Sequent Calculus

This tutorial concerns Boolean operations on predicates, and the semantic notion of satisfaction, $\vDash$. It is **important** for you to have attempted the exercises before going to the tutorial.

Semantics

For any universe U, a predicate is a function `p :: U -> Bool`. Sometimes it is helpful to consider the corresponding subset of the universe $[\![p]\!] = \{x \in U \mid p\ x\}$.

We have defined boolean operations of negation, disjunction, and conjunction on predicates in terms of the corresponding operations on the Boolean type `Bool`.

```
neg :: (u -> Bool) -> (u -> Bool)
neg p x = not (p x)
```

```
(|:|) :: (u -> Bool) -> (u -> Bool) -> (u -> Bool)
(|:|) p q x = p x || q x

(&:&) :: (u -> Bool) -> (u -> Bool) -> (u -> Bool)
(&:&) p q x = p x && q x
```

```
bigOr :: [u -> Bool] -> (u -> Bool)
bigOr ps x = or [p x | p <- ps]

bigAnd :: [u -> Bool] -> (u -> Bool)
bigAnd ps x = and [p x | p <- ps]
```

We use $\neg$, $\vee$, $\bigvee$, $\wedge$, $\bigwedge$, as mathematical notations for these operations on predicates, and $\top, \bot$ as constants denoting the predicates *top,* (`top :: u -> Bool`), and *bottom,* (`bot :: u -> Bool`), defined to be the constant functions, `top _ = True` and `bot _ = False`, corresponding to the entire set U and the empty set $\emptyset$. These constants and operations on predicates are distinguished, by their types, from the corresponding Boolean constants and operations.

The semantic notion of satisfaction is a relation, written as a *sequent*, $\Gamma \vDash \Delta$, that may or may not hold between two finite sets (Γ and Δ) of predicates in a given universe U. We define $\Gamma \vDash \Delta$ in terms of the inclusion relation $a \vDash b$ between single predicates.

$$\Gamma \vDash \Delta \textbf{ iff } \bigwedge \Gamma \vDash \bigvee \Delta$$

This means that we treat the antecedents (predicates before the turnstile) as a conjunction, and the succedents (predicates after the turnstile) as a disjunction.

The following rules are sound:

$$\frac{}{\Gamma, a \vDash a, \Delta}\ I$$

$$\frac{\Gamma \vDash a, \Delta}{\Gamma, \neg a \vDash \Delta}\ \neg L \qquad \frac{\Gamma, a \vDash \Delta}{\Gamma \vDash \neg a, \Delta}\ \neg R$$

$$\frac{\Gamma, a, b \vDash \Delta}{\Gamma, a \wedge b \vDash \Delta}\ \wedge L \qquad \frac{\Gamma \vDash a \quad \Gamma \vDash b, \Delta}{\Gamma \vDash a \wedge b, \Delta}\ \wedge R$$

$$\frac{\Gamma, a \vDash \Delta \quad b \vDash \Delta}{\Gamma, a \vee b \vDash \Delta}\ \vee L \qquad \frac{\Gamma \vDash a, b, \Delta}{\Gamma \vDash a \vee b, \Delta}\ \vee R$$

Here, variables $a, b, \ldots$ are predicates, while Γ, Δ are (finite) sets of predicates. To make the notation less cluttered, we write Γ, p for $\Gamma \cup \{p\}$ and q, Δ for $\{q\} \cup \Delta$. The double inference line signifies a two-way rule: the conclusion is valid iff all of the premises are valid. For the immediate rule, with no premises, this means that the conclusion is ally valid.

