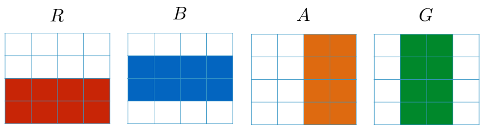
INF1a-CL

CNF KM SequentCalculus Tseytin



RB \ AG	00	01	11	10
00	0	0	0	0
01	0	0	0	0
11	1	1	1	1
10	1	1	1	1

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$$r \leftrightarrow (a \vee b)$$

$$(\neg r \vee a \vee b) \wedge (r \vee \neg a) \wedge (r \vee \neg b)$$

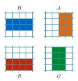
Karnaugh Maps

to produce a CNF / POS
identify blocks of 0s
and write a sum for each

RB \ AG	00	01	11	10
00	1	0	0	0
01	1	1	0	0
11	0	1	1	1
10	0	0	1	1

(R′+A+G)(B+A+G′)(R+A′)

$$(\neg R \vee A \vee G) \wedge (B \vee A \vee \neg G) \wedge (R \vee \neg A)$$



4 atoms:16 states: 64Ki subsets

a	00	01	11	10
00	0	0	0	0
01	1	1	0	1
11	1	1	1	0
10	1	0	0	0

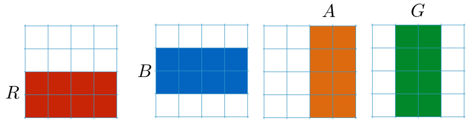
b	00	01	11	10
00	1	0	0	0
01	0	1	1	1
11	0	1	0	1
10	1	0	1	0

c	00	01	11	10
00	0	0	0	1
01	0	0	1	1
11	0	0	1	1
10	1	0	1	1

d	00	01	11	10
00	1	1	1	1
01	1	1	1	1
11	1	1	0	0
10	0	0	0	0

e	00	01	11	10
00	1	0	1	1
01	1	0	0	1
11	0	0	0	1
10	0	0	0	0

f	00	01	11	10
00	0	0	0	0
01	0	0	1	1
11	1	0	0	1
10	0	0	1	0



$$\frac{}{\Gamma, a \vDash a, \Delta}\ (I)$$

$$\frac{\Gamma \vDash a, \Delta \quad \Gamma \vDash b, \Delta}{\Gamma, a \to b \vDash \Delta}\ (\to L) \qquad \frac{\Gamma a, \vDash b, \Delta}{\Gamma \vDash a \to b, \Delta}\ (\to R)$$

$$\frac{\Gamma, a, b \vDash \Delta}{\Gamma, a \wedge b \vDash \Delta}\ (\wedge L) \qquad \frac{\Gamma \vDash a, \Delta \quad \Gamma \vDash b, \Delta}{\Gamma \vDash a \wedge b, \Delta}\ (\wedge R)$$

$$\frac{\Gamma, a \vDash \Delta \quad \Gamma, b \vDash \Delta}{a \vee b \vDash \Delta}\ (\vee L) \qquad \frac{\Gamma \vDash a, b, \Delta}{\Gamma \vDash a \vee b, \Delta}\ (\vee R)$$

$$\frac{\Gamma \vDash a, \Delta}{\Gamma, \neg a \vDash \Delta}\ (\neg L) \qquad \frac{\Gamma, a \vDash \Delta}{\Gamma \vDash \neg a, \Delta}\ (\neg R)$$

$$\frac{}{\Gamma, a \vDash \Delta, a}\ (I)$$

$$\frac{\Gamma, a, b \vDash \Delta}{\Gamma, a \wedge b \vDash \Delta}\ (\wedge L) \qquad \frac{\Gamma \vDash a, b, \Delta}{\Gamma \vDash a \vee b, \Delta}\ (\vee R)$$

$$\frac{\Gamma, a \vDash \Delta \quad \Gamma, b \vDash \Delta}{\Gamma, a \vee b \vDash \Delta}\ (\vee L) \qquad \frac{\Gamma \vDash a, \Delta \quad \Gamma \vDash b, \Delta}{\Gamma \vDash a \wedge b, \Delta}\ (\wedge R)$$

$$\frac{\Gamma \vDash a, \Delta}{\Gamma, \neg a \vDash \Delta}\ (\neg L) \qquad \frac{\Gamma, a \vDash \Delta}{\Gamma \vDash \neg a, \Delta}\ (\neg R)$$

