Informatics 1

Introduction to Computation

Lecture 9

Expression Trees as Algebraic Data Types

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Part I

Expression Trees

Expression Trees

```
data Exp  =  Lit Int
          |  Add Exp Exp
          |  Mul Exp Exp

evalExp :: Exp -> Int
evalExp (Lit n)     =  n
evalExp (Add e f)   =  evalExp e + evalExp f
evalExp (Mul e f)   =  evalExp e * evalExp f

showExp :: Exp -> String
showExp (Lit n)     =  show n
showExp (Add e f)   =  par (showExp e ++ "+" ++ showExp f)
showExp (Mul e f)   =  par (showExp e ++ "*" ++ showExp f)

par :: String -> String
par s  =  "(" ++ s ++ ")"
```

Expression Trees

```
e0, e1 :: Exp
e0 = Add (Lit 2) (Mul (Lit 3) (Lit 3))
e1 = Mul (Add (Lit 2) (Lit 3)) (Lit 3)

*Main> showExp e0
"(2+(3*3))"

*Main> evalExp e0
11

*Main> showExp e1
"((2+3)*3)"

*Main> evalExp e1
15
```

Expression Trees, Infix

```
data  Exp  =  Lit Int
           |  Exp `Add` Exp
           |  Exp `Mul` Exp

evalExp :: Exp -> Int
evalExp (Lit n)    =  n
evalExp (e `Add` f)  =  evalExp e + evalExp f
evalExp (e `Mul` f)  =  evalExp e * evalExp f

showExp :: Exp -> String
showExp (Lit n)      =  show n
showExp (e `Add` f)  =  par (showExp e ++ "+" ++ showExp f)
showExp (e `Mul` f)  =  par (showExp e ++ "*" ++ showExp f)

par :: String -> String
par s  =  "(" ++ s ++ ")"
```

Expression Trees, Infix

```
e0, e1 :: Exp
e0 = Lit 2 `Add` (Lit 3 `Mul` Lit 3)
e1 = (Lit 2 `Add` Lit 3) `Mul` Lit 3

*Main> showExp e0
"(2+(3*3))"

*Main> evalExp e0
11

*Main> showExp e1
"((2+3)*3)"

*Main> evalExp e1
15
```

Expression Trees, Symbols

```
data  Exp  =  Lit Int
           |  Exp :+: Exp
           |  Exp :*: Exp

evalExp :: Exp -> Int
evalExp (Lit n)     =  n
evalExp (e :+: f)   =  evalExp e + evalExp f
evalExp (e :*: f)   =  evalExp e * evalExp f

showExp :: Exp -> String
showExp (Lit n)     =  show n
showExp (e :+: f)   =  par (showExp e ++ "+" ++ showExp f)
showExp (e :*: f)   =  par (showExp e ++ "*" ++ showExp f)

par :: String -> String
par s  =  "(" ++ s ++ ")"
```

Expression Trees, Symbols

```
e0, e1 :: Exp
e0 = Lit 2 :+: (Lit 3 :*: Lit 3)
e1 = (Lit 2 :+: Lit 3) :*: Lit 3

*Main> showExp e0
"(2+(3*3))"

*Main> evalExp e0
11

*Main> showExp e1
"((2+3)*3)"

*Main> evalExp e1
15
```

Part II

Propositions

Propositions

```
type Name = String
data Prop = Var Name
          | F
          | T
          | Not Prop
          | Prop :|: Prop
          | Prop :&: Prop
          deriving (Eq, Ord)

type Names = [Name]
type Env = [(Name,Bool)]
```

Showing a proposition

```
showProp :: Prop -> String
showProp (Var x)    =  x
showProp F          =  "F"
showProp T          =  "T"
showProp (Not p)    =  par ("~" ++ showProp p)
showProp (p :|: q)  =  par (showProp p ++ "|" ++ showProp q)
showProp (p :&: q)  =  par (showProp p ++ "&" ++ showProp q)

par :: String -> String
par s  =  "(" ++ s ++ ")"
```

Names in a proposition

```
names :: Prop -> Names
names (Var x)     =  [x]
names F           =  []
names T           =  []
names (Not p)     =  names p
names (p :|: q)   =  nub (names p ++ names q)
names (p :&: q)   =  nub (names p ++ names q)
```

