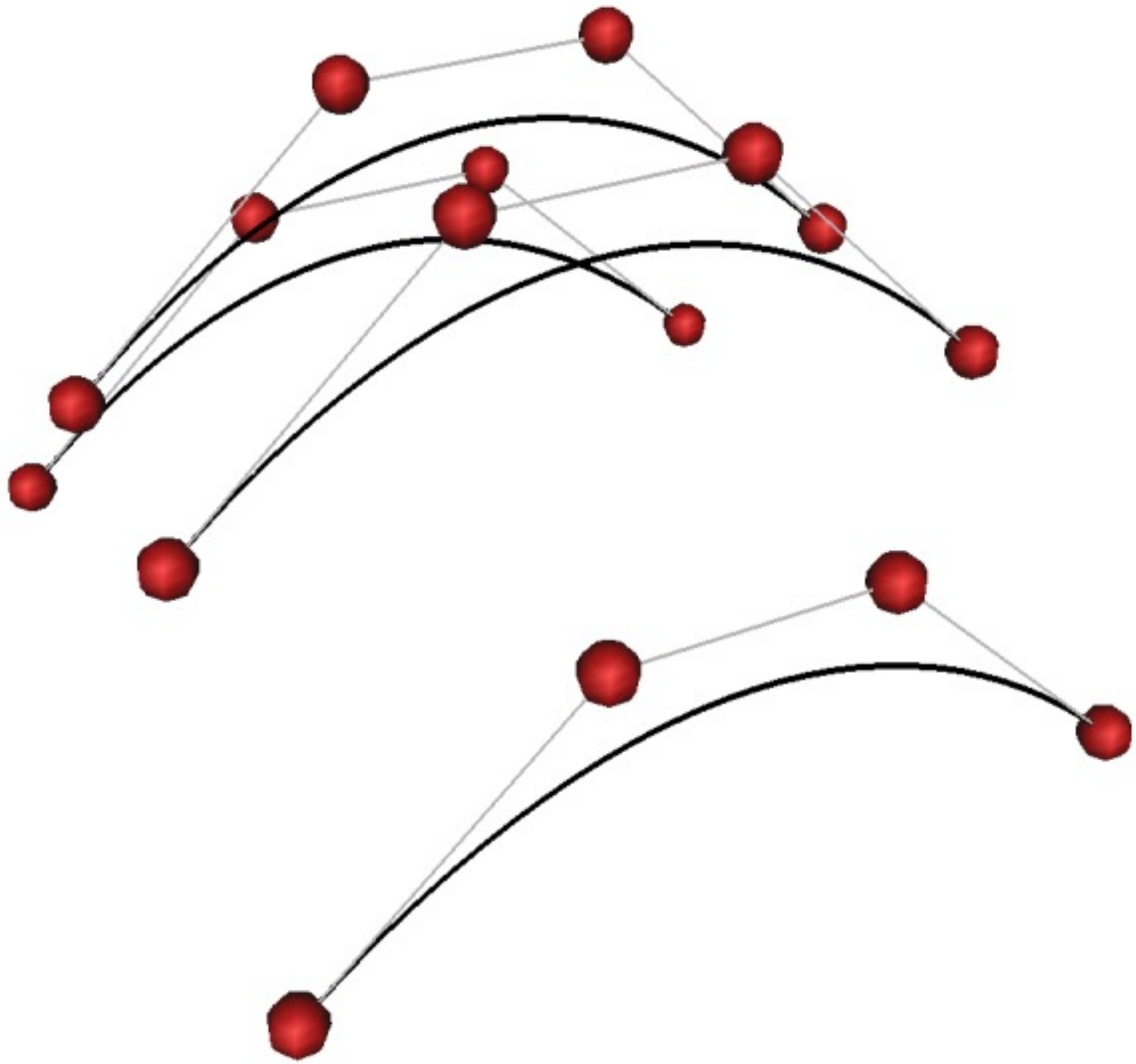
The de Casteljau algorithm



The de Casteljau algorithm



The de Casteljau algorithm



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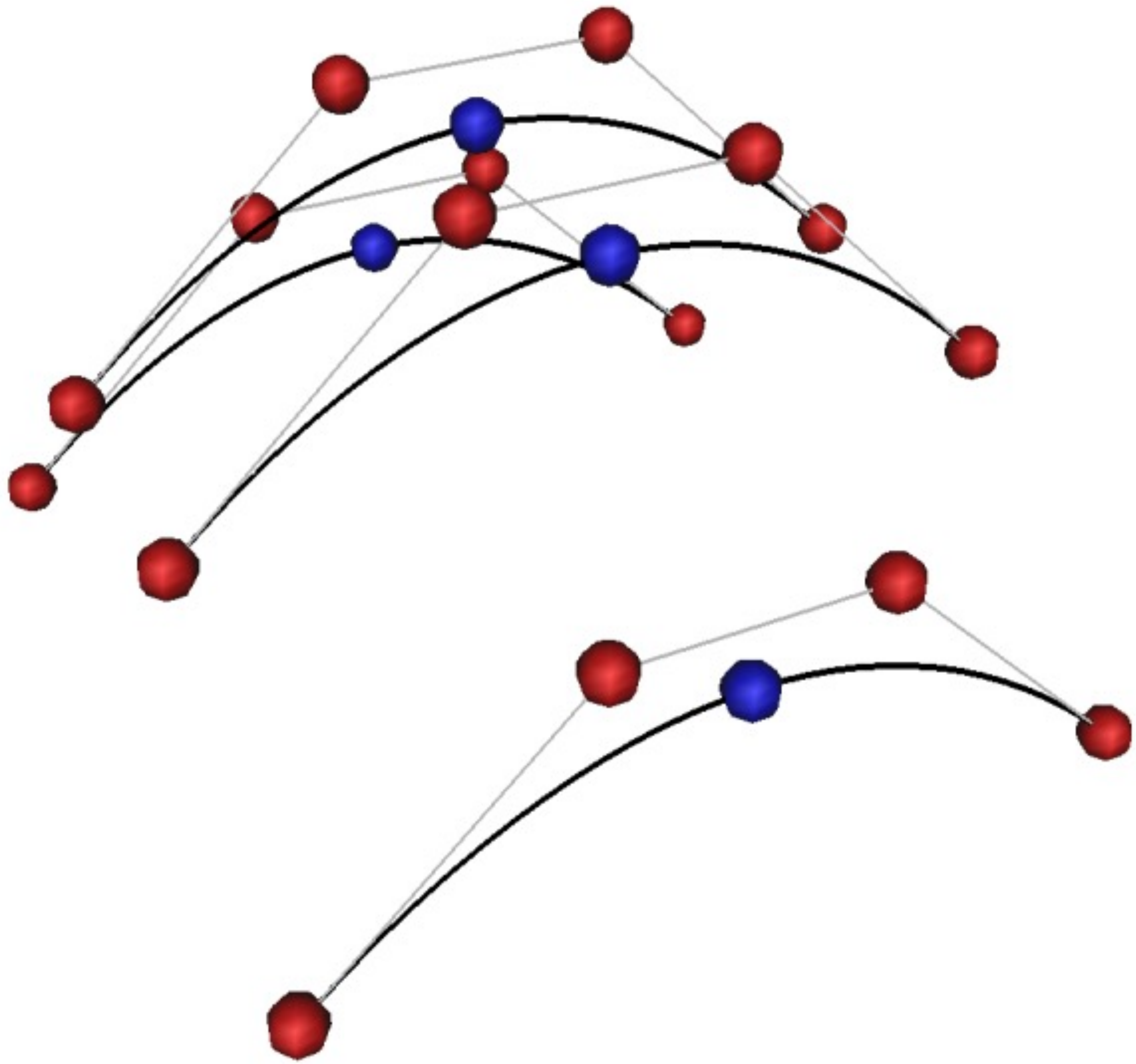

