Computer Aided Geometric Design
–
Béziers Curves

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Bézier Curves

Named after their Inventor **Pierre Bézier**, an ingeneer with the Renault car company.

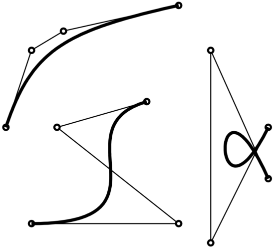


Figure: Cubic Bézier curves

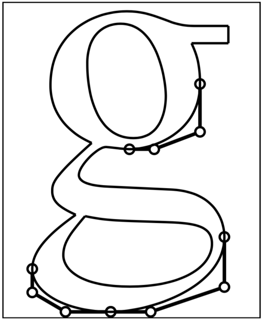


Figure: Font definition using Bézier curves

The Equation of a Bézier Curve

- Parameter $t \in [0,1] \subset \mathbb{R}$
- Bernstein polynomials

$$B_i^n(t) := \binom{n}{i}(1-t)^{n-i}t^i, \;\; i = 0, \ldots, n$$

- Control points:

$$\mathbf{P}_0, \ldots, \mathbf{P}_n \in \mathbb{R}^2$$

- Parameterization:

$$\mathbf{P}(t) := \sum_{i=0}^{n} \mathbf{P}_i \, B_i^n(t)$$

- Degree n Bézier curve
- Observe that $\mathbf{P}(0) = \mathbf{P}_0$, $\mathbf{P}(1) = \mathbf{P}_n$.

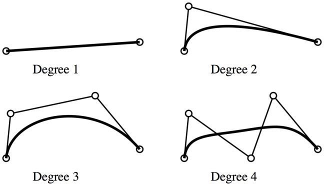


Figure: Bézier curves of various degree

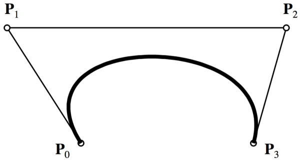


Figure: Control points of a cubic Bézier curve

Arbitrary Parameter Intervals

- In the previous definition, the Bézier curves starts at $t = 0$ and ends at $t = 1$.
- It is useful, especially for fitting together several Bézier curves, to allow an arbitrary parameter interval

$$t \in [t_0, t_1] \subset \mathbb{R}$$

  such that $\mathbf{P}(t_0) = P_0$ and $\mathbf{P}(t_1) = P_n$.
- The modified parameterization is given by

$$\mathbf{P}_{[t_0,t_1]}(t) = \sum_{i=0}^{n} \mathbf{P}_i \binom{n}{i} \left(\frac{t_1 - t}{t_1 - t_0}\right)^{n-i} \left(\frac{t - t_0}{t_1 - t_0}\right)^{i}.$$

- It is obtained by a change of parameter: $t \leftarrow (t - t_0)/(t_1 - t_0)$.

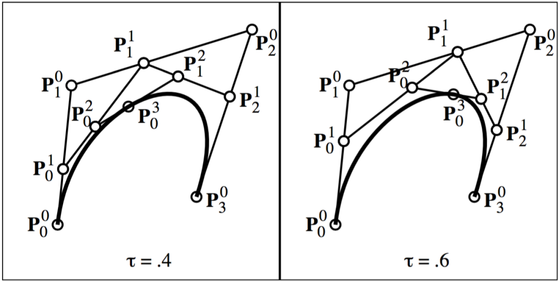
The de Casteljau Algorithm

Subdvision

How to subdivide a Bézier curve $\mathbf{P}_{[t_0,t_2]}$ into two segments $\mathbf{P}_{[t_0,t_1]}$ and $\mathbf{P}_{[t_1,t_2]}$ whose union is equivalent to $\mathbf{P}_{[t_0,t_2]}$?

