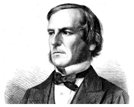
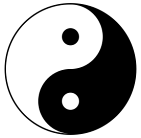


George Boole 1815—1864

Charles Peirce 1839—1914

Beyond Syllogisms

inf1a-cl
Michael Fourman



```haskell
-- Thing is the type of things in our universe
things :: [ Thing ] -- we list every thing

(|=) :: (Thing -> Bool) -> (Thing -> Bool) -> Bool
a |= b = and [ b x | x <- things, a x ]
      -- every a is b

(|/=) :: (Thing -> Bool) -> (Thing -> Bool) -> Bool
a |/= b = not ( a |= b )  -- some a is not b

neg :: (u -> Bool) -> (u -> Bool)
neg a x = not ( a x )
```

$$\frac{a \vDash b \quad b \vDash c}{a \vDash c}\ \textit{barbara} \qquad \frac{a \nvDash c \quad b \vDash c}{a \nvDash b}\ \textit{baroco} \qquad \frac{a \vDash b \quad a \nvDash c}{b \nvDash c}\ \textit{bocardo}$$

$$\frac{a \vDash b \quad b \vDash \neg c}{a \vDash \neg c}\ \textit{celarent} \qquad \frac{a \nvDash \neg c \quad b \vDash \neg c}{a \nvDash b}\ \textit{festino} \qquad \frac{a \vDash b \quad a \nvDash \neg c}{b \nvDash \neg c}\ \textit{disamis}$$

$$\frac{a \vDash b \quad b \vDash \neg c}{c \vDash \neg a}\ \textit{calemes} \qquad \frac{c \nvDash \neg a \quad b \vDash \neg c}{a \nvDash b}\ \textit{fresison} \qquad \frac{a \vDash b \quad c \nvDash \neg a}{b \nvDash \neg c}\ \textit{dimatis}$$

$$\frac{a \vDash b \quad c \vDash \neg b}{a \vDash \neg c}\ \textit{cesare} \qquad \frac{a \nvDash \neg c \quad c \vDash \neg b}{a \nvDash b}\ \textit{ferio} \qquad \frac{a \vDash b \quad a \nvDash \neg c}{c \nvDash \neg b}\ \textit{datisi}$$

$$\frac{a \vDash b \quad c \vDash \neg b}{c \vDash \neg a}\ \textit{camestres} \qquad \frac{c \nvDash \neg a \quad c \vDash \neg b}{a \nvDash b}\ \textit{ferison} \qquad \frac{a \vDash b \quad c \nvDash \neg a}{c \nvDash \neg b}\ \textit{darii}$$

$$\frac{m \models p \quad s \models m}{s \models p}\ \textit{barbara} \qquad \frac{p \models m \quad s \not\models m}{s \not\models p}\ \textit{baroco} \qquad \frac{m \not\models p \quad m \models s}{s \not\models p}\ \textit{bocardo}$$

$$\frac{m \models \neg p \quad s \models m}{s \models \neg p}\ \textit{celarent} \qquad \frac{p \models \neg m \quad s \not\models \neg m}{s \not\models p}\ \textit{festino} \qquad \frac{m \not\models \neg p \quad m \models s}{s \not\models \neg p}\ \textit{disamis}$$

$$\frac{p \models m \quad m \models \neg s}{s \models \neg p}\ \textit{calemes} \qquad \frac{p \models \neg m \quad m \not\models \neg s}{s \not\models p}\ \textit{fresison} \qquad \frac{p \not\models \neg m \quad m \models s}{s \not\models \neg p}\ \textit{dimatis}$$

$$\frac{p \models \neg m \quad s \models m}{s \models \neg p}\ \textit{cesare} \qquad \frac{m \models \neg p \quad s \not\models \neg m}{s \not\models p}\ \textit{ferio} \qquad \frac{m \models p \quad m \not\models \neg s}{s \not\models \neg p}\ \textit{datisi}$$

$$\frac{p \models m \quad s \models \neg m}{s \models \neg p}\ \textit{camestres} \qquad \frac{m \models \neg p \quad m \not\models \neg s}{s \not\models p}\ \textit{ferison} \qquad \frac{m \models p \quad s \not\models \neg m}{s \not\models \neg p}\ \textit{darii}$$

---

rules Aristotle forgot

$$\frac{a \models \neg b}{b \models \neg a}\ \text{contraposition}$$

$$\frac{}{a \models a}\ \text{the \textit{immediate} rule}$$


```haskell
type Pred u = u -> Bool

neg :: (u -> Bool) -> (u -> Bool)
neg a x = not ( a x )

-- Thing is the type of things in our universe
things :: [ Thing ] -- we list every thing

(|=) :: (u -> Bool) -> (u -> Bool) -> Bool
a |= b = and [ b x | x <- things, a x ]
      -- every a is b
```