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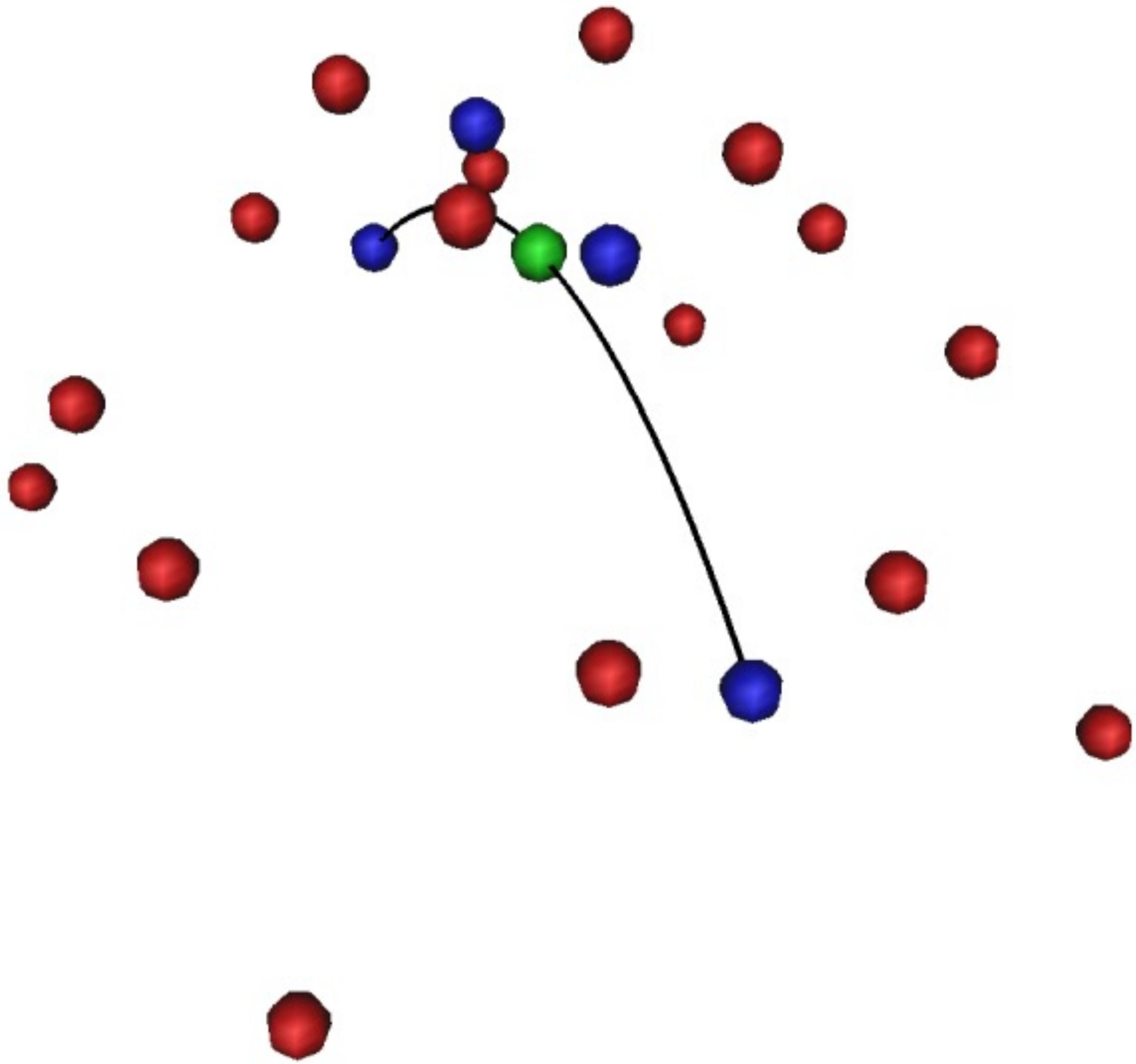