- label the control points as $\mathbf{P}_0^0, \mathbf{P}_1^0, \mathbf{P}_2^0, \mathbf{P}_3^0$; set $\tau := (t_1 - t_0)/(t_2 - t_0)$
- Compute the sequence of points

$$\mathbf{P}_i^j = (1-\tau)\mathbf{P}_i^{j-1} + \tau \mathbf{P}_{i+1}^{j-1}, \; j = 1, \ldots, n, i = 0, \ldots, n-j.$$



- Control points of $\mathbf{P}_{[t_0,t_1]}$

$$\mathbf{P}_0^0, \mathbf{P}_0^1, \ldots, \mathbf{P}_0^n$$

- Control points of $\mathbf{P}_{[t_1,t_2]}$

$$\mathbf{P}_0^n, \mathbf{P}_1^{n-1}, \ldots, \mathbf{P}_n^0$$

- Evaluation : $\mathbf{P}(t_1) = \mathbf{P}_0^n$

The de Casteljau Algorithm

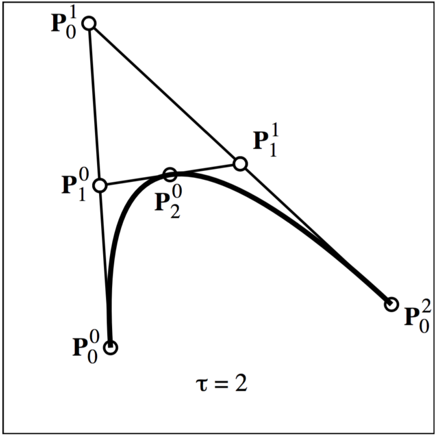


Figure: The de Casteljau Algorithm works even if $\tau \notin [0, 1]$, i.e. $t_1 \notin [t_0, t_2]$. But it is numerically stable only if $t_1 \in [t_0, t_2]$

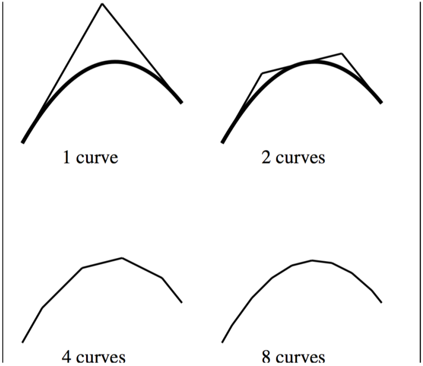


Figure: The collection of control polygons converge to the curve after repeated subdivisions.

Degree Elevation

Any degree n Bézier curve can be exactly represented as a degree $n+1$ Bézier curve.

$$\textbf{New control points :}\ \mathbf{P}_i^* = \frac{i}{n+1}\mathbf{P}_{i-1} + \frac{n+1-i}{n+1}\mathbf{P}_i, \quad i = 0, \ldots, n+1.$$

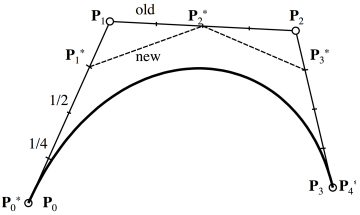


Figure: Degree elevation of a cubic Bézier curve

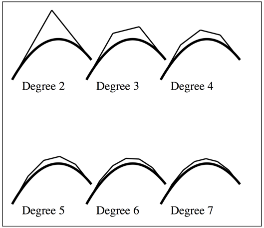
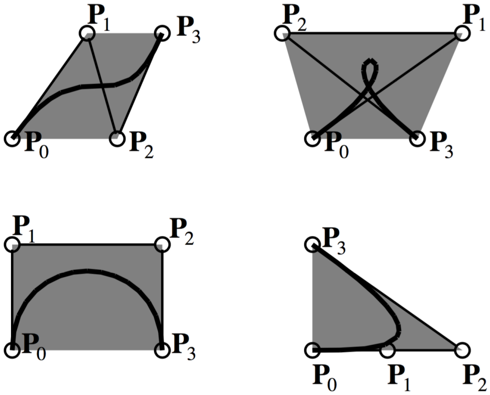


Figure: Repeated degree elevation converges to the curve

$$\mathbf{P}(t) = ((1-t)+t)\,\mathbf{P}(t) = ((1-t)+t)\sum_{i=0}^{n}\mathbf{P}_i B_i^n(t) = \sum_{i=0}^{n+1}\mathbf{P}_i^* B_i^{n+1}(t)$$

The Convex Hull Property of Bézier Curves

- Bézier curves always lie in the **convex hull** of their control points.
- The convex hull is the smallest convex set that contains the control points.
- This is an easy consequence of the definition of a Bézier curve.



Convex set

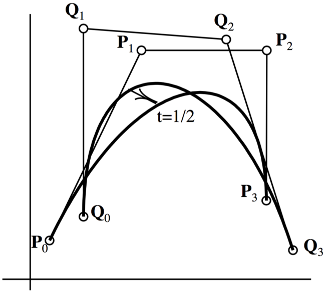
A convex set is a set of points such that, given any two points A, B in that set, the segment $[AB]$ joining them lies entirely within that set.