Evaluating a proposition in an environment

```
eval :: Env -> Prop -> Bool
eval e (Var x)          =  lookUp e x
eval e F                =  False
eval e T                =  True
eval e (Not p)          =  not (eval e p)
eval e (p :|: q)        =  eval e p || eval e q
eval e (p :&: q)        =  eval e p && eval e q

lookUp :: Eq a => [(a,b)] -> a -> b
lookUp xys x  =  the [ y | (x',y) <- xys, x == x' ]
  where
  the [x]  =  x
```

Propositions

```
p0 :: Prop
p0 =  (Var "a" :&: Not (Var "a"))

e0 :: Env
e0 =  [("a",True)]

*Main> showProp p0
(a&(~a))

*Main> names p0
["a"]

*Main> eval e0 p0
False

*Main> lookUp e0 "a"
True
```

How eval works

```
eval e (Var x)          =  lookUp e x
eval e F                =  False
eval e T                =  True
eval e (Not p)          =  not (eval e p)
eval e (p :|: q)        =  eval e p || eval e q
eval e (p :&: q)        =  eval e p && eval e q
```

```
  eval e0 (Var "a" :&: Not (Var "a"))
=
  (eval e0 (Var "a")) && (eval e0 (Not (Var "a")))
=
  (lookup e0 "a") && (eval e0 (Not (Var "a")))
=
  True && (eval e0 (Not (Var "a")))
=
  True && (not (eval e0 (Var "a")))
= ... =
  True && False
=
  False
```

Propositions

```
p1 :: Prop
p1 =  (Var "a" :&: Var "b") :|:
      (Not (Var "a") :&: Not (Var "b"))

e1 :: Env
e1 =  [("a",False), ("b",False)]

*Main> showProp p1
((a&b)|((~a)&(~b)))

*Main> names p1
["a","b"]

*Main> eval e1 p1
True

*Main> lookUp e1 "a"
False
```

All possible environments

```
envs :: Names -> [Env]
envs []       =  [[]]
envs (x:xs)   =  [ (x,False):e | e <- envs xs ] ++
                 [ (x,True ):e | e <- envs xs ]
```

Alternative

```
envs :: Names -> [Env]
envs []       =  [[]]
envs (x:xs)   =  [ (x,b):e | b <- bs, e <- envs xs ]
  where
  bs = [False,True]
```

All possible environments

```
    envs []
=   [[]]

    envs ["b"]
=   [("b",False):[]] ++ [("b",True ):[]]
=   [[("b",False)],
     [("b",True )]]

    envs ["a","b"]
=   [("a",False):e | e <- envs ["b"] ] ++
    [("a",True ):e | e <- envs ["b"] ]
=   [("a",False):[("b",False)],("a",False):[("b",True )]] ++
    [("a",True ):[("b",False)],("a",True ):[("b",True )]]
=   [[("a",False),("b",False)],
     [("a",False),("b",True )],
     [("a",True ),("b",False)],
     [("a",True ),("b",True )]]
```

Satisfiable

```
satisfiable :: Prop -> Bool
satisfiable p  =  or [ eval e p | e <- envs (names p) ]
```

Propositions

```
p1 :: Prop
p1 =  (Var "a" :&: Var "b") :|:
      (Not (Var "a") :&: Not (Var "b"))

*Main> envs (names p1)
[[("a",False),("b",False)],
 [("a",False),("b",True)],
 [("a",True ),("b",False)],
 [("a",True ),("b",True )]]

*Main> [ eval e p1 | e <- envs (names p1) ]
[True,
 False,
 False,
 True]

*Main> satisfiable p1
True
```

Part III

Maybe

The Maybe type

```
data Maybe a = Nothing | Just a
```

Optional argument

```
power :: Maybe Int -> Int -> Int
power Nothing n    =  2 ^ n
power (Just m) n   =  m ^ n
```

Optional result

```
divide :: Int -> Int -> Maybe Int
divide n 0  =  Nothing
divide n m  =  Just (n `div` m)
```

Using an Optional Result

```
divide :: Int -> Int -> Maybe Int
divide n 0  =  Nothing
divide n m  =  Just (n `div` m)


wrong :: Int -> Int -> Int
wrong n m  =  divide n m + 3


right :: Int -> Int -> Int
right n m  =  case divide n m of
                Nothing -> 3
                Just r -> r + 3
```