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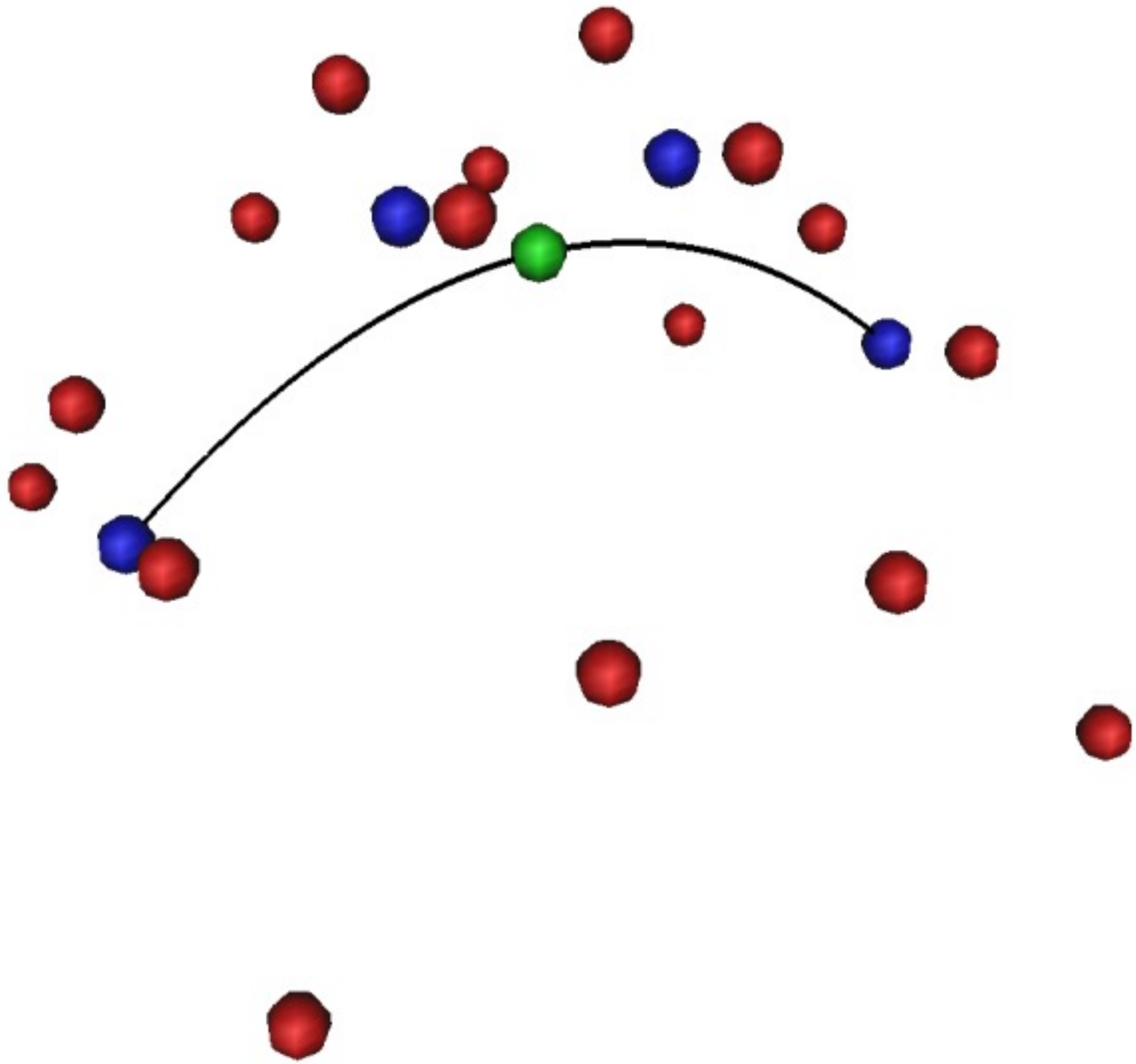
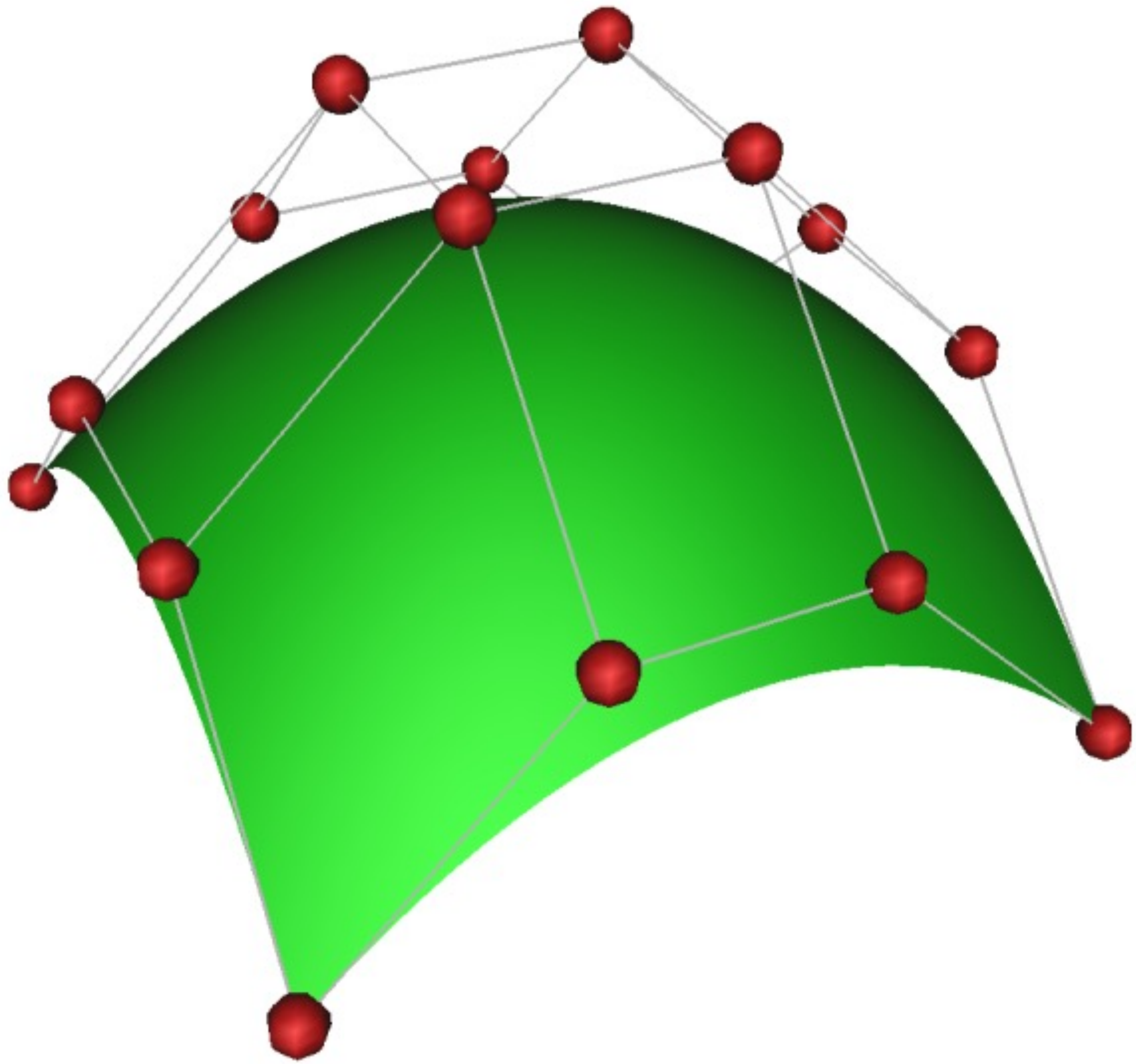
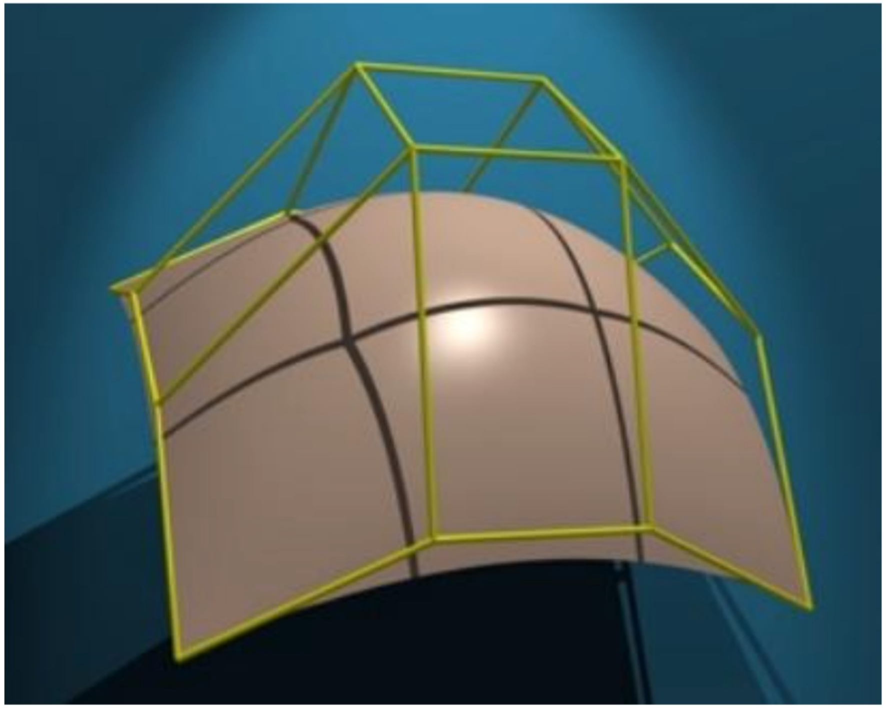


