INDUSTRIAL MATHEMATICS (MATH 6514)
INSTRUCTOR: DR. MARTIN SHORT

Comparing Stochastic Simulations to Partial Differential Equation Models for Isotropic Random Walks

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Introduction

In this paper we will explore two methods for modeling 2-D isotropic random walks. For the sake of simplicity we will make a wide range of assumptions to keep derivations and comparisons as straightforward as possible, but as an extension to this work one could certainly place more realistic constraints or remove assumptions to consider a particular situation of interest. For our purposes, we will assume as has already been noted that the world we consider is two dimensional and isotropic. Additionally we assume that this world extends infinitely in all directions. In this world we observe a single individual whose home is located at the origin of our space. This individual will leave their home everyday at $t = 0$ and embark on an unbiased random walk around their world, in any direction they like. Note that this would be a clear opportunity for extension in which we could consider a New York City style grid pattern of movement instead. To be clear, there are no restrictions on where the individual can walk, and in which direction they can walk, but eventually on each walk they will reach a point (randomly) where they no longer want to keep walking and will immediately return home. This moment in time will be called t_h, and the vector distance they are from their home will be called $\vec{h}$. With all of the necessary context introduced, we will now present two modeling frameworks for two different intensity functions we define for the stopping point, and compare the results.

Discrete Stochastic Simulations

Let us now begin with a discrete stochastic time-step simulation, where each timestep is $\Delta t = 0.1$ and during each timestep the individual walks a distance of $\Delta x = 0.1$. For each step or iteration, the individual randomly chooses a direction out of 360° to walk in and "takes a step" of Δx. However, the person does not walk for all eternity as their stopping process is a Poisson process governed by the intensity function

$$\lambda(t) = \begin{cases} 0 & t \leq t_m \\ \gamma\left(\frac{t}{T}\right)^k & t > t_m \end{cases}$$

Here, t_m denotes the minimum time for a walk (meaning that all walks will have $t_h \geq t_m$), γ is some rate, T a characteristic time, and k satisfies the inequality $k > -1$. Note that for our simulations that we explore and present here, we choose the following set of parameters:

$$t_m = 2,\ \gamma = 1,\ T = 10$$

We now simulate this process using MATLAB. We can simplify the simulation by noting that we can set t_h for each walk before the walk begins given that it is probabilistically determined entirely independently from the current state of the process. We now derive the formula we use in simulation by using the cumulative distribution function and the fact that MATLAB can produce a randomly uniform number between 0 and 1. Given the above intensity function, we have that the probability density function for stopping is given as

$$pdf := \begin{cases} 0, & t \le t_m \\ \gamma(\frac{\tau}{T})^k e^{-\int_{t_m}^{t} \gamma(\frac{q}{T})^k dq}, & t > t_m \end{cases}$$

Note that the integral in the exponential term does not begin at 0 because we know that the intensity of our process is 0 up until we pass our minimum walking time t_m. We will ignore times less than or equal to t_m going forward as we know that we will not calculate any t_h values in that range. With that, we integrate the probability density function to get to the cumulative distribution function as follows:

$$\begin{aligned} cdf &= \int_{t_m}^{t_h} \gamma(\frac{\tau}{T})^k e^{-\int_{t_m}^{t} \gamma(\frac{q}{T})^k dq} d\tau \\ &= \int_{t_m}^{t_h} -\frac{d}{dt}[e^{-\int_{t_m}^{t} \gamma(\frac{q}{T})^k dq}] d\tau \\ &= e^{-\int_{t_m}^{t_m} \gamma(\frac{q}{T})^k dq} - e^{-\int_{t_m}^{t_h} \gamma(\frac{q}{T})^k dq} \\ &= 1 - e^{-\frac{\gamma}{T^k(k+1)}[t_h^{k+1} - t_m^{k+1}]} \end{aligned}$$

We now define u to be a uniformly randomly generated value using the rand function in MATLAB, which we set equal to what we just derived. We now isolate t_h as follows:

$$\begin{aligned} u &= 1 - e^{-\frac{\gamma}{T^k(k+1)}[t_h^{k+1} - t_m^{k+1}]} \\ 1 - u &= e^{-\frac{\gamma}{T^k(k+1)}[t_h^{k+1} - t_m^{k+1}]} \\ \ln(1-u) &= -\frac{\gamma}{T^k(k+1)}[t_h^{k+1} - t_m^{k+1}] \\ t_h &= (t_m^{k+1} - \frac{T^k(k+1)\ln(1-u)}{\gamma})^{\frac{1}{k+1}} \end{aligned}$$

It is important to note that while we subtract the second term from t_m^{k+1}, this term is always negative as $\ln(1-u) < 0$, and thus we guarantee that this will always provide a value of t_h that is greater than t_m, as desired.

With this important caveat fully discussed, we now plot the distribution of the magnitude of h for thousands of simulation runs as we change k. The k values we choose here are -0.5, 0.5, 2, 10, 100. Additionally, we provide a few extra plots that display the average t_h value as we increase k, which further illustrates the behavior of this process as k approaches infinity. From Figures 4 and 5 we can clearly see that as k grows the average value for t_h converges to 10, which is our parameter T, the characteristic time. This result can be inferred by inspecting our intensity function closely, as we note that $(\frac{t}{T})^k$ will converge to 0 if $t < T$ as k goes to infinity, but will diverge towards infinity if $t > T$. Hence, if we take a very large k, we see that the moment the individual has passed time $t = 10$, their intensity for stopping will grow instantaneously, and thus the likelihood that they stop follows directly.

We see in Figures 6 to 10 that with an increase in k, the mean value of $|\vec{h}|$ (i.e. the distance from home at the stopping time t_h) averaged over all our simulations steadily increases: While the average distance $|\vec{h}|$ was only 0.601 for $k = -0.5$, it quickly increased with incremental increases in k, reaching a mean of 0.957 for k = 10. While the first incremental increases were relatively small, the final incrementation is by a factor of 10 to $k = 100$. However, we see that the mean increases but only to 0.979 . Therefore, we can infer from these plots that $|\vec{h}|$ converges to 1 as $k \to \infty$.

We also see that with an increase in k, the histograms become flatter and larger $|\vec{h}|$-values are observed. This could be the case as with an increase in k - as described before - the stopping time t_h increases, which makes it possible for the person to walk further away from home than if he on average has a smaller amount of time to do so.