Part IV

Union of Two Types

Either a or b

```
data Either a b = Left a | Right b


mylist :: [Either Int String]
mylist  =  [Left 4, Left 1, Right "hello", Left 2,
            Right " ", Right "world", Left 17]


addints :: [Either Int String] -> Int
addints []                  =  0
addints (Left n : xs)       =  n + addints xs
addints (Right s : xs)      =  addints xs


addints' :: [Either Int String] -> Int
addints' xs  =  sum [n | Left n <- xs]
```

Either a or b

```
data Either a b = Left a | Right b


mylist :: [Either Int String]
mylist  =  [Left 4, Left 1, Right "hello", Left 2,
            Right " ", Right "world", Left 17]


addstrs :: [Either Int String] -> String
addstrs []                =  ""
addstrs (Left n : xs)     =  addstrs xs
addstrs (Right s : xs)    =  s ++ addstrs xs


addstrs' :: [Either Int String] -> String
addstrs' xs  =  concat [s | Right s <- xs]
```

Part V

Aside:

All sublists of a list

All sublists of a list

```
subs :: [a] -> [[a]]
subs []       =  [[]]
subs (x:xs)   =  subs xs ++ [ x:ys | ys <- subs xs ]
```

All sublists of a list

```
     subs []
=    [[]]

     subs ["b"]
=    subs [] ++ ["b":ys | ys <- subs []]
=    [[]] ++ ["b":[]]
=    [[], ["b"]]

     subs ["a","b"]
=    subs ["b"] ++ ["a":ys | ys <- subs ["b"]]
=    [[], ["b"]] ++ ["a":[], "a":["b"]]
=    [[], ["b"], ["a"], ["a","b"]]
```

Part VI

The Universal Type and Micro-Haskell

The Universal Type and Micro-Haskell

```
data  Univ  =  UBool Bool
            |  UInt Int
            |  UList [Univ]
            |  UFun (Univ -> Univ)

data  Hask  =  HTrue
            |  HFalse
            |  HIf Hask Hask Hask
            |  HLit Int
            |  HEq Hask Hask
            |  HAdd Hask Hask
            |  HVar Name
            |  HLam Name Hask
            |  HApp Hask Hask

type  HEnv  =  [(Name, Univ)]
```

Show and Equality for Universal Type

```
showUniv :: Univ -> String
showUniv (UBool b)    =   show b
showUniv (UInt i)     =   show i
showUniv (UList us)   =
  "[" ++ concat (intersperse "," (map showUniv us)) ++ "]"

eqUniv :: Univ -> Univ -> Bool
eqUniv (UBool b) (UBool c)     =   b == c
eqUniv (UInt i) (UInt j)       =   i == j
eqUniv (UList us) (UList vs)   =
  and [ eqUniv u v | (u,v) <- zip us vs ]
```

Can't show functions or test them for equality.

Micro-Haskell in Haskell

```
hEval :: Hask -> HEnv -> Univ
hEval HTrue r          =  UBool True
hEval HFalse r         =  UBool False
hEval (HIf c d e) r    =
  hif (hEval c r) (hEval d r) (hEval e r)
  where  hif (UBool b) v w  =  if b then v else w
hEval (HLit i) r       =  UInt i
hEval (HEq d e) r      =  heq (hEval d r) (hEval e r)
  where  heq (UInt i) (UInt j) = UBool (i == j)
hEval (HAdd d e) r     =  hadd (hEval d r) (hEval e r)
  where  hadd (UInt i) (UInt j) = UInt (i + j)
hEval (HVar x) r       =  lookUp r x
hEval (HLam x e) r     =  UFun (\ v -> hEval e ((x,v):r))
hEval (HApp d e) r     =  happ (hEval d r) (hEval e r)
  where  happ (UFun f) v  =  f v

lookUp :: HEnv -> Name -> Univ
lookUp x r  =  head [ v | (y,v) <- r, x == y ]
```

Test data

```
h0 =
 (HApp
   (HApp
     (HLam "x" (HLam "y" (HAdd (HVar "x") (HVar "y"))))
     (HLit 3))
   (HLit 4))

test_h0 = eqUniv (hEval h0 []) (UInt 7)
```