If a rule is sound, we can add Γ, Δ to the left, right (respectively) of each turnstile, and still have a sound rule. This follows from the fact, shown in class, that,

$$\Gamma, a \vDash b, \Delta \text{ holds in universe } U \textbf{ iff } a \vDash b \text{ holds in the sub-universe } \{x \in U \mid \bigwedge \Gamma\ x, \neg \bigvee \Delta\ x\}$$

Just as in algebra, where we manipulate arithmetic expressions to solve equations, we will manipulate logical expressions to solve problems in logic. In algebra, variables range over numbers. Expressions such as $\sqrt{b^2 - 4ac}$ are formed from constants 2, 4 and variables a, b, c, using operations $\sqrt{\ }$, $\times$, $-$, $(.)^{(.)}$. In propositional logic, we have variables a, b, c that range over predicates and two constants $\top, \bot$. Expressions are built from variables and constants using operations, called logical *connectives*, that include $\neg$, $\wedge$, $\vee$. Although we can, and will, use Boolean algebra to manipulate equations, it is easier in propositional logic to work with sequents, so we will start with these. We will return later to look at equations.

1 Reduction

As shown in class, we can use these rules to reduce any sequent to a conjunction of *simple* sequents. A sequent $\Gamma \vDash \Delta$ is simple if Γ and Δ are disjoint finite sets of variables, $\Gamma \cap \Delta = \emptyset$, a sequent with no connectives, to which the immediate rule does not apply.

In some cases the conjunction is empty; we reduce the starting sequent to an empty conjunction. This means that the sequent we started from is universally valid.

For example, we can make the following reduction, starting from the sequent, $x \wedge (y \vee z) \vDash (x \wedge y) \vee (x \wedge z)$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{x, y \vDash x, (x \wedge z)} \qquad \dfrac{}{x, y \vDash y, (x \wedge z)}}{x, y \vDash (x \wedge y), (x \wedge z)} \qquad \dfrac{\dfrac{}{x, z \vDash (x \wedge y), x} \qquad \dfrac{}{x, z \vDash (x \wedge y), z}}{x, z \vDash (x \wedge y), (x \wedge z)}}{x, (y \vee z) \vDash (x \wedge y), (x \wedge z)}}{x, (y \vee z) \vDash (x \wedge y) \vee (x \wedge z)}}{x \wedge (y \vee z) \vDash (x \wedge y) \vee (x \wedge z)}$$

To perform the reduction, we simply work backwards, at each step we choose a rule that will eliminate a connective $(\wedge, \vee, \neg)$ from the left or right side of the turnstile. When there is a choice, it doesn't matter which side we do first (but it may save duplication if you use rules with a single premise before those with two premises).

In this example, each branch of the tree ends in an instance of the immediate rule. For the immediate rule, there are no statements above the line, so the conclusion is valid. Working down the tree each inference line corresponds to a sound rule, so we see that the bottom line is valid.

1. Label each inference line in the tree with the name of the rule applied.
2. Reduce the following sequent, $(x \wedge y) \vee (x \wedge z) \vDash x \wedge (y \vee z)$, in similar fashion.
3. Is the equation, $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$, valid in every universe? Explain your answer.

Use reductions to show that the following equations are valid in every universe.

4. $(x \wedge y) \vee (x \wedge z) = x \wedge (y \vee z)$
5. $\neg(x \wedge y) = \neg x \vee \neg y$
6. $\neg x \wedge \neg y = \neg(x \vee y)$

Sometimes the sequent we start from is not universally valid. For example, we can reduce the sequent, $\vDash (a \wedge \neg b) \vee (\neg a \wedge b)$, to derive a new rule. The following reduction:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{a \vDash a,}}{\vDash a, \neg a} \qquad \vDash a, b}{\vDash a, \neg a \wedge b} \qquad \dfrac{\dfrac{a, b \vDash}{\vDash \neg b, \neg a} \qquad \dfrac{\dfrac{}{b \vDash b}}{\vDash \neg b, b}}{\vDash \neg b, \neg a \wedge b}}{\vDash a \wedge \neg b, \neg a \wedge b}}{\vDash (a \wedge \neg b) \vee (\neg a \wedge b)}$$

gives the rule, $$\dfrac{\vDash a, b \qquad a, b \vDash}{\vDash (a \wedge \neg b) \vee (\neg a \wedge b)}.$$

This reduction ends with two simple sequents. These form the premises for a new rule; the conclusion is the original sequent, which is satisfied iff the two premises are satisfied.