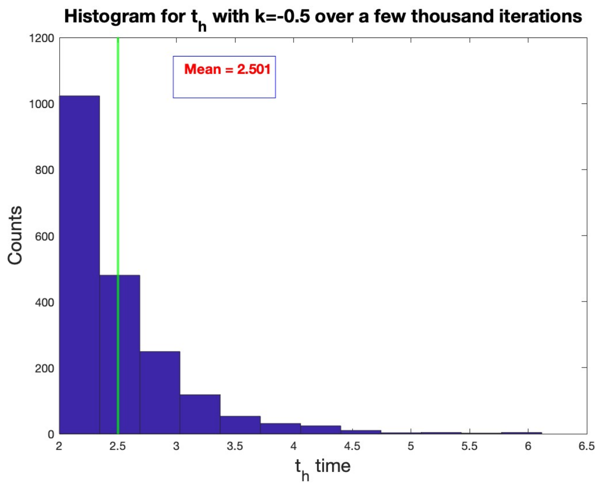

Figure 1: A Histogram of t_h values over many iterations for $k = -0.5$

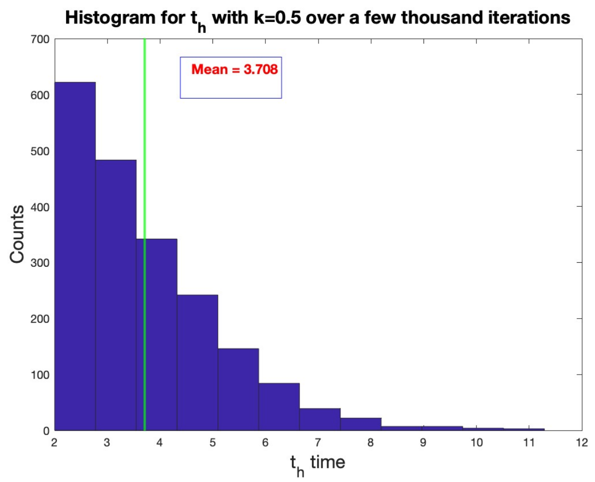

Figure 2: A Histogram of t_h values over many iterations for $k = 0.5$

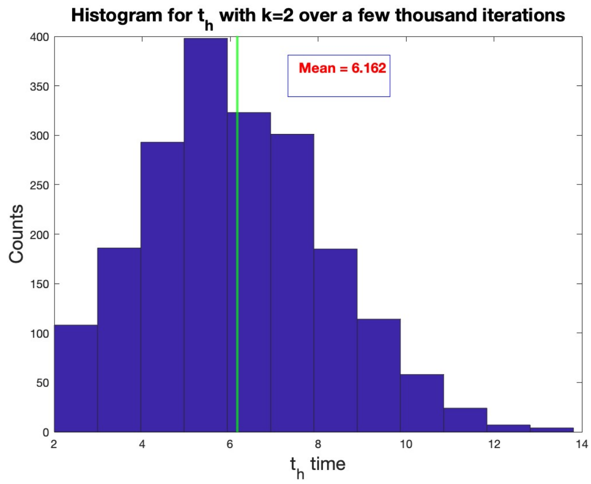

Figure 3: A Histogram of t_h values over many iterations for $k = 2$

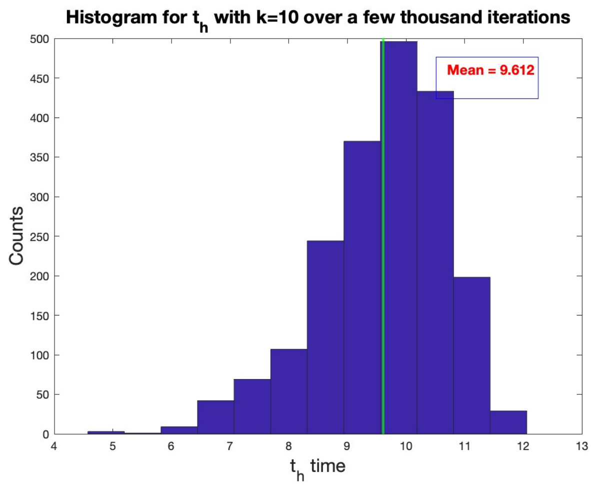

Figure 4: A Histogram of t_h values over many iterations for $k = 10$

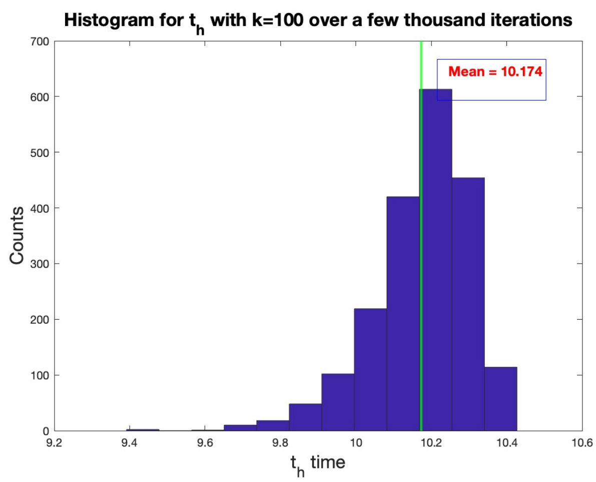

Figure 5: A Histogram of t_h values over many iterations for $k = 100$

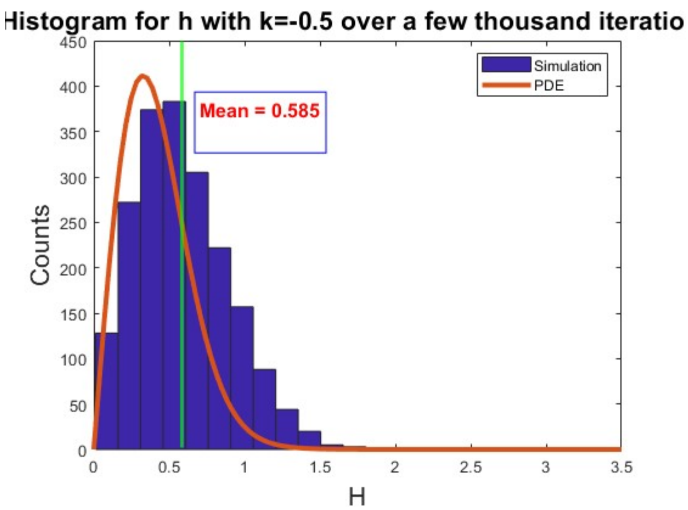

Figure 6: A Histogram of $|h|$ for $k = -0.5$ with the PDE solution overlaid

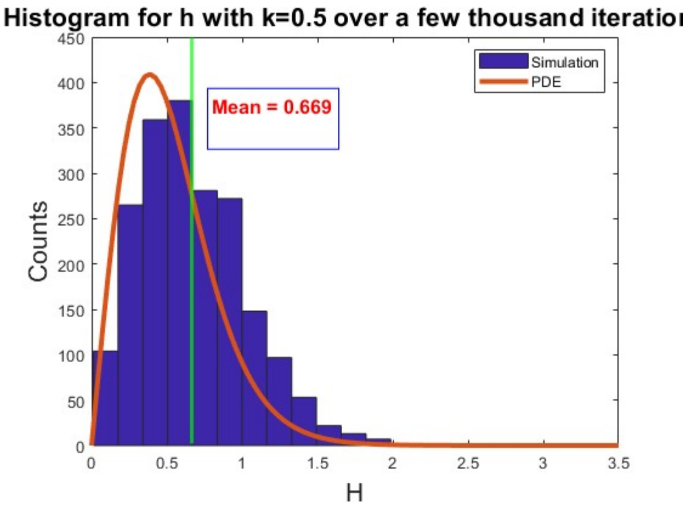

Figure 7: A Histogram of $|h|$ for $k = 0.5$ with the PDE solution overlaid

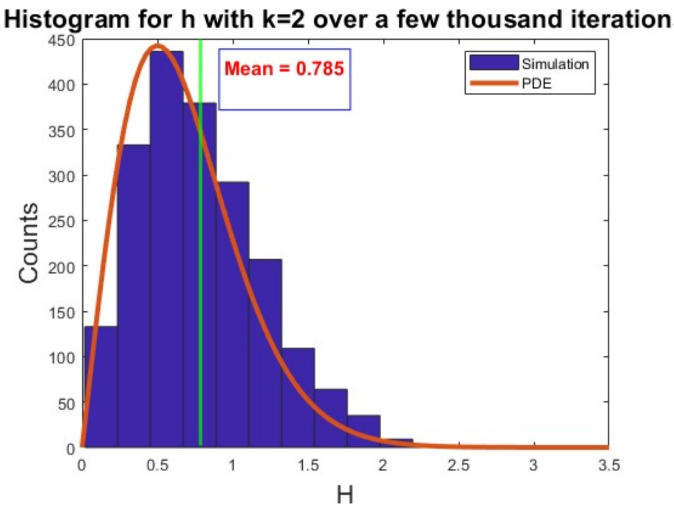

