Hackenbush, Nim and Impartial games: How we can understand winning strategies

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Abstract

Over the course of this essay we will develop methods and techniques to help us simplify and understand combinatorial games – mainly Nim and Hackenbush. We will build up these techniques and eventually use these to help us predict who would win in a complex game of Hackenbush if two rational players face each other.
These techniques will include an understanding of Nimbers, seeing how we can apply Nimbers to all combinatorial games using the Sprague Grundy theorem, and then specifically applying these ideas to Hackenbush through four principles to simplify all Hackenbush positions.

Contents

1 Introduction

Before we start anything, it's probably a good idea to formally define a game.

Definition 1.1. *A combinatorial game[7] consists of two players and contains the following:*

- *A win condition. This could also be defined as not attaining the lose condition.*
- *A set of positions. Positions are the state a game is in. For example a chess board right before we start the game is a position, as is any board that may occur during the game.*
- *A set of rules that define for each and every position what positions each player can move to*

Hackenbush and Nim are two rather different looking games, involving two players. The game of Nim involves us removing objects from one of a set of piles, whereas Hackenbush involves deleting edges of a rooted graph. On first glance one would assume these games share very little in common. However the opposite is true.
Over the course of this Essay we'll encounter the ordinal numbers and via the Sprague-Grundy theorem we'll see how they allow us to classify the status of a whole range of games. Using this and strategies we develop in Nim (a rather simple game) we can create a powerful set of tools to allow us to analyse all impartial games. In the case of Hackenbush, this help will be four theorems that allow us to manipulate any Hackenbush game into a game clearly isomorphic to a Nim game. This will preserve enough information to be able predict which player will win if both know the winning strategy and can help us make winning moves in a game of Hackenbush.

2 A quick aside

The binary expansion of a number is a way to represent it as a sum of unique powers of 2. For this essay our ability to represent any number with a unique binary expansion is very important.

2.1 Binary expansion

Lemma 2.1. *$\forall n \in \mathbb{N}$:*

$$2^0 + 2^1 + \cdots + 2^n = \sum_{k=0}^{n} 2^k = 2^{n+1} - 1$$

Proof.

$$\sum_{k=0}^{n} 2^k = 2\sum_{k=0}^{n} 2^k - \sum_{k=0}^{n} 2^k = \sum_{k=1}^{n+1} 2^k - \sum_{k=0}^{n} 2^k$$
$$= (2 + 4 + ... + 2^n + 2^{n+1}) - (1 + 2 + 4 + ... + 2^n) = 2^{n+1} - 1 \qquad \square$$

Lemma 2.2. *$\forall n \in \mathbb{N}, \exists$ a unique binary expansion of n [1]*

Proof. Obviously the natural numbers $0, 1, 2$ have unique binary representations $0, 2^0, 2^1$ respectively. Assume there exists a unique binary for all natural numbers up to and including $2^k - 1$. Let's show from here it follows that there exists a unique binary expansion for all natural numbers up to and including $2^{k+1} - 1$. Then we can use induction to continue this process forever and show that all numbers have a unique binary expansion. Say we have a natural number M, such that:

$$2^k \leq M \leq 2^{k+1} - 1$$
$$\therefore M - 2^k < 2^k$$

Therefore $M - 2^k$ has a binary representation[1], where all the powers of 2 are less than 2^k. As a result $M - 2^k + 2^k = M$ has a unique binary representation, as does every natural number in the interval $[2^k, 2^{k+1} - 1]$. Therefore by induction every natural number has a unique binary representation. $\square$

2.2 Positions in impartial games

We can denote any position P in any impartial game of 2 players with the following notation: [7]

P=$(a_1, a_2, \ldots, a_x | b_1, b_2, \ldots, b_y)$

Where a_n represents a position player a can move to. Positions in two player impartial games come in one of two types:

- Type N: the next player will win if they are completely rational

[1] We simply just use the new digit of 2^k

- Type P: the previous player will win if they are completely rational

Definition 2.1. *Impartial game [6]*
Let A denote all the positions player 1 can move to and B all the positions player 2 can move to. A game is impartial if $\forall$ positions in the game $P = (A|B)$ implies that $A = B$

Remark. *In layman's terms, both players have the same abilities or moveset.*

Now, we've established some useful results and know what impartial games are. We would obviously like to develop some kind of tool-set to help us understand and manipulate these games. What we will do is try to categorise and simplify all positions using the Sprague Grundy theorem along with other methods specific to Hackenbush. This will allow us to manipulate and understand Hackenbush and Nim games, and to appreciate a relation between these two seemingly unrelated games. To truly start our journey let's consider the game of Nim.