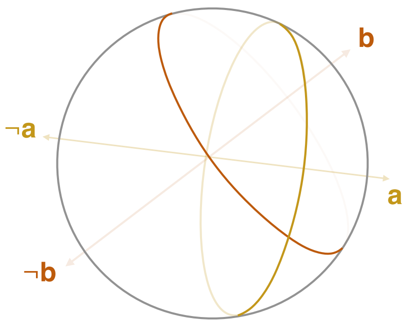
$$\neg\neg a = a$$

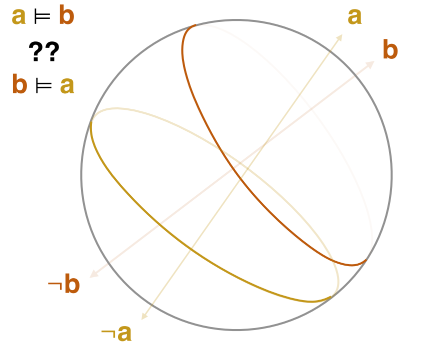
The second rule of boolean logic
(the first is *barbara*)

$$\dfrac{\dfrac{\dfrac{a \vDash b}{\neg b \vDash \neg a}}{\neg\neg a \vDash \neg\neg b}}{a \vDash b} \qquad \frac{a \vDash b}{\neg b \vDash \neg a} \qquad \frac{a \vDash b}{\neg b \vDash \neg a}$$

contraposition

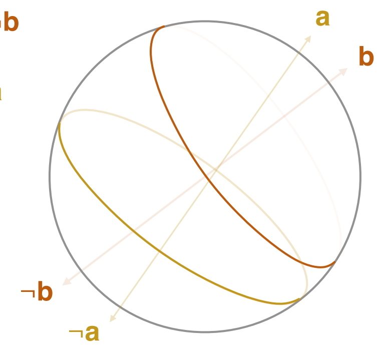
Here we derive the 2-way rule from the single rule.





$\neg a \vDash \neg b$

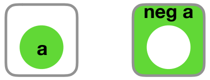
$b \vDash a$



---

predicates are just functions :: U -> Bool
our first operation on predicates is negation

$$(\mathtt{neg}\, a)\, x = \neg(a\, x)$$




```
type Pred u = u -> Bool
neg :: Pred u -> Pred u
neg a x = not ( a x )
```

For Aristotle, these were different syllogisms

for us they are the same syllogism applied to different predicates

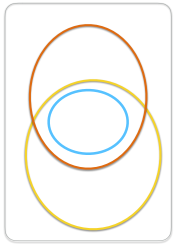
$$\frac{a \vDash b \quad b \vDash c}{a \vDash c}$$

$$\frac{a \vDash b \quad b \vDash \neg c}{a \vDash \neg c}$$

$$(a \wedge b)\, x = a\, x \wedge b\, x$$

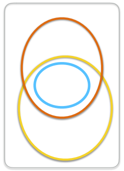
c ⊨ **a** ∧ **b**
iff
c ⊨ **a** and **c** ⊨ **b**

$$\frac{c \vDash a \quad c \vDash b}{c \vDash a \wedge b}$$



∧	⊥	⊤
⊥	⊥	⊥
⊤	⊥	⊤

$$\frac{c \vDash a \quad c \vDash b}{c \vDash a \wedge b}$$



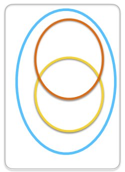
```
(&&) :: Bool -> Bool -> Bool

a :: U -> Bool
b :: U -> Bool

( a &:& b ) x = (a x) && (b x)
(&:&) :: (U -> Bool) -> (U -> Bool) -> (U -> Bool)
```

$$(a \vee b)\, x = a\, x \vee b\, x$$

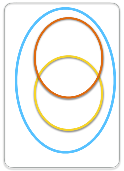
a ∨ **b** ⊨ **c**
iff
a ⊨ **c** and **b** ⊨ **c**



$$\frac{a \vDash c \quad b \vDash c}{a \vee b \vDash c}$$

$\vee$	$\bot$	$\top$
$\bot$	$\bot$	$\top$
$\top$	$\top$	$\top$

$$\dfrac{a \vDash c \quad b \vDash c}{a \vee b \vDash c}$$



```
(||) :: Bool -> Bool -> Bool

a :: U -> Bool
b :: U -> Bool

( a |:| b ) x = (a x) || (b x)
(|:|) :: (U -> Bool) -> (U -> Bool) -> (U -> Bool)
```

$$\dfrac{c \vDash a \quad c \vDash b}{c \vDash a \wedge b} \qquad \dfrac{a \vDash c \quad b \vDash c}{a \vee b \vDash c}$$

$$\dfrac{\dfrac{\dfrac{c \vDash \neg a}{a \vDash \neg c} \quad \dfrac{c \vDash \neg b}{b \vDash \neg c}}{a \vee b \vDash \neg c}}{c \vDash \neg(a \vee b)} \quad \text{and} \quad \dfrac{c \vDash \neg a \quad c \vDash \neg b}{c \vDash \neg a \wedge \neg b} \quad \text{so,} \ \dfrac{c \vDash \neg(a \wedge b)}{c \vDash \neg a \vee \neg b}$$