Figure 8: A Histogram of $|h|$ for $k = 2$ with the PDE solution overlaid

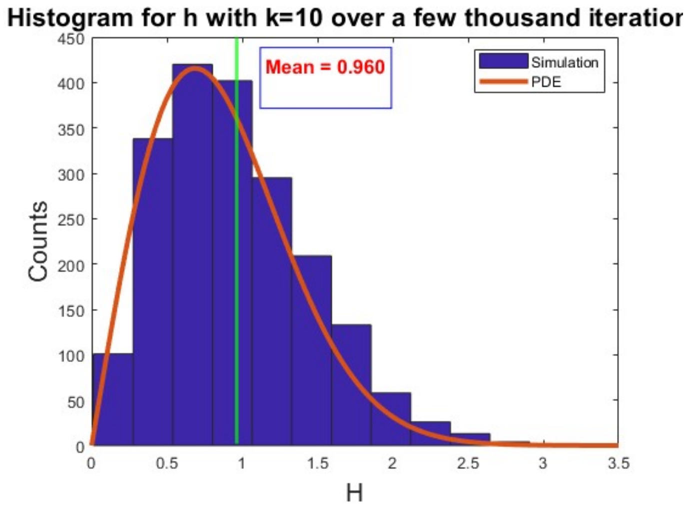

Figure 9: A Histogram of $|h|$ for $k = 10$ with the PDE solution overlaid

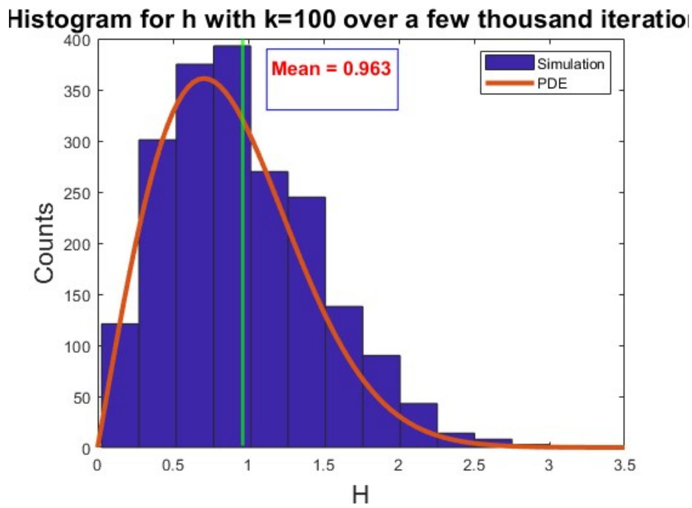

Figure 10: A Histogram of $|h|$ for $k = 100$ with the PDE solution overlaid

A PDE PERSPECTIVE OF OUR RANDOM WALK

PDE Exploration

We can mathematically describe the person's random walk by using polar coordinates (r,θ) to denote his location. Doing so, the probability density $\rho(r,\theta,t)$ that the person is at location (r,θ) at time $0 < t \le t_h$ satisfies the diffusion equation $\frac{\partial\rho}{\partial t} = D\left(\frac{\partial^2\rho}{\partial r^2} + \frac{1}{r}\frac{\partial\rho}{\partial r}\right)$ where $D = \frac{(\Delta x)^2}{4\Delta t}$ is the diffusion coefficient.