3 Nim

3.1 How does one play Nim?

In the game there are some distinct piles of objects. On their turn, a player may remove any number of objects from one pile, but they have to remove at least one object. The first player to not be able to take an object is the loser [2]. So clearly it would be nice to be able to categorize all the information present within the game at a given turn and then be able to use that to our advantage. But first let's think about a simple Nim example. What happens when we play a game where we start out with two equal piles, with completely rational players?

Theorem 3.1. *If player 2 is rational, they will always win if we start the game with 2 equally sized piles*

Proof. Clearly with 2 piles of size 0 player 2 wins. Let's assume this is true for all cases of 2 equally sized piles up to and including 2 of size n. What about 2 piles each of size $n+1$? By the rules of the game player 1 has to take at least 1 object, leaving us with piles of size a and $n+1$, where a is a natural number less than or equal to n. Player 2 can simply mirror what player 1 did to the other pile, giving us 2 piles of size a, and as $a \leq n$, we know player 2 will win the whole game from this position, due to our assumption. Therefore by induction player 2 will always win in the case of 2 equally sized piles. □

One thing to note, is now if we ever reach a point where two rational players have two equal piles[3] in a game we can now predict what will happen, as in Nim (and Hackenbush), the history of the game does not affect future player

[2] Please notice that this is not the same as the last player to take an object being a winner, e.g: consider starting position where there are no objects.

[3] Or a position fundamentally equivalent to the players having 2 equal piles

decisions. Therefore playing moves that create two equal piles could form part of a basic strategy, and would be very effective against an irrational player. For example if one is playing this game in a pub.

3.2 Nimbers - What are they?

This is all well and good, but clearly some kind of notation is necessary to help us think about Nim. We often think of numbers in two ways, as cardinal numbers or (less commonly), as ordinal numbers [2]:

Definition 3.1. *Here we will define the ordinal numbers recursively. Firstly:*
$0 = \{\} = \emptyset$
$\forall$ *ordinal numbers* x, $x + 1 = x \cup \{x\}$. *Where* $x \cup \{x\}$ *is well ordered.*

Similar to cardinal numbers, ordinal numbers have addition and equivalence relations. However we will also define other binary operations on them.

Nimbers are a type of ordinal number, that we can use to denote the amount of objects within a pile, denoted by $*n$ for a pile with n objects [3]. What makes Nimbers different to regular ordinal numbers is the idea of the Nim sum.

3.2.1 Nim addition

We can easily represent a single pile Nim game of n objects by the single Nimber $*n$. From here we can define a notion of adding Nim piles together to create the game of Nim described. When we add piles together each player can choose to make a move in any singular pile. Therefore the expression:

$*4 + *11 + *13$

Represents a Nim game with three piles of 4, 11 and 13.
In fact we aren't just limited to adding singular piles together, but entire games of Nim to each other. This would simply result in a larger game of Nim consisting of the piles from both games. We can even broaden this further and add completely unrelated impartial games to each other, with us simply choosing which game to make a move in each turn. [10].

3.2.2 The Nim sum

The Nim sum is another word for binary XOR addition on 2 numbers. Another way of describing this is as adding the numbers in binary together without carrying the digits. This is often denoted with the symbol $\bigoplus$ [3].

Theorem 3.2. *For all Nimbers* $*a, *b$
$*a + *b = *(a \bigoplus b)$

Lemma 3.3. *In fact if a set of games (or piles) are equivalent to Nimbers* $*a_0 + *a_1 + ... + *a_n$, *addition of these games will give a Nimber equal to* $*(a_0 \bigoplus a_1 \bigoplus ... \bigoplus a_n)$. *[7]*

Proof of this lemma is simple based on the theorem above.
Now we've established enough notation to start to analyse our game a bit further. We will now develop the idea of positions to help us predict who will win in a game of Nim. But first we will need a notion of positions being equal:

3.2.3 Equivalence

Definition 3.2. *2 positions P and P' in 2 games are equivalent if and only if for all positions Q in the games, the 2 positions P+Q and P'+Q are both of the same type (N or P).*

This clearly satisfies the notions of transitivity, symmetry and reflexivity we expect from an equivalence relation. Nimbers and the Nim sum allow us to further categorize Nim positions. Every position in Nim is one of 2 types:

- Balanced
- Unbalanced

A position is balanced if the Nim sum of the piles is 0. Otherwise it is unbalanced.