A « bicubic » Bézier patch



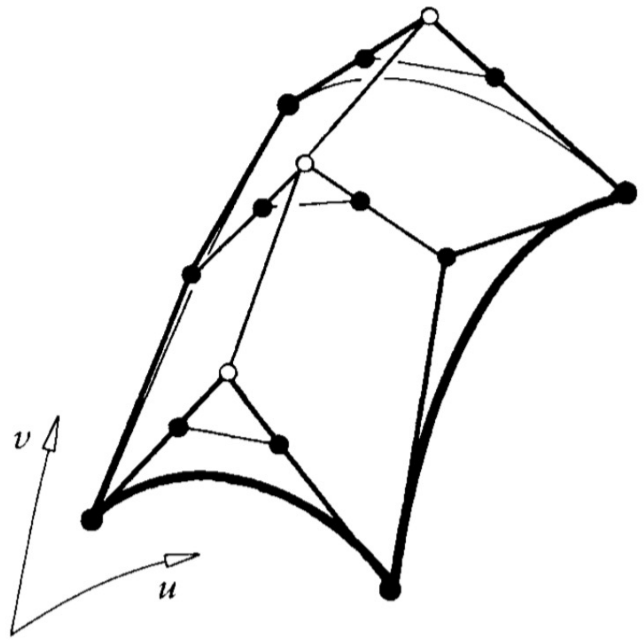
Properties of tensor product Bézier patch

- Affine invariance (follows from the definition)
- Convex hull property : the patch lies in convex hull of its control points
- The patch interpolates the four corner control points
- *Variation diminishing property is NOT inherited from the univariate case*



Degree elevation

Goal : rewrite a Bézier patch of degree *(m,n)* as one of degree *(m+1,n)*



- Find the new coefficients such that

$$\mathbf{b}^{m,n}(u,v) = \sum_{j=0}^{n} \left[\sum_{i=0}^{m+1} \mathbf{b}_{i,j}^{(1,0)} B_i^{m+1}(u) \right] B_j^n(v)$$

- It can be reduced to a series of univariate problems.

$$\mathbf{b}_{i,j}^{(1,0)} = \frac{i}{m+1}\mathbf{b}_{i-1,j} + \left(1 - \frac{i}{m+1}\right)\mathbf{b}_{i,j}; \begin{cases} i = 0, \ldots, m+1 \\ j = 0, \ldots, n. \end{cases}$$

Derivatives

As in the curve case, taking derivatives is accomplished by differencing the control points

- Partial derivative in u-direction:

$$\frac{\partial}{\partial u}\mathbf{b}^{m,n}(u,v) = \sum_{j=0}^{n}\left[\frac{\partial}{\partial u}\sum_{i=0}^{m}\mathbf{b}_{i,j}B_i^m(u)\right]B_j^n(v)$$

$$\frac{\partial}{\partial u}\mathbf{b}^{m,n}(u,v) = m\sum_{j=0}^{n}\sum_{i=0}^{m-1}\Delta^{1,0}\mathbf{b}_{i,j}B_i^{m-1}(u)B_j^n(v) \qquad \Delta^{1,0}\mathbf{b}_{i,j} = \mathbf{b}_{i+1,j} - \mathbf{b}_{i,j}$$

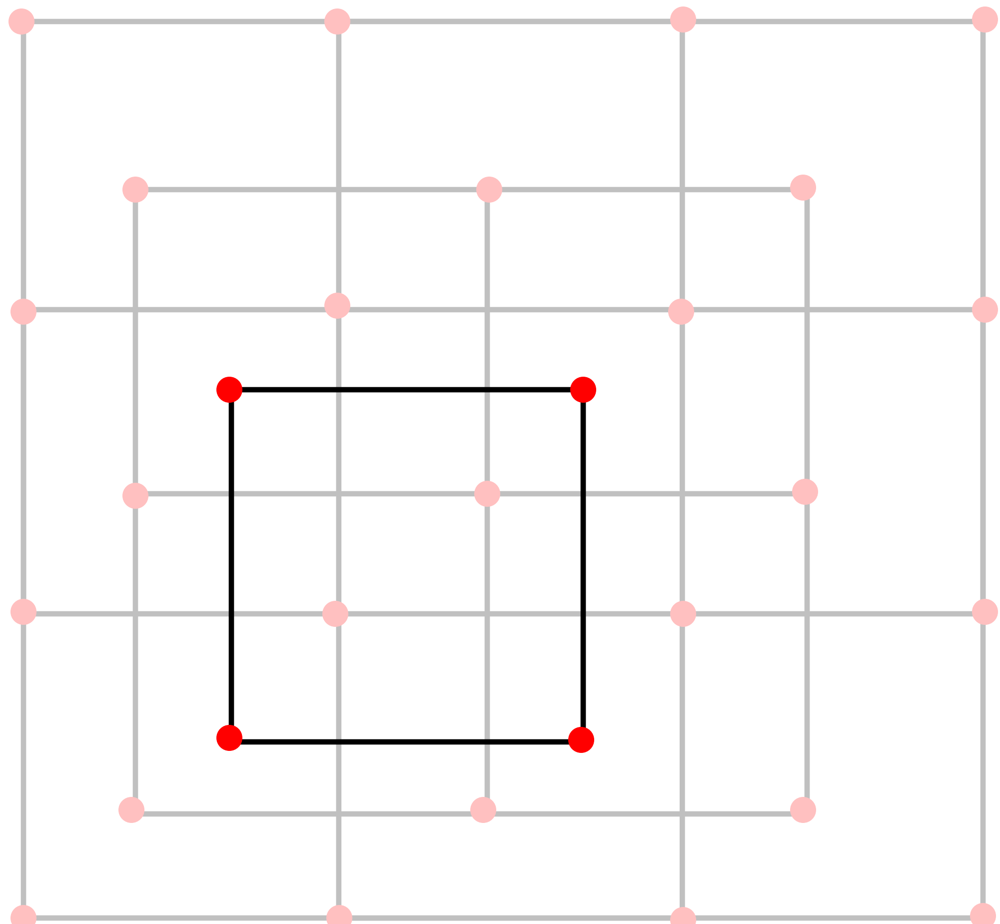
$$\frac{\partial^r}{\partial u^r}\mathbf{b}^{m,n}(u,v) = \frac{m!}{(m-r)!}\sum_{j=0}^{n}\sum_{i=0}^{m-r}\Delta^{r,0}\mathbf{b}_{i,j}B_i^{m-r}(u)B_j^n(v) \quad \Delta^{r,0}\mathbf{b}_{i,j} = \Delta^{r-1,0}\mathbf{b}_{i+1,j} - \Delta^{r-1,0}\mathbf{b}_{i,j}$$

- General formula:

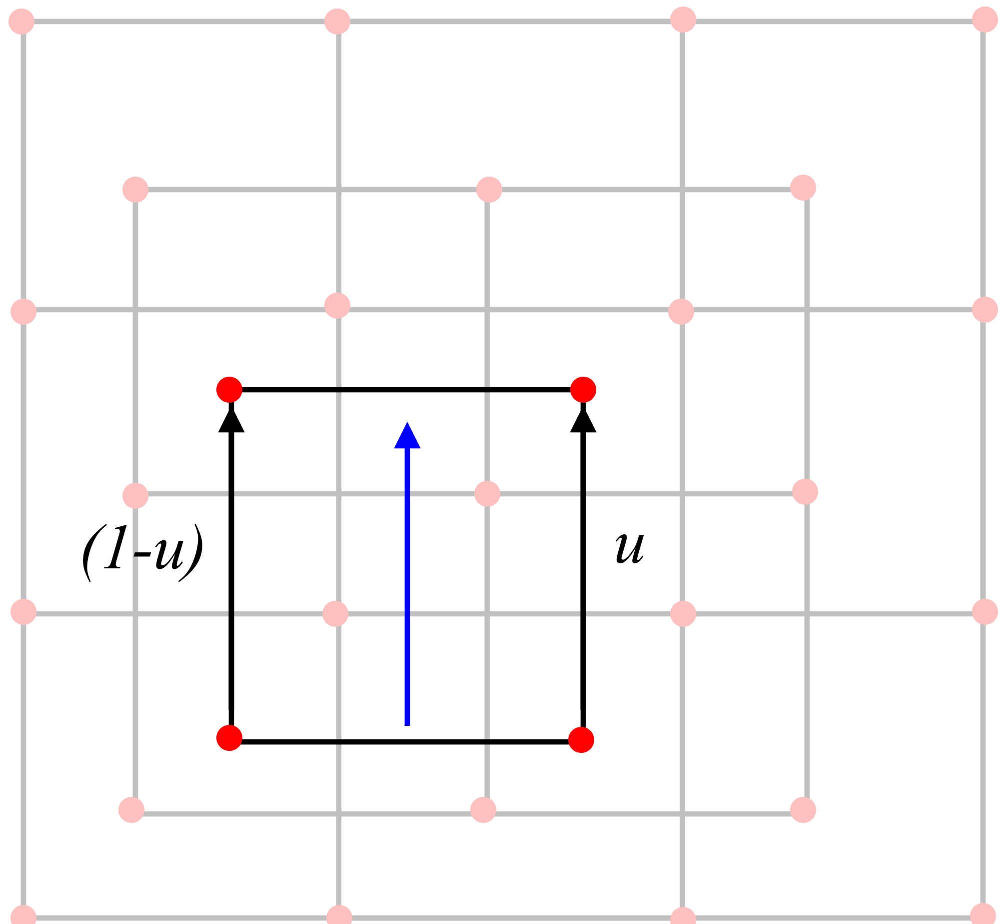
$$\frac{\partial^{r+s}}{\partial u^r \partial v^s}\mathbf{b}^{m,n}(u,v)$$

$$= \frac{m!n!}{(m-r)!(n-s)!}\sum_{i=0}^{m-r}\sum_{j=0}^{n-s}\Delta^{r,s}\mathbf{b}_{i,j}B_i^{m-r}(u)B_j^{n-s}(v)$$

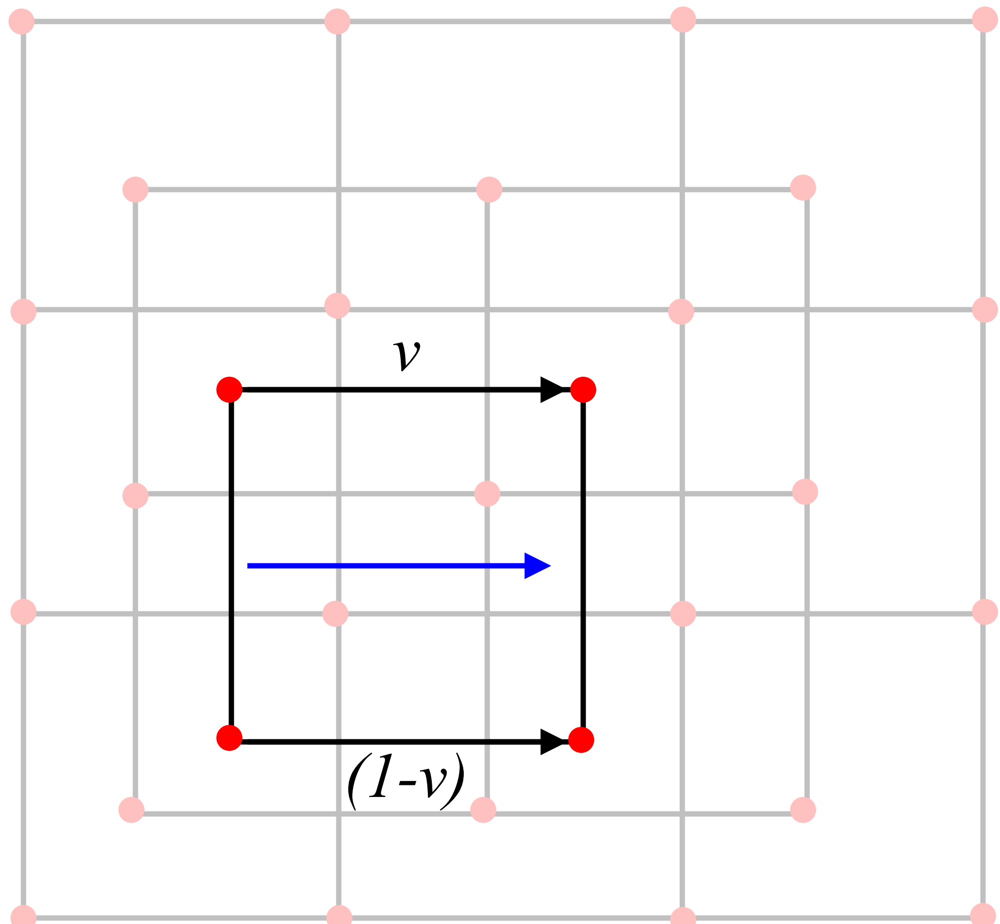
Derivatives using de Casteljau's algorithm



Derivatives using de Casteljau's algorithm



Derivatives using de Casteljau's algorithm



Blossoming for tensor-product patches

$$b(u_1, \ldots, u_m | v_1, \ldots, v_n)$$

- **Symmetry**:

For any permutation q of $(1,\ldots,m)$ and r of $(1,\ldots,n)$:

$$b(u_{q(1)}, \ldots, u_{q(m)} | v_{r(1)}, \ldots, v_{r(n)}) = b(u_1, \ldots, u_m | v_1, \ldots, v_n)$$

- **Multi-affine**:

$$\begin{aligned} b(u_1, \ldots, (1-d)u_k + ds_k, \ldots, u_m | v_1, \ldots, (1-e)v_k + ew_k, \ldots, v_n) \\ = (1-d)(1-e).b(u_1, \ldots, u_k, \ldots, u_m | v_1, \ldots, v_k, \ldots, v_n) \\ +(1-d)e.b(u_1, \ldots, u_k, \ldots, u_m | v_1, \ldots, w_k, \ldots, v_n) \\ +de.b(u_1, \ldots, s_k, \ldots, u_m | v_1, \ldots, w_k, \ldots, v_n) \\ +d(1-e).b(u_1, \ldots, s_k, \ldots, u_m | v_1, \ldots, v_k, \ldots, v_n) \end{aligned}$$

- **Diagonal**:

$$b(u, \ldots, u | v, \ldots, v) = b(u, v)$$

Curves on Bézier surfaces

- Given two points in the domain of a Bézier patch of degree *(n,n)*, they define a straight line

$$\mathbf{u}(t) = (1-t)\mathbf{p} + t\mathbf{q} \qquad \mathbf{p} = (\mathbf{p}_u, \mathbf{p}_v) \text{ and } \mathbf{q} = (\mathbf{q}_u, \mathbf{q}_v)$$

- This line is mapped to a curve on the surface. What are its Bézier control points ?