$$\cfrac{\cfrac{\cfrac{\cfrac{}{\neg a, \neg c \vee b \vDash \neg a, c}\ I \qquad \cfrac{\cfrac{\cfrac{a, b \vDash c}{b, \vDash \neg a, c}\ \neg R}{b, \neg c \vDash \neg a, c}\ \neg L \quad \cfrac{a, b \vDash c}{b, b \vDash \neg a, c}\ \neg R}{b, \neg c \vee b \vDash \neg a, c}\ \vee L}{\cfrac{\neg a \vee b, \neg c \vee b \vDash \neg a, c}{(\neg a \vee b) \wedge (\neg c \vee b) \vDash \neg a \vee c}\ \wedge R}\ \vee L}{\vDash \neg((\neg a \vee b) \wedge (\neg c \vee b)), (\neg a \vee c)}\ \neg R}{\vDash \neg((\neg a \vee b) \wedge (\neg c \vee b)) \vee (\neg a \vee c)}\ \vee R$$

these are valid
in some universe, U

iff this is valid in U

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\neg a, \neg c \lor b \vDash \neg a, c} \quad \dfrac{\dfrac{\dfrac{a, b \vDash c}{b, \vDash \neg a, c}}{b, \neg c \vDash \neg a, c} \quad \dfrac{a, b \vDash c}{b, b \vDash \neg a, c}}{b, \neg c \lor b \vDash \neg a, c}}{\neg a \lor b, \neg c \lor b \vDash \neg a, c}}{(\neg a \lor b) \land (\neg c \lor b) \vDash \neg a \lor c}}{\vDash \neg((\neg a \lor b) \land (\neg c \lor b)), (\neg a \lor c)}}{\vDash \neg((\neg a \lor b) \land (\neg c \lor b)) \lor (\neg a \lor c)}$$

$$\dfrac{a, b \vDash c \quad a, b \vDash c}{\vDash \neg((\neg a \lor b) \land (\neg c \lor b)) \lor (\neg a \lor c)}$$

Our two inference trees
tell two different stories …

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{p \vDash q, p}}{\vDash \neg p, q, p}}{\vDash \neg p \lor q, p} \quad \dfrac{\dfrac{}{p \vDash p}}{\vDash \neg p, p}}{\vDash (\neg p \lor q) \land \neg p, p}}{\vDash ((\neg p \lor q) \land \neg p) \lor p}$$

Every branch is terminated by an immediate rule.

The sequent we started from is valid in every universe!

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\neg a, \neg c \lor b \vDash \neg a, c} \quad \dfrac{\dfrac{\dfrac{a, b \vDash c}{b, \vDash \neg a, c}}{b, \neg c \vDash \neg a, c} \quad \dfrac{a, b \vDash c}{b, b \vDash \neg a, c}}{b, \neg c \lor b \vDash \neg a, c}}{\neg a \lor b, \neg c \lor b \vDash \neg a, c}}{(\neg a \lor b) \land (\neg c \lor b) \vDash \neg a \lor c}}{\vDash \neg((\neg a \lor b) \land (\neg c \lor b)), (\neg a \lor c)}}{\vDash \neg((\neg a \lor b) \land (\neg c \lor b)) \lor (\neg a \lor c)}$$

Some branches lead to *leaves*, sequences with only atoms, in which no atom occurs on both sides of the turnstile.

Our starting sequent is valid in some universe U iff each of these leaves is valid.

It is easy to construct a counterexample to any one of these leaves.

$$\dfrac{}{\Gamma, a \vdash a, \Delta}\ (I)$$

$$\dfrac{\Gamma \vdash a, \Delta \quad \Gamma, b \vdash \Delta}{\Gamma, a \to b \vdash \Delta}\ (\to L) \qquad \dfrac{\Gamma, a \vdash b, \Delta}{\Gamma \vdash a \to b, \Delta}\ (\to R)$$

$$\dfrac{\Gamma, a, b \vdash \Delta}{\Gamma, a \land b \vdash \Delta}\ (\land L) \qquad \dfrac{\Gamma \vdash a, \Delta \quad \Gamma \vdash b, \Delta}{\Gamma \vdash a \land b, \Delta}\ (\land R)$$

$$\dfrac{\Gamma, a \vdash \Delta \quad \Gamma, b \vdash \Delta}{a \lor b \vdash \Delta}\ (\lor L) \qquad \dfrac{\Gamma \vdash a, b, \Delta}{\Gamma \vdash a \lor b, \Delta}\ (\lor R)$$

$$\dfrac{\Gamma \vdash a, \Delta}{\Gamma, \neg a \vdash \Delta}\ (\neg L) \qquad \dfrac{\Gamma, a \vdash \Delta}{\Gamma \vdash \neg a, \Delta}\ (\neg R)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{p \vDash q, p}}{\vDash \neg p, q, p}}{\vDash \neg p \vee q, p} \quad \dfrac{\dfrac{}{p \vDash p}}{\vDash \neg p, p}}{\dfrac{\vDash (\neg p \vee q) \wedge \neg p, p}{\vDash ((\neg p \vee q) \wedge \neg p) \vee p}}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{P \vdash Q, P}}{\vdash \neg P, Q, P}}{\vdash \neg P \vee Q, P} \quad \dfrac{\dfrac{}{P \vdash P}}{\vdash \neg P, P}}{\dfrac{\vdash (\neg P \vee Q) \wedge \neg P, P}{\vdash ((\neg P \vee Q) \wedge \neg P) \vee P}}$$

A proof is a tree of inferences, starting with immediate rules.