Substituting $\neg a \vee \neg b$ and $\neg(a \wedge b)$ for c gives,

$$\dfrac{\overline{\neg a \vee \neg b \vDash \neg a \vee \neg b}}{\neg a \vee \neg b \vDash \neg(a \wedge b)} \qquad \dfrac{\overline{\neg(a \wedge b) \vDash \neg(a \wedge b)}}{\neg(a \wedge b) \vDash \neg a \vee \neg b}$$

$\neg(a \vee b) = \neg a \wedge \neg b \qquad \neg(a \wedge b) = \neg a \vee \neg b \qquad$ (de Morgan)

$$a, b \vDash c$$

every thing in both a and b, is in c

```
-- the following are equivalent
and [ c x | a <- things, b <- things, a x, b x ]
and [ c x | a <- things, b <- things, a x && b x ]
and [ c x | a <- things, b <- things, (a &:& b) x ]
```

$$\dfrac{a, b \vDash c}{a \wedge b \vDash c}$$

```
-- the following are equivalent
and [ c x | a <- things, b <- things, a x, b x ]
and [ c x | a <- things, b <- things, a x && b x ]
and [ c x | a <- things, b <- things, (a &:& b) x ]
```

$$\frac{a, b \vDash c}{\overline{a \wedge b \vDash c}} \qquad \frac{a, b \nvDash c}{\overline{a \wedge b \nvDash c}}$$

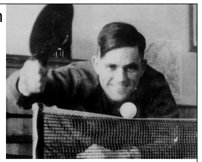
$$a, b \vDash c$$

every thing in both a and b, is in c

How should we treat ∨ on the right?

$$\frac{\frac{\frac{\frac{c \vDash a \vee b}{\overline{\neg(a \vee b) \vDash \neg c}}}{\overline{\neg a \wedge \neg b \vDash \neg c}}\ \text{de Morgan}}{\overline{\neg a, \neg b \vDash \neg c}}}{\overline{c \vDash a, b}}\ ??$$

Gerhard Gentzen
1909—1945



$$\frac{c \vDash a, b}{\overline{c \vDash a \vee b}}$$

every thing in c is in either a or b, or both

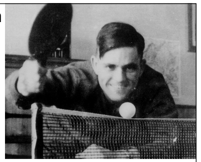
A *sequent*

$$a_0, \ldots, a_{n-1} \vDash s_0, \ldots, s_{m-1}$$

every thing that is in **every** *antecedent*, a_i, is in **some** *succedent*, s_j.

Sequents

Gerhard Gentzen
1909—1945



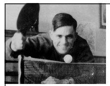
A sequent is valid in a given universe
iff
whenever every antecedent holds
then
some succedent holds

```
gamma |= delta =
  and[ or[ d x | d <- delta ]
     | x <- things, and[ g x | g <- gamma ]]
```

Here, Γ and Δ are finite sets of predicates.

$$\Gamma \vDash \Delta \textbf{ iff } \bigwedge \Gamma \subseteq \bigvee \Delta$$

The operations, $\bigwedge, \bigvee$, on predicates correspond to intersection, $\bigcap$, and union, $\bigcup$, of sets.



$$\frac{}{\Gamma, a \vDash \Delta, a}\ (I)$$

$$\frac{\Gamma, a, b \vDash \Delta}{\Gamma, a \wedge b \vDash \Delta}\ (\wedge L) \qquad \frac{\Gamma \vDash a, b, \Delta}{\Gamma \vDash a \vee b, \Delta}\ (\vee R)$$

$$\frac{\Gamma, a \vDash \Delta \quad \Gamma, b \vDash \Delta}{\Gamma, a \vee b \vDash \Delta}\ (\vee L) \qquad \frac{\Gamma \vDash a, \Delta \quad \Gamma \vDash b, \Delta}{\Gamma \vDash a \wedge b, \Delta}\ (\wedge R)$$

$$\frac{\Gamma \vDash a, \Delta}{\Gamma, \neg a \vDash \Delta}\ (\neg L) \qquad \frac{\Gamma, a \vDash \Delta}{\Gamma \vDash \neg a, \Delta}\ (\neg R)$$

- a and b are predicates from some universe,
- Γ, Δ are finite sets of predicates from some universe,
- Γ, a refers to $\Gamma \cup \{a\}$, and a, Δ refers to $\{a\} \cup \Delta$.