1 Introduction

2 Method

The pendulum is set to follow an angular reference given as a sinusoid

$$\theta^* = A\sin(2\pi f t + \varphi), \tag{1}$$

where the tracking is achieved by a PD - controller, giving the closed-loop system

$$\ddot{\theta} + \zeta\omega_0\dot{\theta} + \omega_0^2\theta = \frac{K_m}{I_f}\left[K_p(\theta^* - \theta) + K_d(\dot{\theta}^* - \dot{\theta})\right] \tag{2}$$

where

$$\omega_0 = \sqrt{\frac{m_f g l_f}{I_f}} \tag{3}$$

is the undamped resonance frequency, ζ is a friction term and K_m is the assumed constant gain of the motor. Rearranging the equation gives

$$\ddot{\theta} + \left(\zeta\omega_0 + \frac{K_m K_d}{I_f}\right)\dot{\theta} + \left(\frac{m_f g l_f + K_m K_p}{I_f}\right)\theta = \frac{K_m}{I_f}\left[K_p\theta^* + K_d\dot{\theta}^*\right] \tag{4}$$

where the resonance frequency is now a function of the proportional control term K_p and motor gain K_m

$$\omega_{K_P}^2 = \left(\omega_0^2 + \frac{K_m K_p}{I_f}\right) = \left(\frac{m_f g l_f + K_m K_p}{I_f}\right) \tag{5}$$

The physical parameters of the pendulum can be further manipulated by attaching weights to each of the legs at fixed known distances from the center of rotation. These weights contributes at torque and moment of inertia according to their attachment point.

$$\begin{aligned} I_w &= J_w + (m_w + m_f)(l_s + l_h/2 + (n_w - 1)l_h)^2 \\ \tau_w &= (m_w + m_f)g(l_s + l_h/2 + (n_w - 1)l_h) \end{aligned} \tag{6}$$

where $n_w \in [0, 8]$ is the mounting hole, l_s is the distance to the first mounting hole, l_h is the distance between mounting holes, m_w and m_f is the mass of the weights and fixtures, and J_w is the moment of inertia around the weights center of gravity. The resonance frequency can then be adjusted as

Table 1: Table of experiments

n_w / K_p	6	7	8	9	10	11	12	15
(0,0)		x	x	x	x	x	x	x
(1,1)			x					
(8,8)			x	x	x			
(5,5)			x	x	x			
(2,8)			x	x	x			
(3,7)	x		x	x	x		x	
(1,0)			x					
(4,0)			x					
(7,0)			x					

$$\omega_{K_P}^2 = \left(\frac{m_f g l_f + \tau_w + K_m K_p}{I_f + I_w}\right) \tag{7}$$

where $I_w = \tau_w = 0$ if $n_w = 0$.

Multiple experiments where then carried out with different K_p and n_w, and each experiment was ran 3 times to increase statistical significance The experiments that were performed with ≥ 3 different K_p values were used to estimate the motor gain K_m by treating $\omega_{K_p}^2$ as a linear function

$$\begin{gathered} \omega_{K_P}^2 = \left(\frac{m_f g l_f + \tau_w + K_m K_p}{I_f + I_w}\right) \\ y = ax + b \\ a = \frac{K_m}{I_f + I_w}, \quad x = K_p, \quad y = \omega_{K_P}^2, \quad b = \left(\frac{m_f g l_f + \tau_w}{I_f + I_w}\right) \end{gathered} \tag{8}$$