We guess a solution for this PDE: $\rho(r,\theta,t) = \frac{1}{4\pi Dt}e^{-\frac{r^2}{4Dt}}$

By differentiating this expression for $\rho(r,\theta,t)$ we'll prove that it indeed satisfies the PDE:

$$\begin{aligned}
\frac{\partial\rho}{\partial t} &= -\frac{4\pi D}{(4\pi Dt)^2}e^{-\frac{r^2}{4Dt}} + \frac{1}{4\pi Dt}\left(\frac{4r^2D}{(4Dt)^2}\right)e^{-\frac{r^2}{4Dt}} = e^{-\frac{r^2}{4Dt}}\left(-\frac{4\pi D}{16\pi^2D^2t^2} + \frac{4r^2D}{64\pi D^3t^3}\right)\\
&= e^{-\frac{r^2}{4Dt}}\left(\frac{r^2}{16\pi D^2t^3} - \frac{1}{4\pi Dt^2}\right)\\
\frac{\partial\rho}{\partial r} &= -\frac{2r}{4Dt}\cdot\frac{1}{4\pi Dt}e^{-\frac{r^2}{4Dt}} = -\frac{r}{8\pi D^2t^2}e^{-\frac{r^2}{4Dt}}\\
\frac{\partial^2\rho}{\partial r^2} &= -\frac{1}{8\pi D^2t^2}e^{-\frac{r^2}{4Dt}} - \frac{r}{8\pi D^2t^2}\cdot\left(-\frac{2r}{4Dt}\right)e^{-\frac{r^2}{4Dt}} = e^{-\frac{r^2}{4Dt}}\left(-\frac{1}{8\pi D^2t^2} + \frac{r^2}{16\pi D^3t^3}\right)
\end{aligned}$$

Using these results, we obtain:

$$\begin{aligned}
D\left(\frac{\partial^2\rho}{\partial r^2} + \frac{1}{r}\frac{\partial\rho}{\partial r}\right) &= D\frac{\partial^2\rho}{\partial r^2} + \frac{D}{r}\frac{\partial\rho}{\partial r}\\
&= De^{-\frac{r^2}{4Dt}}\left(-\frac{1}{8\pi D^2t^2} + \frac{r^2}{16\pi D^3t^3}\right) - \frac{D}{r}\frac{r}{8\pi D^2t^2}e^{-\frac{r^2}{4Dt}}\\
&= e^{-\frac{r^2}{4Dt}}\left(-\frac{1}{8\pi Dt^2} + \frac{r^2}{16\pi D^2t^3}\right) - \frac{1}{8\pi Dt^2}e^{-\frac{r^2}{4Dt}}\\
&= e^{-\frac{r^2}{4Dt}}\left(\frac{r^2}{16\pi D^2t^3} - \frac{1}{4\pi Dt^2}\right)\\
&= \frac{\partial\rho}{\partial t}
\end{aligned}$$

The Distribution of $|\vec{h}|$

In this section we analytically derive the probability density function for the distance from home at the stopping time t_h of the random walk ($|\vec{h}|$) to hereby gain a mathematical perspective on the plots we obtained from our simulation.

As the distance from home at the time t_h obviously depends on the time t_h, we first need to derive the probability distribution for t_h, which we will denote with $\chi(t_h)$:

Using that for a Poisson-Process with rate $\lambda(t)$ and starting point t_a, the probability density function is $\chi(t_1) = \lambda(t_1) exp\left(-\int_{t_a}^{t_1} \lambda(t) dt\right)$ we obtain as the pdf for our stopping time t_h:

For $t_h \leq t_m$: $\chi(t_h) = \lambda(t_h) exp\left(-\int_0^{t_h} \lambda(t) dt\right) = 0 \cdot exp\left(-\int_0^{t_h} 0 dt\right) = 0$

For $t_h > t_m$:

$$\begin{aligned}
\chi(t_h) &= \lambda(t_h) exp\left(-\int_0^{t_h} \lambda(t) dt\right) \\
&= \gamma \left(\frac{t_h}{T}\right)^k exp\left(-\int_0^{t_h} \lambda(t) dt\right) \\
&= \gamma \left(\frac{t_h}{T}\right)^k exp\left(-\int_0^{t_m} \lambda(t) dt - \int_{t_m}^{t_h} \lambda(t) dt\right) \\
&= \gamma \left(\frac{t_h}{T}\right)^k exp\left(-\int_0^{t_m} 0 dt - \int_{t_m}^{t_h} \gamma \left(\frac{t}{T}\right)^k dt\right) \\
&= \gamma \left(\frac{t_h}{T}\right)^k exp\left(-\left[\frac{\gamma}{k+1} \frac{t^{k+1}}{T^k}\right]_{t_m}^{t_h}\right) \\
&= \gamma \left(\frac{t_h}{T}\right)^k exp\left(-\frac{\gamma}{T^k (k+1)} (t_h^{k+1} - t_m^{k+1})\right)
\end{aligned}$$

In conclusion we get as the pdf for t_h:

$$\chi(t_h) = \begin{cases} 0 & t_h \leq t_m \\ \gamma \left(\frac{t_h}{T}\right)^k exp\left(-\frac{\gamma}{T^k (k+1)} (t_h^{k+1} - t_m^{k+1})\right) & t_h > t_m \end{cases}$$

Using this result, we can derive the pdf for $|\vec{h}|$ by calculating the probability that we have a distance $|\vec{h}| = r$ from home at the stopping time t_h and while being in the direction ϕ integrated over all possible stopping times and all possible angles $[0, 2\pi)$:

