INDUSTRIAL MATHEMATICS (MATH 6514)
INSTRUCTOR: DR. MARTIN SHORT

Analyzing Shocks in a Traffic Flow Problem with a Freeway Onramp

December 12, 2023

Marie Bertsch
Johnathan Corbin
Emmanuel Lyngberg

Introduction

In this problem, we model the following freeway onramp traffic scenario: There exists a single lane of cars on which cars can only move in the positive x direction. Between 0 and $\sqrt{2}$ there is an onramp onto the road, so that cars can enter the freeway.

At time $t = 0$, there are no cars on the road, so the density of cars is $\rho(x,0) = 0$ for all $x \in \mathbf{R}$. However, as time evolves, cars enter the road via the onramp which happens at a rate $\beta(x,t) = x$. In the areas, where there is no onramp (i.e. $x < 0,\ x > \sqrt{2}$), no cars can enter or exit the road.

Furthermore, we assume that the velocity of cars at location x at time t is

$$u(\rho(x,t)) = u_{max}\left(1 - \frac{\rho(x,t)}{\rho_{max}}\right) = 1 - \frac{\rho(x,t)}{2}$$

where we used $u_{max} = 1,\ \rho_{max} = 2$ in the last equation.

Positions and Densities of characteristics

We now try to solve for the positions $x(t)$ and densities $\rho(t)$ of characteristics.
To do so, we must distinguish between the two different areas:

1. Area without onramp ($x < 0,\ x > \sqrt{2}$)

2. Onramp ($0 \le x \le \sqrt{2}$)

Area without onramp

For the positions $x(t)$ and densities $\rho(t)$ of characteristics in an area without an onramp the following must hold

$$\frac{dx}{dt} = c(\rho(x,t)) \tag{1}$$
$$\frac{d\rho}{dt} = 0 \tag{2}$$

where $c(x,t)$ denotes the speed.

Solving $\frac{d\rho}{dt} = 0$ yields: $\rho(t) = \rho(0) = 0$ as we used our initial condition $\rho(x,0) = 0$ in the last equation.

Now, solving $\frac{dx}{dt} = c(\rho(x,t)) = c(0)$ where we inserted our solution for ρ in the last equation, we obtain:

$$x(t) = c(0)\,t + x_0 = u_{max}\left(1 - \frac{2\cdot 0}{\rho_{max}}\right)t + x_0 = u_{max}\,t + x_0 = t + x_0$$

So, to sum up our results: For $x \notin [0, \sqrt{2}]$ the position of the characteristics is $x(t) = x_0 + t$ and the density is $\rho(t) = 0$.

Area with Onramp

For the positions $x(t)$ and densities $\rho(t)$ of characeteristics it must hold that

$$\frac{d\rho}{dt} = \beta(x,t) = x$$
$$\frac{dx}{dt} = c(\rho(x,t))$$

We notice, that $\frac{dx}{dt} = \frac{d^2\rho}{dt^2}$, so this system of coupled ODE's is equivalent to the following:

$$\frac{d\rho}{dt} = \beta(x,t) = x$$
$$\frac{d^2\rho}{dt^2} = c(\rho(x,t)) = u_{max}\left(1 - \frac{2\rho}{\rho_{max}}\right) = 1 - \rho$$

We solve this second order differential equation using an online ODE solver and obtain as our solution

$$\rho(t) = c_1 cos(t) + c_2 sin(t) + 1$$

Using the initial condition $\rho(0) = 0$ as for all x it holds that $\rho(x,0) = 0$, we get:

$$\rho(0) = c_1 cos(0) + c_2 sin(0) + 1 = c_1 + 1 = 0 \;\Rightarrow c_1 = -1$$

We substitute this constant in. Then, differentiating once, we get:

$$\rho(t) = c_1 cos(t) + c_2 sin(t) + 1 = -cos(t) + c_2 sin(t) + 1$$
$$\frac{d\rho}{dt} = sin(t) + c_2 cos(t)$$