Theorem 3.4. *If we start with a balanced position the next position must be unbalanced.*

Proof. Start with a position consisting of n piles of sizes $a_1, a_2, ...a_n$. Without loss of generality say we remove items from the pile of size a_n to form a pile of size $a_{n'}$. We know $*a_1 \bigoplus *a_2 \bigoplus ... \bigoplus *a_n = 0$. Suppose $*a_1 \bigoplus *a_2 \bigoplus ... \bigoplus *a_{n-1} \bigoplus *a'_n = 0$. We know by the definition of XOR addition that $*a \bigoplus *b = 0$ can only hold if $*a = *b$. Therefore $*a_1 \bigoplus *a_2 \bigoplus ... \bigoplus *a_{n-1} = *a_n$, as $*a_1 \bigoplus *a_2 \bigoplus ... \bigoplus *a_{n-1} \bigoplus *a_n = 0$. Therefore as $*a_1 \bigoplus *a_2 \bigoplus ... \bigoplus *a_{n-1} \neq *a'_n$
$*a_1 \bigoplus *a_2 \bigoplus ... \bigoplus *a_{n-1} * a_{n'} \neq 0$. ☐

These 2 states are useful to understand to help us predict who will win. But it would be useful if we had an easy way to make the next position balanced if we start with an unbalanced position.

Theorem 3.5. *We can balance an unbalanced game by following this set proceedure:*

- *Consider the binary expansion of each pile and break each pile down into subpiles of size 2^n using our binary expansion*[4]*. We can do this as every number has a singular unique binary expansion.*
- *Start writing down a running total T for how many objects need to be removed*

[4] E.g: a pile of size 7, would have subpiles of size 1, 2 and 4

- *Find the largest n for 2^n for which there's an odd amount of subpiles of that size. We will call this value of n N. Label this pile A.*
- *Add the amount of objects in A, to our running total T*
- *Now pretend A doesn't exist in our current Nim game. For every value of $n \leq N$ where there is an odd number of subpiles of size 2^n remove 2^n from the total*
- *Remove this total from pile A*

Proof. Clearly the balancing procedure results in a legal well defined move, as $0 \leq T \leq a$. $T \leq a$, is fairly obvious. $0 \leq T$ is true as $\sum_{k=0}^{N-1} 2^k = 2 \leq 2^N$, so even if we removed all the subpiles of size smaller than 2^N, $0 \leq T$. Now we just need to check the procedure gives us a balanced game.
The balancing procedure will result in a new pile A', which was $A - T$. By construction $A - T$ can only contain all the subpiles that appear an odd number of times if we consider our Nim game without A or A'. Considering the Nim sum[5] of all the piles we will clearly get 0. Therefore the game is balanced. □

In fact if we have only two piles a and b, the game will only be balanced if $a = b$. This is because each number has 1 unique binary representation, so if $a \neq b$ their binary expansions will be different, and thus there won't be an even number of subpiles for each power of 2. This means their Nim sum cannot be 0. This is all well and good, but why do we care about balanced and unbalanced positions?

Theorem 3.6. *If we assume every player is rational, every balanced position is type P and every unbalanced position is type N. [7]*

Proof. Consider a balanced position. The first player will have to make it an unbalanced position. The second player can balance the position using our procedure. Player 1 is forced to unbalance the position and player 2 can rebalance it. If Player 2's strategy is to apply the balancing procedure, regardless of the history of the game, this cycle will continue indefinitely. The 0 position[6] is clearly balanced and player 2 will thus be the one to move to it. Therefore with rational players all balanced positions are type P.
We can do a similar procedure with unbalanced positions, where we start with an unbalanced position and player 1's strategy is to apply the balancing procedure regardless of the history of the game, but obviously our method will result in Player 1 winning if they're rational. Implying all unbalanced positions are of type N. □

We've found a useful set of tools to help us understand Nim and easily predict who will win if we have 2 rational players. These are incredibly useful

[5]Remember this is the same as binary XOR addition

[6]This is the position where all piles have 0 objects, or we have 0 piles. It is not hard to see that regardless of how many piles one has all of these are equivalent

tools, which we will later see we can even apply to other games like Hackenbush. Next we are going to expand on the idea of Nimbers and show how these ideas can be applied to any impartial combinatorial game with the Sprague Grundy theorem.