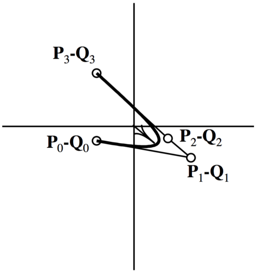
Distance between Two Bézier Curves

- **Difference curve**: Given two Bézier curves $\mathbf{P}(t) = \sum_{i=0}^{n} \mathbf{P}_i B_i^n(t)$ and $\mathbf{Q}(t) = \sum_{i=0}^{n} \mathbf{Q}_i B_i^n(t)$, define the Bézier curve

$$\mathbf{D}(t) = \mathbf{P}(t) - \mathbf{Q}(t) = \sum_{i=0}^{n} (\mathbf{P}_i - \mathbf{Q}_i) B_i^n(t).$$

- The convex hull property implies that the distance between the two curves is bounded by the largest distance from the origin to any of the control points of $\mathbf{D}(t)$.





- **Attractive** because easy to compute
- **Not very tight** because it depends on the parameterizations

Variation Diminishing Property

Property

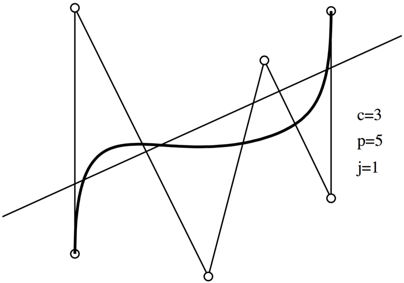
If a straight line intersects a Bézier curve in c number of points and the control polygon in p number of points, then it will always hold that

$$c = p - 2j$$

where j is zero or a positive integer.

Practical interpretation

A Bézier curve will "wiggle" no more than its control polygon.



First derivative – Hodograph

Given a Bézier curve $\mathbf{P}_{[t_0,t_1]}(t) = \sum_{i=0}^{n} \mathbf{P}_i B_i^n(t)$, its first derivative can be expressed as a Bézier curve $\mathbf{P}_{[t_0,t_1]}(t)' = \sum_{i=0}^{n-1} \mathbf{D}_i B_i^{n-1}(t)$ where

$$\mathbf{D}_i = \frac{n}{t_1 - t_0} \left(\mathbf{P}_{i+1} - \mathbf{P}_i \right).$$

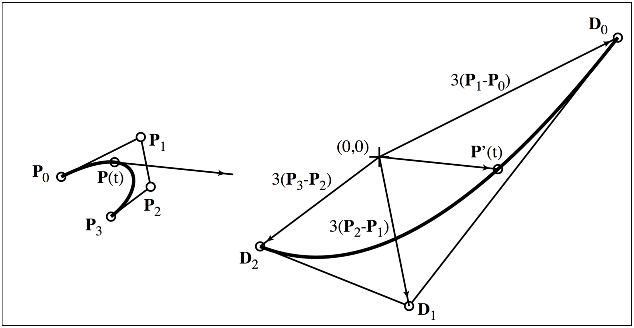


Figure: Hodograph Bézier curve, $[t_0, t_1] = [0, 1]$

Higher Derivatives

- The first derivative curve is know as a hodograph
- The second derivative can be computed as the hodograph of the hodograph, and similarly for the higher derivatives.
- It is convenient to compute them in **tabular form**.

Example

Let $\mathbf{P}_{[0,\frac{1}{2}]}(t)$ be a cubic Bézier curve with the control points

$$\mathbf{P}_0 = (2,1),\ \mathbf{P}_1 = (4,5),\ \mathbf{P}_2 = (8,6),\ \mathbf{P}_3 = (9,2).$$

Then, we get the following hodographs

Curve	Control Points			
$\mathbf{P}(t)$	$(2,1)$	$(4,5)$	$(8,6)$	$(9,2)$
$\mathbf{P}'(t)$	$(12,24)$	$(24,6)$	$(6,-24)$	
$\mathbf{P}''(t)$	$(48,-72)$	$(-72,-120)$		
$\mathbf{P}'''(t)$	$(-240,-96)$			

Three Dimensional Bézier Curves

- Bézier control points are defined in the 3D space
- The resulting curve is hence a 3D curve
- **All the previous discussion extend to 3D without modification**

Remark : conic Bézier curves are always planar. This is because they are defined by only three control points.