We notice that $x(t) = \frac{d\rho}{dt}$ according to our first ODE, so we have

$$x(t) = sin(t) + c_2 cos(t)$$

Using the initial condition $x(0) = x_0$ we get:

$$x(0) = sin(0) + c_2 cos(0) = c_2 = x_0$$

In conclusion, we get for the density of a characteristic starting at location x_0

$$\rho(t) = -cos(t) + x_0 sin(t) + 1$$

and for its position

$$x(t) = sin(t) + x_0 cos(t)$$

First Shock

A shock forms when two different characteristics hit each other. As the characteristics in the areas $x < 0,\ x > \sqrt{2}$ are perfectly parallel to each other, no shock forms in this area, so we have to have a look at the area $0 \le x \le \sqrt{2}$.

The condition that two different characteristics $x(t)$ and $y(t)$ that start at the two different starting locations x_0 and y_0 $(x_0 \neq y_0)$ hit each other is equivalent to

$$\begin{aligned}
&\exists t \ge 0: \ x(t) = y(t) \\
&\Leftrightarrow \exists t \ge 0: x(t) = sin(t) + x_0 cos(t) = sin(t) + y_0 cos(t) = y(t) \\
&\Leftrightarrow \exists t \ge 0: x_0 cos(t) = y_0 cos(t) \\
&\Leftrightarrow \exists t \ge 0: (x_0 - y_0) cos(t) = 0
\end{aligned}$$

As $x_0 \neq y_0$ this can only hold true if $cos(t) = 0$. The first time, for which $cos(t) = 0$ true is $t = \frac{\pi}{2}$. So, the first shock forms at time $t = \frac{\pi}{2}$. The current position of the shock is then

$$x(t) = sin(t) + x_0 cos(t) = sin\left(\frac{\pi}{2}\right) + x_0 cos\left(\frac{\pi}{2}\right) = 1$$

So, all the different characteristics that originate at an x-location between 0 and 1 turn around before they reach the end of the onramp (so they don't leave the onramp before they are part of the first shock).

As for the position at which the charactersitic turns around, it must hold that $\frac{dx}{dt} = 0$. We find that the time at which the characteristic turns around is

$$\begin{aligned} \frac{dx}{dt} &= 0 \\ cos(t) - x_0 sin(t) &= 0 \\ x_0 = \frac{cos(t)}{sin(t)} &= cot(t) \\ t &= cot^{-1}(x_0) \end{aligned}$$

At this time, the characteristic is at position

$$x(cot^{-1}(x_0)) = sin(cot^{-1}(x_0)) + x_0 cos(cot^{-1}(x_0))$$

We find the maximum x_0 so that the characteristic starting at location x_0 turns around on the onramp by setting the location at which the characteristic turns around equal to $\sqrt{2}$ and find, that $x_0 = 1$ is the maximum starting point so that the characteristic curve turns around on the onramp.

Therefore, all the characteristics with starting point $x_0 \in [0,1]$ are involved in this first shock formation.

The intensity $\rho(t)$ of a characteristic increases with an increase in x_0 (as for $0 \leq x \leq \sqrt{2}$ $sin(x) \geq 0$, the jump in density for this first shock is the jump between the two densities ρ^- and ρ^+ where ρ^- is the density of the characteristic at time $\frac{\pi}{2}$ that started at location $x_0 = 0$ and ρ^+ is the density of the characteristic that started at $x_0 = 1$ at time $t = \frac{\pi}{2}$.
So, using that for a characteristic starting at location x_0 we get for its density $\rho(t) = -cos(t) + x_0 sin(t) + 1$ we calculate:

$$\begin{aligned} \rho^- &= -cos\left(\frac{\pi}{2}\right) + 0 sin\left(\frac{\pi}{2}\right) + 1 = 1 \\ \rho^+ &= -cos\left(\frac{\pi}{2}\right) + 1 sin\left(\frac{\pi}{2}\right) + 1 = 2 \end{aligned}$$

Thus, the jump in densities for the first shock is from $\rho^- = 1$ to $\rho^+ = 2$