4 Simplifying and translating games

In this section, we'll look at Sprague Grudny theorem, which states that every position in any impartial game can be represented as a Nimber. This allows us to apply what we've learnt about Nimbers to any impartial game. However before we can prove the Sprague Grundy theorem, we must prove the MEX principle.

4.1 MEX principle

Definition 4.1. *The MEX: [3] [7]*
The MEX of a subset of a well-ordered set [7] *refers to smallest value belonging to a set, but not to a subset.*

Theorem 4.1. *The MEX principle: [3] [7]*
Suppose our game is in position α, with the ability to move to positions $\alpha_1, \alpha_2, \ldots, \alpha_n$. Suppose that:

$\alpha_i \equiv *\alpha_i$ *for all* $1 \leq i \leq n$, *where* i *and* n *are natural numbers.*

It is true that $\alpha \equiv *\alpha_0$, *where* α_0 *is the MEX of the subset* $\alpha_1, \alpha_2, \ldots, \alpha_n$ *with respect to the set of natural numbers.*

Proof. Consider the game $\alpha + *\alpha_0$. We consider the cases of player 1 taking from the first pile or player 1 taking from the second pile, from the perspective of the 2nd player.
If player 1 moves from the Nimber $*\alpha_0$ to $*\alpha_0'$, where $\alpha_0 > \alpha_0'$, then by the fact α_0 is our MEX α_0', must be one of $\alpha_1, \alpha_2, \ldots, \alpha_n$. Now the second player can move α to match the other pile. We now have a case of 2 equal piles and thus player 2 will win.
If player 1 moves α to some α_i, where $\alpha_i \in (\alpha_1, \alpha_2, \ldots, \alpha_n)$. Clearly as α_0 is the MEX of that set $\alpha_0 \neq \alpha_i$ and thus the overall position $P \neq *0$ and therefore we have a type N position and player 2 will win.
Either way player 2 will win and thus $\alpha + *\alpha_0$ is type P and thus equivalent to $*0$. Therefore $\alpha \equiv *\alpha_0$. □

This seems like an incredibly powerful tool for a game like Nim, where we've shown each position can be reduced to a single Nimber, but what about other other impartial games? Well luckily one of the crown jewels of combinatorial game theory can help us.

[7] The ordinal numbers (and thus the Nimbers) are well-ordered

4.2 Sprague Grundy Theorem

Theorem 4.2. *The* Sprague Grundy Theorem *states every position in any impartial game is equivalent to a single Nimber. [7] [6] [4]*

Proof. Let's prove this by induction.
For our base case let's have a position P where the first player immediately loses. Clearly $P \equiv *0$
For our inductive step, let's assume Sprague Grundy holds $\forall$ all positions, where the maximum number of moves needed to end the game is less than some n. We will show it holds $\forall$ positions P where the maximum number of moves needed to end the game is less than some $n + 1$. All positions we can move to from P clearly have Nimbers associated with them. Therefore we can apply the MEX principle and we see that the MEX M of the set of positions we can move to P. Therefore $P \equiv *M$ $\square$

This actually allows us a generalisation of the case of two piles.

Theorem 4.3. *The case of 2 equal games*
If we have a position, which is the sum of 2 equal positions, the second player will always win.

Proof. The 2 equal games are identical and thus have the same Nimber value (they must have a Nimber value by Sprague Grundy), and this overall game is thus the same as a game of Nim with 2 equal piles. $\square$

Theorem 4.4. *Simplifying Nim piles:*
Each Nim pile can be converted to its corresponding Nimber, and an overall associated Nimber can be found for the game via Nim addition.

For example:
If we have piles $*4 + *11 + *13$, then this game is equivalent to $4 \bigoplus 11 \bigoplus 13 = 2$

Nim is all well and good, but the game is ultimately rather simplistic and not very interesting. That is why we look at the far more interesting game of Hackenbush.