After obtaining the lumped expression for $\frac{K_m}{I_f + I_w}$, the remaining parameters $m_f g l_f$ and I_f could be found by rearranging (7) as

$$\begin{gathered} \omega_{K_P}^2 = \left(\frac{m_f g l_f + \tau_w}{I_f + I_w} + aK_p\right) \\ \begin{bmatrix} 1 & -\omega_{K_P}^2 + aK_p \end{bmatrix} \begin{bmatrix} m_f g l_f \\ I_f \end{bmatrix} = \omega_{K_P}^2 I_w - \tau_w - aK_p I_w \end{gathered} \tag{9}$$

for each experiment in Table 1, and constructing an overdetermined set of equations on the form

$$Ax = b \tag{10}$$

where $x = [m_f g l_f, I_f]^T$, and the solution is given by least-squares fitting as

$$x = \left(A^T A\right)^{-1} b. \tag{11}$$

3 Results

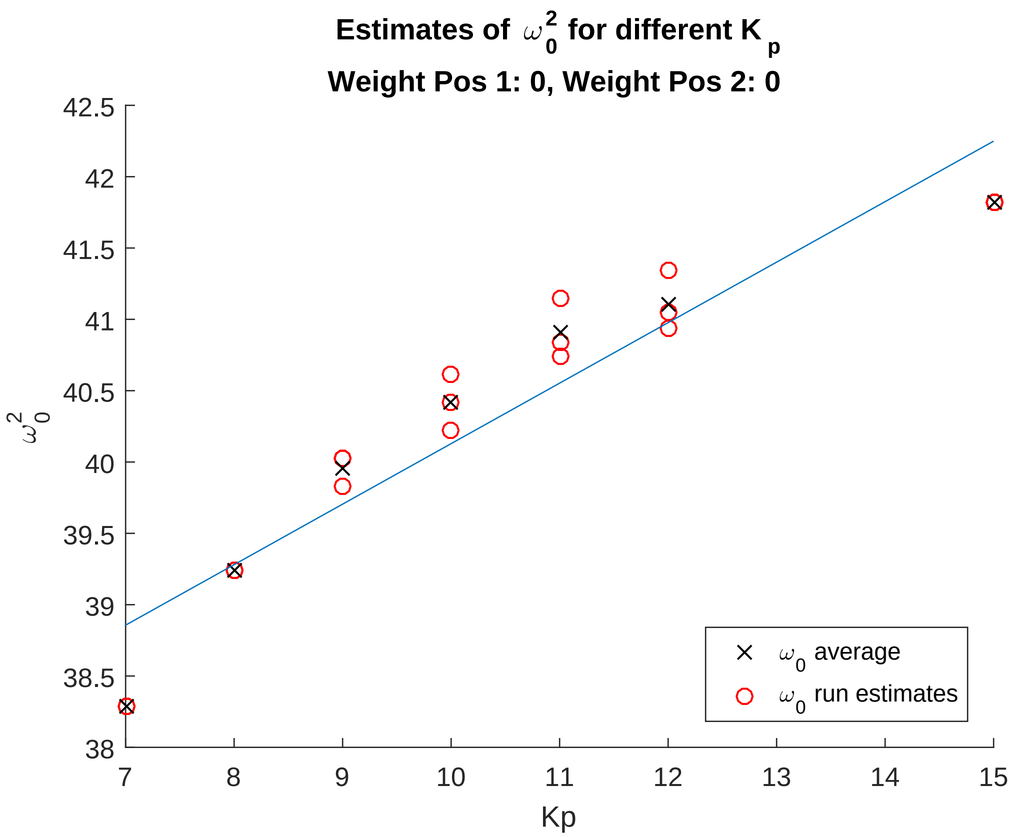

Figure 1: Estimated linear fit for $\frac{K_m}{I_f + I_w}$ parameter for experiments without weights attached.