VISUALIZING THE CHARACTERISTICS AND SHOCK FORMATION

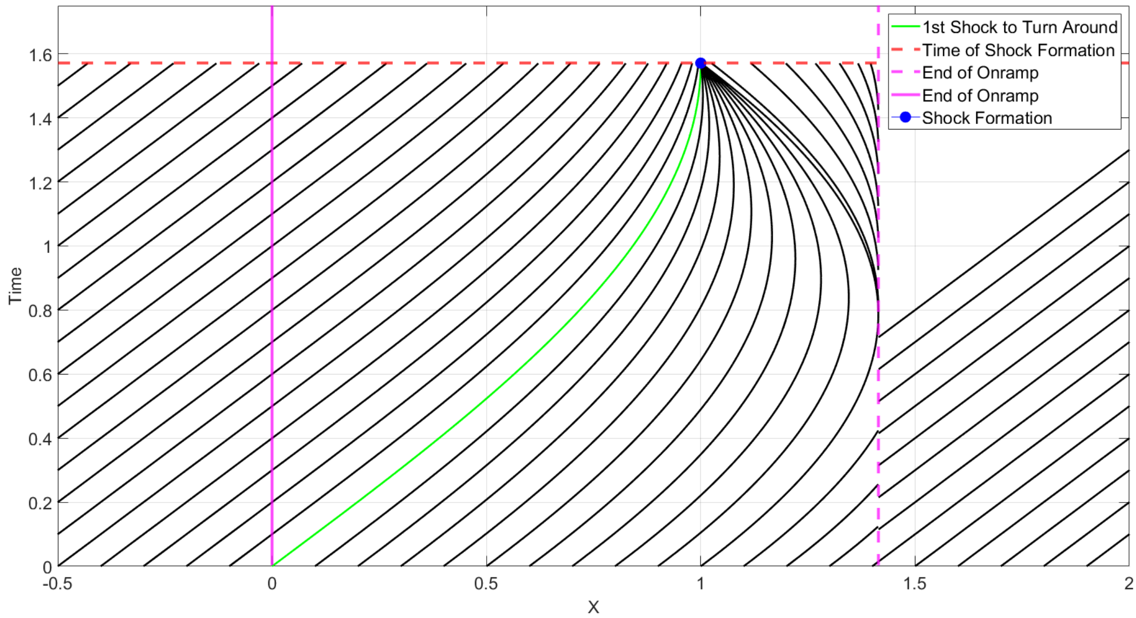

Figure 1: Space-time plot of characteristics before first shock formation

In Figure 1 we provide a detailed visualization of the characteristics observed in our onramp scenario as well as the shock formation that occurs in the onramp area. Describing the figure left to right, we begin with the characteristics to the left of $x = 0$. In this section we have no onramp, and therefore we completely conserve the number of cars at all times, leading to no changes in density along all of the characteristics observed. Once the characteristics reach $x = 0$, they enter the length of the road with the onramp. Here cars enter with rate per unit length $\beta(x, t) = x$, which as we solved for previously yields characteristics that behave in a sinusoidal fashion, and begin to curve upwards eventually. Note that all characteristic curves that seem to originate on the pink $x = 0$ line are identical to the green characteristic that originates at $x = 0$, simply shifted upwards. As we move right, we see that all the characteristics originating between $x = 0$ and $x = 1$ turn back before they can reach the end of the onramp, which inevitably leads to the shock we observe at the blue point $(1, \frac{\pi}{2})$. Note that all of the characteristics we just mentioned converge to that same point in space and time. Moving right once more, we note that all characteristics originating in the onramp between $x = 0$ and $x = \sqrt{2}$ do not turn around before exiting the offramp. Very similar to before the onramp, the characteristics after the onramp are simply lines that originate either from a characteristic in the onramp section that did not turn around, or from a $x > \sqrt{2}$ starting point. It is also important to note that we see characteristics arch left from the vertical line $x = \sqrt{2}$ as they mimick the trajectory of the characteristic originating at $x = 1$ that just reaches the $x = \sqrt{2}$ line, but now shifted upwards. We also note that since these characteristics arch backwards, we have a "dead" region to the left of $x = \sqrt{2}$ above where the characteristic that originates at $x = 1$ meets the $x = \sqrt{2}$ line.