- Using the blossom of the surface, a point on this curve is given by

$$\mathbf{b}[((1-t)\mathbf{p}_u + t\mathbf{q}_u)^{<n>} \mid ((1-t)\mathbf{p}_v + t\mathbf{q}_v)^{<n>}].$$

Curves on Bézier surfaces

$$\sum_{i+j=n} \binom{n}{i,j} (1-t)^i t^j \mathbf{b}[\mathbf{p}_u^{<i>}, \mathbf{q}_u^{<j>} \mid ((1-t)\mathbf{p}_v + t\mathbf{q}_v)^{<n>}].$$

$$\binom{n}{i,j} = \frac{n!}{i!j!}$$

$$\sum_{i+j=n} \sum_{r+s=n} \binom{n}{i,j} \binom{n}{r,s} (1-t)^i t^j (1-t)^r t^s \mathbf{b}[\mathbf{p}_u^{<i>}, \mathbf{q}_u^{<j>} \mid \mathbf{p}_v^{<r>}, \mathbf{q}_v^{<s>}]$$

$$\sum_{i=0}^{n} \sum_{j=0}^{n} \binom{n}{i} \binom{n}{j} (1-t)^{2n-i-j} t^{i+j} \mathbf{b}[\mathbf{p}_u^{<i>}, \mathbf{q}_u^{<n-i>} \mid \mathbf{p}_v^{<j>}, \mathbf{q}_v^{<n-j>}]$$

$$\sum_{k=0}^{2n} \sum_{i+j=k} \frac{\binom{n}{i}\binom{n}{j}}{\binom{2n}{k}} B_k^{2n} \mathbf{b}[\mathbf{p}_u^{<i>}, \mathbf{q}_u^{<n-i>} \mid \mathbf{p}_v^{<j>}, \mathbf{q}_v^{<n-j>}]$$

curve of degree *2n*

Finally, the control points are given by :

$$\mathbf{c}_k = \frac{\binom{n}{i}\binom{n}{j}}{\binom{2n}{k}} \mathbf{b}[\mathbf{p}_u^{<i>}, \mathbf{q}_u^{<n-i>} \mid \mathbf{p}_v^{<j>}, \mathbf{q}_v^{<n-j>}]$$

Normal vectors

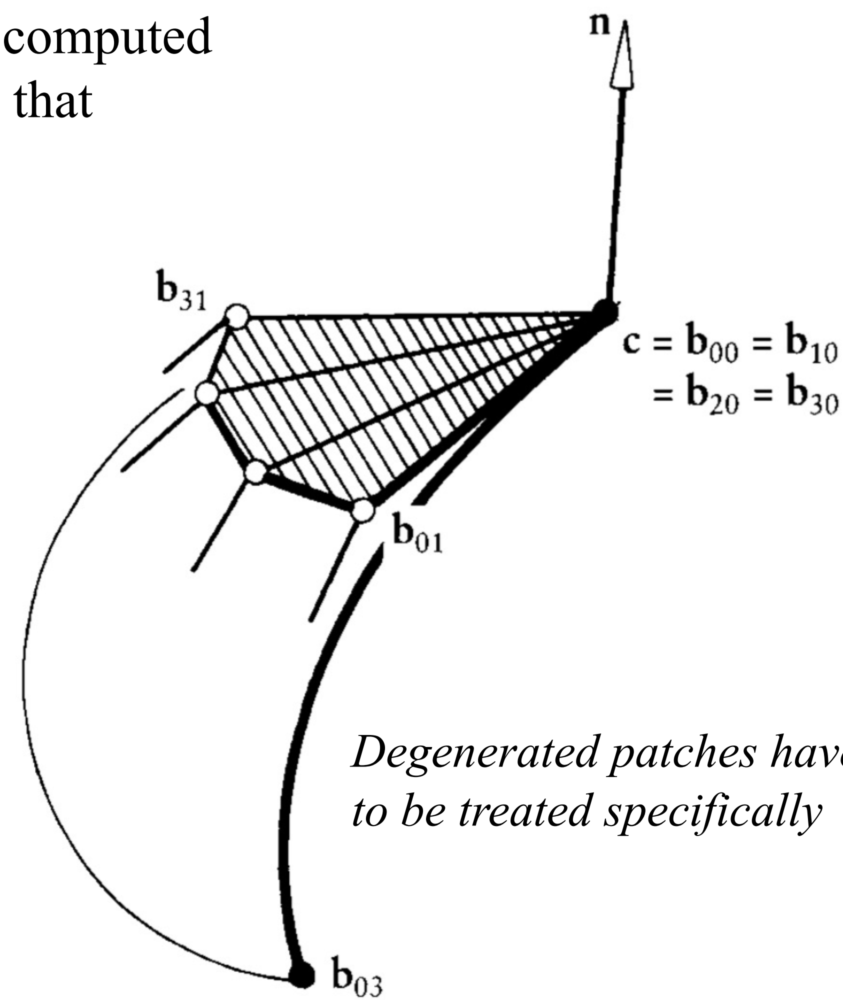
- The normal vector of a surface can be computed from the cross product of any two vectors that are tangent to the surface.

- Using the partial derivatives:

$$\mathbf{n}(u,v) = \frac{\frac{\partial}{\partial u}\mathbf{b}^{m,n}(u,v) \wedge \frac{\partial}{\partial v}\mathbf{b}^{m,n}(u,v)}{\|\frac{\partial}{\partial u}\mathbf{b}^{m,n}(u,v) \wedge \frac{\partial}{\partial v}\mathbf{b}^{m,n}(u,v)\|}$$

- At the four corner points, we have for instance:

$$\mathbf{n}(0,0) = \frac{\Delta^{1,0}\mathbf{b}_{0,0} \wedge \Delta^{0,1}\mathbf{b}_{0,0}}{\|\Delta^{1,0}\mathbf{b}_{0,0} \wedge \Delta^{0,1}\mathbf{b}_{0,0}\|}$$



Degenerated patches have to be treated specifically

The matrix form of a Bézier patch

- The « geometry matrix » of the patch:

$$\mathbf{b}^{m,n}(u,v) = \begin{bmatrix} B_0^m(u) & \dots & B_m^m(u) \end{bmatrix} \begin{bmatrix} \mathbf{b}_{00} & \dots & \mathbf{b}_{0n} \\ \vdots & & \vdots \\ \mathbf{b}_{m0} & \dots & \mathbf{b}_{mn} \end{bmatrix} \begin{bmatrix} B_0^n(v) \\ \vdots \\ B_n^n(v) \end{bmatrix}$$

- Basis transformation to the monomial basis:

$$\mathbf{b}^{m,n}(u,v) = \begin{bmatrix} u^0 & \dots & u^m \end{bmatrix} M^{\mathrm{T}} \begin{bmatrix} \mathbf{b}_{00} & \dots & \mathbf{b}_{0n} \\ \vdots & & \vdots \\ \mathbf{b}_{m0} & \dots & \mathbf{b}_{mn} \end{bmatrix} N \begin{bmatrix} v^0 \\ \vdots \\ v^n \end{bmatrix}$$

where the matrices M and N are given by :

$$m_{ij} = (-1)^{j-i} \binom{m}{j} \binom{j}{i} \qquad n_{ij} = (-1)^{j-i} \binom{n}{j} \binom{j}{i}$$