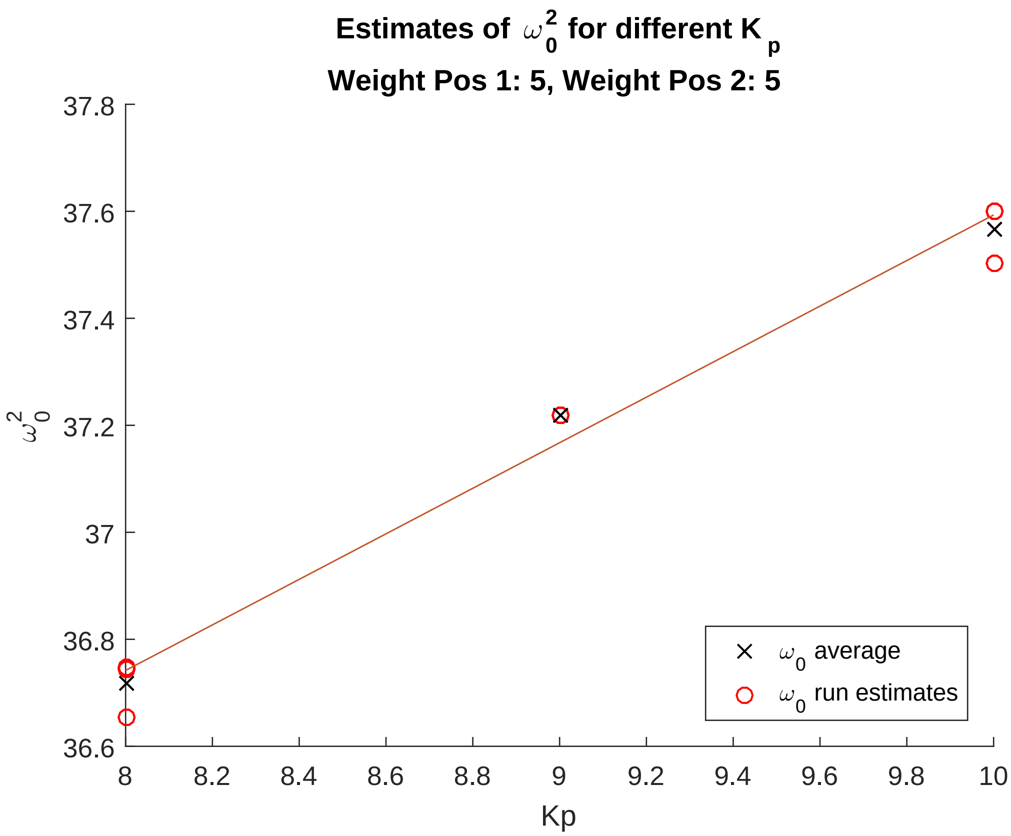

Figure 2: Estimated linear fit for $\frac{K_m}{I_f+I_w}$ parameter for experiments with weights attached at $n_w = (5,5)$.

Table 2: Motor gain estimates

n_w	$\frac{K_m}{I_f+I_w}$
(0,0)	0.424071985869166
(8,8)	0.310548831022341
(5,5)	0.425654308341756
(2,8)	0.374895472210993
(3,7)	0.303311751403483