$$\begin{aligned}
P(|\vec{h}| = r) &= \int_t \int_\theta P(|\vec{h}| = r,\ t_h = t,\ angle = \theta) r d\phi dt \\
&= \int_t \int_\theta \rho(r, \theta, t)\chi(t) r d\phi dt \\
&= \int_0^\infty \int_0^{2\pi} \frac{1}{4\pi Dt} e^{-\frac{r^2}{4Dt}} \chi(t) r d\phi dt \\
&= \int_0^{t_m} \int_0^{2\pi} \frac{1}{4\pi Dt} e^{-\frac{r^2}{4Dt}} \chi(t) r d\phi dt + \int_{t_m}^\infty \int_0^{2\pi} \frac{1}{4\pi Dt} e^{-\frac{r^2}{4Dt}} \chi(t) r d\phi dt \\
&= \int_0^{t_m} \int_0^{2\pi} \frac{1}{4\pi Dt} e^{-\frac{r^2}{4Dt}} \cdot 0 \cdot r d\phi dt \\
&\quad + \int_{t_m}^\infty \int_0^{2\pi} \frac{1}{4\pi Dt} e^{-\frac{r^2}{4Dt}} \gamma \left(\frac{t}{T}\right)^k exp\left(-\frac{\gamma}{T^k(k+1)}(t^{k+1} - t_m^{k+1})\right) r d\phi dt \\
&= \int_{t_m}^\infty \int_0^{2\pi} \frac{1}{4\pi Dt} e^{-\frac{r^2}{4Dt}} \gamma \left(\frac{t}{T}\right)^k exp\left(-\frac{\gamma}{T^k(k+1)}(t^{k+1} - t_m^{k+1})\right) r d\phi dt \\
&= 2\pi \int_{t_m}^\infty \frac{\gamma}{4\pi Dt} \left(\frac{t}{T}\right)^k exp\left(-\frac{\gamma}{T^k(k+1)}(t^{k+1} - t_m^{k+1}) - \frac{r^2}{4Dt}\right) r dt
\end{aligned}$$

We solve this integral numerically using MATLAB for the same k values as we used for our simulations (i.e. $k = -0.5,\ k = 0.5,\ k = 2,\ k = 100$). We overlay the analytic curves we obtain for the distributions of $|\vec{h}|$ over the histograms in figures 6 to 10. We see that for all k our simulations and analytical results look fairly similar. For lower values of k, we obtained a slightly higher mean of $|\vec{h}|$ for the simulations compared to our analytical results but with an increase in k this gap closes.

Now, we have a look at how the pdf for $|\vec{h}|$ changes as $k \to \infty$:
As explained before, for $k \to \infty$ the intensity function

$$\lambda(t) = \begin{cases} 0 & t \le t_m \\ \gamma \left(\frac{t}{T}\right)^k & t > t_m \end{cases}$$

converges to

$$\lambda^*(t) = \begin{cases} 0 & t \le T \\ \infty & t > T \end{cases}$$

What this means is that during the time-interval $[0,T]$ the person has an event from our Poisson process with probability 0 - so the person certainly continues walking until time T. As soon as t is just slightly greater than T, the intensity function suddenly jumps to ∞, meaning that instantaneously after T an event from our Poisson process happens, leading to our person deciding to return home. Since we specified in our model that we only look at the Poisson-process until the first event happens (i.e. the person walks around until he decides at a random time governed by the Poisson-process to stop his walk, and then he returns home and doesn't care about further events from our Poisson-process), we get that the person with 0 probability stops his walk between $[0,T]$, then with probability 1 stops his walk at time T, and for all times after T also decides to return home with probability 0 (as he's already on his way home, so he's already made his decision). If we put this into a mathematical equation we get that the probability density function describing the stopping time as $k \to \infty$ is $\chi(t_h) = \delta(T - t_h)$, where $\delta(x) = \begin{cases} 0 & x \neq 0 \\ \infty & x = 0 \end{cases}$ denotes the Dirac-Delta function.

Now, we need to plug in this new probability density function into the density for $|\vec{h}|$:

$$
\begin{aligned}
&P(|\vec{h}| = r) = \int_t \int_\theta P(|\vec{h}| = r, t_h = t, angle = \theta) r d\phi dt \\
&= \int_0^\infty \int_0^{2\pi} \frac{1}{4\pi D t} e^{-\frac{r^2}{4Dt}} \chi(t) r d\phi dt \\
&= \int_0^\infty \int_0^{2\pi} \frac{1}{4\pi D t} e^{-\frac{r^2}{4Dt}} \delta(T - t) r d\phi dt \\
&= \int_0^{2\pi} \frac{1}{4\pi D T} e^{-\frac{r^2}{4DT}} r d\phi \\
&= \frac{2\pi r}{4\pi D T} e^{-\frac{r^2}{4DT}} \\
&= \frac{r}{2DT} e^{-\frac{r^2}{4DT}}
\end{aligned}
$$

Figure 11 shows the comparison between the general equation for $P(|\vec{h}| = r)$ for k = 100 and the derived expression for $P(|\vec{h}| = r)$ as $k \to \infty$. The shape of both distributions matches quite well. The maxima of the general equation is slightly lower than that of the derived limit, however. If a scaling factor of 1.2 is applied, the curves are perfectly equal. It is possible that if we integrated for a larger k value then the curves would be equal. Regardless, we know that our derived equation as $k \to \infty$ is generally correct.

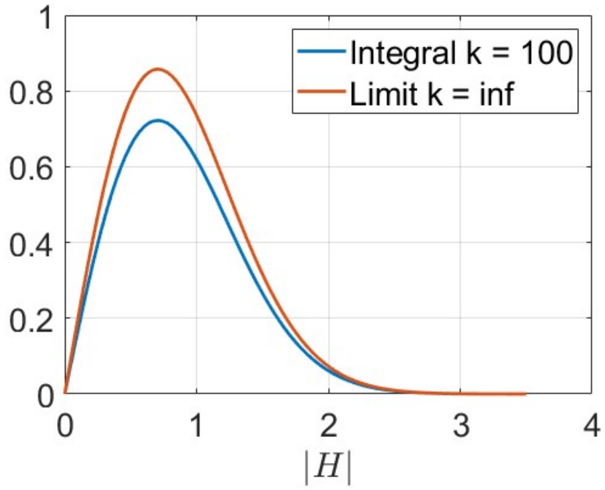

Figure 11: Distribution of $|h|$ from the general equation at $k = 100$ and from the analytical expression for the limit as $k \to \inf$

Discrete Stochastic Simulation for a slightly modified random walk

We now consider a slightly tweaked intensity function, but one that provides us with a different behavior profile that is now dependent upon the location of the individual walking as well. Here we have

$$\lambda(t) = \begin{cases} 0 & r(t) \le r_m \\ \gamma\left(\frac{r(t)}{L}\right)^k & r(t) > r_m \end{cases}$$