Bézier patch of a graph

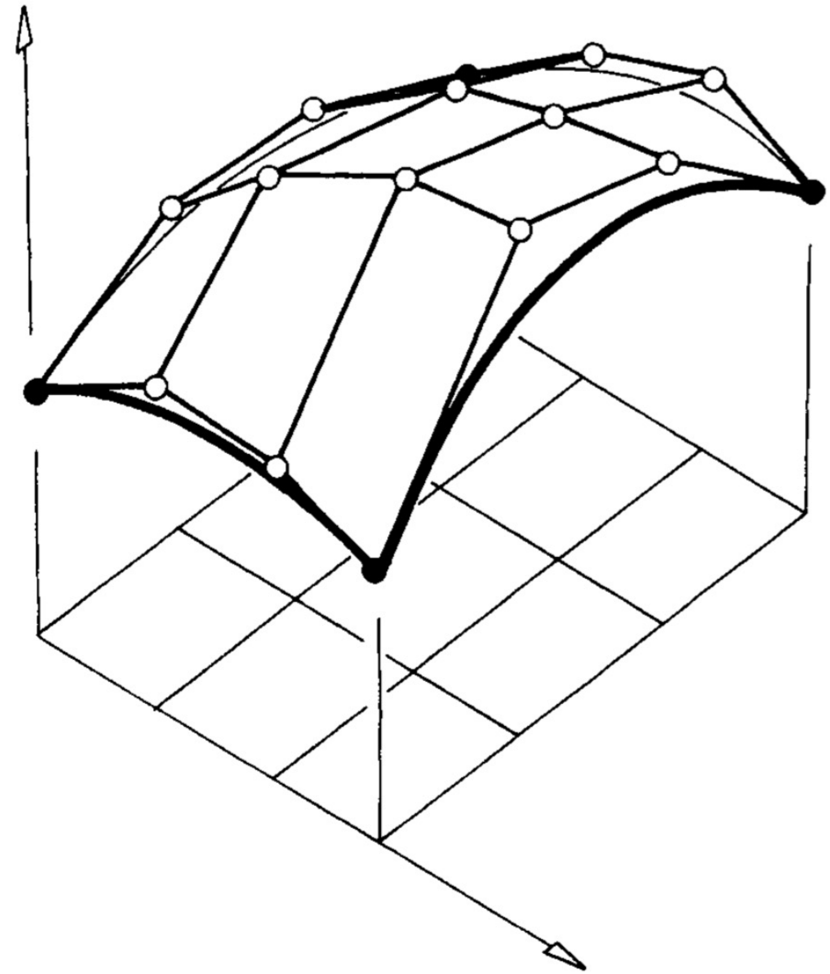
- Parameterization of the form

$$\mathbf{x}(u,v) = \begin{bmatrix} u \\ v \\ f(u,v) \end{bmatrix}$$

- If $f(x,y) = \sum_{i}^{m} \sum_{j}^{n} b_{ij} B_i^m(x) B_j^n(y)$

  then the control points of the patch are given by:

$$\mathbf{b}_{ij} = \begin{bmatrix} i/m \\ j/n \\ b_{ij} \end{bmatrix}$$



The control points are located over a regular partition of the domain rectangle

Composite Bézier surfaces

- Suppose given two Bézier patches of degree *(m,n)*:

left patch :

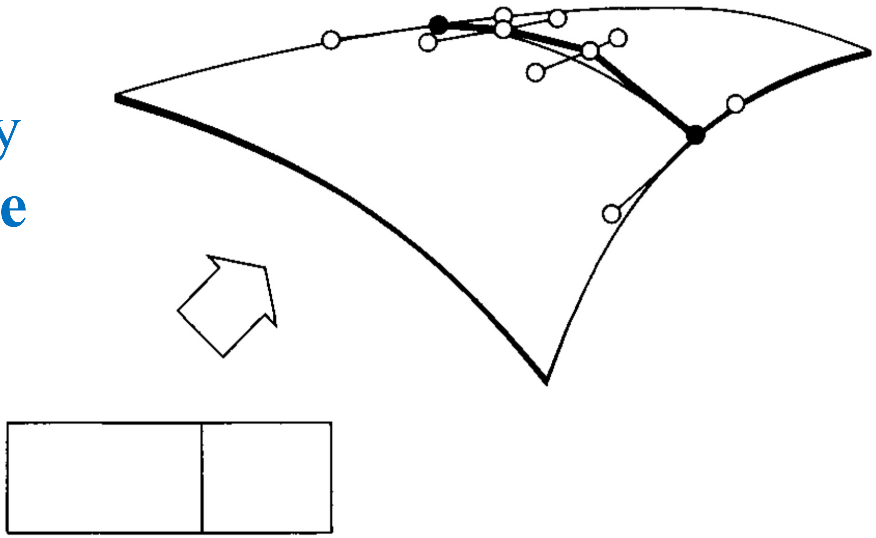
$\{\mathbf{b}_{ij}\};\ 0 \le i \le m,\ 0 \le j \le n$

right patch :

$\{\mathbf{b}_{ij}\};\ m \le i \le 2m,\ 0 \le j \le n$

- To get r times differentiability accross their common boundary, one must evaluate the *u*-partial derivatives.

It is equivalent to the fact that every pair of **adjacent control curves are *r-differentiable***

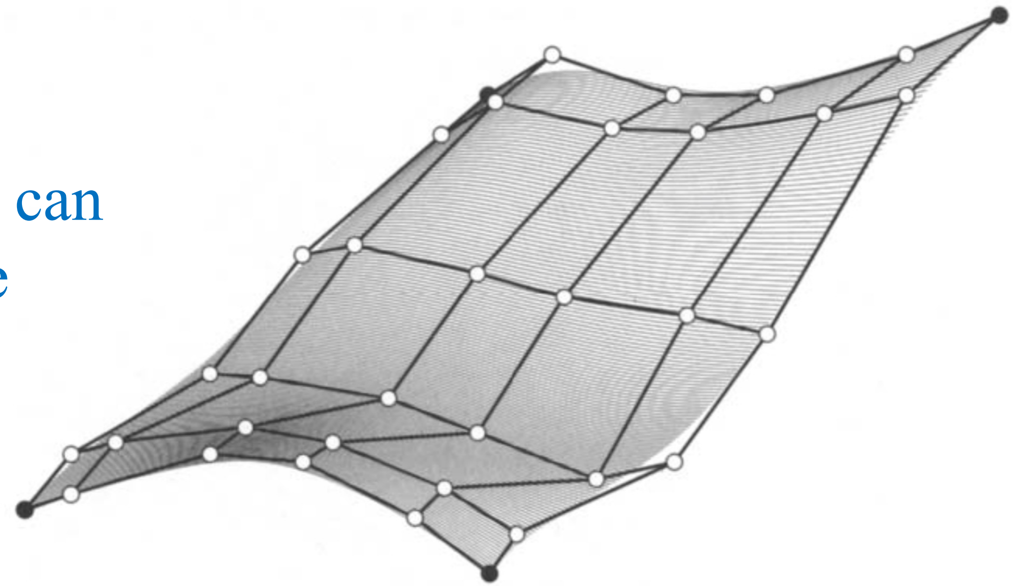


Tensor product B-spline surfaces

$$b(u,v) = \sum_i \sum_j b_{i,j} N_i^m(u) N_j^n(v)$$

- Need two knot sequences : one in the *u*-direction and one in the *v*-direction