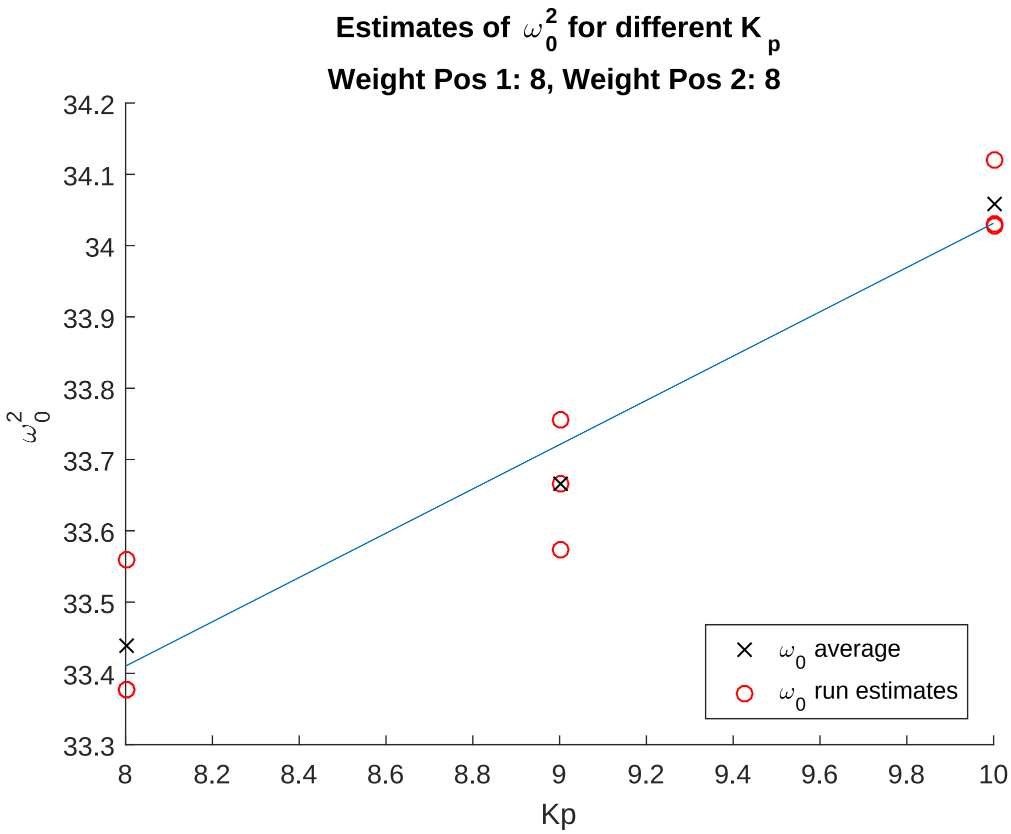

Figure 3: Estimated linear fit for $\frac{K_m}{I_f+I_w}$ parameter for experiments with weights attached at $n_w = (8, 8)$.