Prove the following entailment or if it not provable provide a counterexample

$$P \to (Q \vee R), (Q \wedge R) \to S \vdash P \to S$$

$$\dfrac{}{\Gamma, a \vdash a, \Delta}\ (I)$$

$$\dfrac{\Gamma, \exists x. \vdash \Delta}{\Gamma, A[y/x] \vdash \Delta}\ (\exists L) \qquad \dfrac{\Gamma \vdash \exists x. A, \Delta}{\Gamma \vdash A[t/x], \Delta}\ (\exists R)$$

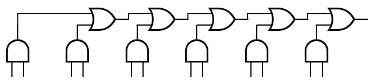
$$\dfrac{\Gamma, \forall x. \vdash \Delta}{\Gamma, A[t/x] \vdash \Delta}\ (\forall L) \qquad \dfrac{\Gamma \vdash \forall x. A, \Delta}{\Gamma \vdash A[y/x], \Delta}\ (\forall R)$$

$$\dfrac{\Gamma \vdash a, \Delta \quad \Gamma, b \vdash \Delta}{\Gamma, a \to b \vdash \Delta}\ (\to L) \qquad \dfrac{\Gamma, a \vdash b, \Delta}{\Gamma \vdash a \to b, \Delta}\ (\to R)$$

$$\dfrac{\Gamma, a, b \vdash \Delta}{\Gamma, a \wedge b \vdash \Delta}\ (\wedge L) \qquad \dfrac{\Gamma \vdash a, \Delta \quad \Gamma \vdash b, \Delta}{\Gamma \vdash a \wedge b, \Delta}\ (\wedge R)$$

$$\dfrac{\Gamma, a \vdash \Delta \quad \Gamma, b \vdash \Delta}{a \vee b \vdash \Delta}\ (\vee L) \qquad \dfrac{\Gamma \vdash a, b, \Delta}{\Gamma \vdash a \vee b, \Delta}\ (\vee R)$$

$$\dfrac{\Gamma \vdash a, \Delta}{\Gamma, \neg a \vdash \Delta}\ (\neg L) \qquad \dfrac{\Gamma, a \vdash \Delta}{\Gamma \vdash \neg a, \Delta}\ (\neg R)$$



$$a \wedge b \vee c \wedge d \vee e \wedge f \vee g \wedge h \vee j \wedge k \vee m \wedge n$$

$$(a \wedge b) \vee (c \wedge d) \vee (e \wedge f) \vee (g \wedge h) \vee (j \wedge k) \vee (m \wedge n)$$

How many clauses in the CNF?

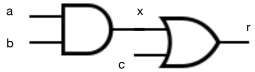
$2^6 = 64$

How many clauses to describe the circuit?

If we start from an expression then
we can draw an equivalent circuit with:

$r = (a \wedge b) \vee c$

a wire for each subexpression,
a logic gate for each operator,
and an input for each variable.



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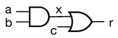

$r \leftrightarrow (a \wedge b)$
$(r \vee \neg a \vee \neg b) \wedge (\neg r \vee a) \wedge (\neg r \vee b)$

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$(\neg r \vee a \vee b) \wedge (r \vee \neg a) \wedge (r \vee \neg b)$

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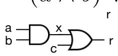

from km

$\boxed{\begin{array}{c} r \leftrightarrow (a \wedge b) \\ (r \vee \neg a \vee \neg b) \wedge (\neg r \vee a) \wedge (\neg r \vee b) \end{array}}$

from km

$\boxed{\begin{array}{c} r \leftrightarrow (a \vee b) \\ (\neg r \vee a \vee b) \wedge (r \vee \neg a) \wedge (r \vee \neg b) \end{array}}$