All methods and algorithms can be reduced to the curve case



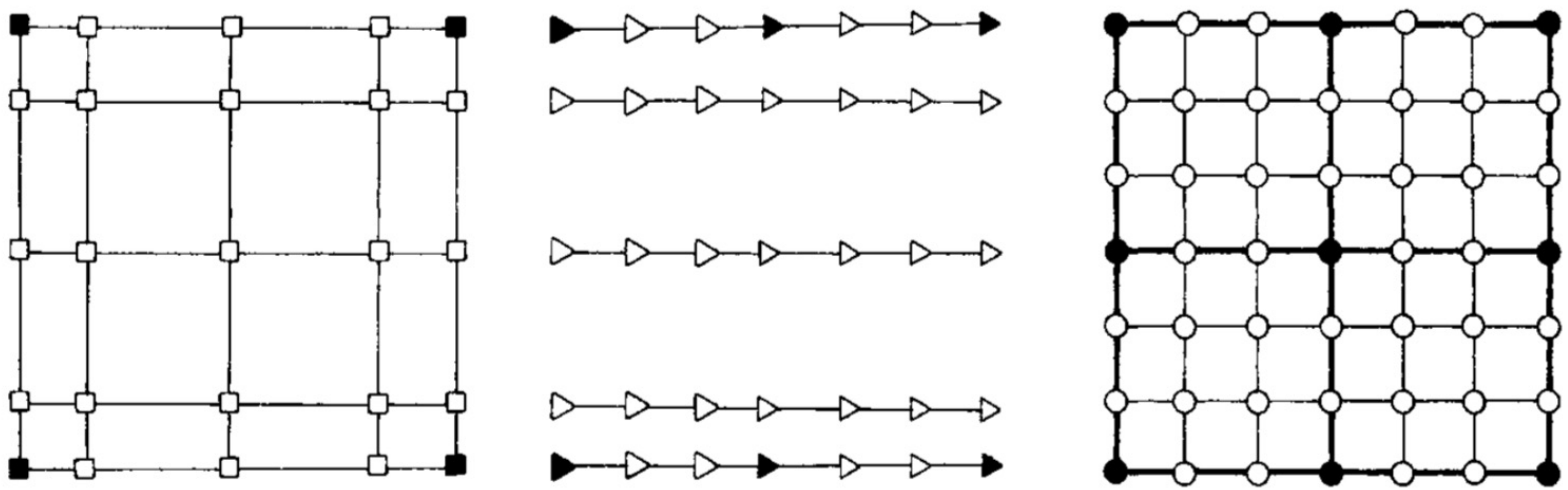
A bicubic B-spline surface consisting of 3x3 Bézier patches

Conversion to Bézier patches

Conversion to Bézier patches is done by using the univariate method :

- Interpret the B-spline control net row by row as univariate B-spline polygons
- convert them to piecewise Bezier form.

- The Bezier points thus obtained may be interpreted, column by column, as B-spline polygons, which we may again transform to Bezier form one by one.



Bringing a bicubic B-spline surface into piecewise bicubic Bézier form: we first perform B-spline–Bézier curve conversion row by row, then column by column.

Rational Bézier and B-spline surfaces

- Bézier and B-spline surfaces can be generalized to their rational counterpart as for curves

- Rational Bézier and B-spline surfaces are projections of a 4D tensor product of B-spline surface

Rational Bézier surface

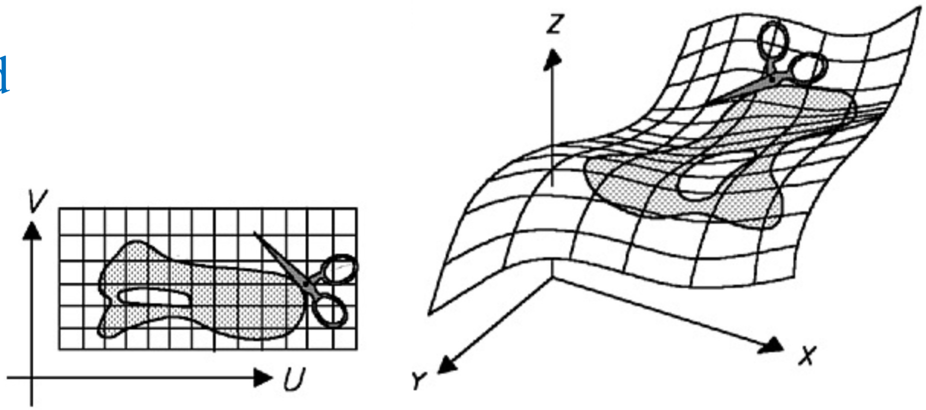
$$\frac{\sum_i \sum_j w_{i,j} b_{i,j} B_i^m(u) B_j^n(v)}{\sum_i \sum_j w_{i,j} B_i^m(u) B_j^n(v)}$$

Rational B-spline surface

$$\frac{\sum_i \sum_j w_{i,j} b_{i,j} N_i^m(u) N_j^n(v)}{\sum_i \sum_j w_{i,j} N_i^m(u) N_j^n(v)}$$

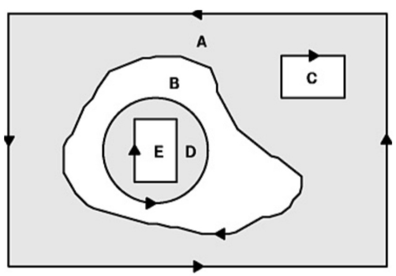
Trimmed surfaces

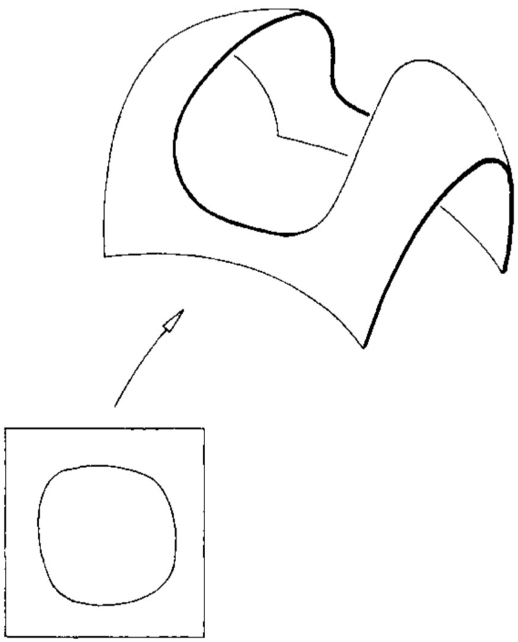
- A parametric curve in the domain is used to trim the surface



- A curve of degree *d* yields a curve of degree *(m+n)d* on the surface

- Orientation of the curve defines the inside/outside points. The test is done by launching a « random » ray.





B-Rep of CAD models

A CAD model : on the left, the trimmed surfaces, on the right the surfaces are printed without trimming