Table 3: Parameter estimates

$m_f g l_f$	7.213988198586391
I_f	0.200275108832547
K_m	0.096860674406427

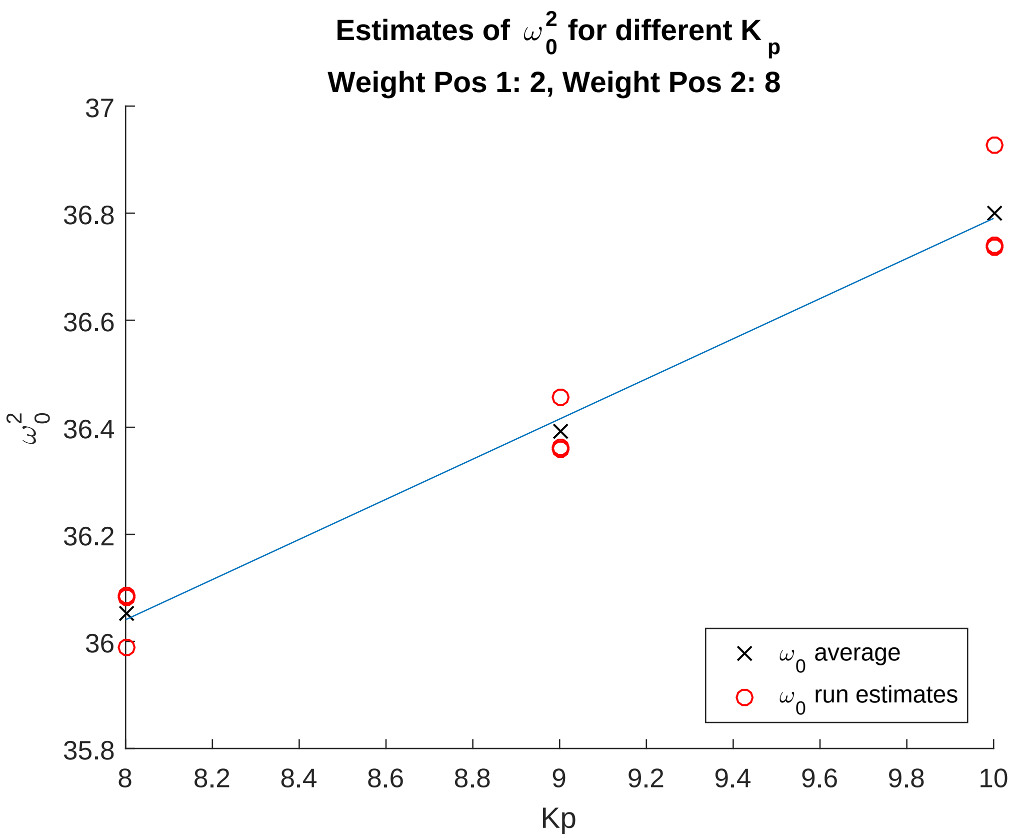

Figure 4: Estimated linear fit for $\frac{K_m}{I_f+I_w}$ parameter for experiments with weights attached at $n_w = (2, 8)$.

Table 4: Verification experiments

n_w	ω_0^*	ω_0	error
(1,1)	6.341697809300672	6.375158252674985	-0.033460443374313
(1,0)	6.297273096333526	6.347945772068390	-0.050672675734864
(4,0)	6.182221951346674	6.221128813138584	-0.038906861791911
(7,0)	6.022659862653705	6.037844301505593	-0.015184438851888

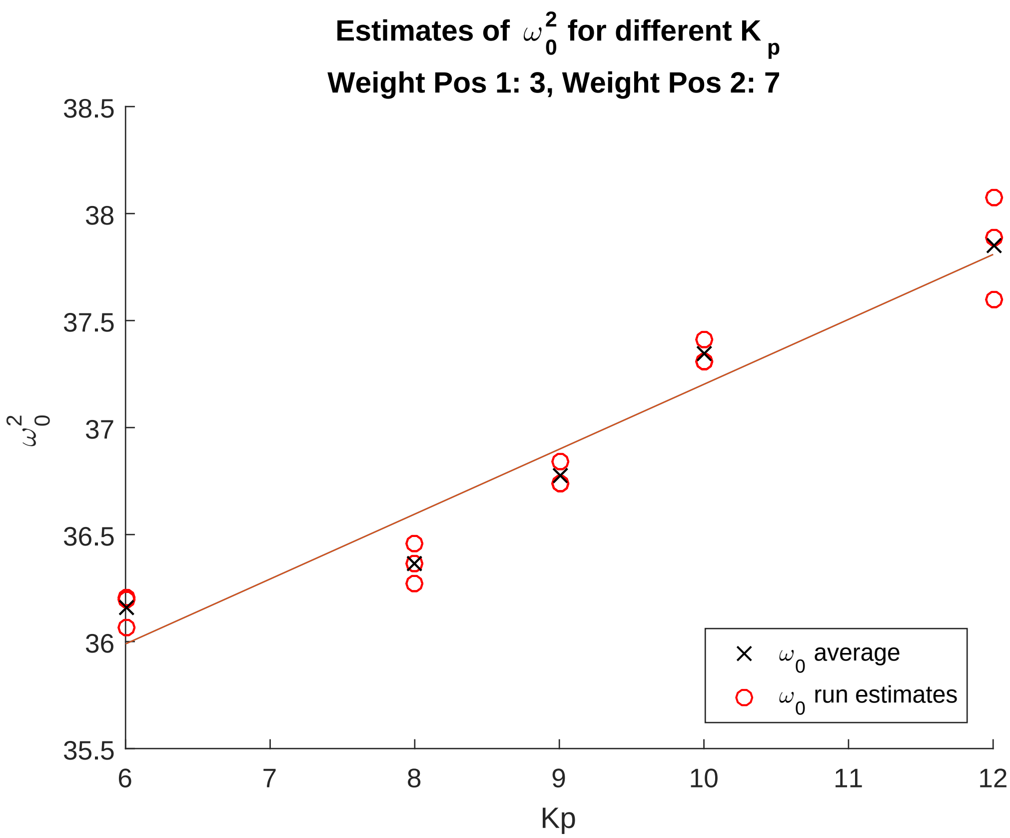

Figure 5: Estimated linear fit for $\frac{K_m}{I_f+I_w}$ parameter for experiments with weights attached at $n_w = (3, 7)$.

4 Conclusion