5 Hackenbush

5.1 How does one play Hackenbush

[9] [4] In Hackenbush[8] we consider a series of rooted[9] graphs[10]. Each player must remove one edge from the graph. Any other edges that are not connected to a rooted part of the graph are immediately removed after any move. Similar to Nim the first player who can't make a legal move loses.

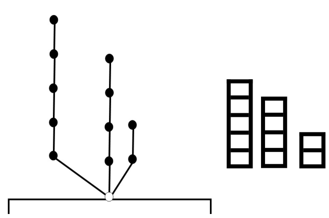
As Hackenbush is a partisan combinatorial game, we can use a similar notion of positions. In fact every Hackenbush position is equivalent to a Nim position, as given by Sprague Grundy theorem. If we can translate every position in Hackenbush to Nim we can apply the same idea of balancing an unbalanced game to gain a winning strategy. However, a look at an illustrated Hackenbush position may make us wonder how exactly we can do that.

You may be happy to hear there are some tools we can develop to simplify the game. And once we have these tools, we can use all our knowledge of Nim to help us win any game of Hackenbush.

5.2 Simplifying Hackenbush positions

Theorem 5.1. *If we have a game of Hackenbush consisting of multiple rooted paths, with one root at a vertex of degree 1. This game is functionally identical to a game of Nim, where each Nim pile has the same amount of objects as each path has vertices. [11] [4]*

This may be hard to see so have a look at the image below:



Proof. To see this consider what happens if we remove a set number of objects n of a Nim pile. This would correspond to creating a cut on the path of a set distance from the root, such that n edges are deleted. In both games we can remove as many edges/objects as we want, but only from one path/pile. The first player with nothing to remove loses. Therefor the moves are functionally identical. □

[8] I shall primarily be discussing the variation of the game referred to in many texts as green Hackenbush. However for this text I shall use the terms interchangeably. If I am referring to a different variation of the game this will be made clear.

[9] Multiple roots are allowed

[10] This does not have to be a simple graph, but must be undirected

Theorem 5.2. *Any edge that connects back to the same node (often referred to as a loop), can simply be replaced by an edge and node connected to the original node. These 2 positions will be equivalent. [11] [4]*

Proof. A single loop requires 1 cut to eliminate and leads onto nothing. Hence if you cut the loop nothing else will be deleted. This is functionally identical to adding a vertex and a single edge. Nothing about gameplay has changed, and thus the 2 positions are identical, as a position is defined in terms of what other positions we can reach from it. □

Theorem 5.3. *The Colon principle*
Third if we have a branching point, where each branch following is a "straight line of edges and nodes" (there are no other branches coming off), then this can be simplified. This Hackenbush position will be equivalent to one with a singular branch coming off that point, where the singular branch is the Nim sum of all the other branches coming out. By Looking at the "highest" branch points first and then going down towards the root, we can reduce the tree to a single Nim value. [11] [4]

Proof. We can think of the branches coming off of the node as a sub graph, with the branching node being a root of sorts. Obviously these 2 branches are essentially separate, as 1 cut on either branch cannot effect both. Therefore we can use our translation between Hackenbush where we just have paths and Nim, discussed in the first point, to reduce both of these to Nim piles and thus Nimbers. Therefore they can be represented by a singular Nimber, or a singular path in Hackenbush and can thus be reduced to one branch. □

These are all well and good and allows us to completely simplify trees. However most graphs in a game of Hackenbush contain circuits, so what can we do?

Theorem 5.4. *The Fusion Principle*
We can drag and fuse multiple nodes together. When fusing 2 nodes the edges joining them form loops, whilst they keep their connections to any other external nodes. This new position will be equivalent to the original. [11] [4] [6]

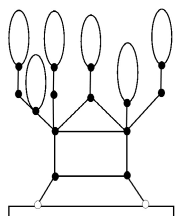
The proof of this theorem is far too long and complex for this essay, but two unique proofs can be found in the books 'On Numbers and Games'[6][11], as well as 'Winning ways for your mathematical plays, volume 1'[4] [12].

By applying theorems 5.1-5.4 we now have a powerful tool-set which will reduce any green Hackenbush position into a set of stalks and thus Nimbers. Therefore from here we can use all our strategies from Nim to help us predict who would win a game of Hackenbush.