In this case we have a new dynamic variable $r(t)$ which is our distance from home. Additionally, we swap t_m for r_m where r_m is a minimum distance the individual must walk before considering the possibility of turning back for home. Finally, we have swapped the characteristic time for the characteristic distance L. Note that in this case we define our intensity based on a dynamic variable, the current distance from home, and therefore our simulation cannot employ the same idea as before. Instead we recalculate the probability of stopping for each iteration and again generate a random value between 0 and 1 to check whether to stop or not. Given that this process is still Poisson, we can quickly determine the probability of stopping by considering the probability of walking on our small timestep Δt

which is governed by

$$P(walk) = e^{-\lambda(t)dt}$$

From there, we consider the negation to determine if the individual stops which gives us

$$P(stop\ walk) = 1 - e^{-\lambda(t)dt}$$

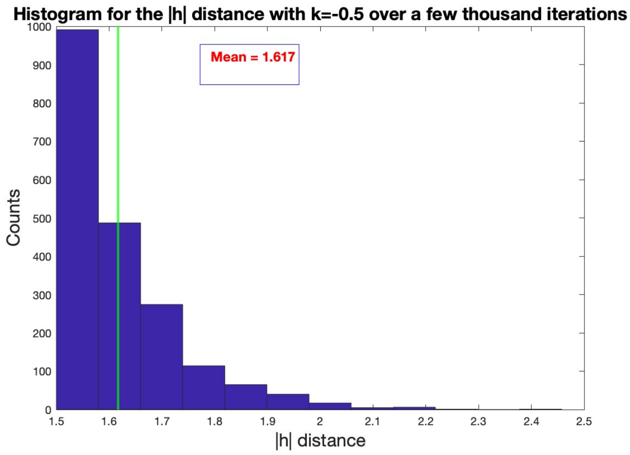

Figure 12: A Histogram of $|h|$ distance values over many iterations for $k = -0.5$

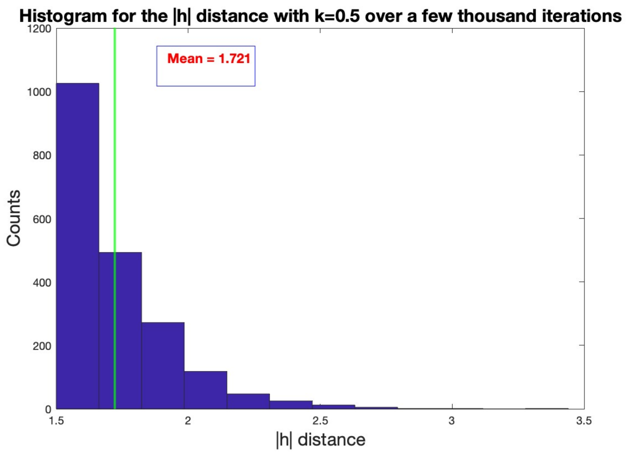

Figure 13: A Histogram of $|h|$ distance values over many iterations for $k = 0.5$

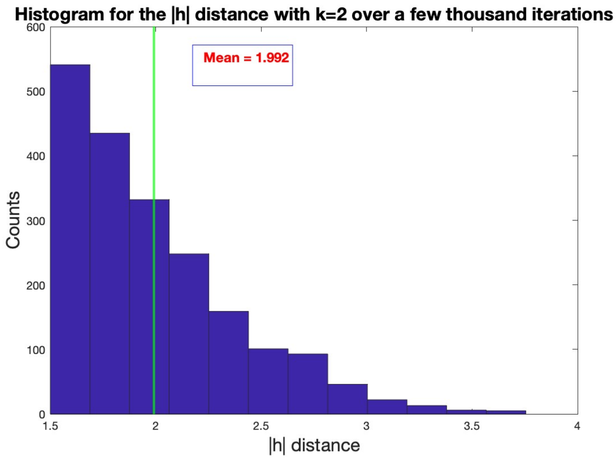

Figure 14: A Histogram of $|h|$ distance values over many iterations for $k = 2$

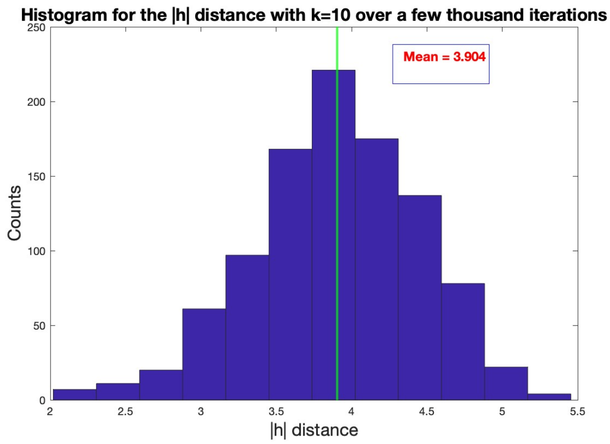

Figure 15: A Histogram of $|h|$ distance values over many iterations for $k = 10$

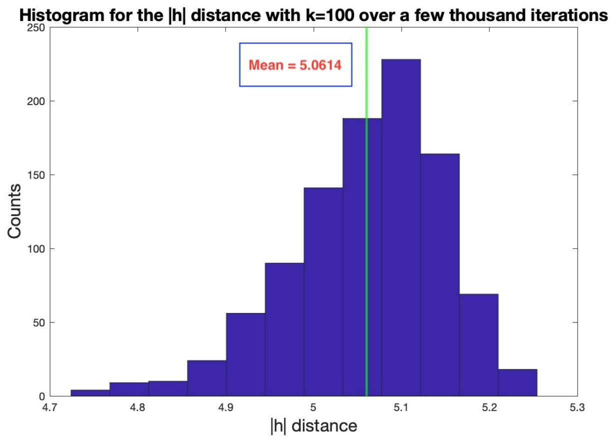

Figure 16: A Histogram of $|h|$ distance values over many iterations for $k = 100$

Deriving a PDE for $\rho(x, y, t)$ for the modified random walk

We now try to derive a partial differential equation for the probability density of the person being at location (x, y) at time t, denoted $\rho(x, y, t)$. To do so, will first simplify our model and assume that the person can no longer walk into any direction of his liking, but can only choose between the four options 1. up, 2. down, 3. left, 4. right.