$x \leftrightarrow (a \wedge b)$

If we start from an expression then
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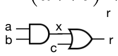

from km

$$r \leftrightarrow (a \wedge b)$$
$$(r \vee \neg a \vee \neg b) \wedge (\neg r \vee a) \wedge (\neg r \vee b)$$

from km

$$r \leftrightarrow (a \vee b)$$
$$(\neg r \vee a \vee b) \wedge (r \vee \neg a) \wedge (r \vee \neg b)$$

$$r \leftrightarrow (a \wedge b)$$

substitute

$r := x \quad a := a \quad b := b$

to give:

$$x \leftrightarrow (a \wedge b)$$

If we start from an expression then
we can draw an equivalent circuit with:

$r = (a \wedge b) \vee c$

a wire for each subexpression,
a logic gate for each operator,
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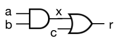

from km

$$r \leftrightarrow (a \wedge b)$$
$$(r \vee \neg a \vee \neg b) \wedge (\neg r \vee a) \wedge (\neg r \vee b)$$

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substitute

$r := x \quad a := a \quad b := b$

to give:

$$x \leftrightarrow (a \wedge b)$$
$$(x \vee \neg a \vee \neg b) \wedge (\neg x \vee a) \wedge (\neg x \vee b)$$

substitute

$r := r \quad a := x \quad b := c$

to give:

$$r \leftrightarrow (x \vee c)$$

If we start from an expression then
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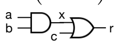

from km

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$$(\neg r \vee a \vee b) \wedge (r \vee \neg a) \wedge (r \vee \neg b)$$

substitute

$r := x \quad a := a \quad b := b$

to give:

$$x \leftrightarrow (a \wedge b)$$
$$(x \vee \neg a \vee \neg b) \wedge (\neg x \vee a) \wedge (\neg x \vee b)$$

substitute

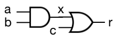
$r := r \quad a := x \quad b := c$

to give:

$$r \leftrightarrow (x \vee c)$$
$$(\neg r \vee x \vee c) \wedge (r \vee \neg x) \wedge (r \vee \neg c)$$

If we start from an expression then we can draw an equivalent circuit with:

$r = (a \wedge b) \vee c$



a wire for each subexpression, a logic gate for each operator, and an input for each variable.

$$r \leftrightarrow (a \wedge b)$$
$$(r \vee \neg a \vee \neg b) \wedge (\neg r \vee a) \wedge (\neg r \vee b)$$

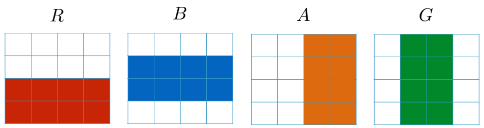
$$r \leftrightarrow (a \vee b)$$
$$(\neg r \vee a \vee b) \wedge (r \vee \neg a) \wedge (r \vee \neg b)$$

$$x \leftrightarrow (a \wedge b)$$
$$(x \vee \neg a \vee \neg b) \wedge (\neg x \vee a) \wedge (\neg x \vee b)$$

$$r \leftrightarrow (x \vee c)$$
$$(\cancel{\neg r \vee} x \vee c) \wedge (\cancel{r \vee \neg x}) \wedge (\cancel{r \vee \neg c})$$

Combine the two CNF, with R = True

$$(x \vee \neg a \vee \neg b) \wedge (\neg x \vee a) \wedge (\neg x \vee b) \wedge (x \vee c)$$



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00	0	0	0	0
01	0	0	0	0
11	1	1	1	1
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RB \ AG	00	01	11	10
00	1	1	0	0
01	0	0	0	0
11	1	1	1	1
10	0	0	1	1

RB \ AG	00	01	11	10
00	0	0	1	1
01	1	1	1	1
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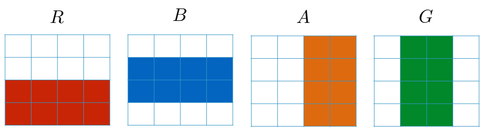

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00	0	0	0	0
01	0	0	0	0
11	1	1	1	1
10	1	1	1	1

RB \ AG	00	01	11	10
00	1	1	1	1
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RB \ AG	00	01	11	10
00	0	0	0	0
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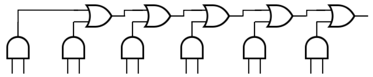
$$r \leftrightarrow (a \wedge b)$$
$$(r \vee \neg a \vee \neg b) \wedge (\neg r \vee a) \wedge (\neg r \vee b)$$



$a \wedge b \vee c \wedge d \vee e \wedge f \vee g \wedge h \vee j \wedge k \vee m \wedge n$

$(a \wedge b) \vee (c \wedge d) \vee (e \wedge f) \vee (g \wedge h) \vee (j \wedge k) \vee (m \wedge n)$

How many clauses in the CNF?

$2^6 = 64$

How many clauses to describe the circuit?

11 x 3 = 33 (before simplification)