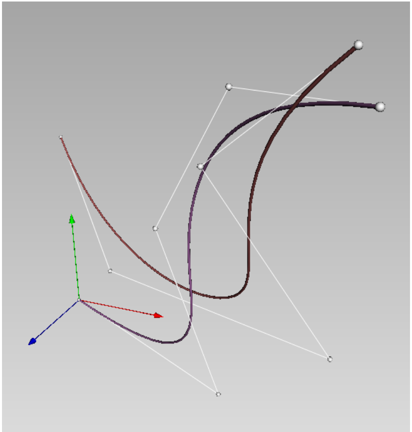


Figure: Intersection of two quartic Bézier curves in space

Rational Bézier Curves

- Each control point $\mathbf{P}_i$ is assigned a scalar weight w_i.
- The equation of a rational Bézier curve of degree n is

$$\mathbf{P}(t) = \frac{\sum_{i=0}^{n} w_i \mathbf{P}_i B_i^n(t)}{\sum_{i=0}^{n} w_i B_i^n(t)}$$

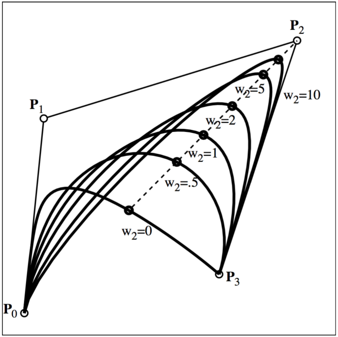


Figure: Impact of the weights

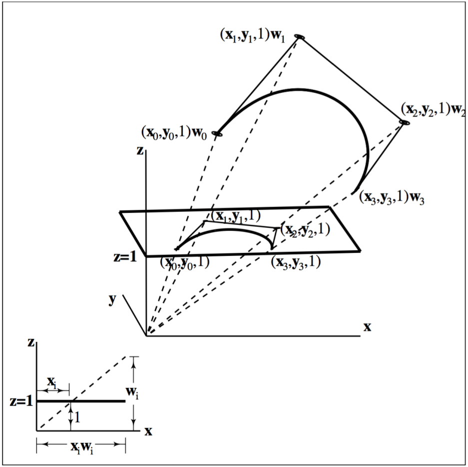
- "Rational" because the parameterization is given by rational functions (ratio of polynomials)
- Provide more control on the shape
- Allow to express exactly conic sections
- If all $w_i = 1$ then rational Bézier curves reduce to (polynomial) Bézier curves

Rational Bézier Curves

A rational Bézier curve can be interpreted as the perspective (or central) projection of a 3D Bézier curve.

- The 3D curve is a Bézier curve
- The 2D curve in the plane $z = 1$ is the perspective of the 3D curve : a rational Bézier curve.
- If $(X(t), Y(t), Z(t))$ denotes the points on the 3D curve, then the points on the 2D curve are given by

$$(x(t), y(t)) = \left(\frac{X(t)}{Z(t)}, \frac{Y(t)}{Z(t)}\right).$$



De Casteljau Algorithm and Degree Elevation

Extension to rational Bézier curves

Both The de Casteljau and the degree elevation algorithms extend easily to rational curves

- First, convert the rational Bézier curve into its corresponding 3D curve,
- Next, perform the algorithm on this 3D curve
- Finally, map the result back to 2D

Careful

The above procedure does not apply for hodographs because of the quotient rule differentiation.

- Nevertheless, we will describe how to compute the first derivative at an endpoint, as well as the curvature
- Used with the de Casteljau algorithm, this allows to compute the first derivative, or the curvature, at any point on a rational Bézier curve.