To derive our PDE, we will first derive mathematical expressions for discrete, finite timesteps Δt and discrete changes of location Δx and at the end take the limits$\Delta t \to 0,\ \Delta x \to 0$ to get a PDE.

So, under these conditions our flat, isotopic 2-D world now consists of an infinite amount of grid points with coordinates (x, y) where the distance between two adjacent gridpoints is dx both in the x as well as the y direction

Let $N(x, y, t)$ denote the number of people at location (x,y) at time t and $A(x, y, t)$ the attractiveness of location (x,y) at time t. Since we assume that our walk is unbiased, all locations at all times are equally attractive, so we set $A(x, y, t) = 1\ \forall x, t$. Due to the unbiasedness we get that for $(x', y'),\ (x, y)$ adjacent gridpoints

$$P(\text{walking from (x', y') to (x,y)}) = \frac{A(x, y, t)}{\sum_{\text{(x'', y'') Neighbors of (x', y')}} A(x'', y'', t)} = \frac{1}{4}$$

always holds.

Since we are looking at discrete timesteps Δt, the probability that a person that walked at time t, keeps walking during the time interval $[t, t + dt]$ is:

$$P(\text{keep walking during } [t, t+dt]) = e^{-\lambda(t)dt} = e^{-\gamma\left(\frac{r(t)}{L}\right)^k dt}$$

Similary,

$$P(\text{stop walking during } [t, t+dt]) = 1 - P(\text{keep walking during}[t, t+dt]) = 1 - e^{-\gamma\left(\frac{r(t)}{L}\right)^k dt}$$

Using these results, we obtain:

$$
\begin{aligned}
N(x,y,t+dt) &= \sum_{|(x',y')-(x,y)|=dx} \text{number of people at (x', y') that keep walking and move from (x', y') to (x, y)}\\
&= \sum_{|(x',y')-(x,y)|=dx} N(x',y',t)e^{-\gamma\left(\frac{r(t)}{L}\right)^k dt}\frac{A(x,y,t)}{\sum_{|(x'',y'')-(x',y')|=dx} A(x'',y'',t)}\\
&= \sum_{|(x',y')-(x,y)|=dx} N(x',y',t)e^{-\gamma\left(\frac{r(t)}{L}\right)^k dt}\cdot\frac{1}{4}\\
&= N(x+dx,y,t)\frac{1}{4}e^{-\gamma\left(\frac{r(t)}{L}\right)^k dt} + N(x-dx,y,t)\frac{1}{4}e^{-\gamma\left(\frac{r(t)}{L}\right)^k dt}\\
&+ N(x,y+dx,t)\frac{1}{4}e^{-\gamma\left(\frac{r(t)}{L}\right)^k dt} + N(x,y-dx,t)\frac{1}{4}e^{-\gamma\left(\frac{r(t)}{L}\right)^k dt}\\
&= \frac{1-\gamma\left(\frac{r(t)}{L}\right)^k dt + O(dt^2)}{4}N(x+dx,y,t) + \frac{1-\gamma\left(\frac{r(t)}{L}\right)^k dt + O(dt^2)}{4}N(x-dx,y,t)\\
&+ \frac{1-\gamma\left(\frac{r(t)}{L}\right)^k dt + O(dt^2)}{4}N(x,y+dx,t) + \frac{1-\gamma\left(\frac{r(t)}{L}\right)^k dt + O(dt^2)}{4}N(x,y-dx,t)\\
&= Q(x+dx,y,t) + Q(x-dx,y,t) + Q(x,y+dx,t) + Q(x,y-dx,t)
\end{aligned}
$$

where we used the Taylor expansion $e^{-\gamma\left(\frac{r(t)}{L}\right)^k dt} = 1-\gamma\left(\frac{r(t)}{L}\right)^k dt + O(dt^2)$ in the second to last equation and defined $Q(x,y,t) = N(x,y,t)\cdot\frac{1-\gamma\left(\frac{r(t)}{L}\right)^k dt + O(dt^2)}{4}$ to simplify the notation.

Now, we Taylor-expand all the terms in the last expression individually and find:

$$
\begin{aligned}
Q(x+dx,y,t) &= Q(x,y,t) + \frac{\partial Q}{\partial x}|_{x,y,t}dx + \frac{1}{2}\frac{\partial^2 Q}{\partial x^2}|_{x,y,t}dx^2 + \frac{1}{6}\frac{\partial^3 Q}{\partial x^3}|_{x,y,t}dx^3 + O(dx^4)\\
Q(x-dx,y,t) &= Q(x,y,t) - \frac{\partial Q}{\partial x}|_{x,y,t}dx + \frac{1}{2}\frac{\partial^2 Q}{\partial x^2}|_{x,y,t}dx^2 - \frac{1}{6}\frac{\partial^3 Q}{\partial x^3}|_{x,y,t}dx^3 + O(dx^4)\\
Q(x,y+dx,t) &= Q(x,y,t) + \frac{\partial Q}{\partial y}|_{x,y,t}dx + \frac{1}{2}\frac{\partial^2 Q}{\partial y^2}|_{x,y,t}dx^2 + \frac{1}{6}\frac{\partial^3 Q}{\partial y^3}|_{x,y,t}dx^3 + O(dx^4)\\
Q(x,y-dx,t) &= Q(x,y,t) - \frac{\partial Q}{\partial y}|_{x,y,t}dx + \frac{1}{2}\frac{\partial^2 Q}{\partial y^2}|_{x,y,t}dx^2 - \frac{1}{6}\frac{\partial^3 Q}{\partial y^3}|_{x,y,t}dx^3 + O(dx^4)
\end{aligned}
$$

We observe, that the Taylor expansion for $Q(x+dx,y,t)$ is similar to that of $Q(x-dx,y,t)$ in the following way: All the terms with dx to an even degree are exactly the same, and all those terms with dx to an odd degree are the same but with opposite signs. Therefore, if we add $Q(x+dx, y, t)$ and $Q(x-dx, y, t)$ all the terms with dx to an odd degree just cancel and those with dx to an even degree are just two times those in $Q(x+dx, y, t)$. The same is the case with $Q(x, y+dx, t)$ and $Q(x, y-dx, t)$. Using this observation we obtain:

$$\begin{aligned}
&Q(x+dx, y, t) + Q(x-dx, y, t) + Q(x, y+dx, t) + Q(x, y-dx, t) \\
&= 2Q(x, y, t) + \frac{\partial^2 Q}{\partial x^2}|_{x,y,t} dx^2 + O(dx^4) + 2Q(x, y, t) + \frac{\partial^2 Q}{\partial y^2}|_{x,y,t} dx^2 + O(dx^4) \\
&= 4Q(x, y, t) + dx^2 \left(\frac{\partial^2 Q}{\partial x^2} + \frac{\partial^2 Q}{\partial y^2} \right) + O(dx^4)
\end{aligned}$$