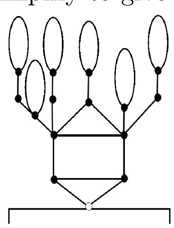
[11] This proof relies on knowledge of the Welter function
[12] In this proof the author tries to create a counter-example

5.2.1 Example of simplifying a Hackenbush game

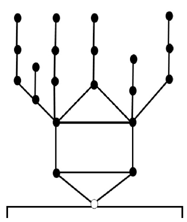
Let's first consider this position[13] of Hackenbush and what we can do to simplify it.



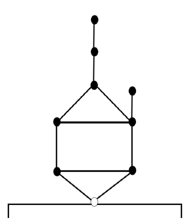
The first thing you might notice is that we have 2 ground nodes, so we can simplify to give us the following image.



We also have loops, which can be simplified to new nodes with edges connecting them.

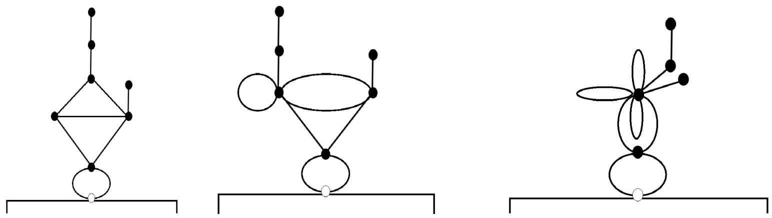


Then at the top of the image, we have these branching structures, that can be simplified via repeated application of the Colon principle and the Nim sum.

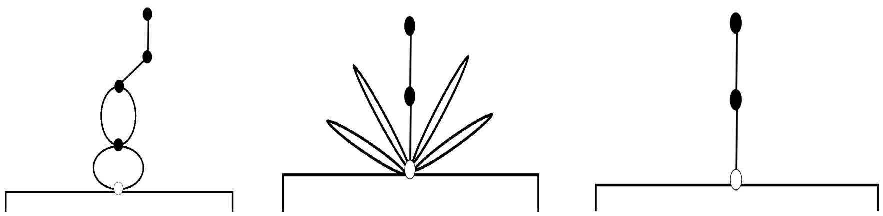


From here, we can repeated apply the fusion rule.

[13]This position was meant to resemble the house from the film Up, but in reality looks far from it



After this we can simplify the loops and the apply the colon rule, then the fusion rule twice. We then simplify the loops and apply the colon rule again.



As you can see we final reach this path of 2 edges and thus the Hackenbush drawing can be assigned Nimber value $*2$. Therefore the starting position is unbalanced, so if both players play completely rationally, Player 1 will win.

5.3 How might Player 2 win?

To think about this, we want to think about a corresponding Nim position at each Hackenbush position and apply the same logic of balancing and unbalancing to see how we could hope to win.
The only way Player 2 could ever hope to win would be by playing against an irrational player. Any move you do to a balanced game will unbalance it and thus as long as Player 1 balances the game, Player 2 will be forced into a position to unbalance the game and Player 1 can continue balancing the game until victory. However if Player 1 fails to balance the game at any point, then Player 2 can adopt this strategy and win. Another idea might be for Player 2 to suggest a game where we duplicate the starting position.

6 Using Nim to understand Hackenbush

As we saw in the last section, by applying our ideas of balancing and unbalancing we can win. But how do we apply the procedure easily to a game of Hackenbush?

- first label all the edges of our game
- we repeatedly apply the fusion principle and simplify all loops (keeping track what vertices the edges originally connected to), we'll get a tree connected from the root node. Note, no edges will have been deleted.
- From here we can apply the colon rule, applied to everywhere the tree branches, except at the root, to get a series of stalks coming from the

root. This results in a Nim position and from here note down what we'd do to balance the position.

- Then look at the position before we simplified this position via the colon rule. To the branch we'd use to balance the position consider what edge we'd remove to give us the same Nimber value as the stalk after being balanced. We can then see where this cut is in our original position (as the edges are labelled) and perform a cut.

This will 'balance' the Hackenbush game.

This may not be something one could easily do when playing a game, but for creating an A.I player this would be ideal.

Although studying these games may not have any real world direct applications, these methods, and ways of thinking often find real world use. For example notation for partisan games of Hackenbush was later found to be a powerful tool in data storage, referred to as Hackenbush numbers. These are an occasionally useful alternative to floating point numbers. [5]

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