First derivative at an endpoint

For a rational Bézier curve of degree n

$$\mathbf{P}_{[t_0,t_1]}(t) = \frac{\sum_{i=0}^{n} w_i \mathbf{P}_i \binom{n}{i} \left(\frac{t_1-t}{t_1-t_0}\right)^{n-i} \left(\frac{t-t_0}{t_1-t_0}\right)^{i}}{\sum_{i=0}^{n} w_i \binom{n}{i} \left(\frac{t_1-t}{t_1-t_0}\right)^{n-i} \left(\frac{t-t_0}{t_1-t_0}\right)^{i}},$$

the first derivative at $t = t_0$ is

$$\mathbf{P}'(t_0) = \frac{w_1}{w_0} \frac{n}{t_1 - t_0} (\mathbf{P}_1 - \mathbf{P}_0)$$

and the second derivative at $t = t_0$ is

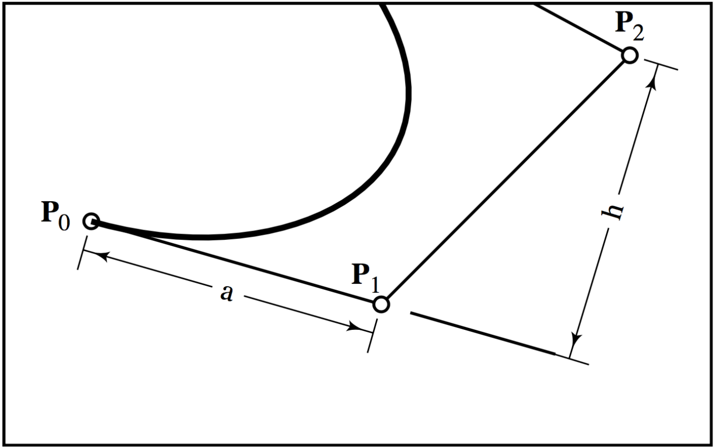
$$\mathbf{P}''(t_0) = \frac{n(n-1)}{(t_1-t_0)^2} \frac{w_2}{w_0} (\mathbf{P}_2 - \mathbf{P}_0) - \frac{2n}{(t_1-t_0)^2} \frac{w_1}{w_0} \frac{nw_1 - w_0}{w_0} (\mathbf{P}_1 - \mathbf{P}_0).$$

Remark: One just applies the quotient rule differentiation and evaluate at $t = t_0$; computations become quickly extremely complicated.

Curvature at an endpoint

- Rational Bézier curve of degree n
- a and h as on the picture

$$\kappa(t_0) = \frac{w_0 w_2}{w_1^2} \frac{n-1}{n} \frac{h}{a^2}$$



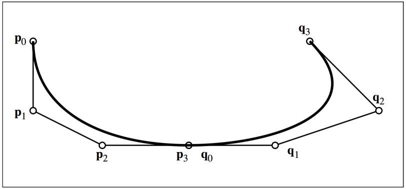
Continuity

Definition

Two curve segments $\mathbf{P}_{[t_0,t_1]}$ and $\mathbf{Q}_{[t_1,t_2]}$ are said to be $\mathcal{C}^k$ continuous if

$$\mathbf{P}(t_1) = \mathbf{Q}(t_1), \mathbf{P}'(t_1) = \mathbf{Q}'(t_1), \ldots, \mathbf{P}^{(k)}(t_1) = \mathbf{Q}^{(k)}(t_1).$$

For example, these two curves are



- $\mathcal{C}^0$ if $\mathbf{p}_3 = \mathbf{q}_0$,
- $\mathcal{C}^1$ if in addition

$$\frac{3}{t_1 - t_0}(\mathbf{p}_3 - \mathbf{p}_2) = \frac{3}{t_2 - t_1}(\mathbf{q}_1 - \mathbf{q}_0)$$

- $\mathcal{C}^2$ if in addition

$$\frac{6}{(t_1 - t_0)^2}(\mathbf{p}_3 - 2\mathbf{p}_2 + \mathbf{p}_1) = \frac{6}{(t_2 - t_1)^2}(\mathbf{q}_2 - 2\mathbf{q}_1 + \mathbf{q}_0)$$

- etc.

Geometric Continuity

- The geometric continuity, denoted G^k, is another method for describing the continuity of two curves, that is independent of their parameterizations.
- The conditions for geometric continuity are less strict compared to the classical continuity.

Definition

Two curves are G^k at a given point P if they can be reparameterized so that they become $\mathcal{C}^k$ at this point P.

- G^0 : the two curves have a common endpoint, but not necessarily with the same parameter value,
- G^1 : the line segments $\mathbf{p}_2 - \mathbf{p}_3$ and $\mathbf{q}_0 - \mathbf{q}_1$ are colinear (i.e. common tangent line),
- G^2 : same tangent line and same center of curvature,
- etc.