Now, by inserting these Taylor-expansions back into our original equation and substituting $Q(x, y, t) = N(x, y, t) \cdot \frac{1-\gamma\left(\frac{r(t)}{L}\right)^k dt + O(dt^2)}{4}$, we get:

$$\begin{aligned}
N(x, y, t+dt) &= Q(x+dx, y, t) + Q(x-dx, y, t) + Q(x, y+dx, t) + Q(x, y-dx, t) \\
&= 4Q(x, y, t) + dx^2 \left(\frac{\partial^2 Q}{\partial x^2} + \frac{\partial^2 Q}{\partial y^2} \right) + O(dx^4) \\
&= 4 \cdot N(x, y, t) \cdot \frac{1-\gamma\left(\frac{r(t)}{L}\right)^k dt + O(dt^2)}{4} \\
&+ dx^2 \left(\frac{\partial^2}{\partial x^2} \left(N(x, y, t) \cdot \frac{1-\gamma\left(\frac{r(t)}{L}\right)^k dt + O(dt^2)}{4} \right) + \frac{\partial^2}{\partial y^2} \left(N(x, y, t) \cdot \frac{1-\gamma\left(\frac{r(t)}{L}\right)^k dt + O(dt^2)}{4} \right) \right) \\
&+ O(dx^4) \\
&= N(x, y, t) \left(1 - \gamma \left(\frac{r(t)}{L} \right)^k dt + O(dt^2) \right) \\
&+ dx^2 \frac{\partial^2}{\partial x^2} \left(N(x, y, t) \right) \frac{1-\gamma\left(\frac{r(t)}{L}\right)^k dt + O(dt^2)}{4} + dx^2 N(x, y, t) \frac{\partial^2}{\partial x^2} \left(\frac{1-\gamma\left(\frac{r(t)}{L}\right)^k dt + O(dt^2)}{4} \right) \\
&+ dx^2 \frac{\partial^2}{\partial y^2} \left(N(x, y, t) \right) \frac{1-\gamma\left(\frac{r(t)}{L}\right)^k dt + O(dt^2)}{4} + dx^2 N(x, y, t) \frac{\partial^2}{\partial y^2} \left(\frac{1-\gamma\left(\frac{r(t)}{L}\right)^k dt + O(dt^2)}{4} \right) \\
&+ O(dx^4)
\end{aligned}$$

$$
\begin{aligned}
N(x,y,t+dt) &= N(x,y,t)\left(1-\gamma\left(\frac{r(t)}{L}\right)^k dt + O(dt^2)\right) + dx^2\frac{\partial^2}{\partial x^2}\left(N(x,y,t)\right)\frac{1-\gamma\left(\frac{r(t)}{L}\right)^k dt + O(dt^2)}{4} \\
&+ dx^2\frac{\partial^2}{\partial y^2}\left(N(x,y,t)\right)\frac{1-\gamma\left(\frac{r(t)}{L}\right)^k dt + O(dt^2)}{4} + O(dx^4) \\
&= N(x,y,t) - \gamma\left(\frac{r(t)}{L}\right)^k dt N(x,y,t) + N(x,y,t)O(dt^2) \\
&+ \frac{dx^2}{4}\left(1-\gamma\left(\frac{r(t)}{L}\right)^k dt + O(dt^2)\right)\left(\frac{\partial^2}{\partial x^2}N(x,y,t) + \frac{\partial^2}{\partial y^2}N(x,y,t)\right) + O(dx^4)
\end{aligned}
$$

Subtracting $N(x,y,t)$ from both sides and ignoring terms of order 3 and above as they're too small, we obtain:

$$
N(x,y,t+dt) - N(x,y,t) = -\gamma\left(\frac{r(t)}{L}\right)^k N(x,y,t)dt + \frac{dx^2}{4}\left(\frac{\partial^2 N}{\partial x^2} + \frac{\partial^2 N}{\partial y^2}\right)
$$

Dividing both sides by dt yields:

$$
\frac{N(x,y,t+dt) - N(x,y,t)}{dt} = -\gamma\left(\frac{r(t)}{L}\right)^k N(x,y,t) + \frac{dx^2}{4dt}\left(\frac{\partial^2 N}{\partial x^2} + \frac{\partial^2 N}{\partial y^2}\right) \tag{1}
$$

As we want to consider a person's density rather than the number of people, we substitute $N(x,y,t) = \rho(x,y,t)dx^2$

$$
\frac{\rho(x,y,t+dt) - \rho(x,y,t)}{dt}dx^2 = -\gamma\left(\frac{r(t)}{L}\right)^k \rho(x,y,t)dx^2 + \frac{dx^4}{4dt}\left(\frac{\partial^2 \rho}{\partial x^2} + \frac{\partial^2 \rho}{\partial y^2}\right)
$$

Dividing both sides by dx^2 we obtain:

$$
\frac{\partial \rho}{\partial t}|_{x,y,t} = -\gamma\left(\frac{r(t)}{L}\right)^k \rho(x,y,t) + \frac{dx^2}{4dt}\left(\frac{\partial^2 \rho}{\partial x^2} + \frac{\partial^2 \rho}{\partial y^2}\right)
$$

Defining $D = \frac{dx^2}{4dt}$ we get as our final PDE:

$$\begin{aligned}
\frac{\partial \rho}{\partial t}|_{x,y,t} &= -\gamma \left(\frac{r(t)}{L}\right)^k \rho(x,y,t) + D\left(\frac{\partial^2 \rho}{\partial x^2} + \frac{\partial^2 \rho}{\partial y^2}\right) \\
&= -\gamma \left(\frac{r(t)}{L}\right)^k \rho(x,y,t) + D\nabla(\nabla \rho(x,y,t)
\end{aligned}$$