The expression, $(a \wedge \neg b) \vee (\neg a \wedge b)$, defines the **exclusive or** function, xor: $a \oplus b \equiv (a \wedge \neg b) \vee (\neg a \wedge b)$; so we have derived the rule, $\dfrac{\vDash a, b \quad a, b \vDash}{\vDash a \oplus b}$, and hence, $\dfrac{\Gamma \vDash a, b, \Delta \quad \Gamma, a, b \vDash \Delta}{\Gamma \vDash a \oplus b, \Delta} \oplus R$.

7. Reduce the sequent $(a \wedge \neg b) \vee (\neg a \wedge b) \vDash$, and use the result to derive a rule $(\oplus L)$.
8. We define $a \to b \equiv \neg a \vee b$. Use reductions of the sequents $\vDash \neg a \vee b$ and $\neg a \vee b \vDash$ to produce rules $(\to R)$ and $(\to L)$.
9. We define, $a \leftrightarrow b \equiv (a \wedge b) \vee (\neg a \wedge \neg b)$. Use reductions to derive rules $(\leftrightarrow R)$ and $(\leftrightarrow L)$
10. The trivial sequent is $\emptyset \vDash \emptyset$, with the empty set on each side of the turnstile. Is the trivial sequent valid in all models, valid in some models (if so describe them), or valid in no models?
11. If our original sequent is non-trivial, and includes no constants, then every sequent in the reduction tree will also be non-trivial. Explain why this is so.

If the expressions in our original sequent include constants, $\top, \bot$, then these will eventually occur, naked, on one or other side of the turnstile.

12. Give sound double-line rules $(\top L)$, $(\top R)$, $(\bot L)$, $(\bot R)$ that eliminate the constants.

We can also use the reduction of a sequent, $\vDash \varphi$, to derive a simple expression equivalent to φ.

We begin by moving every atom in the premises of the derived rule to the right of the turnstile, using negation as necessary. For our $\oplus$ example moving each atom to the right of the turnstile gives the rule,

$$\frac{\vDash a, b \qquad \vDash \neg a, \neg b}{\vDash (a \wedge \neg b) \vee (\neg a \wedge b)}, \qquad \text{which is equivalent to,} \qquad \frac{\vDash a \vee b \qquad \vDash \neg a \vee \neg b}{\vDash (a \wedge \neg b) \vee (\neg a \wedge b)}.$$

This means that semantically, $(a \wedge \neg b) \vee (\neg a \wedge b)$ is equivalent to $(a \vee b) \wedge (\neg a \vee \neg b)$.

In this way we produce a conjunction of disjunctions of literals, where a *literal* is a propositional atom (a variable) or the negation of an atom. We call such a conjunction of disjunctions of literals a conjunctive normal form (CNF).

We can use this technique, starting with any expression on the right-hand side of the turnstile, to find a CNF equivalent to the starting expression.

Use this technique, and your answers to 8 and 9, to give a CNF for the following expressions

13. $r \leftrightarrow (a \wedge b)$
14. $r \leftrightarrow (a \vee b)$
15. $r \leftrightarrow (a \to b)$
16. $r \leftrightarrow (\neg a)$

Use reductions to decide whether the equations below are universally valid (true in every universe).

17. $x \leftrightarrow y = (x \to y) \wedge (y \to x)$
18. $(x \to y) \to z = x \to (y \to z)$
19. $(x \leftrightarrow y) \leftrightarrow z = x \leftrightarrow (y \leftrightarrow z)$
20. $(x \oplus y) \oplus z = x \oplus (y \oplus z)$

This tutorial shows that any sequent can be reduced to a CNF. By falsifying any conjunct in the CNF we can falsify the original sequent. If the CNF is the empty conjunction, then we have shown that the original sequent is universally valid.

21. How could you use the techniques developed in this tutorial to check whether a sequent is satisfiable (valid in some inhabited model)?